# CONTINUOUS INDEX NUMBERS AND QUANTITATIVE STUDY OF THE GENERAL ECONOMY

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## 1. Introduction

The practical economist, concerned with the possible effects of new taxes or other changes in government policy on the structure of the general economy, must resort to index numbers for quantitative evaluation of his qualitative conclusions. Conventional econometric theory, however, upon which he may wish to base his reasoning, finds its most rigorous expression, not in terms of index numbers, but rather in terms of systems of equations in a large and indefinite number of variables, such as the Walrasian equations of the general equilibrium [**5**][1] or in dynamic analogues of the same. The problem of finding some logical or functional connection between the Walrasian and the aggregative points of view is therefore of practical as well as theoretical interest. The present paper represents a reformulation and extension of the author's previous work on this problem [**2**]. It is entirely self-contained, however, and can be followed without reference to the earlier material and with only the most general layman's knowledge of economic theory and terminology.

## 2. Classes of commodities

In what follows, the terms "commodity" and "goods" will be used synonymously to refer to anything which can command a price, including manufactured articles, patents and other rights or titles, labor or other personal services, and the use of land. The present discussion will be concerned primarily with aggregates of commodities obtained by subdividing the set of all possible commodities into mutually exclusive subclasses. The exact manner of subdivision is unimportant except that one preliminary division into four fundamental categories is essential, any convenient subdivision of these basic categories being acceptable.

Most goods are manufactured in one or more of the industrial processes of the economy. Certain goods, however, such as labor, are not. We shall denote the class of all commodities which are produced in the system by the letter "P," suggesting "products." Many goods, moreover, are used in the process of producing other goods. Let us denote the class of all such commodities by the letter "F," suggesting "factors of production."

The classes P and F are not exclusive, and the class of commodities $P \cdot F$, common to both, is customarily referred to as the class of "capital goods." Moreover, the class $\overline{P} \cdot \overline{F}$, composed of goods which belong to neither P nor F,

[1] Boldface numbers in brackets refer to references at the end of the paper (see p. 221).

is not vacuous, since it may be regarded as including various direct personal services, such as that of domestic servants.

We may thus use the classes P and F to distinguish four mutually exclusive categories as follows:

*a*) The class $R = P - P \cdot F$ of consumer's or consumption goods, produced in the system but not used in further production, and consisting principally of retail sales.
*b*) The class $C = P \cdot F$ of capital goods.
*c*) The class $L = F - PF$ of primary factors of production consisting principally of labor and the services of land.
*d*) The class $O = \bar{P} \cdot \bar{F}$ of primary consumption goods consisting principally of direct personal services.

The last of these classes is not important in a quantitative sense economically and is of little theoretical interest for the present discussion. For these reasons, and without loss of generality, it will be neglected. The aggregates to be considered may thus be regarded as obtained from arbitrary subdivision of the basic categories R, C, and L into mutually exclusive subclasses.

## 3. Production functions

Profits are given by the difference between selling value and cost. In industries with a fairly routine production process and for which the prices of both products and factors are stable, it may be possible to relate total costs to the number of units produced by means of a cost function, say:

$$\text{Cost} = C(q), \tag{1}$$

where $q$ is the number of units produced. In general, however, in the course of time and as prices change, it may be possible and desirable to substitute some factors of production for others. The simplest description of the technical production situation may then be to assume that $q$, the quantity of product turned out, is a function of $q_1, \cdots, q_n$, the amounts of each of $n$ different factors of production used. We then have

$$q = q(q_1, \cdots, q_n), \tag{2}$$

with costs being given by

$$\text{Cost} = \sum_{j=1}^{j=n} p_j q_j, \tag{3}$$

where $p_1, \cdots, p_n$ are the prices per unit of the corresponding factors of production.

Unfortunately it is not often possible to adjust the rates of application of all factors of production independently. Men without tools will not contribute to increased production. Other more complicated linkages may exist among the factors. In general we may have, in addition to equation (2), other restraints:

$$F_k = F_k(q_1, \cdots, q_n) = 0; \qquad k = 1, \cdots, r, \tag{4}$$

where $r < n$. The possible existence of joint factors[2] or other such factor linkages will complicate considerably certain portions of the subsequent discussion.

At this point it will be convenient to return to the classes of commodities P and F and define for them relationships of the type (2), (3), and (4). Certain notational conventions will be useful. Relations of type (2) will be assumed to hold for each commodity of class P, and these relations will be written as follows:

$$q_i = q_i[q_j^i; j \epsilon \mathrm{F}]; \qquad i \epsilon \mathrm{P}. \tag{5}$$

In this notation $q_j^i$ represents the quantity of factor $j$ used in the production of quantity $q_i$. The notation $j \epsilon \mathrm{F}$ indicates that $q_i$ is a function of all the $q_j^i$ corresponding to factors of production, that is, of the inputs of all possible commodities of class F. The notation $i \epsilon \mathrm{P}$ indicates that (5) is not a single equation but represents rather a set of production equations including one for each commodity from class P.

In a similar way costs will be defined as

$$\text{Cost of } q_i = \sum_{j \epsilon \mathrm{F}} p_j q_j^i; \qquad i \epsilon \mathrm{P}, \tag{6}$$

where $p_j$ is the price per unit of factor $j$.

Finally the possible restraints on factor allocation will be written

$$0 = F_i^k = F_i^k[q_j^i; j \epsilon \mathrm{F}]; \qquad i \epsilon P, \qquad k = 1, \cdots, \tau_i, \tag{7}$$

where $\tau_i$ is the number of such restraints that figure in the production of commodity $i$.

## 4. Class indices

Divisia [**3**] has introduced the notion of continuous index numbers for measuring the change over time of the average price and quantity of an aggregate of commodities. This type of index will be used here for comparisons made along any hypothetical path of transformation of the system, not necessarily the actual transformation in time.

[2] The possible existence of joint products is still more troublesome, and will not be considered here. In order to discuss this, one must distinguish two spaces: an "output space" and an "input space." The coördinates of a point in the output space would represent the quantities of the various products turned out, and the coördinates of a point in the input space would represent the quantities of the various factors of production used. The technical situation involved could then be described in terms of a mapping or correspondence of points in the one space on points in the other. In general each point in the output space would correspond to (or could be regarded as produced by) a set of points in the input space and lying on some hypersurface in that space. Conversely each point in the input space would represent an allocation of factors which could be used to produce any one of a set of possible joint outputs represented by a set of points lying on a hypersurface in the output space.

Let $X$ denote the aggregate of commodities under consideration. The corresponding price and quantity indices will be defined from the following relations:

$$V_X d \log P_X = \sum_{i \in X} v_i d \log p_i, \tag{8}$$

$$V_X d \log Q_X = \sum_{i \in X} v_i d \log q_i, \tag{9}$$

where the $d \log p_i$ and the $d \log q_i$ represent fractional changes in the corresponding $p_i$ and $q_i$ along any path of displacement, where $v_i = p_i q_i$, and where

$$V_X = \sum_{i \in X} v_i. \tag{10}$$

The index differentials thus defined may be integrated along any given path of displacement to give true index numbers. If $\lambda$ is some parameter of displacement, for example, we have

$$P_X(\lambda) = P_X(\lambda_0) \exp \int_{\lambda_0}^{\lambda} \sum_{i \in X} \left\{ \frac{v_i}{V_X} \frac{d \log p_i}{d\lambda} \right\} d\lambda. \tag{11}$$

It is easy to verify from (8), (9), and (10) that

$$d \log V_X = d \log P_X + d \log Q_X = d \log (P_X Q_X). \tag{12}$$

Moreover, it is easy to prove that the Divisia indices are the only ones whose differentials are linear in the differentials of prices and quantities such that price indices remain constant when only quantities are changed and vice versa, and such that the product of price and quantity indices is an index of value. These properties make the Divisia indices particularly suitable for the present development.

## 5. Aggregate production functions

Applying the formula for a total differential to equations (5), after transformation to logarithmic form, we have

$$d \log q_i = \sum_{j \in F} \frac{\partial \log q_i}{\partial \log q_j^i} d \log q_j^i; \qquad i \in P. \tag{13}$$

Logarithmic derivatives of the type appearing in (13) are frequently called "elasticities" by economists. In order to direct attention to this and to simplify subsequent notation, it will be convenient to introduce this abbreviation $E[q_i : q_j^i]$ for the partials in question. In general the symbol $E[x : y]$ will be used to denote a derivative (either total or partial, as the case may be) of $\log x$ with respect to $\log y$.

Now substitution of (13) into (9) gives

$$V_X d \log Q_X = \sum_{i \epsilon X} \sum_{j \epsilon \mathrm{F}} v_i E\,[q_i : q_j^i\,] d \log q_j^i, \tag{14}$$

or

$$V_X d \log Q_X = \sum_{Y c \mathrm{F}} \sum_{i \epsilon X} \sum_{j \epsilon Y} v_i E\,[q_i : q_j^i\,] d \log q_j^i, \tag{15}$$

where the notation $Y\, c\, \mathrm{F}$ denotes summation over all subclasses $Y$ of F. Fixing attention on some particular subclass $Y$, we may define a quantity $(d \log Q_X)_Y$. representing the differential change in $Q_X$ corresponding to changes in the allocation of factors belonging to $Y$ only, writing

$$V_X (d \log Q_X)_Y = \sum_{i \epsilon X} \sum_{j \epsilon Y} v_i E\,[q_i : q_j^i\,] d \log q_j^i. \tag{16}$$

Moreover, in analogy with (9), we may define an index $Q_Y^X$ representing the amount of factors from $Y$ used in producing products from $X$. We may write

$$V_Y^X\, d \log Q_Y^X = \sum_{i \epsilon X} \sum_{j \epsilon Y} v_j^i\, d \log q_j^i, \tag{17}$$

where

$$V_Y^X = \sum_{i \epsilon X} \sum_{j \epsilon Y} v_j^i. \tag{18}$$

The ratio of $(d \log Q_X)_Y$ to $d \log Q_Y^X$ represents the rate at which $\log Q_X$ changes with respect to variation in $\log Q_Y^X$, with the $q_j^i$ for $i \epsilon X$ and $j \epsilon \mathrm{F}\text{-}Y$ held constant. But this ratio bears sufficient resemblance to the usual definition of a partial derivative to justify the use of the partial derivative or elasticity notations, namely,

$$\frac{(d \log Q_X)_Y}{d \log Q_Y^X} = \frac{\partial \log Q_X}{\partial \log Q_Y^X} = E\,[Q_X : Q_Y^X]. \tag{19}$$

Equation (15) may now be written

$$V_X\, d \log Q_X = \sum_{Y c F} V_X\, E\,[Q_X : Q_Y^X\,]\, d \log Q_Y^X, \tag{20}$$

or simply

$$d \log Q_X = \sum_{Y c F} E\,[Q_X : Q_Y^X\,]\, d \log Q_Y^X \tag{21}$$

in analogy with (13) and the formulas for total partials, *as though* $\log Q_X$ were a function of the $\log Q_Y^X$.

A few remarks may be in order concerning the generalized partials $E\,[Q_X : Q_Y^X\,]$. In the first place, these quantities are not functions of the $Q_Y^X$ as ordinary partials would be (and therefore $\log Q_X$ is *not* a function of the log

$Q_Y{}^X$). On the contrary they are functions of the actual path of displacement, since they involve the $d \log q_j{}^i$ as well as the $v_i$, the $v_j{}^i$, and the partials, $E[q_i : q_j{}^i]$. Moreover, they may not be defined at all in the event $d \log Q_Y{}^X = 0$ since $d \log Q_Y{}^X$ may vanish although $(d \log Q_X)_Y$ does not. In general, however, they represent a sort of mean value for the sums of partials

$$s_Y{}^i = \sum_{j \in Y} E[q_i : q_j{}^i]; \qquad i \in X c \mathrm{P}, \tag{22}$$

since $(d \log Q_X)_Y$ is a weighted arithmetic mean of the sums

$$t_Y{}^i = \sum_{j \in Y} E[q_i : q_j{}^i] d \log q_j{}^i; \qquad i c X \in \mathrm{P}, \tag{23}$$

using the appropriate $v_i$ as weights, and since division by $d \log Q_Y{}^X$ produces a deflation of $(d \log Q_X)_Y$ with respect to a weighted arithmetic mean of the $d \log q_j{}^i$ using the $v_j{}^i$ as weights. If the $d \log q_j{}^i$ are all equal, $E[Q_X : Q_Y{}^X]$ is an internal mean of the $s_Y{}^i$. In general, however, the mean may be external. Later sections will throw further light on the possible values for the aggregate elasticities $E[Q_X : Q_Y{}^X]$ when the $d \log q_j{}^i$ are given by actual displacements with time.

## 6. The hypothesis of strict competition

Profits for the industry producing commodity $i$ may be written

$$\pi_i = p_i q_i - \sum_{j \in \mathrm{F}} p_j q_j{}^i; \qquad i \in \mathrm{P}, \tag{24}$$

where $p_i$ is the price of the product and the $p_j$, as before, are the prices of the corresponding factors. It is the business of the individual entrepreneur to make these profits as large as possible. The manner in which he attempts to do this will depend on which variables he assumes to be subject to his control. If he assumes that no change in his level of output or in his rate of application of any of the factors could appreciably affect the relevant prices, and if he attempts to maximize his profits accordingly, he will then be said to operate under the "hypothesis of strict competition." This is equivalent to assuming (*a*) that all of his possible output could be sold at prevailing prices but that nothing could be sold above these prices, and (*b*) that all amounts of the factors of production needed could be obtained at prevailing prices but that none would be forthcoming at lower ones. Many alternative hypotheses are possible and one of these, the case of monopoly, will be considered in section 8.

Assuming the existence of production functions of the type (5) but neglecting the possibility of additional restraints (7), we have as necessary conditions for maximum profits the following:

$$p_i \, \partial q_i / \partial q_j{}^i - p_j = 0; \qquad i \in \mathrm{P}, \qquad j \in \mathrm{F}, \tag{25}$$

obtained from (24) by equating to zero the partial derivatives of the form $\partial \pi_i / \partial q_j{}^i$. Because of the hypothesis of strict competition, the $p$'s are regarded as constant for this differentiation. It will be convenient to transform (25) into logarithmic form involving the elasticities $E[q_i : q_j^i]$. We have, obviously,

$$v_i E[q_i : q_j^i] = v_j^i; \qquad i \in \mathrm{P}, \quad j \in \mathrm{F}. \tag{26}$$

Now, assuming that all entrepreneurs act in accordance with the hypothesis of strict competition, we may substitute (26) into (16), obtaining

$$V_X (d \log Q_X)_Y = \sum_{i \in X} \sum_{j \in Y} v_j^i d \log q_j^i = V_Y{}^X d \log Q_Y{}^X, \tag{27}$$

from which

$$V_X E[Q_X : Q_Y{}^X] = V_Y{}^X; \qquad X c \mathrm{P}, \quad Y c \mathrm{F}, \tag{28}$$

in analogy with (26).

## 7. Sufficient conditions for profit maxima

In order that a solution of equations (25) should provide a true set of maxima for the corresponding profits $\pi_i$ for fixed prices, it is sufficient that the quadratic forms

$$\sum_{j \in \mathrm{F}} \sum_{k \in \mathrm{F}} \frac{\partial^2 q_i}{\partial q_j^i \partial q_k^i} \Delta q_j^i \Delta q_k^i; \qquad i \in \mathrm{P}, \tag{29}$$

be negative definite. As an example of a production function for which these sufficient conditions are satisfied for the critical values of the independent variables, we may cite the case for which the $q_i$ are given by

$$\log q_i = A_i + \sum_{j \in \mathrm{F}} a_j^i \log q_j^i; \qquad i \in \mathrm{P}, \tag{30}$$

where the $A_i$ and the $a_j^i$ are constants, where the $a_j^i \geqq 0$, and where

$$\sum_{j \in \mathrm{F}} a_j^i < 1; \qquad i \in \mathrm{P}. \tag{31}$$

Obviously, $a_j^i = E[q_i : q_j^i]$. An elementary proof, which will not be reproduced here, suffices to show that the forms (29) are negative definite, under strict competition, for the production functions defined by (30) and (31).

Now returning to the index variables and letting $\lambda$ be a parameter defining position along a given path of displacement, we have

$$\frac{V_X}{Q_X} \frac{dQ_X}{d\lambda} = \sum_{i \in X} \frac{v_i}{q_i} \frac{dq_i}{d\lambda}, \tag{32}$$

and after differentiation with respect to $\lambda$

$$\frac{V_X}{Q_X} \frac{d^2 Q_X}{d\lambda^2} = \sum_{i \in X} \frac{v_i}{q_i} \frac{d^2 q_i}{d\lambda^2} + \rho_X [p_i, q_i], \tag{33}$$

where

$$\rho_X[p_i, q_i] = \sum_{i\epsilon X} v_i \left(\frac{d \log p_i}{d\lambda} - \frac{d \log P_X}{d\lambda}\right)\left(\frac{d \log q_i}{d\lambda} - \frac{d \log Q_X}{d\lambda}\right) \tag{34}$$

If we restrict ourselves to displacements for which the $p_i$ are unchanged, $\rho_X[p_i, q_i] = 0$ and we have

$$\frac{V_X}{Q_X}\frac{d^2Q_X}{d\lambda^2} = \sum_{i\epsilon X}\frac{v_i}{q_i}\left[\sum_{j\epsilon \mathrm{F}}\sum_{k\epsilon \mathrm{F}} \frac{\partial^2 q_i}{\partial q_j^i \partial q_k^i}\frac{dq_j}{d\lambda}\frac{dq_k}{d\lambda}\right] \tag{35}$$

We thus have $d^2Q_X/d\lambda^2 \leqq 0$ provided the quadratic forms (29) are negative definite for all $i \epsilon X$. This result will be of interest later.

## 8. The case of monopoly

As remarked previously, there are many hypotheses as to entrepreneurial behavior alternative to the hypothesis of strict competition. We shall consider only the case in which the individual entrepreneurs assume knowledge of the demand functions for their respective products, that is, of the way in which the amount which can be sold falls off as the price increases. Specifically we shall assume that the monopolist regards $p_i$ as a known function of $q_i$ in (24),the expression for profits. The maximizing equations, (25), will thus be replaced by the following.

$$(p_i + dp_i/dq_i)\,\partial q_i/\partial q_j^i - p_j = 0; \qquad i \epsilon \mathrm{P}, \quad j \epsilon \mathrm{F}. \tag{36}$$

In logarithmic form this becomes

$$v_i(1 + E[p_i : q_i])\,E[q_i : q_j^i] = v_j^i; \qquad i \epsilon \mathrm{P}, \quad j \epsilon \mathrm{F}, \tag{37}$$

or

$$v_i E[q_i : q_j^i] = \mu_i v_j^i; \qquad i \epsilon \mathrm{P}, \quad j \epsilon \mathrm{F}, \tag{38}$$

where $\mu_i$ is the reciprocal of $1 + E[p_i : q_i]$ and may be regarded as a measure of the degree of monopoly in the corresponding industry. It may be noted that, whereas $E[q_i : q_j^i]$ may be safely assumed to be positive, $E[p_i : q_i]$ is equally certain to be negative. For this reason economists usually affix a minus sign to the quantity $E[p_i : q_i]$ in defining a demand elasticity.

Passing now to the index variables, we readily see that relations (28) take the form

$$V_X E[Q_X : Q_Y^X] = M_Y^X V_Y^X; \qquad X c \mathrm{P}, \quad Y c \mathrm{F}, \tag{39}$$

where

$$M_Y^X = \frac{\sum_{i\epsilon X}\sum_{j\epsilon Y} \mu_i v_j^i \, d \log q_j^i}{\sum_{i\epsilon X}\sum_{j\epsilon Y} v_j^i \, d\log q_j^i} \tag{40}$$

$M_Y{}^X$ is thus a weighted arithmetic mean of the $\mu_i$, using the expressions $\sum_{j\epsilon Y} v_j{}^i\, d \log q_j{}^i$ as weights. This quantity is thus defined separately for each $Y c \mathrm{F}$. A more suitable measure of aggregate degree of monopoly might be given by $M_X$ defined by the relation

$$M_X \sum_{i\epsilon X} \sum_{j\epsilon \mathrm{F}} v_j{}^i\, d \log q_j{}^i = \sum_{i\epsilon X} \sum_{j\epsilon \mathrm{F}} \mu_i v_j{}^i\, d \log q_j{}^i \tag{41}$$

or the equivalent relation

$$M_X \sum_{Y c \mathrm{F}} V_Y{}^X\, d \log Q_Y{}^X = \sum_{Y c \mathrm{F}} M_Y{}^X\, V_Y{}^X\, d \log Q_Y{}^X. \tag{42}$$

Since $M_X$ is a mean of the $M_Y{}^X$, in the event the $M_Y{}^X$ for $Yc\mathrm{F}$ do not vary much with $Y$, it may be satisfactory to replace the $M_Y X$ by $M_X$ in relations (39), giving

$$V_X \quad E\,[Q_X \ : \ Q_Y{}^X] \ = \ M_X \quad V_Y{}^X. \tag{43}$$

## 9. Production functions with restraints

In the event the $q_j{}^i$ involved in the production functions are not independent but are restricted to vary in accordance with (7), the definition of aggregate variables may be complicated or entirely impossible along the lines developed thus far. If we solve equations (7) for the first $r_i$ of the $q_j{}^i$ for each $i$, the necessary conditions for a maximum become

$$p_i \frac{\partial q_i}{\partial q_j{}^i} - p_i + \sum_{k=1}^{k=r_i} \left[ p_i \frac{\partial q_i}{\partial q_k{}^i} - p_k \right] \frac{\partial q_k{}^i}{\partial q_j{}^i} = 0, \tag{44}$$

where $i \epsilon \mathrm{P}$ and $k$ runs over the first $r_i$ commodities of F and $j$ runs over the remaining ones. Moreover, (13) must be replaced by

$$d \log q_i = \sum_{\substack{j>r_i \\ j\epsilon \mathrm{F}}} \left\{ E\,[q_i : q_j{}^i] + \sum_{\substack{k\leq r_i \\ k\epsilon \mathrm{F}}} E\,[q_i : q_k]E\,[q_k{}^i : q_j{}^i] \right\} d \log q_j{}^i. \tag{45}$$

Let us consider a special case for which it happens that we can solve (7) for the $q_j{}^i$ for $j \epsilon Y_X = Y_1 + Y_2 + \cdots + Y_{r_x}$ in terms of the remaining $q_j{}^i$ for $j \epsilon \mathrm{F} - Y_X$. We thus assume the restraints to be of such a number that the dependent variables among the $q_j{}^i$ may be selected to fall in $Y_X$ only, whereas the independent variables all fall in $\mathrm{F} - Y_X$, where $Y_X$ is the sum of several of the factor aggregates $Y$ and where this particular partition is assumed to be possible for each $i \epsilon X$. After transformation to logarithmic form, (44) then becomes

$$v_i E\,[q_i : q_j{}^i] - v_j{}^i + \sum_{k\epsilon Y_X} \{v_i E\,[q_i : q_j{}^i] - v_k{}^i\} E\,[q_k{}^i : q_j{}^i] = 0. \tag{46}$$

Similarly (45) becomes

$$d \log q_i = \sum_{j \in F - Y_x} \{E[q_i : q_j{}^i] + \sum_{k \in Y_x} E[q_i : q_k{}^i] E[q_k{}^i : q_j{}^i]\} d \log q_j{}^i. \tag{47}$$

In index form, we have

$$d \log Q_X \sum_{Y c F - Y_x} \{E[Q_X : Q_Y{}^X] + \sum_{Z c Y_x} E[Q_X : Q_Z{}^X] E[Q_Z{}^X : Q_Y{}^X]\} d \log Q_Y{}^X, \tag{48}$$

where

$$V_X E[Q_X : Q_Z{}^X] d \log Q_Z{}^X = \sum_{i \in X} \sum_{k \in Z} v_i E[q_i : q_k{}^i] d \log q_k{}^i, \tag{49}$$

$$V_Z{}^X E[Q_Z{}^X : Q_Y{}^X] d \log Q_Y{}^X = \sum_{i \in X} \sum_{j \in Y} \sum_{k \in Z} v_k{}^i E[q_k{}^i : q_j{}^i] d \log q_j{}^i, \tag{50}$$

and

$$V_X E[Q_X : Q_Z{}^X] E[Q_Z{}^X : Q_Y{}^X] d \log Q_Y{}^X = \sum_{i \in X} \sum_{j \in Y} \sum_{k \in Z} v_i E[q_i : q_k{}^i] E[q_k{}^i : q_j{}^i] d \log q_j{}^i. \tag{51}$$

Now if we multiply (46) by $d \log q_j{}^i$, sum for $i \in X$, $j \in Y$, and divide by $d \log Q_Y{}^X$, we have

$$V_X E[Q_X : Q_Y{}^X] - V_Y{}^X + \sum_{Z c Y_x} \{V_X E[Q_X : Q_Z{}^X] - V_Z{}^X\} E[Q_Z{}^X : Q_Y{}^X] = 0. \tag{52}$$

in analogy with (46).

No satisfactory index analogues of the restraint equations have been found for the cases in which the special assumptions used to obtain (48) and (52) are not admissible. Something further might be done if functional dependence were replaced by statistical dependence, but because other difficulties arise this possibility will not be explored further at the present time.

## 10. Demand and supply functions

The maximization of profits within the restraints of the given production situation automatically yields enough additional equations to determine completely the allocation of factors, and therefore the outputs produced, in terms of the prices. In fact, the sums

$$q_j = \sum_{i \in P} q_j{}^i; \qquad j \in F, \tag{53}$$

when regarded as functions of the prices, represent the total demand for the various factors of production. In differential form, we have

$$v_j\, d\log q_j = \sum_{i \in \mathrm{P}} v_j{}^i\, d\log q_j{}^i; \qquad j \in \mathrm{F}, \tag{54}$$

from which

$$V_Y\, d\log Q_Y = \sum_{X c \mathrm{P}} V_Y{}^X\, d\log Q_Y{}^X; \qquad Y c\, \mathrm{F}, \tag{55}$$

in analogy with (54).

When the $q_j{}^i$ are expressed in terms of the prices, relations (53) express the demand for individual factors in terms of the price structure. In a similar way, the production functions (5) may be used to obtain the amount of each product that will be supplied for given prices. Not all requisite supply and demand functions, however, can be derived from the production situation and the equations for profit maximization. Certain of these functions must be derived from the psychology or utility (disutility) preference schedules of individual buyers or sellers. For this paper, it will suffice to assume the existence of empirical supply and demand functions for these cases, without going into their underlying psychological bases.

Two groups of such functions are involved: the supply functions for primary factors and the demand functions for consumer's goods. For the former, we shall assume simply that $s_j$, the supply offered, is given by

$$s_j = s_j(p_k; k \in R + \mathrm{C} + \mathrm{L}); \qquad j \in \mathrm{L} = \mathrm{F} - \mathrm{P{\cdot}F}. \tag{56}$$

For the latter, we shall assume that $d_i$, the quantity demanded, is given by

$$p_i d_i = I g_i(p_k; k \in \mathrm{R} + \mathrm{C} + \mathrm{L}); \qquad i \in \mathrm{R} = \mathrm{P} - \mathrm{P{\cdot}F}, \tag{57}$$

where $I$ is total income (total profits plus total payments for primary factors) and $g_i$ is the fraction of $I$ which will be spent on commodity $i$ for the given set of prices. For the static economy, it will be assumed that income must be spent entirely on consumption goods, since no possibility of savings or investment have thus far been introduced. We shall therefore assume that

$$\sum_{i \in \mathrm{R}} g_i \equiv 1. \tag{58}$$

In terms of the index variables, (56) becomes

$$d\log S_Y \sum_{Z c \mathrm{R} + \mathrm{C} + \mathrm{L}} E\,[S_Y : P_Z]\, d\log P_Z; \qquad Y c\, \mathrm{L}, \tag{59}$$

where

$$W_Y E\,[S_Y\,, P_Z]\, d\log P_Z = \sum_{j \in Y} \sum_{k \in Z} w_j E\,[s_j : p_k]\, d\log p_k, \tag{60}$$

and where $w_j = p_j s_j$ and $W_Y = \sum_{j \in Y} w_j$.

Similarly we have from (57)

$$U_X = \sum_{i\epsilon X} p_i d_i = IG_X = I \sum_{i\epsilon X} g_i; \qquad X c \mathrm{R}, \tag{61}$$

and

$$d \log G_X = \sum_{Z c \mathrm{R}+\mathrm{C}+\mathrm{L}} E\,[G_X : P_Z] d \log P_Z; \qquad X c \mathrm{R}, \tag{62}$$

where

$$E\,[G_X : P_Z] d \log P_Z = \sum_{i\epsilon X} \sum_{k\epsilon Z} E\,[g_i : p_k] d \log p_k. \tag{63}$$

## 11. The general static equilibrium

If we now assume that prices move to maintain an equality between supply and demand for all commodities in the system, we have as many additional equations as there are prices, and all variables of the system might be expected to be determinate. It happens, however, that one identity exists among the equations of the system so that at least one variable remains undetermined. Since total income was defined as the sum of profits plus payments for primary factors, we have, after equating supply and demand functions, the following:

$$I = \sum_{i\epsilon \mathrm{P}} (v_i - \sum_{j\epsilon \mathrm{F}} v_j{}^i) + \sum_{j\epsilon \mathrm{L}} v_j, \tag{64}$$

or

$$I = \sum_{i\epsilon \mathrm{R}+\mathrm{C}} v_i - \sum_{j\epsilon \mathrm{C}+\mathrm{L}} v_j + \sum_{j\epsilon \mathrm{L}} v_j = \sum_{i\epsilon \mathrm{R}} v_i. \tag{65}$$

But

$$\sum_{i\epsilon \mathrm{R}} v_i = \sum_{i\epsilon \mathrm{R}} p_i d_i = I \sum_{i\epsilon \mathrm{R}} g_i \equiv I. \tag{66}$$

That one degree of indeterminacy remains among the prices is intuitively reasonable, since we have thus far introduced no unit of price or money and therefore no relations are affected by a change in our unit of value. Since relations (25) or (26) depend on the prices only in ratio form, the system could be solved directly for these ratios and the quantity variables, provided that relations (56) and the functions $g_i$ of (57) involved the prices only in ratio form. In any event, the system is determinant only in terms of some one of the prices or of some function of them, such as total income or an index of the general level of prices.

The accompanying table summarizes the equations of the general static equilibrium as developed in the present paper in terms of indices as well as in terms of the basic variables. In this summary the variables $q$ have been re-

THE EQUATIONS OF THE STATIC EQUILIBRIUM

| BASIC VARIABLES | INDEX VARIABLES |
|---|---|
| a) $d\log s_i = \sum_{j \epsilon F} E[s_i : q_j^i] d\log q_j^i; \quad i \epsilon P$ | A) $d\log S_X = \sum_{Y c F} E[S_X : Q_Y^X d\log Q_Y^X; \quad X c P$ |
| b) $w_i E[s_i : q_j^i] = v_j^i; \quad i \epsilon P, \quad j \epsilon F$ | B) $W_X E[S_X : Q_Y^X] = V_Y^X; \quad X c P, \quad Y c F$ |
| c) $u_j = \sum_{i \epsilon P} v_j^i; \quad j \epsilon F$ | C) $U_Y d\log D_Y = \sum_{X c P} V_Y^X d\log Q_Y^X; \quad Y c F$ |
| d) $d\log s_j = \sum_{k \epsilon R+C+L} E[s_j : p_k] d\log p_k; \quad j \epsilon L$ | D) $d\log S_Y = \sum_{Z c R+C+L} E[S_Y : P_Z d\log P_Z; \quad Y c L$ |
| e) $u_i = I g_i(p_k; k \epsilon R+C+L); \quad i \epsilon R$ | E) $U_X = IG_X; \quad X c R$ |
| f) $u_k = w_k; \quad k \epsilon R+C+L$ | F) $U_Z = W_Z; \quad Z c R+C+L$ |
| where $u_i = p_i d_i; \; w_i = p_i s_i$ | where $U_X = P_X D_X; \; W_X = P_X S_X$ |
| $v_j^i = p_i q_j^i$ | $V_Y^X = P_Y^X Q_Y^X$ |
| $I = \sum_{i \epsilon R} u_i = \sum_{i \epsilon R} w_i$ | $I = \sum_{X c R} U_X = \sum_{X c R} W_X$ |

placed by the symbols $s$ or $d$, as the case may be, to distinguish between supply and demand. For the same reason the value variables $v$ have been replaced by the symbols $w$ or $u$, as indicated in the legend at the bottom of the table.

It will be noted that, although $v_j^i = p_j q_j^i$ and $V_Y^X = \sum_{i \epsilon X} \sum_{j \epsilon Y} v_j^i$, $V_Y^X$ has been written as equal to $P_Y^X Q_Y^X$ rather than $P_Y Q_Y^X$. But $d\log V_Y^X = d\log P_Y^X + d\log Q_Y^X$, where $d\log P_Y^X$ is defined by

$$V_Y^X d\log P_Y^X = \sum_{i \epsilon X} \sum_{j \epsilon X} v_j^i \, d\log p_j, \tag{67}$$

and, although both $d\log P_Y$, as previously defined, and $d\log P_Y^X$ are weighted arithmetic means of the same set of $p_j$, the weights are different and the means may therefore differ also. In any event, we have

$$V_Y \, d\log P_Y = \sum_{X c P} V_Y^X \, d\log P_Y^X. \tag{68}$$

Thus $d\log P_Y$ is a weighted mean of the $d\log P_Y^X$ for $X c$ P and will be equal to each of these in the event they are all equal to each other. This equality holds in particular if the $d\log p_j$ happen to be the same for all $j \epsilon Y$. In general the variability among the $d\log P_Y^X$ for $X c$ P will depend on the variability among the $d\log p_j$ for $j \epsilon Y$. Obviously little can be said in the case of arbitrary virtual displacements. For actual displacements with time, however, it is almost always quite safe to regard the $P_Y^X$ as equal to $P_Y$.

## 12. The basic static model

Of the index models covered by the discussion thus far, the one with the smallest number of variables is that obtained when the three basic categories of commodities are not subdivided at all. Equating supply and demand but abandoning all notational differentiation between them, we find that this basic system acquires the following mathematical formulation:

$$d \log Q_R = E[Q_R : Q_C{}^R] d \log Q_C{}^R + E[Q_R : Q_L{}^R] d \log Q_L{}^R, \tag{69}$$

$$d \log Q_C = E[Q_C : Q_C{}^C] d \log Q_C{}^C + E[Q_C : Q_{LC}] d \log Q_L{}^C, \tag{70}$$

$$V_R E[Q_R : Q_C{}^R] = V_C{}^R, \tag{71}$$

$$V_R E[Q_R : Q_L{}^R] = V_L{}^R, \tag{72}$$

$$V_C E[Q_C : Q_C{}^C] = V_C{}^C, \tag{73}$$

$$V_C E[Q_C : Q_L{}^C] = V_L{}^C, \tag{74}$$

$$V_C d \log Q_C = V_C{}^R d \log Q_C{}^R + V_C{}^C d \log Q_C{}^C, \tag{75}$$

$$V_L d \log Q_L = V_L{}^R d \log Q_L{}^R + V_L{}^C d \log Q_L{}^C, \tag{76}$$

$$\begin{aligned} d \log Q_L &\\ = E[Q_L : P_R] d \log P_R + E[Q_L : P_C] &d \log P_C + E[Q_L : P_L] d \log P_L, \end{aligned} \tag{77}$$

$$V_R = I, \tag{78}$$

where strict competition has been postulated.

If all seven elasticities are taken to be constant and if it is assumed that $P_Y{}^X = P_Y$ for $X$ equal to $R$ or $C$ and $Y$ equal to $C$ or $L$, the foregoing system can be solved algebraically in terms of any one of the variables. In terms of $I$, for example, the logarithm of each of the other variables is linear in $\log I$ with coefficients which depend on the values of the elasticities and constants of integration of the system.

Equations (69) and (70) may be rewritten, making use of equations (71) through (74), in the following form:

$$V_R d \log Q_R = V_C{}^R d \log Q_C{}^R + V_L{}^R d \log Q_L{}^R, \tag{79}$$

$$V_C d \log Q_C = V_C{}^C d \log Q_C{}^C + V_L{}^C d \log Q_L{}^C. \tag{80}$$

From (75), (76), (79), and (80) it follows that

$$V_R d \log Q_R = V_C d \log Q_L. \tag{81}$$

Now if $dQ_L = 0$, then $d \log Q_R = dQ_R = 0$ also. But if the quadratic forms (29) are negative definite for all $i \epsilon \mathrm{R}$ and we consider displacements for which the $p_i$ for $i \in \mathrm{R}$ are unchanged, then $d^2Q_R/d\lambda^2$ is negative for any such set of displacements not all zero. We thus have the following result:

*For fixed* $\mathrm{Q_L}$*, at equilibrium and with profit maximization under strict competition, the index* $\mathrm{Q_R}$ *is a maximum with respect to all possible displacements leaving the individual prices of consumer's goods fixed.*

As a slight generalization, the same result is obtained if prices are allowed to change, but $\rho\,[p_i, q_i] \leqq 0$ for $i \in \mathrm{R}$.

This result is a mathematical statement of a principle dating back to Adam Smith and recurring many times in economic literature. It was obtained in a form similar to that presented here by Evans [3] for a simplified economic system analogous to the model described by equations (69) to (78). Static results of this type are of interest in many other cases; for example, see Shephard's work [4] on the incidence of taxation. They retain some validity even for a dynamic system such as that sketched in the following section.

## 13. Index models with dynamic elements

The real economy is beset with a variety of truly dynamic factors, two of which will be considered here. The first of these arises from a distinction between "circulating capital," which is currently used up in production, and "fixed capital," which is merely tied up in the production process. The second concerns the existence of disequilibrium between current supply and demand permitting a variation in stocks and introducing a distinction between current production and inventories.

Let $q_j{}^i$, for $i \in \mathrm{P}$, $j \in \mathrm{C}$, now represent the amount of circulating capital $j$ *used up* per unit time in producing commodity $i$. Let $\gamma_j{}^i$, for $i \in \mathrm{P}$, $j \in \mathrm{C}$, represent the average amount of fixed capital of type $j$ *tied up* in the process of production for industry $i$. Furthermore, $\gamma_j{}^i$ will be regarded as of infinite durability, the requisite physical maintenance being covered by $q_j{}^i$. This simplification is convenient but not essential for the following development. The technical functions (5) must now be replaced by

$$q_i = q_i(q_j{}^i, j \in \mathrm{F}; \gamma_k{}^i, k \in \mathrm{C}); \qquad i \in \mathrm{P}. \tag{82}$$

Profits will be given a new definition, as follows:

$$\pi_i = p_i q_i - \sum_{j \in \mathrm{F}} p_i q_j{}^i - \sigma \sum_{k \in \mathrm{C}} p_k \gamma_k{}^i; \qquad i \in \mathrm{P}, \tag{83}$$

where $\sigma$ is the rate of interest. The initial cost of fixed capital is thus paid for out of investment and the charge on current profit is covered by the interest on perpetuity bonds or other loans to cover the investment. The equations of strict competition (26) must now be augmented by an additional set,

$$v_i E\,[q_i : \gamma_k{}^i] = \sigma p_k \gamma_k{}^i; \qquad i \in \mathrm{P}, \quad k \in \mathrm{C}. \tag{84}$$

Finally equations (53) become

$$q_k = \sum_{i \in \mathrm{P}} (q_k{}^i + \dot{\gamma}_k{}^i); \qquad k \in \mathrm{C}, \tag{85}$$

where $\mathring{\gamma}_k{}^i$ is the time derivative of $\dot{\gamma}_k{}^i$, and the expenditure equation (57) must be augmented by

$$H = Ih(\sigma; p_k, k \in \mathrm{R} + \mathrm{C} + \mathrm{L}), \tag{86}$$

where $h$ is the fraction of income invested and $H$, where

$$H = \sum_{k \in \mathrm{C}} \sum_{i \in \mathrm{P}} p_k \gamma_k{}^i, \tag{87}$$

is the total investment.

Aside from the inclusion of interest payments in $I$, the total income, and from the fact that

$$h + \sum_{i \in \mathrm{R}} q_i \equiv 1, \tag{88}$$

the balance of the static system can be carried over intact. One degree of indeterminacy remains and thus all variables except one can be determined in terms of this variable as functions of time. The presence of both the $\mathring{\gamma}_k{}^i$ and the $\dot{\gamma}_k{}^i$ in the same system of equations introduces time explicitly into the solution.

The index variables may be introduced by the methods used in the static case with analogous results, except for a slight variant occasioned by the presence of the $\dot{\gamma}_k{}^i$. Equation (85) may be written

$$v_k = \sum_{i \in \mathrm{P}} v_k{}^i + \sum_{i \in \mathrm{P}} v_k{}^i \overset{*}{\gamma}_k{}^i; \qquad k \in C, \tag{89}$$

where $v_k{}^i = p_k \gamma_k{}^i$ and $\overset{*}{\gamma}_k{}^i = d \log \gamma_k{}^i / dt$, the asterisk being used to denote the operator $\dfrac{d \log}{dt}$. If we write

$$N_Z{}^X \, d \log \Gamma_Z{}^X = \sum_{i \in X} \sum_{k \in Z} v_k{}^i \, d \log \gamma_k{}^i; \qquad X c \mathrm{P}, \quad Z c \mathrm{C}, \tag{90}$$

where $N_Z{}^X = \sum_{i \in X} \sum_{k \in Z} v_k{}^i$, we have also

$$N_Z{}^X \overset{*}{\Gamma}_Z{}^X = \sum_{i \in X} \sum_{k \in Z} v_k{}^i \overset{*}{\gamma}_k{}^i; \qquad X c \mathrm{P}, \quad Z c \mathrm{C}. \tag{91}$$

The index analogue of (89) thus becomes

$$V_Z = \sum_{X c \mathrm{P}} V_Z{}^X + \sum_{X c \mathrm{P}} N_Z{}^X \; \overset{*}{\Gamma}_Z{}^X; \qquad Z c \mathrm{C}. \tag{92}$$

Let us now find an alternative to equating supply and demand. For convenience we shall substitute a relation in very special form. The relation chosen, however, is not necessarily the most suitable, and further alternatives will suggest themselves immediately. The particular form chosen will suffice to indicate the general type of relation required.

If $s_i$ is the current rate of production of commodity $i$ and $d_i$ is the amount sold per unit of time, the total inventory on hand at time $t$ will be given by $\int_0^t (s_i - d_i)dt$, where the zero on the time scale is placed prior to the start of production. Let $B_i$, a constant, be the normal amount of stock required for satisfactory operation. Any relation giving $\overset{*}{p}_i$ as a monotone increasing function of $B_i - \int_0^t (s_i - d_i)dt$ would represent an equilibrating mechanism tending to keep the stock on hand from departing excessively from$B_i$. The following is such a relation:

$$s_i \overset{*}{p}_i = \xi\{B_i - \int_0^t (s_i - d_i)dt\}, \tag{93}$$

having the additional property that $\xi$ is dimensionless except for the time variable. Differentiating (93) with respect to time gives

$$s_i\left[\frac{\overset{**}{p}_i}{p_i} - (\overset{*}{p}_i)^2\right] + \overset{*}{s}_i\overset{*}{p}_i = -\xi(s_i - d_i); \qquad i \epsilon \mathrm{R} + \mathrm{C} + \mathrm{L}, \tag{94}$$

or

$$w_i\left[\frac{\overset{**}{p}_i}{p_i} - (\overset{*}{p}_i)^2\right] + w_i\overset{*}{s}_i\overset{*}{p}_i = -\xi(w_i - u_i); \qquad i \epsilon \mathrm{R} + \mathrm{C} + \mathrm{L}. \tag{95}$$

In index form we have

$$W_Z\left\{\frac{\overset{**}{P}_Z}{P_Z} + \overset{*}{S}_Z\overset{*}{P}_Z\right\} - \sum_{i\epsilon Z} w_i(\overset{*}{p}_i)^i = \xi(U_Z - W_Z); \qquad Z c \mathrm{R} + \mathrm{C} + \mathrm{L}. \tag{96}$$

But

$$\sum_{i\epsilon Z} w_i(\overset{*}{p}_i)^2 = W_Z(\overset{*}{P}_Z)^2\{1 + (\eta_Z)^2\}; \qquad Z c \mathrm{R} + \mathrm{C} + \mathrm{L}, \tag{97}$$

where

$$(P_Z\eta_Z)^2 = \frac{\sum_{i\epsilon Z} w_i(\overset{*}{p}_i - \overset{*}{P}_Z)^2}{\sum_{i\epsilon Z} w_i}; \qquad Z c \mathrm{R} + \mathrm{C} + \mathrm{L}. \tag{98}$$

In the event the $p_i$ are not too variable, $\eta_Z$ may be negligible, thus permitting the approximation

$$\sum_{i\epsilon Z} w_i(\overset{*}{p}_i)^2 = W_Z(\overset{*}{P}_Z)^2. \tag{99}$$

Although the introduction of the equations on fixed capital produces a general movement with essentially monotone increase or decrease for the individual variables, the introduction of inventories and the equilibrating equations (94) permits oscillations about this general trend. These oscillations are not necessarily periodic, but maximum amplitude and average time for a complete cycle will both vary inversely with $\xi$. As $1/\xi$ approaches zero, the maximum discrepancy between current supply and demand also goes to zero.

## 14. Fitting models to economic time series

The systems with dynamic elements have solutions for the individual variables as functions of time. It is therefore theoretically possible to fit such models to empirical economic time series. The practical difficulties, however, may be enormous. The simplest dynamic model of the type considered involves eleven variables, and even the simplest static model requires nine. Again the number of parameters to be determined even in the simplest cases of constant aggregate elasticities, etc., will be at least ten. Since this determination must be carried out by some multiple regression or general least-square process, the question arises concerning the method most appropriate statistically. The answer to this question will depend on the nature and possible causes of the errors expected. Certain causes of error may be listed at once:

*a*) Standard economic index numbers are at best only sampling estimates of the indices involved in the model.

*b*) Empirical demand functions change with time. Invention, government action, and other such factors introduce systematic error.

*c*) Even if constant in time, the number of parameters in the basic system are so numerous that the index equivalents must be regarded as random variables depending on statistical distributions of individual parameters.

*d*) Even if entrepreneurs behave as assumed, they will be unable to maintain the validity of relations such as (26), and random errors arise which are functions of time.

Other less obvious causes of error may be of equal importance. Much useful work can be done in investigating the nature of such stochastic and systematic errors, but this will be most difficult even in connection with specific models.

The use of models, fitted to available data by any practical means whatever, is an essential part of quantitative study of the general economy. It is believed that the general development followed in the present paper clarifies the problem of model building and represents a necessary preliminary step toward a more suitable probabilistic treatment, since it provides a type of model with a clearly defined relation to underlying economic theory.

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