A POSTULATIONAL CHARACTERIZATION OF STATISTICS

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1. Introduction

We shall present a system of postulates which will be shown to constitute an adequate basis for statistics. The fundamental elements in the system are variates, and we assume that these variates behave very much like numbers in ordinary algebra. In particular

$$x + y, \qquad x \cdot y, \qquad -x$$

are variates if x and y are variates. An observation of a variate is a number, and an observation of the variate $x + y$ is the sum of the corresponding observations made on the variates x and y. Similarly for the product and the negative. Since some of the observations of a variate may be zero, we should not expect division to be defined. We should, however, expect the presence of a zero variate 0 and a unity 1. Thus our system is just what algebraists describe as a commutative ring with a unity.

A fortuitous event is a special case of a variate. An observation of such a variate is either a success (represented by a 1) or a failure (represented by a 0). Since

$$1 \cdot 1 = 1 \quad \text{and} \quad 0 \cdot 0 = 0,$$

we might expect a fortuitous event x to be an idempotent variate, that is, one such that

$$x \cdot x = x.$$

The product $x \cdot y$ of two events can be interpreted as the conjunction of the events, that is, the event "x and y." Thus an observation of $x \cdot y$ is a 1 (success) if and only if the corresponding observations of the factors are both 1's (successes).

The disjunction of x and y is symbolized by $x \vee y$ (read "x or y") and is defined by the equation

$$x \vee y = x + y - x \cdot y.$$

An observation of $x \vee y$ succeeds if and only if the corresponding observation of at least one member of the disjunction succeeds.

The unity 1 satisfies the equations

$$1 \cdot x = x \cdot 1 = x$$

for all variates x. Each observation of the unity must be the number 1. Hence the unity can be interpreted as the tautology, that is, the event which always succeeds. The event "not-x" is symbolized by $\sim x$ and defined by the equation

$$\sim x = 1 - x .$$

An observation of $\sim x$ succeeds or fails according as the corresponding observation of x fails or succeeds. We prove that the idempotent elements, together with the operators $\cdot$, $\vee$, $\sim$, constitute a Boolean algebra.

Although Boolean algebra was originally designed to meet the needs of probability, it has made no provision for the word "if," which plays such a fundamental role in this theory. It is perhaps because of this deficiency that Boolean algebra has not been extensively used in probability, whereas it has been extensively used in other branches of mathematics. Hence we introduce the operation $x \subset y$ (read "x if y"), the resultant of which is an idempotent variate, provided x and y are idempotent. The operation is not defined unless y is an idempotent variate different from zero, but it is defined for an arbitrary x. This more general case is interpreted as the operation of selection (the "Auswahl" of von Mises). The resultant of two selections y and z is itself a selection denoted by $y \times z$. Thus

$$(x \subset y) \subset z = x \subset (y \times z) .$$

The product $y \times z$ does not possess an inverse, but it does possess the remaining properties of a non-commutative group with 1 as the identity.

An idempotent variate which cannot be decomposed into the disjunction of two distinct non-zero variates is called an atom. If a is an atom and x is an arbitrary variate, then $x \subset a$ is a number which is interpreted as an observation of x.

The expected value of a variate x is denoted by $e(x)$. The average of the observations $x \subset a_1$, $x \subset a_2, \cdots, x \subset a_n$ turns out to be

$$e[x \subset (a_1 \vee a_2 \vee \cdots \vee a_n)] .$$

The limit of this average (if it exists) is $e(x)$.

In the discussion which follows we shall omit proofs of many of the theorems since they are obvious.

2. The classes R and B and the operators $+ , \cdot , - , \vee , \sim$

Postulate 1. R is a commutative ring with a unity (distinct from zero). The elements of R are interpreted as variates.

Definition 1. If x is an element of R and $x \cdot x = x$, then x is said to be idempotent, or x is an element of B. The elements of B are interpreted as fortuitous events.

Definition 2. $\quad \sim x = 1 - x , \qquad x \vee y = x + y - x \cdot y .$

Theorem 1. $\qquad x \vee y = \sim(\sim x \cdot \sim y) .$

THEOREM 2. If x and y are elements of B, then $x \cdot y$, $\sim x$, and $x \vee y$ are elements of B.

THEOREM 3. If x is an element of B, then $x \cdot \sim x = 0$ and $x \vee \sim x = 1$.

It is now easily seen that B is a Boolean algebra. The operators $\cdot$, $\vee$, $\sim$ are defined with respect to all elements of R but are interpreted only for elements of B.

3. The operator $\subset$

POSTULATE 2. If x is an element of R, y an element of B, and $y \neq 0$, then $x \subset y$ is an element of R.

Hereafter, whenever an expression such as $x \subset y$ appears, it is understood that x is an element of R, y an element of B, and $y \neq 0$.

POSTULATE 3. $\quad (x + y) \subset z = (x \subset z) + (y \subset z)\,.$

POSTULATE 4. $\quad (x \cdot y) \subset z = (x \subset z) \cdot (y \subset z)\,.$

POSTULATE 5. $\quad x \subset y = (x \cdot y) \subset y\,.$

POSTULATE 6. $\quad x \subset (y \cdot z) = (x \subset y) \subset (z \subset y)\,.$

Note that postulate 5 can be interpreted as a sensible English sentence, whereas the English language lacks the appropriate punctuation to translate postulate 6 directly. Nevertheless postulate 6 is a valid logical equivalence.

POSTULATE 7. If $x \subset z = y \subset z$, then $x \cdot z = y \cdot z$.

THEOREM 4. If x is an element of B, then $x \subset y$ is an element of B.

THEOREM 5. $\quad 0 \subset x = 0\,.$

For let $0 \subset x = z$. Then $z + z = z$, and hence $z = 0$.

4. The classes N and A

Definition 3. If $x \subset y = x$ for every non-zero element y of B, then x is said to belong to N. The elements of N are shown to be complex numbers.

THEOREM 6. If x and y are elements of N, then $x + y$, $x \cdot y$, and $-x$ are elements of N. Also 0 is an element of N.

POSTULATE 8. If x is an element of N and $x \neq 0$, then there exists an element y of R such that $x \cdot y = 1$. The element y is denoted by $1/x$.

POSTULATE 9. If x and z are elements of N, y is an element of R, $z \neq 0$, and $x \cdot z = y \cdot z$, then $x = y$. Cancellation law.

THEOREM 7. If x belongs to both N and B, then x is 1 or 0.

For if $x \cdot x = x = 1 \cdot x$ and $x \neq 0$, then $x = 1$.

Definition 4. If $x \subset a$ is an element of N for every element x of R, then a is said to belong to A. The elements of A are called atoms, and the element $x \subset a$ is interpreted as an observation of x.

POSTULATE 10. If x and y are elements of R and $x \subset a = y \subset a$ for every a of A, then $x = y$.

THEOREM 8. Class A is not empty.

For otherwise $0 \subset a = 1 \subset a$ for every a of A, and hence $1 = 0$.

THEOREM 9. The element 1 belongs to N.

In order to prove this theorem let $1 \subset a = u$, where a is a member of A. Then u belongs to both N and B and hence is 1 or 0. But if $u = 0 = 0 \subset a$, then $a = a \cdot 1 = a \cdot 0 = 0$ by postulate 7. But this is impossible, and therefore $1 = 1 \subset a$ is a member of N.

THEOREM 10.
$$1 \subset x = x \subset x = 1\,,$$
$$-(x \subset y) = (-x) \subset y\,,$$
$$\sim(x \subset y) = (\sim x) \subset y\,,$$
$$(x \vee y) \subset z = (x \subset z) \vee (y \subset z)\,.$$

THEOREM 11. If x is a member of N and $x \neq 0$, then $y = 1/x$ is a member of N.

For $(x \cdot y) \subset z = (x \subset z) \cdot (y \subset z) = x \cdot (y \subset z) = 1 \subset z = 1 = x \cdot y$. Therefore $y \subset z = y$, and y is a member of N.

POSTULATE 11. Class N contains an ordered subclass of elements called reals satisfying Dedekind's postulate and the usual sign laws. N also contains an element i such that $i \cdot i = -1$.

THEOREM 12. N is the class of all complex numbers.

THEOREM 13. If x is a member of B and a is a member of A, then $x \cdot a = a$, or $x \cdot a = 0$.

For either $x \subset a = 1 = 1 \subset a$ or $x \subset a = 0 = 0 \subset a$, and hence the theorem follows from postulate 7.

THEOREM 14. If a is an element of A, then a is not a disjunction of two distinct non-zero elements of B.

For if $a = x \vee y$, then $a \cdot x = x \cdot (x \vee y) = x$. Therefore x equals a or 0, and similarly for y.

5. Indefinitely decomposable elements and the operator $\times$

Definition 5. An element x of B is said to be indefinitely decomposable if $x \neq 0$ and x is not a finite disjunction of atoms.

POSTULATE 12. If x is indefinitely decomposable and y is an element of R, then there exists an element z of R such that $z \subset x = y$.

POSTULATE 13. If a and b are elements of A, then there exists an indefinitely decomposable x such that either $a \subset x = b$ or $b \subset x = a$.

POSTULATE 14. If a is an element of A, if x, y, z are indefinitely decomposable, and if $[(a \subset x) \subset y] \subset z = a \subset y \neq 0$, then $a \subset x = a$.

POSTULATE 15. If E is a subset of A, then there exists an x of B such that $a \cdot x = 0$ if and only if a belongs to $A - E$.

THEOREM 15. If x is indefinitely decomposable and y belongs to B, then $x \vee y$ is indefinitely decomposable.

For if $x \vee y = a_1 \vee a_2 \vee \cdots \vee a_n$, then $x = x \cdot (x \vee y) = (x \cdot a_1) \vee (x \cdot a_2) \vee \cdots \vee (x \cdot a_n)$.

THEOREM 16. The element 1 is indefinitely decomposable.

In order to prove this theorem, note that $1 = x \vee \sim x$ and that there exists an indefinitely decomposable x by postulate 13 and theorem 8.

Definition 6. If $z \subset x = y$ and $z \cdot x = z$, then $z = x \times y$.

THEOREM 17. If x is indefinitely decomposable and y belongs to B, then $x \times y$ exists and is unique.

For by postulate 12 there exists u such that $u \subset x = y$. Set $z = u \cdot x$. Then $z \subset x = y$ by postulate 5. Also $z \cdot x = z$ and therefore $z = x \times y$. Next if $z_i \subset x = y$ and $z_i \cdot x = z_i$ for $i = 1$ and 2, then $z_1 = z_1 \cdot x = z_2 \cdot x = z_2$ by postulate 7.

THEOREM 18. $$(x \times y) \subset x = y\,.$$

THEOREM 19. $$x \times (y \subset x) = x \cdot y\,.$$

For $x \times (y \subset x) = z$ where $z \subset x = y \subset x$ and $z \cdot x = z$. But $z \cdot x = y \cdot x$ by postulate 7.

THEOREM 20. $(x \subset y) \subset z = x \subset (y \times z)$ if y is indefinitely decomposable.

For $x \subset (y \times z) = x \subset [y \cdot (y \times z)] = (x \subset y) \subset [(y \times z) \subset y] = (x \subset y) \subset z$.

THEOREM 21. $$x \subset 1 = x\,.$$

For $x = (1 \times x) \subset 1 = (1 \times x) \subset (1 \cdot 1) = [(1 \times x) \subset 1] \subset (1 \subset 1) = x \subset 1$.

THEOREM 22. If a belongs to A and x is indefinitely decomposable, and if $a \subset x \neq 0$, then $a \subset x$ belongs to A.

In order to prove this theorem let $a \subset x = b$, c belong to A, and $z = x \times c$. Then $c \cdot b = (z \subset x) \cdot (a \subset x) = (z \cdot a) \subset x = 0$ or $a \subset x = b$. But $c \cdot b = 0$ or c. If $c \cdot b = 0$ for every c belonging to A, then $b = 0$. Since this is impossible, b must belong to A.

THEOREM 23. If a belongs to A and x is indefinitely decomposable, then $x \times a$ belongs to A.

For $[(x \times a) \cdot c] \subset x = [(x \times a) \subset x] \cdot (c \subset x) = a \cdot (c \subset x) = a = (x \times a) \subset x$ or 0. Therefore $(x \times a) \cdot c = (x \times a) \cdot x \cdot c = (x \times a) \cdot c \cdot x = (x \times a) \cdot x = x \times a$ or 0. The remainder of the proof is similar to that of theorem 22.

THEOREM 24. If x and y are indefinitely decomposable, then $x \times y$ is indefinitely decomposable.

For if $x \times y = a_1 \vee a_2 \vee \cdots \vee a_n$, then $y = (a_1 \subset x) \vee \cdots \vee (a_n \subset x)$.

THEOREM 25. $$x \times 1 = 1 \times x = x\,.$$

In order to prove the first half of the theorem let $x \times 1 = y$. Then $y \subset x = 1 = x \subset x$ and therefore $y = x \cdot y = x \cdot x = x$. Next let $1 \times x = z$. Then $z \subset 1 = x = x \subset 1$ and $z = z \cdot 1 = x \cdot 1 = x$.

THEOREM 26. $$(x \times y) \times z = x \times (y \times z)\,.$$

In order to prove this theorem let $x \times y = u$, $u \times z = v$, $x \times (y \times z) = w$, and show that $v = w$. We have the equalities $(y \times z) \subset y = z = v \subset u = v \subset (u \cdot x) = (v \subset x) \subset (u \subset x) = (v \subset x) \subset y$ and hence $w \subset x = y \times z = (y \times z) \cdot y = (v \subset x) \cdot y = (v \subset x) \cdot (u \subset x) = (v \cdot u) \subset x = v \subset x$. Therefore $w = w \cdot x = v \cdot x = v \cdot u \cdot x = v \cdot u = v$.

THEOREM 27. If $x \times y = 1$, then $x = 1$ and $y = 1$.

For $1 = 1 \subset x = y$ and $1 = x \times y = x \times 1 = x$.

The indefinitely decomposable elements do not form a group with respect to the operation $\times$ since the inverses are missing (except for 1). This, however, is the only group property which is lacking.

THEOREM 28. $$x \times (y \vee z) = (x \times y) \vee (x \times z)\,.$$

In order to prove the theorem let $x \times (y \vee z) = u$. Then $u \subset x = y \vee z = [(x \times y) \subset x] \vee [(x \times z) \subset x] = [(x \times y) \vee (x \times z)] \subset x$. Therefore $u = u \cdot x = [(x \times y) \cdot x] \vee [(x \times z) \cdot x] = (x \times y) \vee (x \times z)$.

6. Ordering the class A

Definition 7. If a and b belong to A and there exists an indefinitely decomposable element x such that $b \subset x = a$, then $a \leqq b$. If $a \leqq b$ and $a \neq b$, then $a < b$.

THEOREM 29. A possesses serial order.

For if $a \neq b$, then $a < b$ or $b < a$ by postulate 13. Next if $b \subset z = a$ and $a \subset y = (b \subset z) \subset y = b$, then $a = b \subset z = b$ by postulate 14 and hence it is false that both $a < b$ and $b < a$. Finally, if $b \subset x = a$ and $c \subset y = b$, then $c \subset (y \times x) = a$. Suppose that $a \subset z = c$. Then $c \subset y = a \subset (z \times y) = b$ and hence $a = b = c$. Thus if $a < b$ and $b < c$, then $a < c$.

THEOREM 30. If x is indefinitely decomposable and E is the set of all atoms b such that $b \cdot x = b$, then the transformation $b = a \times x$, together with its inverse $b \subset x = a$, establishes a one-to-one order preserving correspondence between A and E.

First note that if $b = x \times a$, then $b \subset x = a$, and $b \cdot x = b$ belongs to E. Conversely, if $b \cdot x = b$, then $b \subset x = a \neq 0$ and hence a belongs to A. Next if $b_1 < b_2$ and $a_2 = b_2 \subset x < a_1 = b_1 \subset x$, then there exist x and z indefinitely decomposable such that $b_2 \subset y = b_1$ and $a_1 \subset z = a_2$. That is, $[(b_2 \subset y) \subset x] \subset z = b_2 \subset x$. Therefore $b_1 = b_2 \subset y = b_2$ by postulate 14. This is a contradiction, and hence $a_1 < a_2$.

THEOREM 31. A has no last element.

For suppose that a is the last element of A. Then there exists an element b of A such that $b < a$ since there are indefinitely decomposable elements. Therefore there exists an x indefinitely decomposable such that $a \subset x = b$. Let $c = x \times a$. Then $c \subset x = a$ and $c \cdot x = c$. But if $a = c$, then $a = b$. This is a contradiction, and hence there is no last element.

THEOREM 32. A is well ordered.

In order to prove this theorem let a belong to A and let E consist of a and its successors. Let x be such that $x \cdot b = b$ if and only if b belongs to E. Then x is indefinitely decomposable and E has the same order type as A by theorem 30. Therefore A has a first element since E has. Finally, every infinite subset of A has a first element since it has the same order type as A, and hence A is well ordered.

THEOREM 33. A has the order type ω.

For let (1) be the first element of A. We have to prove that every element except (1) has an immediate predecessor. Assume the contrary and let b be the first element other than (1) which has no immediate predecessor. Then the set of elements preceding b has the same order type as A. But this is the order type ω and the contradiction is established.

7. Spectra and periodicity

We shall denote the elements of A by

$$(1), (2), (3), \cdots.$$

The sequence

$$x \subset (1), \quad x \subset (2), \quad \cdots$$

is called the spectrum of x. Thus the spectrum of an element of R is a sequence of complex numbers (see definition 4 and theorem 12). If x belongs to B, the spectrum consists only of 1's and 0's (see theorem 4 and theorem 7). Since the operation of selection is distributive with respect to the operations of addition, multiplication, and disjunction, the spectrum of the sum, product, or disjunction of two elements is immediately determined in terms of the spectra of the two elements. Thus the spectra give the structures of the elements and display the natures of the various operations.

Let x be indefinitely decomposable and let

$$x \times (n) = (\xi_n) .$$

Then the terms $x \subset (\xi_n)$ are the 1's in the spectrum of x since $x \cdot (k) = (k)$ if and only if k is some ξ_n (see theorem 30). Next note that

$$(y \subset x) \subset (n) = y \subset [x \times (n)] = y \subset (\xi_n) .$$

Hence the spectrum of $y \subset x$ is the sequence

$$y \subset (\xi_1) , \qquad y \subset (\xi_2) , \qquad \cdots .$$

The powers of an indefinitely decomposable element x are defined with respect to the cross product as follows:

$$x^0 = 1 \quad \text{and} \quad x^n = x^{n-1} \times x$$

[note the distinction between 1 and (1)]. The negative powers are of course undefined. The set of elements x^n for a given x possesses all the group properties with the exception of inverses. In particular, consider the element

$$a = \sim(1) .$$

We have $(1) \cdot a = 0$ and $(k) \cdot a = (k)$ if $k > 1$ and hence $a \times (k) = (k + 1)$. It follows inductively that

$$a^n \times (k) = (n + k) .$$

An element x of R for which

$$x \subset a^n = x$$

is said to be periodic with period n. For such an element

$$x \subset (k) = (x \subset a^n) \subset (k) = x \subset (n + k) .$$

That is, the spectrum of x has the period n. The periodic elements play a fundamental role in defining expectations. Some of the importance of these elements can also be seen from the consequences of the following postulate. The purpose of this postulate is to determine the spectrum of $x \subset f$ when f is finitely decomposable.

POSTULATE 16. If $b_1, b_2, \cdots, b_n$ belong to A, if $b_1 < b_2 < \cdots < b_n$, and if $f = b_1 \vee b_2 \vee \cdots \vee b_n$, then $(1) \cdot (b_1 \subset f) = (1)$ and $(b_{k+1} \subset f) \subset a = b_k \subset f$, where $k = 1, 2, \cdots, n-1$.

THEOREM 34. If $b_1, b_2, \cdots, b_n, f$ are defined as in postulate 16, then $(b_1 \subset f) \subset a = b_n \subset f$ and $(b_k \subset f) \subset a^n = b_k \subset f$.

Let $b_k \subset f = c_k$. Then $1 = f \subset f = c_1 \vee c_2 \vee \cdots \vee c_n$ and $1 = 1 \subset a = (c_1 \subset a) \vee c_1 \vee c_2 \vee \cdots \vee c_{n-1}$. Also $(c_1 \subset a) \cdot c_k = (c_1 \subset a) \cdot (c_{k+1} \subset a) = (c_1 \cdot c_{k+1}) \subset a = [(b_1 \cdot b_{k+1}) \subset f] \subset a = 0$ if $k = 1, 2, \cdots, n-1$. Therefore $c_1 \subset a = c_n$. The second part of the theorem readily follows by induction.

THEOREM 35. If f is defined as in postulate 16, y is a member of R, and $z = y \subset f$, then $z \subset a^n = z$.

For $z = y \subset f = [y \cdot (b_1 \vee b_2 \vee \cdots \vee b_n)] \subset f = (y \cdot b_1 + y \cdot b_2 + \cdots + y \cdot b_n) \subset f = (y \cdot b_1) \subset f + (y \cdot b_2) \subset f + \cdots + (y \cdot b_n) \subset f$. Let $y \subset b_k = u_k$. Then u_k belongs to N, $y \subset b_k = u_k \subset b_k$, and $y \cdot b_k = u_k \cdot b_k$. Hence $(y \cdot b_k) \subset f = (u_k \subset f) \cdot (b_k \subset f) = u_k \cdot (b_k \subset f)$ and $[u_k \cdot (b_k \subset f)] \subset a^n = (u_k \subset a^n) \cdot [(b_k \subset f) \subset a^n] = u_k \cdot (b_k \subset f)$. Hence $z \subset a^n = z$.

Let

$$(0,n) = (1) \subset \sim a^n, \qquad (r,n) = a^r \times (0,n).$$

Note that $\sim a^n = (1) \vee (2) \vee \cdots \vee (n)$, $(1) \cdot (0,n) = (1)$, $(kn + r + 1) \cdot (r,n) = (kn + r + 1)$, $(r,n) = b_{r+1} \subset f$ (see postulate 16), $(0,1) = 1$, $(r,1) = a^r$. It can also be proved that

$$(r,n) \times (s,m) = (r + sn, nm).$$

Thus the set of elements (r,n) possesses all the group properties with the exception of inverses. Any periodic element of B can be represented as a finite disjunction of the elements (r,n), and any periodic element of R can be represented as a linear combination of these elements.

8. Expected values

POSTULATE 17. If $x \subset a^n = x$, then $e(x)$ exists and is an element of N.

POSTULATE 18. If x belongs to N, y belongs to R, and $e(y)$ exists, then $e(x \cdot y) = x \cdot e(y)$ exists.

POSTULATE 19. $e(1) = 1$ if $e(1)$ exists.

POSTULATE 20. If $e(x)$ and $e(y)$ exist, then $e(x + y) = e(x) + e(y)$.

POSTULATE 21. If $e(x)$ and $e(x \subset a)$ exist, then they are equal.

POSTULATE 22. $e(x)$ exists and equals μ if and only if

(i) for every ϵ and N there exists n such that

$$|e(x \subset \sim a^n) - \mu| < \epsilon \quad \text{and} \quad n > N,$$

(ii) μ is the only number (including $\mu = \infty$) which satisfies (i).

Note that postulate 22 (i) constitutes a series of predictions. That is, corresponding to any pair of positive numbers ϵ, N, we predict that we can find n satisfying condition (i). Such predictions are based on the belief that $e(x) = \mu$. Theoretically any one of these predictions can be verified (if true) but its contradictory cannot. In (ii) we assume that no variate has more than one expectation. Neither this assumption nor its contradictory can be physically verified. It is therefore harmless but extremely useful in that it greatly simplifies the theory.

THEOREM 36. $e(x) = \mu$ if and only if $\lim_{n\to\infty} e(x \subset \sim a^n) = \mu$.

Proof by Bolzano-Weierstrass theorem and postulate 22.

THEOREM 37. If x belongs to N, then $e(x) = x$.

For $1 \subset a^n = 1$ and hence $e(1) = 1$ exists. Furthermore, $e(x) = e(x \cdot 1) = x \cdot e(1) = x$.

THEOREM 38. If $b_1, b_2, \cdots, b_n, f$ are defined as in postulate 16 and y belongs to R, then

$$e(y \subset f) = \frac{1}{n}\sum_{k=1}^{n} y \subset b_k .$$

For $y \subset f = \sum_{k=1}^{n} u_k \cdot (b_k \subset f)$, where $u_k = y \subset b_k$ (see theorem 35). Also $(b_k \subset f) \subset a^n = b_k \subset f$ and hence $e(b_k \subset f)$ exists. Also by postulate 21 $e(b_i \subset f) = e(b_j \subset f)$. Moreover $e(f \subset f) = 1 = \sum_{k=1}^{n} e(b_k \subset f)$ and hence $e(b_k \subset f) = 1/n$. Therefore

$$e(y \subset f) = \sum_{k=1}^{n} u_k \cdot e(b_k \subset f) = \frac{1}{n}\sum_{k=1}^{n} y \subset b_k .$$

This is the average of the observations $y \subset b_k$.

THEOREM 39. $$e(x \subset \sim a^n) = \frac{1}{n}\sum_{k=1}^{n} x \subset (k) .$$

This is the average of the first n observations.

THEOREM 40. If x, y belong to B, then

$$e(x \vee y) = e(x) + e(y) - e(x \cdot y)$$

and

$$e(\sim x) = 1 - e(x) .$$

These expectations are interpreted as probabilities.

THEOREM 41. If x belongs to B, y belongs to R, and $e(x)$ and $e(y \subset x)$ exist, then

$$e(x \cdot y) = e(x) \cdot e(y \subset x) .$$

For $(y \subset x) \subset \sim a^n = y \subset \{x \times [(1) \vee (2) \vee \cdots \vee (n)]\}$
$= y \subset [(\xi_1) \vee (\xi_2) \vee \cdots \vee (\xi_n)]$.

Therefore $e[(y \subset x) \subset \sim a^n] = \frac{1}{n} \sum_{k=1}^{n} y \subset (\xi_k)$

$$= \frac{1}{n} \sum_{k=1}^{n} (x \cdot y) \subset (\xi_k) = \frac{1}{n} \sum_{k=1}^{\xi n} (x \cdot y) \subset (k),$$

since $(x \cdot y) \subset (k) = 0$ if $k \neq \xi_n$. Also $e(x \subset \sim a^{\xi_n}) = n/\xi_n$. Therefore $e(x \subset \sim a^{\xi_n}) \cdot e[(y \subset x) \sim a^n] = e[(x \cdot y) \subset \sim a^{\xi_n}]$. If x is a finite disjunction of atoms, then $e(x) = e(x \cdot y) = 0$ and the theorem is obviously true. If x is indefinitely decomposable, there are infinitely many ξ_n's, that is, $\lim_{n\to\infty} \xi_n = \infty$. Hence it is only necessary to pass to the limit to prove the theorem.

9. Functions

Let $f(x)$ be such that $f(x)$ belongs to N whenever x belongs to N. Then there is a unique extension of this function to elements y of R by means of the relation

$$f[y \subset (n)] = f(y) \subset (n)$$

for every (n) of A. If f is so extended, then

$$f(y \subset z) = f(y) \subset z$$

for every non-zero element z of B.

Let E be a set of elements of N and let

$$\varphi(E,x) = \begin{cases} 1 \text{ if } x \text{ belongs to } E\,, \\ 0 \text{ otherwise}\,. \end{cases}$$

Then $\varphi(E,y)$ belongs to B if y is a member of R. The element $\varphi(E,y)$ is interpreted as the event which succeeds whenever an observation of y belongs to E. Let y be a member of R such that $y \subset (n)$ is real for every n, let I_s be the interval $-\infty < t \leqq s$ and I_s' be the interval $-\infty < t < s$. Finally let

$$F(s+0) = e[\varphi(I_s,y)]\,, \qquad F(s-0) = e[\varphi(I_s',y)]\,,$$
$$F(s) = \frac{1}{2}[F(s+0) + F(s-0)]\,.$$

Then $F(s)$ (if it exists) is called the distribution function of y. In general

$$F(s \pm 0) = \lim_{\epsilon\to 0} F(s \pm \epsilon)\,, \qquad F(-\infty) = 0\,, \qquad F(+\infty) = 1\,,$$

although it is possible to construct pathological elements y for which these properties do not hold.

If x is a real member of the set N, the distribution function $F(s)$ of x is such that $F(s) = 0$ if $s < x$, $F(s) = 1/2$ if $s = x$, and $F(s) = 1$ if $s > x$.

Thus $F(s)$ is a step-function with a single jump of 1 at x. If $u_1, u_2, \cdots, u_n$ are real elements of N and $x_1, x_2, \cdots, x_n$ are mutually exclusive elements of B (that is, $x_i \cdot x_j = 0$ if $i \neq j$) such that $x_1 \vee x_2 \vee \cdots \vee x_n = 1$ and $e(x_i) = p_i$, then the distribution function of

$$x = u_1 \cdot x_1 + u_2 \cdot x_2 + \cdots + u_n \cdot x_n$$

is a step-function with jumps of p_i at the points u_i. In order to construct such an element x when its distribution function is a given step-function, it is only necessary to construct the elements $x_1, x_2, \cdots, x_n$.

First we shall consider the simpler problem of constructing a single element x of B with a given probability $p = e(x)$. We let

$$x \subset (k) = [kp] - [(k-1)p] = 1 \quad \text{or} \quad 0\,,$$

where $[m]$ denotes the largest integer which does not exceed m. Thus the spectrum of x is determined and consequently x is determined. Moreover,

$$e(x \subset \sim a^n) = \frac{1}{n} \sum_{k=1}^{n} x \subset (k) = [np]/n$$

and hence $e(x) = p$.

Next let x be a given element of B and let $e(x) = p = p_1 + p_2$. We propose to construct two mutually exclusive elements x_1 and x_2 such that $x_1 \vee x_2 = x$, $e(x_1) = p_1$, and $e(x_2) = p_2$. Note that $e(x_1 \subset x) = p_1/p$ and hence $x_1 \subset x$ can be constructed in the manner described above. Finally, $x \times (x_1 \subset x) = x \cdot x_1 = x_1$ and $x_2 = x - x_1$ and the construction is completed. By proceeding in this manner we can split the element 1 up into n mutually exclusive elements with given probabilities. We can now construct an element x whose distribution function is a given step-function. Since an arbitrary distribution function can be approximated by a step-function, a simple limiting process enables us to construct an element with an arbitrary distribution function.

By means of this construction it is easy to define Stieltjes integrals in terms of expected values. For let $F(s)$ be an arbitrary distribution function and let $f(s)$ be a function of the real variable s having at most singularities of the first kind. Let x be a variate with the distribution function $F(s)$. Then the Stieltjes integral of f with respect to F is defined by the equation

$$\int_{-\infty}^{+\infty} f(s)dF(s) = e[f(x)]\,.$$

Incidentally, by this procedure the student of statistics can be painlessly introduced to the mysteries of Stieltjes integration. It is then unnecessary to treat discrete variates and continuous variates separately. It should now be clear that the system R constitutes an adequate basis for statistics.