FOUNDATIONS OF THE THEORY OF CONTINUOUS PARAMETER MARKOV CHAINS

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1. Foreword

This paper is exclusively concerned with continuous parameter Markov processes with a denumerably infinite number of states and stationary transition matrix function. The foundations of the proper theory of such processes, as distinguished from that of the discrete parameter version, or of Markov processes which are either more special (for example, a finite number of states) or more general (for example, general state space; nonstationary transition matrix function), were laid by Doob [1], [2], and Lévy [4], [5], [6].[1] Roughly speaking it was Lévy in 1952 who drew, in his inimitable way, the comprehensive picture while Doob, ten years earlier, had supplied the essential ingredients. The present effort aims at a synthesis of the most fundamental parts of the theory, made possible by the contributions of these two authors. While the results given here generally extend and clarify those in the cited literature, immense credit must go to Professors Lévy and Doob for the inspiration of their pioneer work. To them I am also indebted for much valuable discussion through correspondence and conversation. An attempt is made in the presentation to be quite formal and rigorous, in the spirit of Doob's already classic treatise [3]. Further developments of the theory will be published elsewhere.

2. Introduction

We consider a probability space Ω with the generic point ω, a Borel field $\mathcal{B}$ of ω-set including Ω itself, and a (complete) probability measure P defined on $\mathcal{B}$. For general definitions and notations we refer to [3], unless otherwise specified. The notation $x(t, \omega)$, for example, will be used both for the function $x(t, \cdot)$ and its value at ω.

A Markov chain $\{x(t, \omega), 0 \leqq t < \infty\}$ is given as follows. The state space is the set of nonnegative integers. The initial distribution is given by

$$P\{x(0, \omega) = i\} = p_i, \qquad i = 0, 1, 2, \cdots \tag{2.1}$$

where $p_i \geqq 0, \sum p_i = 1$. The stationary transition probability functions are

$$p_{ij}(t) = P\{x(s+t, \omega) = j \mid x(s, \omega) = i\}, \qquad s \geqq 0, t > 0, \tag{2.2}$$

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[1] It is a pleasure to note that Professor Lévy in his paper [4] attributed its origin to a conversation held in the course of the Second Berkeley Symposium.

and satisfy the following conditions: for $i, j = 0, 1, 2, \cdots, t > 0$ and $s > 0$

$$(2.3) \qquad p_{ij}(t) \geqq 0, \qquad \sum_j p_{ij}(t) = 1, \qquad p_{ij}(s+t) = \sum_k p_{ik}(s)\, p_{kj}(t).$$

We assume that for every i, j

$$(2.4) \qquad \lim_{t \to 0} p_{ij}(t) = \delta_{ij}.$$

According to theorem II.2.6 of [3], *there is a standard modification of the process which is separable relative to the closed sets, and measurable.* Moreover, *the denumerable set R satisfying the conditions of the separability definition may be taken to be any set which is everywhere dense* (see pp. 59–60 in [3]). *We shall always take R to be the set of rational numbers of the form $r2^{-n}$.* Another consequence of (2.4) is that *the $p_{ij}(\cdot)$ are all continuous functions,* since

$$(2.5) \qquad |p_{ij}(t+h) - p_{ij}(t)| \leqq 1 - p_{ii}(h)$$

for every positive t and h.

The separability of $\{x(t, \omega)\}$, however, may require the adjunction of the value ∞ to the range of $x(\cdot, \omega)$, so as to make it compact. If we treat this value ∞ as a new state we have then for every finite i and every $t > 0$

$$(2.6) \qquad p_{i\infty}(t) = P\{x(t, \omega) = \infty \mid x(0, \omega) = i\} = 0,$$

since $\sum_{0 \leqq j < \infty} p_{ij}(t) = 1$. Furthermore since $\sum_{0 \leqq i < \infty} p_i = 1$ we have also $P\{x(t, \omega) = \infty\} = 0$ for every $t \geqq 0$. Thus any conditional probability under the hypothesis $x(t, \omega) = \infty$, in particular $p_{\infty j}(t)$, is undefined. We shall call this state the *adjoined* state. *In the following an unspecified state shall mean one which is not adjoined.*

We remark that if we label the states in a different way we may need other adjoined states. For example if the states are all the integers then we may need the two adjoined states $+\infty$ and $-\infty$; if the states form mutually noncommunicating classes it may be more advisable to label them in such a way so as to allow for distinct adjoined states in distinct classes. In one of Lévy's examples (see p. 366 in [4]) the states are all the rational numbers, and the adjoined states all the irrational numbers.

3. A fundamental theorem

It is known (see theorem 9 in [1]) that the limit

$$(3.1) \qquad \lim_{t \to 0} \frac{1 - p_{ii}(t)}{t} = q_i$$

exists for every i, but it may be infinite. The state i is called *stable* or *instantaneous* according as $q_i < \infty$ or $q_i = \infty$. The fundamental property of the two kinds of states is the following. If $P\{x(\tau, \omega) = i\} > 0$, then

$$(3.2) \qquad P\{x(s, \omega) \equiv i \text{ for } \tau \leqq s \leqq \tau + t \mid x(\tau, \omega) = i\} = e^{-q_i t}, \qquad t \geqq 0,$$

where the right member is interpreted as 0 if $q_i = \infty$.

We introduce the general notation: for any state i, inclusive of ∞,

$$(3.3) \qquad S_i(\omega) = \{t : x(t, \omega) = i\};$$

the closure of $S_i(\omega)$ will be denoted by $\overline{S_i(\omega)}$.

We first establish a fundamental theorem concerning stable states. An open interval of

t in which $x(\cdot, \omega) \equiv i$, and which is not properly contained in another such interval, is called an *i-interval of the sample function* $x(\cdot, \omega)$. Its closure is called the corresponding *closed i-interval.*

THEOREM 1. *There is a set* $\Omega_0 \in \mathcal{B}$ *with* $P(\Omega_0) = 1$ *having the following property. If* $\omega \in \Omega_0$ *the set* $\overline{S_i(\omega)}$, *for every stable i, is the union of disjoint closed i-intervals whose number is finite in every finite t-interval.*

PROOF. We prove first that there is a set $\Omega_1 \in \mathcal{B}$ with $P(\Omega_1) = 1$ such that if $\omega \in \Omega_1$, then for every finite $T > 0$, and every stable i, the sample function $x(\cdot, \omega)$ has only a finite number of i-intervals in $(0, T)$.

Consider for each n the numbers

$$\frac{T}{n}, \frac{2T}{n}, \dots, \frac{(n-1)\,T}{n}. \tag{3.4}$$

Define a sequence of random variables as follows:

$\tau_1^{(n)}(\omega) =$ the smallest number in (3.4) which lies in an i-interval of $x(\cdot, \omega)$, if such a number exists; otherwise $\tau_1^{(n)}(\omega) = T$.

(3.5) $r_{s+1}^{(n)}(\omega) =$ the smallest number in (3.4) which exceeds $\tau_s^{(n)}(\omega)$ and which lies in an i-interval of $x(\cdot, \omega)$ distinct from the one containing $\tau_s^{(n)}(\omega)$, if such a number exists; otherwise $\tau_{s+1}^{(n)}(\omega) = T$.

Let N be a positive integer. Consider the ω-set $\Lambda_N^{(n)}$ for which $\tau_1^{(n)}(\omega) < \cdots < \tau_N^{(n)}(\omega) < T$ ($\Lambda_N^{(n)}$ is empty if $n \leqq N$). Let Λ_N be the ω-set for which $x(\cdot, \omega)$ has at least N i-intervals *interior* to $(0, T)$. For each $\omega \in \Lambda_N$ there exists an $n = n(\omega)$ such that $\omega \in \Lambda_N^{(n!)}$. Since $\Lambda_N^{(n!)} \subsetneq \Lambda_N^{([n+1]!)}$ we have then $\Lambda_N \subsetneq \bigcup_{n=1}^{\infty} \Lambda_N^{(n!)}$.

For $1 \leqq k \leqq n-1$ let

$$Q_k = P\left\{\frac{kT}{n} \text{ is the smallest number in (3.4) lying in an } i\text{-interval such that } x(t, \omega) \not\equiv i,\ 0 < t < \frac{kT}{n} \,\middle|\, x(0, \omega) = i\right\}. \tag{3.6}$$

In words, Q_k is the probability that, starting from i at $t = 0$, kT/n is the first one among the sequence of "instants" (3.4) such that $x(t, \omega) = i$ again *after having left i some time before.* By stationarity we have, if $1 \leqq j < k \leqq n-1$,

$$Q_{k-j} = P\left\{\tau_{s+1}^{(n)}(\omega) = \frac{kT}{n} \,\middle|\, \tau_s^{(n)}(\omega) = \frac{jT}{n}\right\}. \tag{3.7}$$

The random variables $\tau_s^{(n)}(\omega)$, $1 \leqq s \leqq N$, form a discrete Markov chain whose states are the numbers in (3.4) and the number T. If $Q(\cdot, \cdot)$ denotes the transition probability function, then

$$\begin{aligned}
&Q\left(\frac{jT}{n}, \frac{kT}{n}\right) = 0, && 1 \leqq k \leqq j,\\
&Q\left(\frac{jT}{n}, \frac{kT}{n}\right) = Q_{k-j}, && 1 \leqq j < k \leqq n-1,\\
&Q\left(\frac{jT}{n}, T\right) = 1 - \sum_{k=j+1}^{n-1} Q_{k-j}, && 1 \leqq j \leqq n-1,\\
&Q(T, T) = 1.
\end{aligned} \tag{3.8}$$

Now we have

$$(3.9)\quad P(\Lambda_N^{(n)}) = \sum_{1\leqq k_1<k_2<\cdots<k_N\leqq n-1} P\left\{\tau_1^{(n)}(\omega) = \frac{k_1T}{n};\ \tau_2^{(n)}(\omega) = \frac{k_2T}{n};\ \cdots;\ \tau_N^{(n)}(\omega) = \frac{k_NT}{n}\right\}$$

$$= \sum_{1\leqq k_1<\cdots<k_N\leqq n-1} P\left\{\tau_1^{(n)}(\omega) = \frac{k_1T}{n}\right\} Q_{k_2-k_1}Q_{k_3-k_2}\cdots Q_{k_N-k_{N-1}}$$

$$\leqq \sum_{k=1}^{n-1} P\left\{\tau_1^{(n)}(\omega) = \frac{kT}{n}\right\}\left(\sum_{k=1}^{n-1} Q_k\right)^{N-1} \leqq \left(\sum_{k=1}^{n-1} Q_k\right)^{N-1}.$$

Since

$$(3.10)\qquad \sum_{k=1}^{n-1} Q_k \leqq P\{x(t,\omega)\not\equiv i,\quad 0<t<T \mid x(0,\omega)=i\}$$

we have by (3.2)

$$(3.11)\qquad P(\Lambda_N^{(n)}) \leqq (1-e^{-q_iT})^{N-1}$$

regardless of n. Hence letting $n\to\infty$ we have

$$(3.12)\qquad P(\Lambda_N) \leqq (1-e^{-q_iT})^{N-1}.$$

Let $\Lambda_\infty(T) = \bigcap_{N=1}^{\infty}\Lambda_N$, then $\Lambda_\infty(T)$ is the set of ω whose sample function has infinitely many i-intervals in $(0, T)$. It follows from the above that $P[\Lambda_\infty(T)] = 0$ for every $T>0$. Let $\Lambda = \bigcup_{n=1}^{\infty}\Lambda_\infty(n)$, then also $P(\Lambda) = 0$. The set $\Omega_1 = \Omega - \Lambda$ fulfills our requirement.

Now it follows from (3.2) that if i is stable, then every rational point τ in $S_i(\omega)$ must be contained in an i-interval, with probability one [see theorem 2 (i) below]. Hence we may choose a subset Ω_0 of Ω_1 with $P(\Omega_0) = 1$ such that if $\omega\in\Omega_0$, then every rational point in every $S_i(\omega)$ with i stable is contained in an i-interval. Let $\omega\in\Omega_0$ and let τ be an irrational point of $S_i(\omega)$. Because of separability there are rational points r_n of $S_i(\omega)$ in any neighborhood of τ. By the choice of Ω_0 there must then be i-intervals intersecting any neighborhood of τ. Unless τ itself is contained in a closed i-interval there must be an infinite number of distinct i-intervals in its neighborhood. This is impossible by what we have proved. Hence each point of $S_i(\omega)$ is contained in a closed i-interval. Since there is only a finite number of i-intervals in every finite t-interval the closure $\overline{S_i(\omega)}$ is the union of the closed i-intervals. They are disjoint since the possibility of two abutting i-intervals is precluded by separability. Theorem 1 is proved.

For a discussion of theorem 1, see the remarks after theorem 5.

For each $\omega\in\Omega_0$ given in theorem 1, and for each stable state i, we can now define the first, second, $\cdots$, nth, $\cdots$ i-intervals of each sample function $x(\cdot,\omega)$. However, for given ω and n there may not exist an nth i-interval of $x(\cdot,\omega)$. We denote by $\Delta_n(i)$ the set of ω for which the nth i-interval exists. For each $\omega\in\Delta_n(i)$, there is then a finite nonnegative number $\tau_n(\omega, i)$ which is the beginning of the nth i-interval, a positive "number" $\tau_n'(\omega, i)$, possibly ∞, which is the end of the same interval, and a third positive number $\lambda_n(\omega, i) = \tau_n'(\omega, i) - \tau_n(\omega, i)$, possibly ∞, which is the length of the interval. Furthermore, for each $\omega\in\Delta_{n+1}(i)$, $n\geqq 1$, there is a finite positive number $\rho_n(\omega, i) =$

$\tau_{n+1}(\omega, i) - \tau_n(\omega, i)$. A single-valued function $\xi(\omega)$ will be called a random variable if its range is $[-\infty, +\infty]$ and its domain of definition is a set $\Delta \in \mathcal{B}$ such that for every real a, the set of $\omega \in \Delta$ for which $\xi(\omega) \leqq a$ belongs to $\mathcal{B}$. It can then be shown, but we omit the proof, that $\tau_n(\omega, i)$, $\tau_n'(\omega, i)$, $\lambda_n(\omega, i)$ and $\rho_n(\omega, i)$ as functions of ω are random variables in the above sense. We shall call them *the nth entrance, exit, sojourn* and *return times of the state i*, respectively.

4. Further properties

The first three properties, enumerated below, apply to a stable state as well as an instantaneous state. In the former case however more definitive results have been established in theorem 1.

(i) If $x(t, \omega) = i$, then t is a point of density of $S_i(\omega)$ for almost all ω. More precisely:

$$(4.1)\qquad \begin{aligned} &P\{\lim_{\epsilon\downarrow 0} \epsilon^{-1} m\,[S_i(\omega) \cap (t, t+\epsilon)\,] = 1 \mid x(t, \omega) = i\} = 1\,. \\ &P\{\lim_{\epsilon\downarrow 0} \epsilon^{-1} m\,[S_i(\omega) \cap (t-\epsilon, t)\,] = 1 \mid x(t, \omega) = i\} = 1\,. \end{aligned}$$

PROOF. By Fubini's theorem,

$$(4.2)\qquad E\{\epsilon^{-1} m\,[S_i(\omega) \cap (t, t+\epsilon)\,] \mid x(t, \omega) = i\} = \frac{1}{\epsilon}\int_0^\epsilon p_{ii}(t)\,dt.$$

Similarly, if $P\{x(0, \omega) = h\} = 1$, then

$$(4.3)\qquad E\{\epsilon^{-1} m\,[S_i(\omega) \cap (t-\epsilon, t)\,] \mid x(t, \omega) = i\} = \frac{1}{\epsilon}\,\frac{\int_0^\epsilon p_{hi}(t-u)\,p_{ii}(u)\,du}{p_{hi}(t)}.$$

Thus both conditional expectations tend to 1 as $\epsilon \downarrow 0$; the second assertion being true for every h remains true under any initial distribution.

Let $\{\delta_n\}$ and $\{\epsilon_n\}$ be two sequences of positive numbers decreasing steadily to 0. It follows from (4.2) that

$$(4.4)\qquad \begin{aligned} p_n &= P\{\epsilon_n^{-1} m\,[S_i(\omega) \cap (t, t+\epsilon_n)\,] < 1 - \delta_n \mid x(t, \omega) = i\} \\ &\leqq \frac{1}{\delta_n}\Big\{1 - \frac{1}{\epsilon_n}\int_0^{\epsilon_n} p_{ii}(t)\,dt\Big\}. \end{aligned}$$

We have

$$(4.5)\qquad 1 - \frac{1}{\epsilon_n}\int_0^{\epsilon_n} p_{ii}(t)\,dt = \int_0^1 [1 - p_{ii}(\epsilon_n s)\,]\,ds\,.$$

Since

$$(4.6)\qquad 1 - p_{ii}(\epsilon_n s) \leqq 1 - p_{ii}(s)^{\epsilon_n} \leqq \epsilon_n |\log p_{ii}(s)\,|$$

we have

$$(4.7)\qquad p_n \leqq \delta_n^{-1}\epsilon_n \int_0^1 |\log p_{ii}(s)\,|\,ds\,.$$

Thus if we choose δ_n and ϵ_n so that $\sum \delta_n^{-1}\epsilon_n < \infty$ we have by the Borel-Cantelli lemma

$$(4.8)\qquad P\{\epsilon_n^{-1} m\,[S_i(\omega) \cap (t, t+\epsilon_n)\,] < 1 - \delta_n \text{ for infinitely many } n \mid x(t, \omega) = i\} = 0\,.$$

If in addition we choose ϵ_n so that $\epsilon_{n+1}/\epsilon_n \to 1$ we obtain the first relation in (4.1). The proof of the second relation is analogous.

A t-set S will be called *metrically dense in itself* if every open interval which contains a point of S contains a subset S of positive measure.

(ii) For almost all ω, the set $S_i(\omega)$ is metrically dense in itself.

PROOF. Since the process is separable relative to the closed sets, there exists $\Omega_2 \in \mathcal{B}$ with $P(\Omega_2) = 1$ such that if $\omega \in \Omega_2$, then $x(\cdot, \omega)$ has the following property: any open interval which contains a point of $S_i(\omega)$ contains also a point of $R \cap S_i(\omega)$, where R is defined in section 2. For let A be the set of nonnegative integers, except i, closed at ∞. Let $\{I_n\}$ be the denumerable set of all open intervals with rational endpoints. Then Ω_2 may be so chosen that

$$(4.9)\quad \Omega_2 \cap \{\omega : x(r, \omega) \in A,\ r \in R \cap I_n\} - \Omega_2 \cap \{\omega : x(t, \omega) \in A,\ t \in I_n\} = 0$$

for every n. Hence if $\omega \in \Omega_2$ and $I_n \cap S_i(\omega) \neq 0$, then $I_n \cap R \cap S_i(\omega) \neq 0$. Since any open interval contains an interval I_n, this proves the assertion regarding Ω_2. Applying property (i) to all $t = r \in R$, we see that for almost all ω such that $x(r, \omega) = i$, any open interval containing r contains a set of $S_i(\omega)$ of positive measure (in fact, on both sides of r). Combining this with the property of Ω_2 stated above, we obtain (ii).

In the sequel we shall make use of an important result due to Doob (see theorem 12 in [1]), which we shall refer to as theorem D.

THEOREM D. *There is a set $\Omega_3 \in \mathcal{B}$ with $P(\Omega_3) = 1$ such that if $\omega \in \Omega_3$, then $x(\cdot, \omega)$ has the following property: as $t \downarrow \tau$ or $t \uparrow \tau$, $x(t, \omega)$ has at most one finite limiting value.*

(iii) For every $j \neq i$ and almost all ω, $\overline{S_i(\omega)} \cap \overline{S_j(\omega)}$ is a finite set in every finite t-interval. Consequently, $\overline{S_i(\omega)} - S_i(\omega)$ is at most denumerable.

PROOF. By theorem D, for almost all ω, a point of $B(\omega) = \overline{S_i(\omega)} \cap \overline{S_j(\omega)}$ must be a limit point of $S_i(\omega)$ and of $S_j(\omega)$ from different sides. If there is a point ω_o for which $B(\omega_o)$ is infinite in a finite t-interval, then there is a limit point τ and an infinite sequence of points of $B(\omega_o)$, all on the same side of τ and converging to it. Thus as $t \to \tau$ from this side, $x(t, \omega_o)$ has the two limiting values i and j. This can happen only for a set of ω_o with probability zero, by theorem D. Thus the first part of (iii) is proved; the second is an immediate consequence since $\overline{S_i(\omega)} - S_i(\omega) \subseteq \bigcup_{j\neq i} [\overline{S_i(\omega)} \cap \overline{S_j(\omega)}]$.

(iv) If i is instantaneous, then for almost all ω the set $S_i(\omega)$ is nowhere dense.[2]

PROOF. It follows from (3.1) with $q_i = \infty$ that for almost all ω the set $S_i(\omega)$ does not contain any open interval with rational endpoints. Hence for almost all ω, $S_i(\omega)$ does not contain any open interval. If $\overline{S_i(\omega)}$ should contain an open interval then this interval would intersect some $S_j(\omega)$ with $j \neq i$ and thus by property (ii) contain a subset of $S_j(\omega)$ of positive measure. But by property (iii) the set $\overline{S_i(\omega)} - S_i(\omega)$ is at most denumerable. Hence $\overline{S_i(\omega)}$ does not contain any open interval. This is equivalent to the assertion that $S_i(\omega)$ is nowhere dense.

Properties (ii) and (iv) are given by Lévy (see p. 373 in [4]), and property (iii) in [6]. Our proofs of (iii) and (iv) are quite different from Lévy's and are gathered from conversations with Doob, who discovered (iii) independently.

5. Nature of sample functions

The following theorem is due to Doob (see p. 457 in [2] and compare theorem 11 in [1] for a less specific statement) and is repeated here for the sake of completeness.

THEOREM 2. *Let $\tau \geq 0$ be fixed. Then the following is true for almost all ω:*

(i) *If $x(\tau, \omega) = i$ where i is stable, then $x(t, \omega) \to i$ as $t \to \tau$;*

[2] *Added in proof.* According to McKean and Feller (communication by letter) the union $\bigcup_i S_i(\omega)$ for all instantaneous i may be everywhere dense with probability one. Thus all states may be instantaneous.

(ii) *If* $x(\tau, \omega) = i$ *where* i *is instantaneous, then* $x(t, \omega)$ *has exactly two limiting values* i *and* ∞ *as* $t \to \tau$;

(iii) $x(\tau, \omega) \not\equiv \infty$.

PROOF. Part (i) follows from (3.1). More precisely, if $P\{x(0, \omega) = h\} = 1$,

$$(5.1)\quad \lim_{\epsilon \to 0} P\{x(t, \omega) \equiv i,\ \tau - \epsilon \leqq t \leqq \tau + \epsilon \mid x(0, \omega) = i\} = \lim_{\epsilon \to 0} \frac{p_{hi}(\tau - \epsilon)\, e^{-2q_i \epsilon}}{p_{hi}(\tau)} = 1.$$

Part (ii) follows from properties (i) and (iv) in section 4 and theorem D. Part (iii) is mentioned in section 2.

We can now give a general description of the sample functions of the process. For the sake of convenience we shall say that $x(t, \omega)$ *has a discontinuity* at $t = \tau$ *if* $\lim_{t \to \tau} x(t, \omega) = \infty$ *whether* $x(\tau, \omega) = \infty$ *or not.*

THEOREM 3. *There is a set* $\Omega_4 \in \mathcal{B}$ *with* $P(\Omega_4) = 1$ *such that if* $\omega \in \Omega_4$, *then the sample function* $x(\cdot, \omega)$ *defined on* $(0, \infty)$ *has the following properties. The set of its discontinuities* $D(\omega)$ *is a closed set which consists of the union of a possibly empty and at most denumerable set of perfect, nowhere dense sets* $\overline{S_i(\omega)}$ *with* i *instantaneous; and a set of measure zero which is contained in the closure of the set of jumps. Each of the open intervals whose union is the (possibly empty) complement of* $D(\omega)$ *belongs to a certain* $S_i(\omega)$ *with* i *stable, and two adjacent ones belong to different* $S_i(\omega)$. *For each fixed stable* i, *the number of intervals belonging to* $S_i(\omega)$ *is finite in every finite* t*-interval. The sets* $\overline{S_i(\omega)}$ *for all* i *are disjoint except for an at most denumerable set and each* $\overline{S_i(\omega)}$ *differs from* $S_i(\omega)$ *by an at most denumerable set. On each* $S_i(\omega)$, $x(\cdot, \omega)$ *is constant and equal to* i.

PROOF. Let $C(\omega)$ be the union of all $\overline{S_i(\omega)}$ with i instantaneous. By property (iv), section 4, $C(\omega) \subseteq D(\omega)$. If $\tau \in D(\omega) - C(\omega)$ and i is an instantaneous state then there is a neighborhood of τ in which $x(\cdot, \omega) \not\equiv i$. Hence by theorem D we have as $t \to \tau$ from one side, the following possibilities with probability one:

(i) $\lim_{t \uparrow \tau} x(t, \omega) = i \neq j = \lim_{t \downarrow \tau} x(t, \omega)$ where i and j are stable;

(ii) $\lim_{t \uparrow \tau} x(t, \omega) = i < \infty = \lim_{t \downarrow \tau} x(t, \omega)$ or $\lim_{t \uparrow \tau} x(t, \omega) = \infty > i = \lim_{t \downarrow \tau} x(t, \omega)$ where i is stable;

(iii) $\lim_{t \to \tau} x(t, \omega) = \infty$;

(iv) $x(t, \omega)$ has two limiting values i and ∞ where i is stable.

The set N_1 of ω for which (iv) is true for some τ has probability zero, by theorem 1. Because of separability we have $x(\tau, \omega) = \infty$ in case (iii) if τ does not belong to the denumerable set satisfying the conditions of the separability definition; while if it belongs to this set then case (iii) has probability zero by theorem 3 applied to all τ in the denumerable set. Now the set E of (τ, ω) for which $x(\tau, \omega) = \infty$ is measurable and for each fixed τ the ω-set $\{\omega: (\tau, \omega) \in E\}$ has probability zero by theorem 2 (iii). Hence by Fubini's theorem if ω is not in a set N_2 with $P(N_2) = 0$ the set of τ for which (iii) is true has measure zero. Finally let Ω_0 be the set specified in theorem 1. Then for each $\omega \in \Omega_0$ the set of τ for which either (i) or (ii) is true may be put into an at most 2-1 correspondence with the set of stable intervals of $x(\cdot, \omega)$. Hence it is a denumerable set by theorem 1. We have thus proved that for every $\omega \in \Omega_0 - N_1 - N_2 = \Omega_4$, $m[D(\omega) - C(\omega)] = 0$. Since the possibility (iv) has been excluded, each of the remaining possibilities presents a discontinuity τ which is either a jump or a limit point of jumps. The other assertions in theorem 3 merely give a résumé of some of the previous results.

Theorem 3 reduces to a result by Lévy (see p. 349 in [4]) if there are no instantaneous states.

Note that the value of $x(\cdot, \omega)$ is not uniquely prescribed by separability at certain points, for example, at a point of $\overline{S_i(\omega)} \cap \overline{S_j(\omega)}$, $i \neq j$. *For the adjoined state* ∞ *the set* $S_\infty(\omega)$ *may be further specified as follows:* $\tau \in S_\infty(\omega)$ if and only if $\lim_{t \to \tau} x(t, \omega) = \infty$. In fact, at any other point separability does not prevent us from changing the value of $x(\tau, \omega)$ to one of the finite limiting values of $x(t, \omega)$ as $t \to \tau$. The resulting process remains measurable since changes are made only on a (t, ω) set of measure zero.

We are now in a position to strengthen theorem D as follows:

THEOREM 4. *For almost all* ω, *the sample function* $x(\cdot, \omega)$ *has the following properties. For every* τ, *as* $t \uparrow \tau$ *or* $t \downarrow \tau$ *we have one of the following possibilities:*

(i) $x(t, \omega) \to i$ *where* i *is stable;*

(ii) $x(t, \omega)$ *has exactly two limiting values* i *and* ∞ *where* i *is instantaneous;*

(iii) $x(t, \omega) \to \infty$.

Furthermore, if $x(\tau, \omega) = i$ *where* i *is stable then* (i) *is true with the same* i *as* $t \to \tau$ *from at least one side; if* $x(\tau, \omega) = i$ *where* i *is instantaneous then* (ii) *is true with the same* i *as* $t \to \tau$ *from at least one side; if* $x(\tau, \omega) = \infty$ *then* (iii) *must be true as* $t \to \tau$ *from both sides.*

PROOF. According to theorem D these are the three possibilities with the state i in (i) and (ii) yet unspecified. Now by property (iv) of section 4 the probability is zero that $x(t, \omega) \to i$ with i instantaneous, hence in (i) the state i must be stable. By theorem 1 the probability is zero that a stable i is a limiting value of $x(t, \omega)$ as $t \to \tau$ from one side without being the limit. Hence in (ii) the i must be instantaneous. The remaining assertions follow from the separability of the process and the remark preceding theorem 4.

6. System theorems

In this section we prove several results concerning the optional starting, stopping and splitting of the process. Such theorems have their origin in so-called gambling systems, hence the name "system theorems." They were frequently regarded as obvious and used without comment.

Let $\mathcal{F}\{x(s, \omega), s < t\}$, or $\mathcal{F}\{x(s, \omega), s \geqq t\}$, be the Borel field of ω-sets generated by the random variables $x(s, \omega)$ with $s < t$, or $s \geqq t$. Let $\alpha(\omega)$ be a nonnegative random variable such that for every $t > 0$

$$\{\alpha(\omega) < t\} \in \mathcal{F}\{x(s, \omega), s < t\}. \tag{6.1}$$

For each positive integer n, ω-sets of the form

$$\bigcup_{r=0}^{\infty} [\Lambda_r \cap \{r 2^{-n} \leqq \alpha(\omega) < (r+1) 2^{-n}\}] \tag{6.2}$$

where $\Lambda_r \in \mathcal{F}\{x(s, \omega), s < (r+1)2^{-n}\}$ form a Borel field $\mathcal{F}_n$. It is clear that $\mathcal{F}_n \geqq \mathcal{F}_{n+1}$. We define

$$\mathcal{F}\{x(s, \omega), s \leqq \alpha(\omega)\} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{6.3}$$

Similarly, ω-sets of the form (6.2) where $\Lambda_r \in \mathcal{F}\{x(s, \omega), s \geqq r2^{-n}\}$ form a Borel field $\mathcal{G}_n$ with $\mathcal{G}_n \geqq \mathcal{G}_{n+1}$. We define

$$\mathcal{F}\{x(s, \omega), s \geqq \alpha(\omega)\} = \bigcap_{n=1}^{\infty} \mathcal{G}_n. \tag{6.4}$$

Informally speaking, a set in $\mathcal{G}\{x(s, \omega), s \leqq \alpha(\omega)\}$, or $\mathcal{G}\{x(s, \omega), s \geqq \alpha(\omega)\}$, is determined by conditions on $x(s, \omega)$ for $s \leqq \alpha(\omega)$, or $s \geqq \alpha(\omega)$. We remark that even though $\alpha(\omega)$ may be ∞ or undefined with positive probability, every set in $\mathcal{G}\{x(s, \omega), s \leqq \alpha(\omega)\}$ or $\mathcal{G}\{x(s, \omega), s \geqq \alpha(\omega)\}$ is contained in the set $\{\alpha(\omega) < \infty\}$.

Naturally, we define $\mathcal{G}\{x(s, \omega), s < \alpha(\omega)\}$ to be the smallest Borel field containing all $\mathcal{G}\{x(s, \omega), s \leqq \alpha(\omega) - 1/n\}$, $n \geqq 1$; and $\mathcal{G}\{x(s, \omega), s > \alpha(\omega)\}$ to be the smallest Borel field containing all $\mathcal{G}\{x(s, \omega), s \geqq \alpha(\omega) + 1/n\}$, $n \geqq 1$.

THEOREM 5. *Let $\tau(\omega) = \tau_n(\omega, i)$, $\tau'(\omega) = \tau_n'(\omega, i)$ and $\lambda(\omega) = \lambda_n(\omega, i)$ be the nth entrance, exit and sojourn times of the stable state i with $q_i > 0$. Then for every finite $a > 0$, and any two sets $\Lambda_1 \in \mathcal{G}\{x(s, \omega), s \leqq \tau(\omega)\}$ and $\Lambda_2 \in \mathcal{G}\{x(s, \omega), s \geqq \tau'(\omega)\}$, we have under any initial distribution*

$$(6.5) \qquad P\{\tau(\omega) < \infty\}P\{\Lambda_1; \lambda(\omega) \geqq a; \Lambda_2\} = P(\Lambda_1)P(\Lambda_2)e^{-q_i a}.$$

PROOF. For each $t \geqq 0$ define $\tau'(t, \omega)$ on the set $\{\omega: x(t, \omega) = i\}$ to be the least exit time of i that exceeds t. Let $\Lambda_2(t)$ be defined with respect to $\tau'(t, \omega)$ in exactly the same way as Λ_2 is defined with respect to $\tau'(\omega)$. Then $P\{\Lambda_2(t) | x(t, \omega) = i\}$ is a number P_0 not depending on t whenever $P\{x(t, \omega) = i\} > 0$. We evaluate $P(\Lambda_2)$ as follows, letting $h = 2^{-n}$, $n \to \infty$:

$$(6.6) \qquad \begin{aligned} P(\Lambda_2) &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{(r-1)h \leqq \tau(\omega) < rh; \Lambda_2\} \\ &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{(r-1)h \leqq \tau(\omega) < rh < \tau'(\omega); \Lambda_2(rh)\} \end{aligned}$$

since $0 < \tau'(\omega) - \tau(\omega) < \infty$ with probability one, and on the set $\{\omega: \tau(\omega) < rh < \tau'(\omega)\}$ the set Λ_2 becomes $\Lambda_2(rh)$. Now the set $\{\omega: \tau'(\omega) > rh\}$ differs from a set in $\mathcal{G}\{x(s, \omega), s < rh\}$ by a set of probability zero; hence we may use the Markovian property to obtain

$$(6.7) \qquad \begin{aligned} P(\Lambda_2) &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{(r-1)h \leqq \tau(\omega) < rh < \tau'(\omega); x(rh, \omega) = i\} \\ &\qquad \cdot P\{\Lambda_2(rh) \mid x(rh, \omega) = i\} \\ &= P\{\tau(\omega) < \infty\}P_0. \end{aligned}$$

We can now evaluate the left member of (6.5) as follows:

$$(6.8) \qquad \begin{aligned} P\{\Lambda_1; \lambda(\omega) > a; \Lambda_2\} &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{\Lambda_1; (r-1)h \leqq \tau(\omega) < rh < \tau'(\omega); \\ &\qquad x(t, \omega) \equiv i, rh \leqq t \leqq rh + a; \Lambda_2(rh + a)\} \\ &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{\Lambda_1; (r-1)h \leqq \tau(\omega) < rh < \tau'(\omega); x(rh, \omega) = i\} \\ &\qquad \cdot P\{x(t, \omega) \equiv i, rh \leqq t \leqq rh + a \mid x(rh, \omega) = i\} \\ &\qquad \cdot P\{\Lambda_2(rh + a) \mid x(rh + a, \omega) = i\} \\ &= \lim_{h\to 0}\sum_{r=1}^{\infty} P\{\Lambda_1; (r-1)h < \tau(\omega) < rh < \tau'(\omega)\} e^{-q_i a}P_0 \\ &= P(\Lambda_1) e^{-q_i a}P_0. \end{aligned}$$

This is equivalent to (6.5) by (6.7).

COROLLARY 1. *Suppose that $P\{\tau(\omega) < \infty\} = 1$. Let $\mathcal{G}$ be the smallest Borel field con-*

taining both $\mathcal{F}\{x(s, \omega), s \leqq \tau(\omega)\}$ *and* $\mathcal{F}\{x(s, \omega), s \geqq \tau'(\omega)\}$. *Then for every* $a > 0$ *and any set* $\Lambda \in \mathcal{G}$ *we have*

$$(6.9) \qquad P\{\lambda(\omega) > a; \Lambda\} = P(\Lambda)\, e^{-q_i a}.$$

PROOF. Consider a set of the form

$$(6.10) \qquad \Lambda_0 = \bigcup_{m=1}^{\infty} \Lambda_m^{(1)} \Lambda_m^{(2)}$$

where the $\Lambda_m^{(1)}$ and $\Lambda_m^{(2)}$ are disjoint sets in $\mathcal{F}\{x(s, \omega), s \leqq \tau(\omega)\}$ and $\mathcal{F}\{x(s, \omega), s \geqq \tau'(\omega)\}$ respectively. Such sets form a field $\mathcal{G}_0$ containing both these Borel fields. We have

$$(6.11) \qquad P\{\lambda(\omega) > a; \Lambda_0\} = \sum_m P\{\lambda > a; \Lambda_m^{(1)}; \Lambda_m^{(2)}\}$$

$$= \sum_m P(\Lambda_m^{(1)}) P(\Lambda_m^{(2)})\, e^{-q_i a}.$$

Putting $a = 0$ we obtain

$$(6.12) \qquad P(\Lambda_0) = \sum_m P(\Lambda_m^{(1)}) P(\Lambda_m^{(2)}).$$

Hence we have for every $\Lambda_0 \in \mathcal{G}_0$,

$$(6.13) \qquad P\{\lambda(\omega) > a; \Lambda_0\} = P(\Lambda_0)\, e^{-q_i a}.$$

Thus (6.9) is true for every set in $\mathcal{G}_0$ and consequently also for every set in $\mathcal{G}$.

COROLLARY 2. *Suppose that all* $\tau_n(\omega, i)$, $\tau_n'(\omega, i)$ *and* $\lambda_n(\omega, i)$ *for all stable* i *and* n *are defined with probability one. Then every finite set of random variables* $\lambda_{n_\nu}(\omega, i_\nu)$, $v = 1, \cdots, N$, *such that if* $\mu \neq v$ *then* $i_\mu \neq i_\nu$ *or* $n_\mu \neq n_\nu$, *is a set of independent random variables.*

This follows from repeated application of corollary 1. Corollary 2 is stated by Lévy (see p. 349 in [4]) and is essential for much of his work there.

Because of the relative recent growth of rigor in the discussion of stochastic processes we permit ourselves the following remarks. Consider, for example, the successive $\lambda_n(\omega, i)$, $n \geqq 1$, for a fixed i. By an abuse of the Markov property it seems obvious that these random variables are independent and identically distributed. It would then be easy, for example, to deduce theorem 1, namely, that their number is finite in every finite t-interval. However, we wish to stress the point that such a procedure cannot possibly be justified. In fact, theorem 1 must precede corollary 2 above in the logical order, because the random variables $\lambda_n(\omega, i)$ cannot be defined without theorem 1. Let us suppose for the sake of argument that the latter theorem merely asserted that there is at most a denumerable number of i-intervals in every finite t-interval. It would then be impossible to define the $\lambda_n(\omega, i)$ unless the set of i-intervals were first shown to be well ordered. Perhaps this is one reason why Doob in his 1945 paper [2] restricted himself to processes whose discontinuities are well ordered in time.

THEOREM 6. *Let* $\alpha(\omega)$ *be the least* t *for which* $t \in \overline{S_i(\omega)}$. *Suppose that the initial distribution is such that* $P\{\alpha(\omega) < \infty\} = 1$. *Let* $\Lambda \in \mathcal{F}\{x(s, \omega), s \leqq \alpha(\omega)\}$, $0 < t_1 < t_2 < \cdots < t_N$, $i_1, \cdots, i_N$, *be any states. We have then*

$$(6.14) \qquad P\{\Lambda; x[\alpha(\omega) + t_\nu, \omega] = i_\nu, 1 \leqq \nu \leqq N\}$$

$$= P(\Lambda)\, p_{i i_1}(t_1)\, p_{i_1 i_2}(t_2 - t_1) \cdots p_{i_{N-1} i_N}(t_N - t_{N-1}).$$

PROOF. Let $S_i^+(\omega)$, or $S_i^-(\omega)$, be the set of t such that $(t, t+\epsilon) \cap S_i(\omega) \neq 0$, or $(t-\epsilon, t) \cap S_i(\omega) \neq 0$, for every $\epsilon > 0$. Note that $\overline{S_i(\omega)} = S_i(\omega) \cup S_i^+(\omega) \cup S_i^-(\omega)$. For given t_ν and i_ν, it follows from theorem 3 that the following sets have the same probability, under any initial distribution.

$$(6.15)\qquad \{\omega: t_\nu \in S_{i_\nu}^+(\omega) \cap S_{i_\nu}^-(\omega)\},\ \{\omega: t_\nu \in S_{i_\nu}^+(\omega) \cup S_{i_\nu}^-(\omega)\},\ \{\omega: x(t_\nu, \omega) = i\}.$$

Therefore we have, omitting a proof that the ω-sets below belong to $\mathcal{B}$,

$$\begin{aligned}(6.16)\quad &\lim_{\epsilon\downarrow 0} P\{S_{i_\nu}(\omega) \cap (t_\nu - \epsilon, t_\nu) \neq 0;\ S_{i_\nu}(\omega) \cap (t_\nu, t_\nu+\epsilon) \neq 0;\\ &\qquad\qquad 1 \leqq \nu \leqq N \mid x(0, \omega) = i\}\\ &= \lim_{\epsilon\downarrow 0} P\{S_{i_\nu}(\omega) \cap (t_\nu - \epsilon, t_\nu+\epsilon) \neq 0;\ 1 \leqq \nu \leqq N \mid x(0, \omega) = i\}\\ &= p_{ii_1}(t_1)\, p_{i_1 i_2}(t_2 - t_1) \cdots p_{i_{N-1} i_N}(t_N - t_{N-1}).\end{aligned}$$

Now owing to separability of the process we have

$$\begin{aligned}(6.17)\quad &P\{\Lambda; a(\omega) + t_\nu \in S_{i_\nu}^+(\omega) \cap S_{i_\nu}^-(\omega);\ 1 \leqq \nu \leqq N\} \leqq P\{\Lambda; x[a(\omega) + t_\nu, \omega] = i_\nu;\\ &\qquad 1 \leqq \nu \leqq N\} \leqq P\{\Lambda;\ a(\omega) + t_\nu \in S_{i_\nu}^+(\omega) \cup S_{i_\nu}^-(\omega);\ 1 \leqq \nu \leqq N\}.\end{aligned}$$

The first member of (6.17) is equal to, if $h = 2^{-n}$, $n \to \infty$,

$$\begin{aligned}(6.18)\quad &\lim_{\epsilon\downarrow 0} P\{\Lambda;\ S_{i_\nu}(\omega) \cap [a(\omega) + t_\nu - \epsilon, a(\omega) + t_\nu] \neq 0;\ S_{i_\nu}(\omega) \cap [a(\omega) + t_\nu, a(\omega) + t_\nu\\ &\quad + \epsilon] \neq 0;\ 1 \leqq \nu \leqq N\} \geqq \lim_{\epsilon\downarrow 0} \lim_{h\to 0} \sum_{m=0}^{\infty} P\{\Lambda;\ x(rh, \omega) \neq i,\ 0 \leqq r < m;\\ &\quad x(mh, \omega) = i\} P\{S_{i_\nu}(\omega) \cap [mh + t_\nu - \epsilon, (m-1)h + t_\nu] \neq 0;\\ &\quad S_{i_\nu}(\omega) \cap [mh + t_\nu, (m-1)h + t_\nu + \epsilon] \neq 0;\\ &\qquad\qquad 1 \leqq \nu \leqq N \mid x(mh, \omega) = i\}\\ &= \lim_{\epsilon\downarrow 0} P\{S_{i_\nu}(\omega) \cap (t_\nu - \epsilon, t_\nu) \neq 0;\ S_{i_\nu}(\omega) \cap (t_\nu, t_\nu + \epsilon) \neq 0;\ 1 \leqq \nu \leqq N \mid\\ &\quad x(0, \omega) = i\} \lim_{h\to 0} \sum_{m=0}^{\infty} P\{\Lambda;\ x(rh, \omega) \neq i,\ 0 \leqq r < m;\ x(mh, \omega) = i\}.\end{aligned}$$

The inequality above follows from the fact that if $(m-1)h < a(\omega) \leqq mh$ and $h < \epsilon$, then $[a(\omega) + t_\nu - \epsilon, a(\omega) + t_\nu] \supset [mh + t_\nu - \epsilon, (m-1)h + t_\nu]$, etc. The last-written limit is equal to $P(\Lambda)$ since $P[a(\omega) < \infty] = 1$; the limit preceding it is given by (6.16); hence

$$\begin{aligned}(6.19)\quad P\{\Lambda;\ x[a(\omega) + t_\nu, \omega] = i_\nu\} \geqq P(\Lambda)\, p_{ii_1}(t_1)\, p_{i_1 i_2}(t_2 - t_1) \cdots\\ p_{i_{N-1} i_N}(t_N - t_{N-1}).\end{aligned}$$

The third member of (6.17) is evaluated in a similar way and seen not to exceed the right side of (6.19). Therefore, by (6.17), equality sign holds in (6.19) and the theorem is proved.

COROLLARY. *There is a standard modification of the process* $\{y(t, \omega), t > 0\}$ *defined by*

$$(6.20)\qquad y(t, \omega) = x[a(\omega) + t, \omega], \qquad t > 0,$$

which is separable relative to the closed sets and measurable. It is a Markov chain with the *same states and the same transition probability functions as* $\{x(t, \omega), t > 0\}$ *and whose initial distribution may be taken to be* $P\{y(0, \omega) = i\} = 1$.

This is a consequence of theorem 6 and theorem II.2.6 of [3].

If i is a stable state this reduces to a theorem of Doob (see theorem 2.1 in [2]) except for the inclusion of the set Λ which is necessary for certain applications.

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