A CONTRIBUTION TO THE THEORY OF STOCHASTIC PROCESSES

HARALD CRAMÉR
UNIVERSITY OF STOCKHOLM

1. Introduction

Let ω denote a point or element of an arbitrary space Ω, where a probability measure $\Pi(\Sigma)$ is defined for every set Σ belonging to a certain additive class of sets in Ω, the Π-measurable sets. The probability distribution in Ω defined by $\Pi(\Sigma)$ will be referred to as the *probability field* (Π, Ω). The points ω will be denoted as the *elementary events* of the field, while any set Σ corresponds to an *event*, the probability of which is equal to $\Pi(\Sigma)$.

A complex valued Π-measurable function

$$x = g(\omega)$$

constitutes a *random variable*, defined on the field (Π, Ω). The *mean value* of x is defined by the relation

$$Ex = \int_\Omega g(\omega)\, d\Pi. \tag{1}$$

Throughout the paper, we shall always assume that, for every random variable considered, we have

$$Ex = \int_\Omega g(\omega)\, d\Pi = 0\,, \qquad E|x|^2 = \int_\Omega |g(\omega)|^2 d\Pi < \infty\,.$$

The first condition introduces some formal simplification, but does not imply any restriction of the generality of our considerations, while the second condition is essential. Two variables x and y are considered as identical, if $E|x - y|^2 = 0$.

Consider a complex valued function $x(t, \omega)$ such that, for every fixed t belonging to some specified set T, the function $x(t, \omega)$ is a Π-measurable function of ω, and thus defines a random variable $x(t)$ on the field (Π, Ω). When t ranges over T, we thus obtain a family of random variables, depending on the parameter t. On the other hand, to any fixed elementary event ω there corresponds a function

$$x(t) = x(t, \omega)\,,$$

defined for all t belonging to T, and to any event Σ there corresponds a set of functions $x(t)$ having the probability $\Pi(\Sigma)$. The function $x(t)$ will be denoted as a *random function*, defined on the field (Π, Ω).

Throughout this paper, the set T will be assumed to be the real axis, $-\infty < t < +\infty$. However, most of our considerations may easily be extended to more general spaces.

In the majority of applications, t will represent the time, and $x(t)$ will then denote some variable quantity attached to a system, the temporal development of which is subject to random influences. With reference to this type of applications, the random function $x(t)$, or the family of random variables $x(t)$, will be said to constitute a *random* or *stochastic process*.

According to (1), the mean value

$$r(t, u) = E x(t) \overline{x(u)}$$

exists for all t and u. The function $r(t, u)$ is known as the *covariance function* of the process. We always have

$$r(u, t) = \overline{r(t, u)}, \qquad r(t, t) \geqq 0 .$$

Consider the family $L(x)$ of all random variables of the form

$$c_1 x(t_1) + c_2 x(t_2) + \ldots + c_n x(t_n), \tag{2}$$

where the c_i are complex constants. Closing the set $L(x)$ with respect to convergence in the mean, we obtain an extended set $L_2(x)$. The elements of $L_2(x)$ are random variables expressible in the form (2), or as limits in the mean of random variables of the form (2). If the inner product of two arbitrary elements y and z of $L_2(x)$ is defined by the relation

$$(y, z) = E y \bar{z},$$

it is known, Karhunen [5], Cramér [3], that $L_2(x)$ is a Hilbert space. We shall call $L_2(x)$ the *linear space* of the process. If, for any t, we have

$$\underset{t_n \to t}{\text{l.i.m.}}\; x(t_n) = x(t),$$

we shall say that the process is continuous in the mean. A necessary and sufficient condition for continuity in the mean is that the covariance function $r(t, u)$ should be continuous in every point of the line $t = u$. If this condition is satisfied, $r(t, u)$ is continuous for all t and u, and the space $L_2(x)$ is separable.

Integrals of the types

$$J_1 = \int_a^b g(t)\, x(t)\, dt, \qquad J_2 = \int_a^b g(t)\, dx(t),$$

where $g(t)$ denotes a nonrandom function, can be defined for example as limits in the mean of certain sequences of approximating sums formally associated with the integrals. See, for example, Doob [4], Cramér [1], Karhunen [5], Loève [6]. Both integrals are random variables with zero mean values, and we have

$$E|J_1|^2 = \int_a^b \int_a^b g(t)\, \overline{g(u)}\, r(t, u)\, dt\, du, \tag{3}$$

$$E|J_2|^2 = \int_a^b \int_a^b g(t)\, \overline{g(u)}\, d^2 r(t, u). \tag{4}$$

Suppose first that (a, b) is a finite interval. As soon as the double integral occurring in the expression for $E|J_i|^2$ exists ($i = 1, 2$), the corresponding integral J_i

exists and possesses the ordinary formal properties of an integral. If the double integral converges when extended over the whole plane, the corresponding J_i will converge in the mean as $a \to -\infty$, $b \to +\infty$, and the limit will be defined as the integral over $(-\infty, \infty)$.

In the present paper, we shall first investigate the properties of certain *additive random set functions* attached to a class of stochastic processes. We shall then deduce a theorem concerning the representation of a stochastic process by means of integrals of the type J_2, and show that this theorem includes several previously known representations as particular cases.

2. Additive random set functions

In many types of applications, it is natural to consider the value $x(t)$ assumed by a random function at the instant t as built up additively by the successive increments of the function during the development of the process up to the instant t. In such a case we may conceive the process as an *impulse process*, the value of the impulse received during the time interval $(t, t + \Delta t)$ being $\Delta x(t) = x(t + \Delta t) - x(t)$. Thus the impulse corresponding to any time interval is a random variable associated with the interval, or a *random interval function.*

For any finite half open interval $I = [t_1, t_2)$, this interval function $X(I)$ is defined by the relation

$$X(I) = x(t_2) - x(t_1) \, . \tag{5}$$

If $I_1, I_2, \ldots, I_n$ are finite disjoint half open intervals, and $I = I_1 + \ldots + I_n$, this definition may be consistently extended by writing

$$X(I) = X(I_1) + X(I_2) + \ldots + X(I_n) \, . \tag{6}$$

It is natural to ask if the definition could be further extended, so that we could define the impulse $X(S)$ received, for example, during an arbitrary Borel set S of time points. Such an impulse function $X(S)$ should obviously possess the additive property which is a direct generalization of (6), and we should even like to extend this property in some reasonable way to the sum of an infinite sequence of disjoint sets.

It will be convenient in the first place to restrict our considerations to *bounded* sets. In this connection, we shall lay down the following definition. An *additive random set function* is a family of random variables such that:

1) For every bounded Borel set S of real numbers, $X(S)$ is a uniquely defined random variable.

2) If $S_1, S_2, \ldots$ are disjoint Borel sets, such that $S = S_1 + S_2 + \ldots$ is bounded, then $X(S) = X(S_1) + X(S_2) + \ldots$, where the series converges in the mean.

As usual we assume $EX(S) = 0$ and $E|X(S)|^2 < \infty$. Our problem is now if, given a stochastic process $x(t)$, we can find an additive random set function $X(S)$ such that, whenever S is a half open interval $I = [t_1, t_2)$, we shall have $X(S) = X(I)$ as defined by (5).

Consider first the case of a bounded set $I = I_1 + I_2 + \ldots$, where the I_n are disjoint half open intervals, $I_n = [t_n, t_n + h_n)$. Writing

$$X(I) = X(I_1) + X(I_2) + \dots, \tag{7}$$

we must then require that the series should converge in the mean. We have

$$E\left|\sum_{\mu=m}^{n} X(I_\mu)\right|^2 = \sum_{\mu,\nu=m}^{n} EX(I_\mu)\overline{X(I_\nu)} \tag{8}$$

$$= \sum_{\mu,\nu=m}^{n} \Delta^2_{I_\mu * I_\nu} r(t, u)$$

where

$$\Delta^2_{I_\mu * I_\nu} r(t, u) = r(t_\mu + h_\mu, t_\nu + h_\nu) - r(t_\mu, t_\nu + h_\nu) - r(t_\mu + h_\mu, t_\nu) + r(t_\mu, t_\nu).$$

Suppose now that the covariance function $r(t, u)$ is of bounded variation in every finite domain D, in the sense that

$$\sum_{k=1}^{N} |\Delta^2_{i_k * j_k} r(t, u)| < C \tag{9}$$

for any finite sequences of half open one dimensional intervals i_k and j_k such that the $i_k * j_k$ are disjoint two dimensional intervals belonging to D.

It then follows from (8) that the series $\Sigma X(I_\mu)$ converges in the mean, so that (7) defines a random variable $X(I)$. We shall see below that this definition is unique.

Under the condition (9), the covariance function $r(t, u)$ determines a complex valued additive set function $R(W)$ defined for all bounded Borel sets W in the (t, u)-plane, and such that

$$R(i * j) = \Delta^2_{i*j} r(t, u) \tag{10}$$

for any pair of finite half open one dimensional intervals i, j. We have

$$R = R_1^{(+)} - R_1^{(-)} + i(R_2^{(+)} - R_2^{(-)}),$$

where $R_1^{(+)}, \dots$ are nonnegative additive set functions, which are finite for any bounded Borel set W. For an *unbounded* W, the functions $R_1^{(+)}, \dots$ may be infinite, and R may become indeterminate.

Let now S be a bounded one dimensional Borel set, and let $\epsilon_1, \epsilon_2, \dots$ be a decreasing sequence of positive numbers tending to zero. We can then always find a sequence of bounded one dimensional sets $I_1, I_2, \dots$, such that

$$I_n = i_{n1} + i_{n2} + \dots, \tag{11}$$

where the i_{nk} are disjoint half open intervals, while

$$I_1 \supset I_2 \supset \dots \supset S, \tag{12}$$

$$R_1^{(+)}(S * S) \leqq R_1^{(+)}(I_n * I_n) < R_1^{(+)}(S * S) + \epsilon_n, \tag{13}$$

and similar relations for $R_1^{(-)}$, $R_2^{(+)}$ and $R_2^{(-)}$. Writing

$$z_n = X(I_n) = \sum_{k=1}^{\infty} x(i_{nk}),$$

we obtain by some calculation, using (10) and (11),

$$E|z_m - z_n|^2 = E\left|\sum_{k=1}^{\infty} [x(i_{mk}) - x(i_{nk})]\right|^2$$

$$= R(I_m * I_m) - R(I_m * I_n) - R(I_n * I_m) + R(I_n * I_n).$$

We have, however, according to (12) and (13),

$$|R(I_m * I_n) - R(S * S)| < \epsilon_q \sqrt{2},$$

where $q = \inf(m, n)$, and hence it follows that the sequence $\{z_n\}$ converges in the mean. A similar argument shows that the random variable l.i.m. z_n is independent of the choice of the approximating sequence $\{I_n\}$, the representation (11), and the sequence $\{\epsilon_n\}$. Thus if we write

$$X(S) = \underset{n\to\infty}{\text{l.i.m.}}\ z_n = \underset{n\to\infty}{\text{l.i.m.}} \sum_{k=1}^{\infty} x(i_{nk}),$$

the random variable $X(S)$ will be uniquely defined for all bounded Borel sets S. Obviously $X(S) = X(I)$ whenever S is a bounded half open interval I.

It follows easily from the definition that we have $EX(S) = 0$, and

$$EX(S_1)\overline{X(S_2)} = R(S_1 * S_2) = \int_{S_1}\int_{S_2} d^2 r(t, u).$$

In particular,

$$E|X(S)|^2 = R(S * S) = \int_S \int_S d^2 r(t, u).$$

Hence we immediately obtain for any bounded and disjoint S_1 and S_2

$$E|X(S_1 + S_2) - X(S_1) - X(S_2)|^2 = 0,$$

that is,

$$X(S_1 + S_2) = X(S_1) + X(S_2).$$

Suppose now $S = S_1 + S_2 + \ldots$, where S is bounded, and the S_k are disjoint. Then,

$$X(S) = X(S_1) + \ldots + X(S_{n-1}) + X(S_n + \ldots),$$

$$E|X(S) - X(S_1) - \ldots - X(S_{n-1})|^2 = E|X(S_n + \ldots)|^2$$

$$= R[(S_n + \ldots) * (S_n + \ldots)] \to 0,$$

as the limiting set of $(S_n + \ldots) * (S_n + \ldots)$ is empty. Thus

$$X(S_1 + S_2 + \ldots) = X(S_1) + X(S_2) + \ldots$$

in the sense of convergence in the mean. We have thus proved the following theorem.

If the covariance function $r(t, u)$ of the stochastic process $x(t)$ is of bounded variation in every finite domain, in the sense expressed by (9), *there exists an additive random set function $X(S)$, uniquely defined for all bounded Borel sets S, and such that $X(S) = x(t_2) - x(t_1)$ when S is a half open interval (t_1, t_2).*

Conversely, suppose that an additive random set function $X(S)$ is defined for all bounded Borel sets, and is such that $EX(S) = 0$ and

$$EX(S_1)\overline{X(S_2)} = \int_{S_1}\int_{S_2} d^2 r(t, u)$$

where $r(t, u)$ is of bounded variation in every finite domain. Writing

$$(14) \qquad x(t) = \begin{cases} X[(t_0, t)], & t > t_0 \cdot \\ -X[(t, t_0)], & t < t_0, \end{cases}$$

we have $Ex(t) = 0$, and the covariance function becomes

$$Ex(t)\overline{x(u)} = r(t, u) - r(t_0, u) - r(t, t_0) + r(t_0, t_0).$$

The definition of the integral

$$J_2(S) = \int_S g(t)\, dx(t) = \int_S g(t)\, dX$$

can now be extended to any bounded Borel set S, and we find that this integral is an additive random set function such that $EJ_2(S) = 0$ and

$$EJ_2(S_1)\overline{J_2(S_2)} = \int_{S_1}\int_{S_2} g(t)\overline{g(u)}\, d^2 r(t, u).$$

We shall now apply our results to some simple particular cases. Consider first a process which is *orthogonal to its increments*, so that

$$Ex(t)[\overline{x(t+h) - x(t)}] = 0$$

for all t and all $h > 0$. Writing $E|x(t)|^2 = F(t)$, where $F(t)$ is a nonnegative and never decreasing function, the covariance function becomes

$$(15) \qquad r(t, u) = Ex(t)\overline{x(u)} = F(\inf t, u),$$

and the R "distribution" is a distribution of real and positive mass on the line $t = u$, such that the segment of the line consisting of the points (τ, τ) with $\tau \leqq t$ contains the mass $F(t)$. The corresponding set function

$$X(S) = \int_S dx(t)$$

is defined for all Borel sets S having a finite upper bound, and is such that

$$EX(S_1)\overline{X(S_2)} = \int_{S_1 S_2} dF(t).$$

Thus in particular for two disjoint sets S_1 and S_2, the random variables $X(S_1)$ and $X(S_2)$ are orthogonal. Random set functions of this type have been used by Karhunen [5].

Consider finally a *stationary process*, that is, a process such that the covariance function $r(t, u)$ is a function of the difference $t - u$:

$$Ex(t)\overline{x(u)} = r(t, u) = r(t-u).$$

It is then well known that we have

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda),$$

where $F(\lambda)$ is real, never decreasing and bounded. Let us suppose

$$\int_{-\infty}^{\infty} \lambda^2 dF(\lambda) < \infty.$$

Then,

$$R(W) = \iint_W d^2 r(t-u) = -\iint_W r''(t-u)\, dt du$$

$$= \iint_W dt du \int_{-\infty}^{\infty} \lambda^2 e^{i(t-u)\lambda} dF(\lambda).$$

Thus the "density" of the R "distribution" at the point (t, u) is

$$-r''(t-u) = \int_{-\infty}^{\infty} \lambda^2 e^{i(t-u)\lambda} dF(\lambda),$$

which is constant on every line $t - u =$ const. Let, for example, S denote the rectangle bounded by the lines $t + u = a$, $t + u = b$ and $t - u = \pm h$, then it will be seen that the "mass" in this rectangle is equal to

$$(b-a)\int_{-\infty}^{\infty} \lambda \sin h\lambda\, dF(\lambda).$$

On the other hand, the infinite rectangle $t \leqq t_0$, $u \leqq t_0$ carries the constant "mass"

$$r(t, t) = r(0) = \int_{-\infty}^{\infty} dF(\lambda).$$

The corresponding random set function $X(S)$ is such that

$$EX(S_1)\overline{X(S_2)} = -\int_{S_1}\int_{S_2} r''(t-u)\, dt du.$$

3. The linear space of an additive random set function

Suppose that we are given an additive random set function $Z(S)$, defined for all bounded Borel sets S, and such that

$$EZ(S_1)\overline{Z(S_2)} = \int_{S_1}\int_{S_2} d^2 \rho(\lambda, \mu)$$

where $\rho(\lambda, \mu)$ is of bounded variation in every finite domain. Denote by $L(Z)$ the set of all random variables of the form $c_1 Z(S_1) + \ldots + c_n Z(S_n)$, where the S_k are bounded Borel sets, and the c_k are complex constants, and let $L_2(Z)$ denote the closure of $L(Z)$ with respect to convergence in the mean. We define the inner product of two arbitrary elements z and u of $L_2(Z)$ by the relation $(z, u) = Ez\bar{u}$, and denote $L_2(Z)$ as the *linear space* of the set function $Z(S)$. Like $L_2(x)$ in section 1, $L_2(Z)$ is a Hilbert space.

For any $g(\lambda)$ such that

$$(16)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda)\overline{g(\mu)}\,d^2\rho(\lambda,\mu) = \lim_{\substack{a_1, a_2 \to -\infty \\ b_1, b_2 \to \infty}} \int_{a_1}^{b_1}\int_{a_2}^{b_2} g(\lambda)\overline{g(\mu)}\,d^2\rho(\lambda,\mu)$$

is finite, the integral

$$(17)\quad z = \int_{-\infty}^{\infty} g(\lambda)\,dZ$$

will be a well defined element in $L_2(Z)$, and we shall have

$$(18)\quad E z_1 \overline{z_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\lambda)\overline{g_2(\mu)}\,d^2\rho(\lambda,\mu).$$

Consider now the set $\Lambda_2(\rho)$ of all measurable complex valued functions $g(\lambda)$, defined for $-\infty < \lambda < \infty$, and such that the integral (16) exists. Two functions g_1 and g_2 will be considered as identical, if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [g_1(\lambda) - g_2(\lambda)][\overline{g_1(\mu)} - \overline{g_2(\mu)}]\,d^2\rho(\lambda,\mu) = 0.$$

If we define the inner product of two elements in $\Lambda_2(\rho)$ by the integral in the second member of (18), the set $\Lambda_2(\rho)$ will have all the properties of Hilbert space, except possibly the completeness property. In fact, it is obvious that (g_1, g_2) has the ordinary bilinear and Hermite symmetric properties, and further

$$(g, g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda)\overline{g(\mu)}\,d^2\rho(\lambda,\mu) = E\left|\int_{-\infty}^{\infty} g(\lambda)\,dZ\right|^2 \geqq 0,$$

and $(g, g) = 0$ only when

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\lambda)\overline{g(\mu)}\,d^2\rho(\lambda,\mu) = 0,$$

that is, when $g(\lambda)$ is identical with 0. It follows that we have the Schwarz inequality,

$$|(g_1, g_2)|^2 \leqq (g_1, g_1)(g_2, g_2).$$

If the space $\Lambda_2(\rho)$ is not complete, it can be made complete by adjunction, von Sz. Nagy [7, p. 4], and we shall suppose that this has been done, so that $\Lambda_2(\rho)$ is a Hilbert space.

To any function $g(\lambda)$ in $\Lambda_2(\rho)$ there corresponds by (17) a uniquely defined element z in $L_2(Z)$, and it follows from (18) that we have $(z_1, z_2) = (g_1, g_2)$ and hence, defining the norms in the usual way,

$$(19)\quad \|z_1 - z_2\| = \|g_1 - g_2\|.$$

An "ideal" element in $\Lambda_2(\rho)$ is a sequence of functions $g_1, g_2, \ldots$ convergent in norm, and according to (19) the sequence $z_1, z_2, \ldots$ of the corresponding elements in $L_2(Z)$ converges in the mean, and thus defines a unique limiting element z in $L_2(Z)$. Consequently to every element in $\Lambda_2(\rho)$ there corresponds one element in $L_2(Z)$, uniquely defined by (17). The transformation is obviously linear and, according to (19), isometric.

We shall show that, conversely, to every element in $L_2(Z)$, there corresponds one and only one element in $\Lambda_2(\rho)$, so that (17) defines a one to one linear and isometric correspondence between the two spaces. Suppose first that $z = c_1 Z(S_1) + \ldots + c_n Z(S_n)$ is an element of $L(Z)$. Then obviously the unique corresponding element in $\Lambda_2(\rho)$ is the function $c_1 e_{S_1}(\lambda) + \ldots + c_n e_{S_n}(\lambda)$, where $e_S(\lambda)$ denotes the characteristic function of the set S. Now any other element of $L_2(Z)$ is the limit in the mean of a sequence $z_1, z_2, \ldots$ of elements in $L(Z)$, and according to (19) the corresponding elements $g_1, g_2, \ldots$ in $\Lambda_2(\rho)$ will converge in norm, and will thus define an element of $\Lambda_2(\rho)$ such that (19) holds. It finally follows from (19) that this element is unique.

Obviously the elements $Z(S)$ form a base of the space $L_2(Z)$, and similarly the elements $e_S(\lambda)$ form a base of the space $\Lambda_2(\rho)$. It follows that whenever we are concerned with a one to one linear correspondence between the two spaces, such that $Z(S)$ and $e_S(\lambda)$ are always corresponding elements, this will coincide with the transformation defined by (17). We shall use this remark in the following paragraph.

4. Integral representation of a stochastic process

Consider now a function $g(t, \lambda)$ such that, for every fixed real t, $g(t, \lambda)$ belongs to $\Lambda_2(\rho)$. The integral

$$x(t) = \int_{-\infty}^{\infty} g(t, \lambda)\, dZ \tag{20}$$

is then defined for every real t, and we have

$$r(t, u) = Ex(t)\overline{x(u)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t, \lambda)\overline{g(u, \mu)}\, d^2\rho(\lambda, \mu). \tag{21}$$

Conversely, if we know that the covariance function of a given stochastic process is of the form (21), it can be shown that the random function associated with the process can be expressed in the form (20). We have, in fact, the following theorem.

Let $x(t)$ be the random function associated with a stochastic process such that $Ex(t) = 0$ and the covariance function $r(t, u)$ is given by the expression (21), *where $\rho(\lambda, \mu)$ is known to be a covariance function which is of bounded variation in every finite domain, in the sense expressed by* (9). *Then there exists an additive random set function $Z(S)$ such that $EZ(S) = 0$ and*

$$EZ(S_1)\overline{Z(S_2)} = \int_{S_1}\int_{S_2} d^2\rho(\lambda, \mu),$$

and such that (20) *holds for every real t. Further, we have $L_2(Z) = L_2(x)$ when and only when there does not exist any element in $\Lambda_2(\rho)$ different from zero, which is orthogonal to $g(t, \lambda)$ for all real t.*

According to (21), the correspondence $x(t) \rightleftarrows g(t, \lambda)$ defines a one to one linear and isometric correspondence between the space $L_2(x)$ and the subspace of $\Lambda_2(\rho)$

spanned by the functions $g(t, \lambda)$ when the parameter t ranges through the whole real axis.

Suppose first that the set of functions of λ obtained from $g(t, \lambda)$ when t ranges through the real axis forms a base of the space $\Lambda_2(\rho)$. Then the correspondence defined by $x(t) \rightleftarrows g(t, \lambda)$ will extend to the whole spaces $L_2(x)$ and $\Lambda_2(\rho)$. Let S be any bounded Borel set, and let $Z(S)$ denote the element of $L_2(x)$ that corresponds to the function $e_S(\lambda)$ in $\Lambda_2(\rho)$. Then $EZ(S) = 0$, and by (21) we have

$$EZ(S_1)\overline{Z(S_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{S_1}(\lambda)\, e_{S_2}(\mu)\, d^2\rho(\lambda, \mu) = \int_{S_1}\int_{S_2} d^2\rho(\lambda, \mu).$$

It is also easily seen that $Z(S)$ is a completely additive function of S, in the sense of convergence in the mean. Thus $Z(S)$ defines an additive random set function. Since $Z(S)$ is an element of $L_2(x)$, we have $L_2(Z) \subset L_2(x)$. On the other hand, the elements $e_S(\lambda)$ form a base of $\Lambda_2(\rho)$, and thus the corresponding elements $Z(S)$ in $L_2(x)$ form a base of the latter space, so that we have $L_2(Z) = L_2(x)$. Thus the correspondence $x(t) \rightleftarrows g(t, \lambda)$ defines a one to one linear correspondence between $L_2(Z)$ and $\Lambda_2(\rho)$, such that $Z(S)$ and $e_S(\lambda)$ are corresponding elements. According to the concluding remark of the preceding paragraph, the element $x(t)$ in $L_2(Z) = L_2(x)$ corresponding to the element $g(t, \lambda)$ in $\Lambda_2(\rho)$ is then for every t given by the relation (20).

In the case when the $g(t, \lambda)$ do not form a base of $\Lambda_2(\rho)$, we have to adjoin a conveniently chosen set of functions $h(u, \lambda)$ in order to obtain a base, and the proof can then be completed in the same manner as in the case treated by Karhunen [5, p. 47]. It should be observed that, in this case, we have to perform an extension of the basic probability field in order to prove the existence of $Z(S)$, and accordingly we have $L_2(Z) \supset L_2(x)$.

The truth of the last assertion of the theorem follows now from the fact that a nonzero element of $\Lambda_2(\rho)$ which is orthogonal to all the $g(t, \lambda)$ exists when and only when the $g(t, \lambda)$ do not form a base of the space $\Lambda_2(\rho)$.

5. Applications to some particular cases

Taking first $g(t, \lambda) = e^{it\lambda}$, and supposing that $\rho(\lambda, \mu)$ is of bounded variation over the whole plane, we obtain the class of stochastic processes denoted by Loève [6] as *harmonizable processes*, and (20) becomes the spectral representation

$$x(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ$$

of a process belonging to this class. In this case, the functions $e^{it\lambda}$ obviously form a base of the space $\Lambda_2(\rho)$, and we have $L_2(x) = L_2(Z)$.

Suppose, on the other hand, that $\rho(\lambda, \mu)$ in (21) is the covariance function of an orthogonal process. (Compare section 2.) The relation (21) then reduces to

$$r(t, u) = \int_{-\infty}^{\infty} g(t, \lambda)\, \overline{g(u, \lambda)}\, dF(\lambda), \tag{22}$$

and (20) gives the representation due to Karhunen [5] of a stochastic process having a covariance function of the form (22). In the particular case when $g(t, \lambda) = e^{it\lambda}$

and $F(\lambda)$ is bounded, this reduces to the well known spectral representation of a stationary process.

Consider in particular the case of a stationary process with an absolutely continuous *spectral function* $F(\lambda)$. Then (22) can be written in the form

$$r(t, u) = r(t-u) = \int_{-\infty}^{\infty} e^{i(t-u)\lambda} F'(\lambda)\, d\lambda$$
$$= \int_{-\infty}^{\infty} e^{it\lambda} \sqrt{F'(\lambda)} \cdot \overline{e^{iu\lambda} \sqrt{F'(\lambda)}}\, d\lambda .$$

where $F'(\lambda)$ and $\sqrt{F'(\lambda)}$ are nonnegative. Since $\sqrt{F'(\lambda)}$ is quadratically integrable over the real axis, it has a Fourier transform which we denote by $g(\tau)$. Then $g(t+\tau)$ is the Fourier transform of $e^{it\lambda}\sqrt{F'(\lambda)}$, and the Parseval formula gives

$$r(t, u) = \int_{-\infty}^{\infty} e^{it\lambda} \sqrt{F'(\lambda)}\, \overline{e^{iu\lambda} \sqrt{F'(\lambda)}}\, d\lambda = \int_{-\infty}^{\infty} g(t+\tau)\, \overline{g(u+\tau)}\, d\tau .$$

Hence we obtain by the theorem of the preceding paragraph

$$x(t) = \int_{-\infty}^{\infty} g(t+\tau)\, dZ$$

where [5, p. 72] $Z(S)$ is an additive random set function such that

$$EZ(S_1)\overline{Z(S_2)} = m(S_1S_2),$$

where m denotes the ordinary Lebesgue measure of the product set S_1S_2.

REFERENCES

[1] H. Cramér, "On the theory of stationary random processes," *Annals of Math.*, Vol. 41 (1940), pp. 215–230.

[2] ———, "On harmonic analysis in certain functional spaces," *Ark. Mat. Astr. Fys.*, Vol. 28 B (1942), pp. 1–7.

[3] ———, "Problems in probability theory," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 165–193.

[4] J. L. Doob, "Stochastic processes depending on a continuous parameter," *Trans. Amer. Math. Soc.*, Vol. 42 (1937), pp. 107–140.

[5] K. Karhunen, "Über lineare Methoden in der Wahrscheinlichkeitsrechnung," *Ann. Acad. Scient. Fennicae*, Ser. AI, No. 37 (1947), pp. 1–79.

[6] M. Loève, "Fonctions aléatoires du second ordre," supplement to P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948. This work contains references to the earlier papers on the subject by M. Loève.

[7] B. von Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Springer, Berlin, 1942.