

STATISTICAL MECHANICS OF A CONTINUOUS MEDIUM (VIBRATING STRING WITH FIXED ENDS)

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1. For the past twenty years there has been much discussion in fluid mechanics about the statistical theory of turbulence. It was J. Boussinesq (1872) and O. Reynolds (1895), in their pioneering work, who expressed the idea that the turbulent velocity fluctuations of a fluid were much too complicated (changing rapidly from one time and one point to another) to be known in all their details; we must be satisfied to study only some convenient mean values. The systematic use of statistical methods has led, since 1930, to very important results, in the fundamental works of Sir Geoffrey Taylor, Th. von Karman and A. N. Kolmogoroff, for instance.

But, if we look carefully at all the results so far obtained, we see at first glance that they are not at all in the same close relation with the theoretical equations of fluid mechanics as the classical statistical mechanics bears to the Hamilton-Jacobi equations for a dynamical system having a finite number of degrees of freedom. For such a system the main features of statistical mechanics are the following:

(a) definition of the phase space Ω ; every state of the system is characterized by a point $\omega \in \Omega$, whose $2k$ coordinates are the Lagrangian parameters q_1, \dots, q_k and the conjugate moments p_1, \dots, p_k .

(b) proof of a uniqueness theorem: starting, at the initial time $t = 0$, from a given initial state ω , all the subsequent or prior states $T_t \omega$, where t varies from $-\infty$ to $+\infty$, are perfectly well determined and describe, in Ω , a curve, the trajectory Γ_ω .

(c) definition of a measure in Ω , invariant under the transformation $T_t \omega$. (Liouville's theorem.)

(d) proof of the ergodic theorem (existence of time averages computed along a trajectory Γ_ω); the time averages and statistical averages can be validly considered as equal only in the special case in which the transformation $T_t \omega$ of Ω into itself is metrically transitive.

As we have pointed out in [4], at the time being, all the work done in what is called statistical theory of turbulence deals only with some kind of time averages; so far as we know, a convenient phase space Ω has never been introduced, in order to represent all the possible states ω of a fluid; this phase space must be, as we have shown in [5], a function space, quite easy to define. But, so little is known about the solutions of the nonlinear equations of fluid mechanics (for a perfect as well as for a viscous fluid), that immediately after the definition of Ω , we are stopped by

the lack of a uniqueness theorem and, especially, by the lack of any method leading to the definition of an invariant measure in Ω .

To point the way to what has to be done in order to build a real statistical mechanics of turbulence, it has seemed worth while to us to sketch a statistical mechanics for a continuous medium, much simpler than a fluid. For a vibrating string, due to the linear character of the equations, the problem can be completely solved; in [5] we have treated the case of the infinite string; we want to discuss here the string, with fixed ends, which has much more interesting features than the infinite one.

2. In the manner classical since Bernoulli, Euler and D'Alembert, let us consider a vibrating string as a material curve moving in a plane Oxy ; when at rest the string is the straight line

$$0 \leq x \leq l, \quad y = 0.$$

Every point x of the string moves parallel to the Oy axis and we represent by $y(x, t)$ its *displacement* at the time t .

We will make the four following assumptions:

- (2.1) $y(x, t)$ is a continuous function of (x, t) ,
- (2.2) $y(0, t) = y(l, t) = 0$ for every t ,
- (2.3) $y(x, t)$ has continuous derivatives up to the second order,
- (2.4) $y(x, t)$ is a solution of $y_{xx}''(x, t) = y_{tt}''(x, t)$.

For the sake of brevity, let us put

- (2.5) $p(x, t) = y_x'(x, t)$,
- (2.6) $v(x, t) = y_t'(x, t)$,
- (2.7) $j(x, t) = y_{tt}''(x, t)$;

v and j represent respectively the *velocity* and the *acceleration* of the point x of the string at the time t .

The motion of the string, from $t = -\infty$ to $t = +\infty$, is completely determined by the knowledge of the displacement and the velocity for every point x at any given time t .

In fact let us introduce two functions of x , $y(x)$ and $v(x)$, such that:

- (2.8) the displacement $y(x)$ is a continuous function for $0 \leq x \leq l$,
- (2.9) $y(0) = y(l) = 0$,
- (2.10) the first two derivatives $y'(x)$ and $y''(x)$ exist and are continuous;
- (2.11) the velocity $v(x)$ is a continuous function for $0 \leq x \leq l$,
- (2.12) $v(0) = v(l) = 0$,
- (2.13) the first derivative $v'(x)$ exists and is continuous.

From these functions, we derive a function $f(x)$ defined for $-\infty < x < +\infty$ by:

$$(2.14) \quad f(x) = \frac{y'(x) + v(x)}{2}, \quad 0 \leq x \leq l,$$

$$(2.15) \quad f(x) = \frac{y'(-x) - v(-x)}{2}, \quad -l \leq x \leq 0,$$

$$(2.16) \quad f(x + 2l) = f(x).$$

It is obvious that $f(x)$ is a continuous function of x , having a continuous derivative $f'(x)$, and that

$$(2.17) \quad \int_0^{2l} f(x) dx = 0.$$

The unique displacement function $y(x, t)$, satisfying (2.1) to (2.4) and such that

$$y(x, \tau) = y(x), \quad v(x, \tau) = v(x)$$

is given by the classical formula

$$(2.18) \quad y(x, t) = \int_{-x}^x f(t - \tau + s) ds,$$

from which we obtain immediately

$$(2.19) \quad p(x, t) = f(t - \tau + x) + f(t - \tau - x),$$

$$(2.20) \quad v(x, t) = f(t - \tau + x) - f(t - \tau - x),$$

$$(2.21) \quad j(x, t) = f'(t - \tau + x) - f'(t - \tau - x).$$

3. According to the preceding well known results, we shall take as phase space Ω the following function space: Ω is the set of all real valued functions $f(s)$ of a real variable s , $-\infty < s < +\infty$, satisfying the four conditions:

$$(3.1) \quad f(s) \text{ is periodic, } f(s + 2l) = f(s),$$

$$(3.2) \quad f(s) \text{ is continuous,}$$

$$(3.3) \quad f(s) \text{ has a continuous derivative } f'(s),$$

$$(3.4) \quad \int_0^{2l} f(s) ds = 0.$$

Every function $f(s)$ satisfying these assumptions is called a point ω of the space Ω .

Every state of the vibrating string, being well defined by the knowledge of the displacement function $y(x)$ and the velocity function $v(x)$, is represented in the phase space Ω by the point ω corresponding to the function $f(s)$ computed according to (2.14), (2.15), and (2.16). Conversely, from every point $\omega \in \Omega$ we can deduce a displacement function $y(x)$ and a velocity function $v(x)$ by solving (2.14) and (2.15):

$$(3.5) \quad y(x) = \int_{-x}^x f(s) ds,$$

$$(3.6) \quad v(x) = f(x) - f(-x);$$

hence to every $\omega \in \Omega$ corresponds a completely determined state of the string.

There is thus a one to one correspondence between the states of the string and the points of the phase space Ω .

We may note that Ω is obviously a separable Banach space, if we define the norm by:¹

$$(3.7) \quad \|\omega\| = \max_{0 \leq s \leq 2l} |f(s)|.$$

4. Let us now introduce a one parameter Abelian group of one to one transformations of Ω into itself:

$$(4.1) \quad \omega \rightarrow T_t \omega, \quad -\infty < t < +\infty;$$

ω corresponding to $f(s)$, we define its transform $T_t \omega$ as the point corresponding to $f(t+s)$.

It is clear that for every $\omega \in \Omega$,

$$(4.2) \quad T_{t'}(T_{t''}\omega) = T_{t''}(T_{t'}\omega) = T_{t'+t''}\omega,$$

$$(4.3) \quad T_0 \omega = \omega,$$

$$(4.4) \quad (T_t)^{-1} \omega = T_{-t} \omega,$$

$$(4.5) \quad T_{t+2l} \omega = T_t \omega.$$

With the definition (3.7) of the norm, the transformation is isometric:

$$(4.6) \quad \|T_t \omega\| = \|\omega\|.$$

If we consider a fixed point $\omega \in \Omega$, the set of its transforms $T_t \omega$, when t varies from $-\infty$ to $+\infty$, is a *closed curve*,² which we shall call the trajectory Γ_ω . The trajectory Γ_0 corresponding to the point $\omega = 0$ contains only that point; except for this particular trajectory Γ_0 , we can establish a one to one correspondence between the points of the trajectory Γ_ω and the points of a circle C of radius l/π ; let us fix the position of a point on C by the length γ of the arc ending at this point,

$$(4.7) \quad 0 \leq \gamma < 2l;$$

then we obtain the one to one correspondence

$$\gamma \rightarrow T_\gamma \omega.$$

Thus it is clear that, when t varies from $-\infty$ to $+\infty$, the transforms $T_t \omega$ of ω describe the trajectory Γ_ω infinitely many times.

¹ Usually for the Banach space of the continuous functions $f(s)$ having a continuous derivative $f'(s)$, the norm is taken as

$$\|\omega\| = \max_{0 \leq s < 2l} |f(s)| + \max_{0 \leq s < 2l} |f'(s)|,$$

but here, because of (3.4), the second term need not be introduced.

² The name "curve" is justified by the fact that the continuity condition

$$\lim_{h \rightarrow 0} \|T_{t+h} \omega - T_t \omega\| = 0$$

is satisfied for every t .

Using the geometrical language we have introduced, we can restate the uniqueness theorem of section 2, in the following terms: when t varies from $-\infty$ to $+\infty$, all the states of the vibrating string, corresponding to given initial conditions, are represented in the phase space Ω by the unique trajectory Γ_ω , which passes through the particular point ω representing the initial state.

The fact that Γ_ω is a closed curve means that after a period $2l$ the state of the string is again the initial state: the motion is periodic.

The singular trajectory Γ_0 corresponds to the case in which

$$y(x) = 0, \quad v(x) = 0;$$

the string is always at rest and the unique point $\omega = 0$ represents all its states.

5. In what follows we shall represent by a symbol like

$$F[\omega | \lambda_1, \dots, \lambda_n]$$

a functional defined on Ω , depending on n parameters: to every point $\omega \in \Omega$ and to every set of values of the parameters $\lambda_1, \dots, \lambda_n$ corresponds a real number.

Of special interest are the linear functionals; in our particular space Ω , they are given by the F. Riesz theorem:

$$(5.1) \quad L[\omega | \lambda_1, \dots, \lambda_n] = \int_0^{2l} f(s) d\Phi(s, \lambda_1, \dots, \lambda_n),$$

where $\Phi(s, \lambda_1, \dots, \lambda_n)$ is supposed to be a function of bounded variation of s for every set of values of the parameters $\lambda_1, \dots, \lambda_n$.

We shall have, in particular, to consider the linear functionals:

$$(5.2) \quad Y[\omega | x] = \int_{-x}^x f(s) ds, \quad 0 \leq x \leq l,$$

$$(5.3) \quad P[\omega | x] = f(x) + f(-x),$$

$$(5.4) \quad V[\omega | x] = f(x) - f(-x).$$

Obviously, according to (2.18), (2.19), and (2.20), along the trajectory Γ_ω (where ω represents the state of the string at $t = 0$):

$$(5.5) \quad Y[T_t \omega | x] = y(x, t),$$

$$(5.6) \quad P[T_t \omega | x] = p(x, t),$$

$$(5.7) \quad V[T_t \omega | x] = v(x, t).$$

6. Now we have to define an invariant measure m in the phase space Ω ; an invariant measure is a set function $m(E)$ defined for every subset E of Ω belonging to a σ -subalgebra S of the Boolean algebra of all subsets of Ω (Borel field), with the following properties:

$$(6.1) \quad 0 \leq m(E) \leq 1 \quad \text{for every } E \in S;$$

$$(6.2) \quad m(\Omega) = 1;$$

$$(6.3) \quad m\left[\bigcup_n E_n\right] = \sum_n m(E_n)$$

for every finite or countable sequence of disjoint sets $E_n \in S$, $E_j \cap E_k = 0$ for $j \neq k$;

$$(6.4) \quad \text{if } E \in S, \text{ then } T_t E \in S \text{ for every } t \text{ and } m(T_t E) = m(E).$$

We can obtain the measure m by a well known process due to Kolmogoroff [6].

We start with cylinder sets; a cylinder set in Ω is the set of all ω corresponding to functions $f(s)$ satisfying a condition such as

$$(6.5) \quad f(s_1) < \xi_1, \dots, f(s_n) < \xi_n;$$

we may take for n all the integer values $1, 2, \dots$, for (s_1, \dots, s_n) any set of n different real numbers and for (ξ_1, \dots, ξ_n) any set of real numbers.

Now we define the measure of the cylinder sets, by choosing an arbitrary sequence of distribution functions³

$$\mathcal{Q}_1(\xi_1), \mathcal{Q}_2(\xi_1, \xi_2, s_2 - s_1), \dots, \mathcal{Q}_n(\xi_1, \dots, \xi_n, s_2 - s_1, \dots, s_n - s_1), \dots,$$

and by putting⁴

$$(6.6) \quad m \mathbb{E}_{\omega} [f(s_1) < \xi_1, \dots, f(s_n) < \xi_n] \\ = \mathcal{Q}_n(\xi_1, \dots, \xi_n, s_2 - s_1, \dots, s_n - s_1).$$

This being done, we have only to extend the definition of the measure to the σ -subalgebra S generated by the cylinder sets; it is known that this extension is unique [1], [2].

The invariance of the measure m thus defined is a consequence of the fact that the distribution functions do not depend on the s_1, \dots, s_n , but only on the differences $s_2 - s_1, \dots, s_n - s_1$, which are invariant under any translation, thus

$$m \mathbb{E}_{\omega} [f(t + s_1) < \xi_1, \dots, f(t + s_n) < \xi_n] = m \mathbb{E}_{\omega} [f(s_1) < \xi_1, \dots, f(s_n) < \xi_n];$$

now it is quite evident that

$$T_t \mathbb{E}_{\omega} [f(s_1) < \xi_1, \dots, f(s_n) < \xi_n] = \mathbb{E}_{\omega} [f(t + s_1) < \xi_1, \dots, f(t + s_n) < \xi_n],$$

which establishes the invariance of the measure of the cylinder sets under the transformation $T_t \omega$.

³ As is well known, these functions must satisfy a set of consistency conditions:

$\mathcal{Q}_1(\xi_1)$ is a nondecreasing function of ξ_1 , such that

$$\mathcal{Q}_1(-\infty) = 0, \quad \mathcal{Q}_1(+\infty) = 1;$$

$\mathcal{Q}_2(\xi_1, \xi_2, s_2 - s_1) (s_2 - s_1 \neq 0)$ is a nondecreasing function of (ξ_1, ξ_2) such that

$$\mathcal{Q}_2(-\infty, \xi_2, s_2 - s_1) = \mathcal{Q}_2(\xi_1, -\infty, s_2 - s_1) = 0,$$

$$\mathcal{Q}_2(+\infty, \xi_2, s_2 - s_1) = \mathcal{Q}_1(\xi_2),$$

$$\mathcal{Q}_2(\xi_1, +\infty, s_2 - s_1) = \mathcal{Q}_1(\xi_1),$$

$$\mathcal{Q}_2(+\infty, +\infty, s_2 - s_1) = 1;$$

and so on (for the details see [6]).

⁴ By the symbol $\mathbb{E}_{\omega}[P]$ we mean the subset of Ω , for every point of which a given condition P is satisfied.

But, with this method⁵ the difficulties begin when we try to take account of the fact that Ω is not the space of all real functions, but only of functions subject to the conditions (3.1) to (3.4); these conditions impose on the distribution functions $\mathcal{Q}_1, \mathcal{Q}_2, \dots$ some particular properties; for instance, as we shall see later, in order to satisfy (3.4) we must have

$$(6.7) \quad \int_{-\infty}^{+\infty} \xi d\mathcal{Q}_1(\xi) = 0.$$

But it does not seem at all easy to find directly all the conditions resulting for $\mathcal{Q}_1, \dots, \mathcal{Q}_n, \dots$ from (3.1) to (3.4).

To avoid this difficulty, we shall take another course, making sure from the beginning that every function corresponding to a point ω of Ω does in fact satisfy (3.1) to (3.4); this is quite easy using the representation, due to Norbert Wiener, of a random function, for the particular case of Brownian motion.

Let us take an abstract measure space A , for which the measure μ is such that

$$\mu(A) = 1.$$

Let us consider a real valued function $f(s, \alpha)$ defined over the product space $R \times A$, where R means the set of all real numbers $-\infty < s < +\infty$. We suppose that for every $\alpha \in A$, as a function of s , $f(s, \alpha)$ belongs to some given function space Ω (for instance, the space of all bounded functions, or of all continuous functions, etc.); further we suppose that for every real s , as a function of α , $f(s, \alpha)$ is measurable (with respect to the measure μ).

When α describes A , the set of all functions $f(s, \alpha)$ is, in general, only a subset Ω^* of the function space Ω ; for every $E^* \subset \Omega^*$ we shall denote by $A(E^*)$ the set of all α whose images are in E^* (it must be noted that the correspondence between A and Ω^* is not necessarily one to one).

For any subset $E \subset \Omega$ we shall say that it is measurable if $A[E \cap \Omega^*]$ is measurable; if this is the case, we take for the measure of E , by definition,

$$(6.8) \quad m(E) = \mu[A(E \cap \Omega^*)].$$

It can be proved [3] that the set function thus defined is effectively a measure on a subalgebra S of subsets of Ω .

From (6.8) we get

$$(6.9) \quad m(\Omega) = m(\Omega^*) = \mu[A(\Omega^*)] = 1;$$

for any E such that $\Omega^* \subset E$

$$m(E) = 1,$$

and for any E such that $\Omega^* \cap E = \emptyset$

$$m(E) = 0.$$

⁵ Every step could be interpreted in the language of probability if we assume that we pick at random a point ω in Ω , the probability being defined by

$$Pr[\omega \in E] = m(E), \quad E \in S;$$

the definition of the measure that we have sketched corresponds exactly to the definition of a random stationary function of s .

As a byproduct of this definition of m , we obtain the distribution functions, by observing that the set

$$E = \int_{\omega} [f(s_1) < \xi_1, \dots, f(s_n) < \xi_n]$$

has the same measure as the subset of Ω^*

$$E \cap \Omega^* = \int_{\omega} [f(s_1, a) < \xi_1, \dots, f(s_n, a) < \xi_n],$$

and the measure of this set is simply the measure of the subset of A for which

$$f(s_1, a) < \xi_1, \dots, f(s_n, a) < \xi_n;$$

but we need no longer worry about special conditions for the distribution functions, being sure *a priori* that all the functions $f(s, a)$ we have introduced possess just the desired properties.

In order to define an invariant measure m in the phase space Ω of the vibrating string, it is very convenient to take for A a product space

$$A = B \times C,$$

where C is the circle already considered in (4.7) and B is any measure space; we take on C the ordinary Lebesgue measure and we call ν the measure defined on B ; the measure μ on A is then defined in the ordinary way for a product space.

Now let us define on A a real valued function

$$g(\beta, \gamma), \quad \beta \in B, \quad \gamma \in C,$$

with the following properties:

$$(6.10) \quad \text{for every } \gamma, g \text{ is a measurable function of } \beta;$$

$$(6.11) \quad \text{for every } \beta, g \text{ is periodic in } \gamma, g(\beta, \gamma + 2l) = g(\beta, \gamma);$$

$$(6.12) \quad \text{for every } \beta, g \text{ is a continuous function of } \gamma;$$

$$(6.13) \quad \text{for every } \beta, g \text{ has a continuous derivative } g'_\gamma(\beta, \gamma);$$

$$(6.14) \quad \text{for every } \beta,$$

$$\int_0^{2l} g(\beta, \gamma) d\gamma = 0.$$

If we put

$$(6.15) \quad f(s) = g(\beta, s + \gamma), \quad \beta \in B, \quad \gamma \in C,$$

the set Ω^* of all these functions $f(s)$ is obviously a subset of the phase space Ω , since the conditions (6.11) to (6.14) imply the conditions (3.1) to (3.4).

The meaning of β and γ is clear; the set of points ω , corresponding to a given value of β , and to all the values of γ in the interval $0 \leq \gamma < 2l$ is simply a trajectory Γ ; thus Ω^* is really a set of trajectories, each value of β defining a trajectory. It follows immediately that Ω^* is transformed into itself by T_l :

$$(6.16) \quad T_l \Omega^* = \Omega^*.$$

By the method just described above, starting from the space A and the functions $g(\beta, s + \gamma)$ we define a measure m in Ω by the measure μ in A ; let us observe that the transformation from ω to $T_t \omega$ corresponds to the change of $g(\beta, s + \gamma)$ into $g(\beta, t + s + \gamma)$; this means that the transformation T_t of Ω into itself induces in A a very simple transformation: a rotation of amplitude $\pi t/l$ of the circle C . The Lebesgue measure on C being invariant under a rotation, it follows that the measure μ on A is invariant, and further that the measure m on Ω is also invariant.

Thus the measure m fulfills all the necessary conditions (6.1) to (6.4).

The invariance of the measure m under the transformation $T_t \omega$ of the phase space Ω into itself is just the equivalent of Liouville's theorem in classical statistical mechanics; it is a necessary condition for the introduction of probability. The full sequence of all the states of the vibrating string is completely determined, according to the uniqueness theorem, by its state at any given time t . A definition of the probability of picking at random a state of the string makes sense only if it gives the same value for the probability along a trajectory Γ , independently of the time t of our choice. We can now make a perfectly coherent definition of the probability of picking at random a state of the vibrating string by setting

$$(6.17) \quad Pr[\omega \in E] = m(E) \quad \text{for every } E \in S,$$

because along the trajectory Γ_ω we have at any time t

$$Pr[\omega \in T_t E] = m(E).$$

7. We shall say that a functional $F[\omega]$, defined over the phase space Ω , belongs to the class L , if its Lebesgue integral, computed with the measure m , exists.

For every functional $F[\omega] \in L$, we define the *statistical average* by

$$(7.1) \quad \overline{F[\omega]} = \int_{\Omega} F[\omega] dm = \int_{\Omega^*} F[\omega] dm.$$

If we observe that, on Ω^* , any functional could be put in the form

$$(7.2) \quad F[\omega] = \Phi(\beta, \gamma),$$

we get

$$(7.3) \quad \overline{F[\omega]} = \int_A \Phi(\beta, \gamma) d\mu;$$

since A is a product space, we have by the Fubini theorem

$$(7.4) \quad \overline{F[\omega]} = \int_B \left[\frac{1}{2l} \int_0^{2l} \Phi(\beta, \gamma) d\gamma \right] d\nu.$$

Noting that

$$(7.5) \quad F[T_t \omega] = \Phi(\beta, t + \gamma)$$

and that

$$(7.6) \quad \int_0^{2l} \Phi(\beta, t + \gamma) d\gamma = \int_0^{2l} \Phi(\beta, \gamma) d\gamma,$$

we get the following result:

$$(7.7) \quad \overline{F[T_t \omega]} = \overline{F[\omega]} \quad \text{for any given } t.$$

As in classical statistical mechanics, we have also to consider a second, quite different, type of average, the time average; this average is equal to the mean of all the values $F[T_t \omega]$ along a given trajectory Γ_ω :

$$(7.8) \quad \mathcal{M}_t F[\omega] = \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T F[T_t \omega] dt.$$

From (7.5) we get

$$\mathcal{M}_t F[\omega] = \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T \Phi(\beta, t + \gamma) dt;$$

thus, for every functional $F[\omega] \in L$ and for every trajectory Γ_ω , the time average exists and has the value

$$(7.9) \quad \mathcal{M}_t F[\omega] = \frac{1}{2l} \int_0^{2l} \Phi(\beta, \gamma) d\gamma;$$

this result is the particular form of the ergodic theorem for the statistical mechanics of the vibrating string.

Any time average $\mathcal{M}_t F[\omega]$ is itself a functional, say $M[\omega]$; this functional, being independent of γ according to (7.9), is invariant along every trajectory Γ :

$$(7.10) \quad M[T_t \omega] = M[\omega].$$

Combining (7.4) and (7.9), we get

$$(7.11) \quad \overline{F[\omega]} = \int_B \mathcal{M}_t F[\omega] d\nu,$$

proving that, except perhaps for some particular functionals (those for which $\mathcal{M}_t F[\omega] = \text{const.}$ for every ω) or for some particular choice of the measure m the two kinds of averages are not equal; this result means that in general the transformation T_t of Ω into itself is not metrically transitive; there are subsets E of Ω , with a measure different from 0 or 1, invariant under the transformation T_t .

We obtain two interesting examples, by taking for the measure space B the two typical cases:

(a) B is a finite or a countably infinite set of points β_n ; we define a measure ν in B , by assigning to every point β_n a weight $\nu_n \geq 0$, in such a way that

$$\sum_n \nu_n = 1.$$

Then the measure of any subset B_1 of B is simply the sum of the ν_n corresponding to the $\beta_n \in B_1$.

In this case

$$\overline{F[\omega]} = \frac{1}{2l} \sum_n \nu_n \int_0^{2l} \Phi(\beta_n, \gamma) d\gamma,$$

and for the trajectory corresponding to β_n

$$\mathcal{M}_t F[\omega_n] = \frac{1}{2l} \int_0^{2l} \Phi(\beta_n, \gamma) d\gamma.$$

(b) B is the interval $0 \leq \beta \leq 1$ and ν is the ordinary Lebesgue measure; then

$$\overline{F[\omega]} = \frac{1}{2l} \int_0^1 \left[\int_0^{2l} \Phi(\beta, \gamma) d\gamma \right] d\beta,$$

$$\mathcal{M}_t F[\omega] = \frac{1}{2l} \int_0^{2l} \Phi(\beta, \gamma) d\gamma.$$

It is obvious from these two examples that, in the statistical mechanics of the vibrating string with fixed ends, it is impossible in general to replace a time average by a statistical average. Yet there is one remarkable exception: for any linear functional $L[\omega]$ the statistical average and the time average are both equal to zero:

$$(7.12) \quad L[\omega] = \mathcal{M}_t L[\omega] = 0.$$

According to (7.11) it is enough to prove that $\mathcal{M}_t L[\omega] = 0$; here the formula (5.1) gives for any linear functional

$$L[\omega] = \int_0^{2l} g(\beta, s + \gamma) d\phi(s),$$

whence

$$\mathcal{M}_t L[\omega] = \frac{1}{2l} \int_0^{2l} \left[\int_0^l g(\beta, s + \gamma) d\phi(s) \right] d\gamma;$$

inverting the order of integration and taking account of (6.11) and (6.14), the result follows immediately.

As an application of (7.12) let us consider

$$L[\omega|x] = f(x) = g(\beta, x + \gamma);$$

if we remark that by the very definition of the Lebesgue integral

$$\int_{\Omega} L[\omega|x] dm = \int_{-\infty}^{+\infty} \xi d\mathcal{Q}_1(\xi),$$

we get immediately the condition (6.7).

8. An interesting application is the computation of the statistical average of the functional

$$G[\omega|x] = Y[T_t \omega|x]^k P[T_t \omega|x]^m V[T_t \omega|x]^n;$$

according to (5.5), (5.6) and (5.7), this average gives the value of the moment

$$\overline{y(x, t)^k p(x, t)^m v(x, t)^n}.$$

For the three first order moments, since Y, P, V are linear functionals, we have simply:

$$(8.1) \quad \overline{y(x, t)} = 0,$$

$$(8.2) \quad \overline{p(x, t)} = 0,$$

$$(8.3) \quad \overline{v(x, t)} = 0.$$

In order to compute the six second order moments we must introduce a function $\rho(h)$ defined in the following way:

$$(8.4) \quad \rho(h, \beta) = \frac{1}{2l} \int_0^{2l} g(\beta, \gamma) g(\beta, \gamma + h) d\gamma,$$

$$(8.5) \quad \rho(h) = \int_B \rho(h, \beta) d\nu.$$

We shall call $\rho(h, \beta)$ the time correlation function and $\rho(h)$ the statistical correlation function.

It is quite easy to prove that:

$$(8.6) \quad \rho(h) \text{ is periodic, } \rho(h + 2l) = \rho(h);$$

$$(8.7) \quad \rho(h) \text{ is continuous};$$

$$(8.8) \quad \int_0^l \rho(h) dh = 0;$$

$$(8.9) \quad \rho(-h) = \rho(h);$$

$$(8.10) \quad |\rho(h)| \leq \rho(0);$$

$$(8.11) \quad \rho(h) \text{ is positive definite}^6;$$

$$(8.12) \quad \rho(h) = \sum_1^{+\infty} C_n \cos n \frac{\pi h}{l},$$

the series being uniformly convergent and

$$C_n = \frac{1}{2} \int_B [a_n(\beta)^2 + b_n(\beta)^2] d\nu,$$

where

$$a_n(\beta) = \frac{1}{l} \int_0^{2l} g(\beta, \gamma) \cos n \frac{\pi \gamma}{l} d\gamma,$$

$$b_n(\beta) = \frac{1}{l} \int_0^{2l} g(\beta, \gamma) \sin n \frac{\pi \gamma}{l} d\gamma.$$

To every function g satisfying (6.10) to (6.14)—that is to every convenient choice of an invariant measure m in the phase space Ω —there corresponds a statistical correlation function $\rho(h)$ with the properties (8.6) to (8.12). By very easy computations we can now express the six second order moments as follows:

$$(8.13) \quad \overline{y(x, t)^2} = \int_{-x}^x \int_{-x}^x \rho(s_2 - s_1) ds_1 ds_2, \quad 0 \leq x \leq l,$$

$$(8.14) \quad \overline{p(x, t)^2} = 2 [\rho(0) + \rho(2x)],$$

$$(8.15) \quad \overline{v(x, t)^2} = 2 [\rho(0) - \rho(2x)],$$

$$(8.16) \quad \overline{y(x, t) p(x, t)} = \frac{1}{2} \int_{-x}^x [\rho(s+x) + \rho(s-x)] ds,$$

⁶ Positive definite means that for every set of values h_1, \dots, h_n and for every set of complex numbers X_1, \dots, X_n , the hermitian form

$$\sum_{i,k} X_i X_k^* \rho(h_i - h_k)$$

is nonnegative.

$$(8.17) \quad \overline{y(x, t) v(x, t)} = 0,$$

$$(8.18) \quad \overline{p(x, t) v(x, t)} = 0.$$

As an application of these formulas let us consider the functional sometimes called the total energy of the vibrating string:

$$\begin{aligned} E[T_t \omega] &= -2 \int_0^l f(t+x) f(t-x) dx \\ &= \frac{1}{2} \int_0^l [v(x, t)^2 - p(x, t)^2] dx; \end{aligned}$$

we have, from (8.14), (8.15) and (8.8),

$$(8.19) \quad \overline{E[T_t \omega]} = 0.$$

9. As we have seen, the time average and the statistical average of a nonlinear functional are in general different in the statistical mechanics of the vibrating string.

The only case in which we have metric transitivity for the transformation of the phase space Ω into itself corresponds to the case that $g(\beta, \gamma)$ does not depend on β :

$$(9.1) \quad g(\beta, \gamma) = g(\gamma).$$

The meaning of this very particular choice of g is quite simple: in this case our subset Ω^* is identical with a given trajectory Γ_1 . In other terms, for the corresponding measure m in the phase space Ω , we have

$$m(\Gamma_1) = 1,$$

and for the set Ω_1 of all the other trajectories

$$m(\Omega_1) = 0.$$

In the language of probability we assume that all the states of the vibrating string not represented by points ω belonging to the particular trajectory Γ_1 have not the slightest chance of being observed.

If (9.1) is true, for every functional $F[\omega] \in L$ we can put

$$F[\omega] = \Phi(\gamma);$$

then (7.4) becomes

$$F[\omega] = \frac{1}{2l} \int_0^{2l} \Phi(\gamma) d\gamma,$$

which, according to (7.9), gives obviously

$$(9.2) \quad \overline{F[\omega]} = \mathcal{M} F[\omega].$$

10. As a final remark, we should like to say: if someday, starting from the equations of a perfect or a viscous fluid, the moments of the functions involved could be computed by the same method we have applied to obtain (8.1) to (8.3) and

(8.13) to (8.18), then it would be all right to speak of a statistical mechanics of turbulence.

What has been done so far, fruitful and important as it has been for the development of fluid mechanics, is a quite different thing. Since no invariant measure has been introduced in the phase space representing the states of a fluid, we are always concerned with more or less empirical definitions of some kind of time average; often the only connection of these averages with the theoretical equations of fluid mechanics is through dimensional analysis; sometimes a really powerful physical intuition seems to have illuminated the whole field and one can not help admiring the advances which have been made. But from the theoretical point of view, we dare say that only the first steps have been taken: there is still a long way to a statistical mechanics of turbulence.

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