RANDOM DISTRIBUTIONS WITH AN APPLICATION TO TELEPHONE ENGINEERING

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1. General Poisson distributions

Recently, A. Blanc-Lapierre and I remarked that we were not aware of any definition of a general Poisson distribution, that is to say, of a Poisson distribution not, as usual, on the straight line or on some Euclidean space, but on a perfectly general space. Such a definition may be useful, and can be given in the following obvious way.

Let $\mathcal{X}$ be any space of elements x, $\mathcal{B}$ a Borel field of subsets e of $\mathcal{X}$, and $m(e)$ a measure on $\mathcal{B}$ (not necessarily bounded or finite). A random family F of elements of $\mathcal{X}$ is a Poisson distribution on $\mathcal{X}$ [with respect to $\mathcal{B}$ and $m(e)$] if, $M(e)$ being the number of elements of F belonging to $e \in \mathcal{B}$, we have the following properties [1]:

1) If $m(e) < +\infty$, the random variable $M(e)$ is almost certainly finite and its distribution function is the Poisson law with parameter $m(e)$.

2) If k is any integer and if $e_1, e_2, \cdots, e_k$ are any disjoint sets belonging to $\mathcal{B}$, with $m(e_j) < +\infty, j = 1, 2, \cdots, k$, the k random variables $M(e_j)$ are independent.

The classical properties of Poisson distributions on the straight line remain true. For instance, it is easy to see the following.

1) If $e \in \mathcal{B}$ with $0 < m(e) < +\infty$, then conditionally when $M(e) = k$, with k any integer >0, the distribution on e of the k elements of F belonging to e is statistically equivalent to the choice at random, independently, of k elements x on e, with $Pr\{x \in e'\} = m(e')/m(e)$, where e' is any subset of e belonging to $\mathcal{B}$.

2) Let e_t be a family of sets belonging to $\mathcal{B}$, $0 \leqq t < +\infty$, such that (a) $e_t \subset e_\tau$ if $t < \tau$; (b) e_0 reduces to an element $x_0 \in \mathcal{X}$; (c) $m(e_t)$, as a function of t, is continuous even for $t \to +0$ with $m(e_0) = 0$. Let T be the random variable defined by the following. If $t < T$, no element of F belongs to e_t; if $t > T$, at least one element of F belongs to e_t. The distribution function of T is

$$1 - e^{-m(e_t)} . \tag{1.1}$$

3) Let $\mathfrak{Y}$ be a second space, $\mathcal{C}$ a Borel field of subsets $\omega \subset \mathfrak{Y}$, let $p(x; \omega)$ be a probability measure on $\mathcal{C}$ corresponding to every $x \in \mathcal{X}$, and let $Y(x)$ be a random element taking its values on $\mathfrak{Y}$ obeying the law $p(x; \omega)$. I assume that the different $Y(x)$ for different x are mutually independent. Let $e \in \mathcal{B}$ with $m(e) < +\infty$, let $X_1, X_2, \cdots, X_j, \cdots$ be the elements of F belonging to e. Then the $Y(X_j)$ are Poisson distributed on $\mathfrak{Y}$ (with respect to $\mathcal{C}$) and the mathematical expectation of the number of the $Y(X_j)$ belonging to ω is

$$\int_e p(x; \omega)\, m(dx) . \tag{1.2}$$

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This last property is useful in some physical problems (in such problems, usually $\mathcal{X}$ and $\mathcal{Y}$ are the time axis, or some Euclidean spaces).

2. Random distributions (by Gelfand)

We saw in section 1 that it is possible to define, and to handle directly, Poisson distributions. It may be expected that the same is true for all random distributions which are almost surely (a.s.) purely discrete distributions (these distributions, in the case of distributions on the straight line, were called "point processes" by H. Wold). But for some problems it may be easier to use an indirect way. On the other hand, one encounters, in many questions, distributions which are not purely discrete distributions, and in such a case it seems that we have no way except an indirect one. Roughly speaking, we can consider a distribution as a linear functional on some vectorial function space $\mathfrak{D}$; this must be the starting point. As an example of what can be done in this direction, I take a recent paper by Gelfand [2], which is concerned with distributions on the time axis, $-\infty < t < +\infty$.

As a space $\mathfrak{D}$, he takes L. Schwartz's $\mathfrak{D}$ space of the functions $f(t)$ which are indefinitely differentiable and equal to 0 outside a compact set with the usual pseudotopology (see p. 24 and p. 66 in [3]). Any continuous linear functional F on this space $\mathfrak{D}$ is an L. Schwartz distribution. The space of these distributions is the dual space $\mathfrak{D}'$ of $\mathfrak{D}$. Let us consider also two other spaces: the space $\mathfrak{D}^*$ consisting of all the linear functionals on $\mathfrak{D}$, which are continuous or not, and the space $\mathcal{E}$ consisting of all the functionals on $\mathfrak{D}$, which are linear or not, continuous or not. We then have $\mathfrak{D}' \subset \mathfrak{D}^* \subset \mathcal{E}$.

If we think of a random element F taking its values in $\mathcal{E}$, we can consider the family of the random variables $X_f = \langle F, f \rangle$, where $\langle F, f \rangle$ is the number obtained by applying to $f \in \mathfrak{D}$ the functional $F \in \mathcal{E}$; and we can say that F, as a random element, is defined by the family of the k-dimensional distribution functions $\mathcal{Q}^{(k)}[f_1, f_2, \cdots, f_k]$ of the k-dimensional random variable $\{X_{f_1}, X_{f_2}, \cdots, X_{f_k}\}$ for every integer k and for every system $\{f_1, \cdots, f_k\}$ of k elements of $\mathfrak{D}$. This family of distribution functions is something analogous to the "temporal law" of an ordinary random function, and under an obvious necessary and sufficient consistency condition it defines a measure on $\mathcal{E}$.

If, more particularly, F takes a.s. its values in $\mathfrak{D}^*$, it is a.s. that F is a linear functional on $\mathfrak{D}$. That is to say, it is a.s. that for all fixed k, all fixed $f_1, f_2, \cdots, f_k$, and all fixed numbers $a_1, \cdots, a_k$, if $f = \sum_{j=1}^{k} a_j f_j$, then we have

$$X_f = \sum_{j=1}^{k} a_j X_{f_j}. \tag{2.1}$$

A weak consequence of this is that $\mathcal{Q}^{(k)}[f_1, \cdots, f_k]$ have the following consistency property.

Consistency property or property (A). For any fixed k, if $f = \sum_{j=1}^{k} a_j f_j$, $\mathcal{Q}_f^{(1)}$ is related in an obvious way to $\mathcal{Q}^{(k)}[f_1, \cdots, f_k]$.

If, more specially, F takes a.s. its values in $\mathfrak{D}'$, it is a.s. that for all fixed k and all fixed $f_1, f_2, \cdots, f_k$, if

$$\lim_{n \to +\infty} f_j^{(n)} = f_j, \qquad j = 1, 2, \cdots, k, \tag{2.2}$$

then we have

(2.3) $$\lim_{n\to+\infty}\{X_{f_1^{(n)}},\cdots,X_{f_k^{(n)}}\}=\{X_{f_1},\cdots,X_{f_k}\}.$$

A weak consequence of this is that the $\mathcal{Q}^{(k)}[f_1,\cdots,f_k]$ have the following continuity property.

Continuity property or property (B). For any fixed k and fixed $f_1,\cdots,f_k$, $\lim_{n\to+\infty}\mathcal{Q}^{(k)}[f_1^{(n)},\cdots,f_k^{(n)}]=\mathcal{Q}^{(k)}[f_1,\cdots,f_k]$ in the sense of the usual convergence of distribution functions.

Under the name of "general random process" (G.R.P.), Gelfand considers random elements F in $\mathcal{E}$ having properties (A) and (B). These properties together do not necessarily imply that F takes its values in $\mathcal{D}'$, or even in $\mathcal{D}^*$, but it is worth noticing that they imply that for any fixed k and fixed $f_1,\cdots,f_k$, if

(2.4) $$\lim_{n\to+\infty} f_j^{(n)}=f_j\,,\qquad j=1,2,\cdots,n\,,$$

then $\{X_{f_1^{(n)}},\cdots,X_{f_k^{(n)}}\}$ tends *in probability* to $\{X_{f_1},\cdots,X_{f_k}\}$.

Gelfand defines as the characteristic functional, $\varphi(f)$, of F the functional (defined on $\mathcal{D}$)

(2.5) $$\varphi(f)=E(e^{iX_f}).$$

It is not difficult to prove that the necessary and sufficient condition for a functional $\varphi(f)$ on $\mathcal{D}$ to be the characteristic functional of a G.R.P. F is that $\varphi(f)$ be positive definite and continuous.

An interesting feature presented by G.R.P.'s is that they always possess a derivative. By one obvious application of the derivation of L. Schwartz's distributions [3], we can say that the derivative of the G.R.P. F, which has characteristic functional $\varphi(f)$, is the G.R.P. F', which has characteristic functional

(2.6) $$\varphi^{(1)}(f)=\varphi(-f')\,,$$

f' being the derivative of f.

Gelfand applies his notion of general random process to three problems:

1) G.R.P.'s of the second order, that is to say, those for which $E[|X_f|^2]<+\infty$ for every $f\in\mathcal{D}$. A particular case of a G.R.P. of the second order is the case of a Gaussian G.R.P.

2) Stationary G.R.P.'s or more generally G.R.P.'s which are stationary of order n, that is to say, G.R.P.'s the nth derivatives of which are stationary G.R.P.'s.

3) G.R.P.'s with independent significations which constitute a natural extension of ordinary random functions with independent increments.

For these three problems, Gelfand obtains many very interesting results that I shall not reproduce here (proofs are not given in Gelfand's paper). It may be interesting to compare Gelfand's work with papers by E. Mourier and myself (see, for instance, E. Mourier [4], R. Fortet and E. Mourier [5]), which concern random elements in Banach spaces. What appears from these different papers is that a random distribution, as a random element, can be defined by its characteristic functional, which itself is often a suitable tool. Generally speaking, this characteristic functional is useful and easy to handle in linear problems, but only in linear problems.

3. An application to telephone engineering

Unfortunately, we are faced with many applications involving nonlinearity. As an example, I take the following.

Let us consider a telephone exchange $\mathcal{C}$. The subscribers make calls at instants t_j that have to be considered as random instants, the distribution of which on the time axis is a random distribution N. More precisely, we put $N(\tau) - N(t)$ equal to the number of calls occurring during (t, τ), with $t < \tau$. Sometimes we have to consider N as the sum of several random distributions N_i. For instance, N_1 may be the distribution of calls asking for a subscriber of the same exchange, N_2 the distribution of out-of-town calls, and so on. We assume the N_i are mutually independent. For each call belonging to N_i and arising at time t, we have to consider its holding time $T_i(t)$. This is a random positive number with distribution function $F_i(u) = Pr\{T_i(t) < u\}$. We assume that the different $T_i(t'), T_j(t''), \cdots$ for any $i, j, \cdots$ and different $t', t'', \cdots$ are mutually independent, and independent of the N_k and of the state of the exchange. Also, for a call in N_i arising at time t, there may be a nonnegligible interval between the time t and the time τ of the beginning of the conversation. I call $\tau - t = \Theta(t)$ the orientation duration. The $\Theta_i(t)$ are also random variables, mutually independent, independent of the N_i and of the $T_i(t)$. Assuming the possible states e_j of the exchange are denumerable (but this is only to simplify the writing), we call $G_{i,j}(u) = Pr\{\Theta_i(t) < u\}$ the distribution function of $\Theta_i(t)$ if at time t the state of the exchange is e_j.

As the telephone exchange has only a finite number of links, whenever too many people put in calls at about the same time there is some congestion. Either some calls are delayed, and we have a queuing problem, or a call is lost if it arises at a time when the exchange is in some "blocking" state (for instance, if all the links are busy). In what follows, I consider only the second case.

In general, engineers or probabilists interested in this question have treated problems of the following kind. Under some assumptions on the N_i, $T_i(t)$, and $\Theta_i(t)$ (for example, assuming stationarity for the N_i's), they have tried to compute, for instance, the probability P_i (probability of loss) that a call arising at time t will find the exchange in a state such that the call is lost.

Generally speaking, they have tried to compute mathematical expectations concerning the possible states of the exchange. But other probabilities or information may be required, for instance, concerning the random distribution $\mathfrak{A}$ on the time axis of the calls that are not lost, or the distribution $\mathcal{R}$ of the calls that are lost, or the number as a function of t of the conversations held at time t, etc. In order to be studied, such random elements have to be taken into account in the reasoning. This can be done in the following way.

Let $X_i(t)$ be a random function with value 1 if a call belonging to N_i and arising at time t is not lost, and 0 in the contrary case. Let $Y_i(t)$ and $Z_i(t)$ be the number of conversations and the number of orientations in calls belonging to N_i and held at time t. Let $R_i(t; \tau)$ and $S_i(t; \tau)$ be random functions with values

$$(3.1) \qquad R_i(t;\tau) = \begin{cases} 1 & \text{if } \tau - t \in [\Theta_i(t), \Theta_i(t) + T_i(t)], \\ 0 & \text{if } \tau - t \notin [\Theta_i(t), \Theta_i(t) + T_i(t)]; \end{cases}$$

$$(3.2) \qquad S_i(t;\tau) = \begin{cases} 1 & \tau - t \in [0, \Theta_i(t)], \\ 0 & \text{if } \tau - t \notin [0, \Theta_i(t)]. \end{cases}$$

Finally, let $P_i(t)$ be the probability of loss at time t of a call belonging to N_i. Then we have

$$(3.3)\qquad Y_i(t) = \int_{t_0}^{t-0} X_i(u) R_i(u;\, t)\, dN_i(u)\,,$$

$$(3.4)\qquad Z_i(t) = \int_{t_0}^{t-0} X_i(u)\, S_i(u;\, t)\, dN_i(u)$$

(here we assume that the process is beginning at time t_0). Obviously these two equations are not sufficient to describe the whole process. We have three unknown elements, $X_i(t)$, $Y_i(t)$, $Z_i(t)$, and only two equations. To get a third equation we have to specify the kind of exchange we are studying.

However, from equations (3.3) and (3.4) it is already possible to get easily certain results that are difficult to obtain by classical procedures, and even some results that are new. For example, assuming that the N_i are stationary Poisson distributions with parameters μ_i and with $t_0 = -\infty$, the $P_i(t)$ are some constants P_i. We notice that the value of $X_i(t)$ depends only on the state of the exchange at time t, and that we have $E[X_i(t)] = 1 - P_i$. Let p_j be the probability that at time u the exchange is in state e_j and let $X_{i,\,j}$ be the value of $X_i(u)$ if at time u the exchange is in state e_j. Then

$$(3.5)\qquad \lambda_i = \int_0^{+\infty} t dF_i(t) = \int_0^{+\infty} [1 - F_i(t)]\, dt\,,$$

$$(3.6)\qquad \gamma_{i,\,j} = \int_0^{+\infty} t dG_{ij}(t) = \int_0^{+\infty} [1 - G_{ij}(t)]\, dt\,.$$

Of course, we assume that the integrals (3.5) and (3.6) exist ($< +\infty$).

If we compute $E[R_i(u;\, t)dN_i(u)]$ conditionally when the exchange is in state e_j at time u, we get

$$(3.7)\qquad E\,[R_i(u;\, t)\, dN_i(u)] = \mu_i E\,[R_i(u;\, t)]\, du\,,$$

with

$$(3.8)\qquad E\,[R_i(u;\, t)] = \int_0^{t-u} [1 - F_i(t - u - a)]\, dG_{ij}(a)\,.$$

Here we have *a priori*

$$\begin{aligned}(3.9)\quad E\,[Y_i(t)] &= \mu_i \int_{-\infty}^{t} \Big[\sum_j p_j X_{i,\,j} \int_0^{t-u} [1 - F_i(t - u - a)]\, dG_{ij}(a) \Big] du \\ &= \mu_i \sum_j p_j X_{i\ \,j} \int_0^{+\infty} \Big[\int_{-\infty}^{t-a} [1 - F_i(t - a - u)]\, du \Big] dG_{ij}(a) \\ &= \mu_i \gamma_i \Big[\sum_j p_j X_{ij} \Big] \\ &= \mu_i \gamma_i (1 - P_i)\,.\end{aligned}$$

By a similar computation, we get

$$(3.10)\qquad E\,[Z_i(t)] = \mu_i \sum_j p_j X_{ij} \gamma_{ij}\,,$$

and if we assume, as usual, that $\gamma_{ij} = \gamma_i$ does not depend on j, we have

$$(3.11)\qquad E\,[Z_i(t)] = \mu_i \gamma_i (1 - P_i)\,.$$

The useful formulas (3.9) and (3.11) have been known for particular cases only.

4. On a stochastic integral equation

In many cases it appears that the assumption that the N_i's are Poisson distributions does not agree with observation. We have to investigate whether equations like (3.3) and (3.4) can give results without precise assumptions about the N_i. To do this we simplify the model, reducing it to the classical Erlang model; that is to say, (a) we neglect the $\Theta_i(t)$ and (b) we assume $F_i(u)$ and $X_{i,j}$ are independent of i. It is no longer necessary to distinguish between N_1 and N_2, and so on. Let $Y(t)$ be the total number of conversations held at time t, and let $X(t)$ be the random function with value 1 if a call belonging to N and arising at time t is not lost, and 0 in the contrary case.

We have

$$Y(t) = \int_{t_0}^{t-0} X(u) R(u; t)\, dN(u). \tag{4.1}$$

On the other hand, in the Erlang model, there is a positive integer n (number of links) such that

$$X(t) = \begin{cases} 1 & \text{if } Y(t) < n, \\ 0 & \text{if } Y(t) = n, \end{cases} \tag{4.2}$$

and such that $Y(t)$ cannot be larger than n. If $V(x)$ is any function such that $V(k) = 1$ when the integer $k = 0, 1, 2, \cdots, n-1$, and $V(x) = 0$ for $x \geqq n$, then we can write

$$X(t) = V[Y(t)]. \tag{4.3}$$

Assuming t_0 finite (for instance, $t_0 = 0$) with initial condition $X(0) = 1$, or $Y(0) = 0$, equations (4.1) and (4.3) can determine $X(t)$ and $Y(t)$. I do not say that $X(t)$ and $Y(t)$ are the only elements we need, for practical purposes, but I restrict myself to this problem, as an example.

From a theoretical point of view, the first question that arises concerns the existence and the unicity of the solution of the system (4.1) to (4.3), but this question is immediately answered. The mathematical procedure to prove existence and unicity of the solution is the same as the mechanical procedure of telephone exchanges, and we get the following.

Under the assumption that a.s. $N(t) - N(0)$ *is finite for every finite* t, *the system* (4.1) *to* (4.3) *has a.s. a unique solution.*

This is one way to build the solution, another way is the following. Let $Y(t)$ be the unique solution of the system that can be reduced to the equation

$$Y(t) = \int_0^{t-0} V[Y(u)] R(u; t)\, dN(u) \tag{4.4}$$

with

$$Y(0) = 0. \tag{4.5}$$

We can choose

$$V(x) = \begin{cases} 1 & \text{if } x \leqq n-1 \\ n - x & \text{if } n-1 \leqq x, \end{cases} \tag{4.6}$$

which satisfies the following Lipschitz condition

$$|V(x') - V(x'')| \leqq |x' - x''|. \tag{4.7}$$

We put

$$Y_0(t) = \int_0^{t-0} R(u;\, t)\, dN(u)\,. \tag{4.8}$$

We are sure that

$$Y_0(t) \geqq Y(t) \qquad \text{for every } t\,, \tag{4.9}$$

and we define

$$Y_{k+1}(t) = \int_0^{t-0} V\,[Y_k(u)]\,R(u;\, t)\, dN(u)\,, \tag{4.10}$$

$$D_k(t) = |\, Y_k(t) - Y_{k-1}(t)\,| \qquad \text{with} \qquad D_0(t) \leqq 1 \tag{4.11}$$

Hence we have with (4.7)

$$D_{k+1}(t) \leqq \int_0^{t-0} D_k(u) R(u;\, t)\, dN(u)\,. \tag{4.12}$$

Since $R(u, t) \leqq 1$, we find, on iterating (4.10),

$$D_k(t) \leqq \frac{N(t)^k}{k!}\,, \qquad k = 1, 2, \cdots, \tag{4.13}$$

and we have *a.s.* $Y_k(t)$ *tends to* $Y(t)$ *uniformly in t on every finite interval.*

We can notice that, since $V(x)$ is a nonincreasing function, we have $Y_k(t) \leqq Y(t)$ if k is odd, and $Y_k(t) \geqq Y(t)$ if k is even,

$$Y_{2k}(t) \leqq Y_{2(k-1)}(t)\,, \qquad Y_{2k+1}(t) \geqq Y_{2k-1}(t)\,. \tag{4.14}$$

Now if we want to consider the case $t_0 = -\infty$ we can start with t_0 finite. Making t_0 tend to $-\infty$ we have $Y(t)$ tending to a limit in some stochastic sense when we make a convenient assumption on $N(t)$, and the limit process is the suitable solution of (4.4).

5. The stationary case with $n = 1$

As an example, we can treat in this way the case $n = 1$, assuming that $N(t)$ is stationary (but not necessarily a Poisson distribution) and that $t_0 = -\infty$. This particular case ($n = 1$) is of interest in the theory of Geiger counters. It is also of interest in telephone engineering because sometimes it is possible to solve the general case when the case $n = 1$ has been solved (see C. Palm [6]).

If $n = 1$, we can choose $V(x) = 1 - x$, and the limit $Y(t)$ of the $Y_k(t)$ is given by

$$\begin{aligned} Y(t) = {} & \int_{-\infty}^{t-0} R(u;\, t)\, dN(u) - \int_{-\infty}^{t-0}\int_{-\infty}^{u_2-0} R(u_1;\, u_2) R(u_2;\, t)\, dN(u_1)\, dN(u_2) \\ & + \cdots + (-1)^{k+1} \int_{-\infty}^{t-0}\int_{-\infty}^{u_k-0}\cdots\int_{-\infty}^{u_2-0} R(u_1;\, u_2) R(u_2;\, u_3) \cdots R(u_k;\, t)\, dN(u_1) \\ & \cdots dN(u_k) + \cdots\,. \end{aligned} \tag{5.1}$$

However, if we try to deduce some useful information from this formula (5.1), we have to specify the probability law of $N(t)$. Let $\rho_k(t_1;\, t_2; \cdots;\, t_k;\, \tau)$ be defined by the relation that $\rho_k(t_1;\, t_2; \cdots;\, t_k;\, \tau)\, d\tau$ is equivalent, as $d\tau \to +0$, to the probability that a call will arise at time τ, knowing that calls arose at times $t_1, t_2, \cdots, t_k$, with $t_1 < t_2 < \cdots < t_k < \tau$.

The probability law of $N(t)$ is defined if the ρ_k's are given. Since $N(t)$ is stationary, ρ_k depends only on the differences, $t_2 - t_1, \cdots, \tau - t_k$, and we write it in the following way,

$$\rho_k(t_2 - t_1; t_3 - t_2; \cdots; t_k - t_{k-1}; \tau - t_k); \tag{5.2}$$

ρ_0 is a constant (the average number of calls per unit time).

From this we obtain, for instance,

$$E[Y(t)] = \lambda\rho_0 \sum_k (-1)^k \int_0^{+\infty} \cdots \int_0^{+\infty} G(a_1)G(a_2)\cdots G(a_k)\, \rho_1(a_1)\, \rho_2(a_1; a_2) \cdots \rho_k(a_1; a_2; \cdots; a_k)\, da_1 da_2 \cdots da_k, \tag{5.3}$$

where

$$G(a) = E[R(t-a; t)], \qquad \lambda = \int_0^{+\infty} G(a)\, da. \tag{5.4}$$

Noticing that $Y(t)dN(t)$ is the distribution of lost calls, and computing in the same way $E[Y(t)dN(t)]$, we obtain

$$E[Y(t)\, dN(t)] = \left[\rho_0 - \frac{E|Y(t)|}{\lambda}\right] dt. \tag{5.5}$$

In the particular but important case where $N(t)$ is such that the intervals between two consecutive calls are independent random variables with the same distribution function (see C. Palm [6]), we have

$$\rho_k(a_1; a_2; \cdots; a_k) = \rho_1(a_k), \tag{5.6}$$

and consequently

$$E[Y(t)] = \frac{\lambda\rho_0}{1 + \int_0^{+\infty} G(a)\, \rho_1(a)\, da}. \tag{5.7}$$

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