ON A CLASS OF PROBABILITY SPACES

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1. Introduction

Kolmogorov's model for probability theory [10], in which the basic concept is that of a probability measure P on a Borel field $\mathcal{B}$ of subsets of a space Ω, is by now almost universally considered by workers in probability and statistics to be the appropriate one. In 1948, however, three somewhat disturbing examples were published by Dieudonné [2], Andersen and Jessen [1], and Doob [3] and Jessen [9], as follows.

A. (Dieudonné). There exist a pair $(\Omega, \mathcal{B})$, a probability measure P on $\mathcal{B}$, and a Borel subfield $\mathcal{A} \subset \mathcal{B}$ for which there is no function $Q(\omega, E)$ defined for all $\omega \in \Omega$, $E \in \mathcal{B}$ with the following properties: Q is for fixed E an $\mathcal{A}$-measurable function of ω, for fixed ω a probability measure on $\mathcal{B}$, and for every $A \in \mathcal{A}$, $E \in \mathcal{B}$, we have

$$\int_A Q(\omega, E)\, dP(\omega) = P(A \cap E). \tag{1}$$

B. (Andersen and Jessen). There exist a sequence of pairs $(\Omega_n, \mathcal{B}_n)$ and a function P defined for all sets of $\cup \mathcal{A}_n$, where $\mathcal{A}_n$ consists of all subsets of the infinite product space $\Omega_1 \times \Omega_2 \times \cdots$ in the Borel field determined by sets of the form $B_1 \times \cdots \times B_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots$, $B_i \in \mathcal{B}_i$, $i = 1, \cdots, n$, such that P is countably additive on each $\mathcal{A}_n$ but not on $\cup \mathcal{A}_n$.

C. (Doob, Jessen). There exist a pair $(\Omega, \mathcal{B})$, a probability measure P on $\mathcal{B}$, and two real-valued $\mathcal{B}$-measurable functions f, g on Ω such that

$$P\{\omega\colon f \in F,\ g \in G\} = P\{\omega\colon f \in F\} P\{\omega\colon g \in G\} \tag{2}$$

holds for every two linear Borel sets F, G but not for every two linear sets F, G for which the three probabilities in (2) are defined.

In each case Ω is the unit interval, $\mathcal{B}$ is the Borel field determined by the Borel sets and one or more sets of outer Lebesgue measure 1 and inner Lebesgue measure 0, and P consists of a suitable extension of Lebesgue measure to $\mathcal{B}$. The fact that A, B, C cannot happen if Ω is a Borel set in a Euclidean space and $\mathcal{B}$ consists of the Borel subsets of Ω is known. For A, the proof was given by Doob [4], for B by Kolmogorov [10], and for C by Hartman [7].

To the extent that A, B, C violate one's intuitive concept of probability, they suggest that the Kolmogorov model is too general, and that a more restricted concept, in which A, B, C cannot happen, is worth considering. In their book [5], Gnedenko and Kolmogorov propose a more restricted concept, that of a *perfect* probability space, which is a triple $(\Omega, \mathcal{B}, P)$ such that for any real-valued $\mathcal{B}$-measurable function f and any linear set A for which $\{\omega\colon f(\omega) \in A\} \in \mathcal{B}$, there is a Borel set $B \subset A$ such that

$$P\{\omega\colon f(\omega) \in B\} = P\{\omega\colon f(\omega) \in A\}. \tag{3}$$

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As noted by Doob [4] (see appendix in [5]) in perfect spaces A, C cannot happen, and it then follows from a theorem of Ionescu Tulcea [8] that B cannot happen in perfect spaces.

The concept introduced by the writer here is that of a *Lusin space*, which is a pair $(\Omega, \mathcal{B})$ such that (a) $\mathcal{B}$ is *separable*, that is, there is a sequence $\{B_n\}$ of elements of $\mathcal{B}$ such that $\mathcal{B}$ is the smallest Borel field containing all B_n, and (b) the range of every real-valued $\mathcal{B}$-measurable function f on Ω is an *analytic set*, that is, a set which is the continuous image of the set of irrational numbers. The concept of Lusin space is more restricted than that of perfect space in the sense that if $(\Omega, \mathcal{B})$ is a Lusin space and P is any probability measure on $\mathcal{B}$, then $(\Omega, \mathcal{B}, P)$ is perfect.

It is shown below that for Lusin spaces none of A, B, C can occur. The primary property of Lusin spaces which ensures this regularity, and which fails for the example of A, B, C mentioned above, is that the only events whose occurrence or nonoccurrence is determined by specifying which events in a sequence $E_1, E_2, \cdots$ occur are the events in the Borel field determined by the sequence $\{E_n\}$. This property permits the identification of the concepts, for real-valued $\mathcal{B}$-measurable functions f, g, "f is a function of g" and "f is a Baire function of g," the nonequivalence of which in general is a technical nuisance to say the least.

2. Preliminaries

In this section we list some definitions and some known properties of analytic sets to be used in later sections.

If M is a metric space, the sets in the smallest Borel field containing all open sets will be called the *Borel sets* of M.

A Borel field $\mathcal{B}$ of subets of a space Ω will be called *separable* if there is a sequence $\{B_n\}$ of sets in $\mathcal{B}$ such that $\mathcal{B}$ is the smallest Borel field containing all B_n. Thus if Ω is a separable metric space, the class of Borel sets is a separable Borel field, though not conversely. If $\mathcal{B}$ is a separable Borel field of subsets of Ω and $\{B_n\}$ is a sequence determining $\mathcal{B}$, the sets of the form $\cap C_n$, where each C_n is either B_n or $\Omega - B_n$, are called the *atoms* of $\mathcal{B}$. Any two nonidentical atoms are disjoint and every set in $\mathcal{B}$ is a union of atoms, so that the class of atoms of $\mathcal{B}$ is independent of the particular sequence $\{B_n\}$.

A metric space A will be called *analytic* if A is the continuous image of the set of irrational numbers. We shall use the following properties of analytic sets, due to Lusin [11].

I. If $\{A_n\}$ is a sequence of analytic sets in a metric space M, then $\cup A_n$, $\cap A_n$ if nonempty, the product space $A_1 \times A_2$, and the infinite product space $A_1 \times A_2 \times \cdots$ are analytic sets.

II. If A is analytic, so is every Borel subset of A.

III. Every Borel set of Euclidean n-space is analytic.

IV. If A, B are disjoint analytic subsets of a metric space M, there is a Borel set D of M such that $D \supset A$ and D is disjoint from B.

V. If f is a Borel-measurable mapping of an analytic set A into a separable metric space M, that is, the inverse image of any open set in M is a Borel set of A, then fA, the range of f, is an analytic set.

We shall also use the following property pointed out to the writer by A. P. Morse.

VI. If P is a probability measure on the Borel sets of a metric space M and A is an analytic subset of M then for every $\epsilon > 0$ there is a compact C inside A with $P(C) > \mu - \epsilon$, where $\mu = \min P(B)$ as B varies over all Borel sets of M which contain A.

3. Lusin spaces and analytic sets

The main content of theorems 1 and 2 is that, apart from the unessential difference that the atoms of a Lusin space need not be points, Lusin spaces are identical with pairs $(\Omega, \mathcal{B})$ where Ω is analytic and $\mathcal{B}$ is the class of Borel sets of Ω.

THEOREM 1. *If Ω is analytic and $\mathcal{B}$ is the class of Borel sets of Ω, then $(\Omega, \mathcal{B})$ is a Lusin space.*

PROOF. Separability of $\mathcal{B}$ follows from the separability of Ω, and that the range of every $\mathcal{B}$-measurable real-valued f is analytic is the special case of V with M the real line.

THEOREM 2. *If $(\Omega, \mathcal{B})$ is a Lusin space whose atoms are points and $\{E_n\}$ is any sequence determining $\mathcal{B}$ then there is a metric on Ω with respect to which Ω is an analytic set, $\mathcal{B}$ consists of the Borel sets of Ω, and every E_n is both open and closed.*

PROOF. Say $\{E_n\}$ determines $\mathcal{B}$ and let $f(\omega) = \sum_n e_n(\omega)/3^n$ where e_n is the characteristic function of E_n. Then f is a 1–1 $\mathcal{B}$-measurable map of Ω onto an analytic subset A of the line. Let $d(\omega_1, \omega_2) = 1/k(\omega_1, \omega_2)$, where k is the smallest n for which $e_n(\omega_1) \neq e_n(\omega_2)$. Then f is bicontinuous between Ω and A, and every E_n is open and closed, since any point not in E_n has distance at least $1/n$ from E_n. Finally, to identify $\mathcal{B}$ with the class $\mathcal{D}$ of images of Borel sets of A under f^{-1}, the $\mathcal{B}$-measurability of f implies $\mathcal{B} \supset \mathcal{D}$, and we need only show $E_n \in \mathcal{D}$ to conclude $\mathcal{D} \supset \mathcal{B}$. Since E_n is the image under f^{-1} of the set of numbers in A whose nth ternary digit is 1, the proof is complete.

4. Set theoretic properties of Lusin spaces

THEOREM 3. *If $(\Omega, \mathcal{B})$ is a Lusin space, $\mathcal{C}$ is a separable Borel field of $\mathcal{B}$-sets and $A \in \mathcal{B}$ is a union of atoms of $\mathcal{C}$, then $A \in \mathcal{C}$.*

PROOF. Say $\{C_n\}$ determines $\mathcal{C}$ and let $f(\omega) = \sum_n c_n(\omega)/3^n$. Then f maps $\mathcal{C}$-atoms into points, and different $\mathcal{C}$-atoms into different points. Then fA and $f(\Omega - A)$ are disjoint analytic linear sets, so that from property IV there is a linear Borel set D such that $D \supset fA$ and D is disjoint from $f(\Omega - A)$. Consequently $f^{-1}D = A$, so that, since f is $\mathcal{C}$-measurable, $A \in \mathcal{C}$.

COROLLARY 1. *If $(\Omega, \mathcal{B})$ is a Lusin space, two separable Borel fields of $\mathcal{B}$-sets with the same atoms are identical.*

COROLLARY 2. *Let $(\Omega, \mathcal{B})$ be a Lusin space and let f map Ω onto an arbitrary space Z. If there is a separable Borel field $\mathcal{C} \subset \mathcal{B}$ whose atoms are the sets $f^{-1}(z)$, $z \in Z$, then $\mathcal{C}$ is identical with the class of all sets in $\mathcal{B}$ of the form $f^{-1}D$, $D \subset Z$, and $(Z, \mathcal{D})$ is a Lusin space, where $\mathcal{D}$ consists of all $D \in \mathcal{D}$ for which $f^{-1}D \in \mathcal{B}$.*

PROOF. The mapping f^{-1} is a 1–1 mapping between the points $z \in Z$ and the atoms of $\mathcal{C}$. Thus every $C \in \mathcal{C}$ has the form $f^{-1}D$ for some $D \subset Z$. Conversely if $A = f^{-1}D$ for some $D \subset Z$, A is a union of atoms of $\mathcal{C}$, so that, from the theorem, $A \in \mathcal{B}$ implies $A \in \mathcal{C}$. Thus $D \in \mathcal{D}$ if and only if $f^{-1}D \in \mathcal{C}$. It follows that if $\{C_n\}$ generates $\mathcal{C}$, then $\{fC_n\}$ generates $\mathcal{D}$, so that $\mathcal{D}$ is separable. Finally, if h is any real-valued $\mathcal{D}$-measurable function on Z, then hf is a $\mathcal{C}$-measurable function on Ω whose range is the same as the range of h. Since $(\Omega, \mathcal{B})$ is a Lusin space, this range is analytic and the proof is complete.

COROLLARY 3. *Let $(\Omega, \mathcal{B})$ be a Lusin space, let f map Ω onto an arbitrary space Z, and denote by $\mathcal{S}$ the Borel field of all Z-sets S for which $f^{-1}S \in \mathcal{B}$. For any separable $\mathcal{D} \subset \mathcal{S}$,*

$(Z, \mathcal{D})$ is a Lusin space. If in addition every $S \in \mathcal{S}$ is a union of atoms of $\mathcal{D}$, then $\mathcal{S} = \mathcal{D}$, so that $(Z, \mathcal{S})$ is a Lusin space.

PROOF. The first conclusion of the corollary follows immediately from the definition of a Lusin space. The second conclusion follows from the first and theorem 3.

COROLLARY 4. *If $(\Omega, \mathcal{B})$ is a Lusin space and f is a $\mathcal{B}$-measurable function from Ω into a separable metric space M, then for every set $A \subset M$ for which $f^{-1}A \in \mathcal{B}$ there is a Borel set B of M such that $f^{-1}B = f^{-1}A$.*

PROOF. The class $\mathcal{C}$ of all sets of the form $f^{-1}B$, where B is a Borel set of M, is a separable Borel subfield of $\mathcal{B}$. Every set $f^{-1}A$ is a union of atoms of $\mathcal{C}$, and thus is in $\mathcal{C}$ if it is in $\mathcal{B}$. Thus if $f^{-1}A \in \mathcal{B}$, there is a Borel set B of M for which $f^{-1}B = f^{-1}A$.

Theorem 3 identifies, for Lusin spaces, the concepts "an event A depends only on events in $\mathcal{C}$" and "$A \in \mathcal{C}$." The following theorem extends this to functions.

THEOREM 4. *Let $(\Omega, \mathcal{B})$ be a Lusin space, let f, g be $\mathcal{B}$-measurable functions from Ω into separable metric spaces Y, Z, and denote by $\mathcal{B}_f(\mathcal{B}_g)$ the class of all sets of the form $f^{-1}S$ $(g^{-1}T)$ where $S(T)$ is a Borel set in $Y(Z)$.*

(a) *If there is a function ϕ from Y into Z such that $\phi f = g$, then g is measurable with respect to the Borel field of f-sets, that is, $\mathcal{B}_g \subset \mathcal{B}_f$.*

(b) *If $\mathcal{B}_g \subset \mathcal{B}_f$, then there is a Borel-measurable function ψ from Y into Z such that $\psi f = g$.*

PROOF. (a) Separability of Y implies separability of $\mathcal{B}_f$. Every set in $\mathcal{B}_g$ is a union of atoms of $\mathcal{B}_f$, so that (a) follows from theorem 3. The hypothesis that $(\Omega, \mathcal{B})$ is a Lusin space is not necessary for (b); the proof by Doob [4] for Y, Z Euclidean spaces extends easily to arbitrary separable metric spaces Y, Z.

5. Conditional probability distributions, Kolmogorov extension

THEOREM 5. *Let $(\Omega, \mathcal{B})$ be a Lusin space, let P be a probability measure on $\mathcal{B}$, and let $\mathcal{A}$ be a separable Borel subfield of $\mathcal{B}$. There is a real-valued function $Q(\omega, B)$ defined for all $\omega \in \Omega$, $B \in \mathcal{B}$ such that*

(a) *for fixed $B \in \mathcal{B}$, Q is an $\mathcal{A}$-measurable function of ω,*

(b) *for fixed ω, Q is a probability distribution on $\mathcal{B}$,*

(c) *for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\int_A Q(\omega, B)dP = P(A \cap B)$, and*

(d) *there is a set $N \in \mathcal{A}$ with $P(N) = 0$ such that $Q(\omega, A) = 1$ for $\omega \in A$, $\omega \notin N$.*

PROOF. We may suppose that the atoms of $\mathcal{B}$ are points. Choose $F_n \in \mathcal{B}$ so that $\{F_n\}$ determines $\mathcal{B}$ and a subsequence of $\{F_n\}$ determines $\mathcal{A}$, and (theorem 2) metrize Ω so that Ω is analytic, $\mathcal{B}$ consists of the Borel sets, and each F_n is open and closed. Choose C_n compact so that $C_n \subset C_{n+1}$, $P(C_n) \to 1$, and denote by $\mathcal{G}$, $\mathcal{G}$ the fields determined by $F_1, F_2, \cdots$ and $F_1, F_2, \cdots, C_1, C_2, \cdots$ respectively.

Let $Q_1(\omega, B)$ be defined so as to satisfy (a) and (c) for $B \in \mathcal{G}$. Since $\mathcal{G}$ is countable, there is a set $N \in \mathcal{A}$ with $P(N) = 0$ such that for $\omega \notin N$,

$$Q_1 \text{ is additive and nonnegative on } \mathcal{G}, \tag{4}$$

$$Q_1(\omega, \Omega) = 1, \tag{5}$$

$$Q_1(\omega, A) = 1 \quad \text{for} \quad A \in \mathcal{A}, \quad A \in \mathcal{G} \tag{6}$$

and

$$Q_1(\omega, C_n) \to 1 \quad \text{as} \quad n \to \infty. \tag{7}$$

Then Q_1 is countably additive on $\mathcal{G}$, for if $\{H_n\}$ is a sequence of disjoint sets of $\mathcal{G}$ with $\cup H_n = \Omega$, for every n, since C_n is compact and the H_n are open and closed, there is an M such that $\bigcup_1^M H_i \supset C_n$. Finite additivity of Q_1 on $\mathcal{G}$ yields, for $\omega \notin N$, $Q_1(\omega, C_n) \leqq \sum_1^M Q_1(\omega, H_i) \leqq \sum_1^\infty Q_1(\omega, H_i)$. Letting $n \to \infty$ and using (7) yields $\sum_1^\infty Q_1(\omega, H_i) \geqq 1$. Additivity yields the reverse inequality, so that, for $\omega \notin N$, Q_1 is countably additive on $\mathcal{G}$. For $\omega \notin N$, we define $Q(\omega, B)$ as the (unique) countably additive extension of Q_1 from $\mathcal{G}$ to $\mathcal{B}$. For $\omega \in N$, we define $Q(\omega, B) = P(B)$. Then (b) holds, and the class of sets B for which (a) and (c) hold is a monotone class containing $\mathcal{G}$, so coincides with $\mathcal{B}$ [6]. To verify (d) let $A \in \mathcal{A}$, $\omega \in A$, $\omega \notin N$, and denote by I the $\mathcal{A}$-atom containing ω. Then $I \subset A$ and, since a subsequence of $\{F_n\}$ determines $\mathcal{A}$, there is a sequence $J_n \in \mathcal{G}$ for which $\cap J_n = I$. From (2), $Q(\omega, J_n) = 1$ for all n, so that $Q(\omega, I) = 1$. This completes the proof.

THEOREM 6. *Let $\{(\Omega_n, \mathcal{B}_n)\}$ be a sequence of Lusin spaces, let Ω be the infinite product space $\Omega_1 \times \Omega_2 \times \cdots$, and let $\mathcal{A}_n$ be the Borel field determined by all sets $A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots$, $A_i \in \mathcal{B}_i$. A function P defined on $\cup \mathcal{A}_n$ which is a probability measure on each $\mathcal{A}_n$ is countably additive on $\cup \mathcal{A}_n$.*

PROOF. Let A_n be a decreasing sequence of sets in $\cup \mathcal{A}_n$ with $P(A_n) \to 2\delta > 0$. We must show that $\cap A_n$ is not empty. We may suppose that the atoms of $\mathcal{B}_n$ are points and metrize Ω_n so that it becomes analytic and $\mathcal{B}_n$ consists of the Borel sets of Ω_n. We may also suppose that $A_n \in \mathcal{A}_n$. From property VI there is a set $D_n \in \mathcal{A}_n$ such that $D_n \subset A_n$, $P(D_n) > P(A_n) - \delta/2^n$, and $D_n = C_n \times \Omega_{n+1} \times \Omega_{n+2} \times \cdots$, where C_n is a compact subset of $\Omega_1 \times \cdots \times \Omega_n$. Since $P(D_1 \cap \cdots \cap D_N) > P(A_1 \cap \cdots \cap A_N) - \delta \sum_1^N 2^{-n} > \delta > 0$, $D_1 \cap \cdots \cap D_N$ is nonempty for each N. If $\omega_N = (x_{N1}, x_{N2}, \cdots) \in D_1 \cap \cdots \cap D_N$, we have $(x_{N1}, \cdots, x_{Nk}) \in C_k$ for all $N \geqq k$, so that there is a subsequence of ω_N which converges coordinatewise to a point $\omega^* = (x_1^*, x_2^*, \cdots)$, and $(x_1^*, \cdots, x_k^*) \in C_k$ for each k. Thus $\omega^* \in D_k \subset A_k$ for all k and the proof is complete.

6. Independence, perfection

THEOREM 7. *If $(\Omega, \mathcal{B})$ is a Lusin space, P is any probability measure on $\mathcal{B}$, and f, g are any two $\mathcal{B}$-measurable functions from Ω into separable metric spaces X, Y such that*

$$(8) \qquad P\{\omega: f \in A,\ g \in B\} = P\{\omega: f \in A\} P\{g \in B\}$$

for all Borel sets A, B in X, Y, then (8) *holds for all sets A, B in X, Y for which the terms are defined.*

PROOF. The theorem follows immediately from corollary 4 of theorem 3.

THEOREM 8. *If $(\Omega, \mathcal{B})$ is a Lusin space, P is any probability measure on $\mathcal{B}$ and f is any $\mathcal{B}$-measurable function from Ω into a separable metric space, then inside any $A \subset M$ for which $f^{-1}A \in \mathcal{B}$ there is a Borel set B of M with $P(f^{-1}B) = P(f^{-1}A)$.*

PROOF. We may suppose $A \subset R$, the range of f. If $\mathcal{C}$ consists of the Borel sets of R, then $(R, \mathcal{C})$ is a Lusin space and, from corollary 4 of theorem 3, $A \in \mathcal{C}$. The function $\phi(C) = P(f^{-1}C)$ is a probability measure on $\mathcal{C}$, and from property VI there is inside A a union B of compact sets with $\phi(B) = \phi(A)$.

7. Some unsolved problems

Problem 1. If $\mathcal{B}$ is a separable Borel field of subsets of a space Ω such that every separable Borel subfield of $\mathcal{B}$ with the same atoms is identical with $\mathcal{B}$, is $(\Omega, \mathcal{B})$ a Lusin space?

Problem 2. If $\mathcal{B}$ is a separable Borel field of subsets of a space Ω such that $(\Omega, \mathcal{B}, P)$ is perfect for every probability measure P on $\mathcal{B}$, is $(\Omega, \mathcal{B})$ a Lusin space?

Problem 3. Can the exceptional set N be eliminated from (d) of theorem 5? That is, given an analytic set N and a separable Borel subfield $\mathcal{A}$ of the class $\mathcal{B}$ of Borel sets of N, does there exist a function $Q(\omega, B)$ defined for all $\omega \in N$ and all $B \in \mathcal{B}$ which (i) for fixed B is an $\mathcal{A}$-measurable function of ω, (ii) for fixed ω is a probability distribution on $\mathcal{B}$ and (iii) for which $Q(\omega, A) = 1$ for all $A \in \mathcal{A}$, $\omega \in A$?

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