12. UNSOLVED PROBLEMS.

We shall list here some problems which seem of interest to us, in the order of the chapters to which they refer. It appears that the most significant problem is problem 8. We know little about how the problems compare in difficulty, but some of the problems are only of technical interest.

To Chapter 3.

Problem 1. Prove that for bond-percolation on the triangular lattice with three parameters, as discussed in Application 3.4 (iii) the critical surface is

$$p(1) + p(2) + p(3) - p(1)\, p(2)\, p(3) = 1 . \tag{12.1}$$ ///

Sykes and Essam (1964) conjectured that (12.1) gives the critical surface for this bond-percolation problem, and we mentioned several strong indications for the truth of this in Application 3.4 (iii). We also mentioned without proof that we can prove that for this problem

$$\begin{aligned} \theta(p) = 0 \;, \quad \text{whenever} \quad p \gg 0 \quad \text{and} \\ p(1) + p(2) + p(3) - p(1)\, p(2)\, p(3) \le 1. \end{aligned} \tag{12.2}$$

The proof of this fact is based on the following theorem.

Theorem 12.1. *Let* $(\mathcal{G},\mathcal{G}^*)$ *be a matching pair of periodic graphs in in* $\mathbb{R}^2$ *and let* P_p *be a* λ*-parameter periodic probability measure on the occupancy configurations of* $\mathcal{G}$ *based on the partition* $\mathcal{V}_1,\dots\mathcal{V}_\lambda$ *of the vertices of* $\mathcal{G}$ *(cf. Sect. 3.2). Assume that*

$$P_p\{v \text{ is occupied}\} > 0 \quad \text{for all} \quad v.$$

Assume also that at least one of the following two symmetry conditions holds:

(i) the first or second coordinate axis is an axis of symmetry for $\mathcal{G}$ *as well as for the partition* $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ *(cf. Def. 3.4),*

(ii) $\mathcal{G}$ <u>and</u> P_p <u>are symmetric with respect to the origin, i.e., if</u> $v(e)$ <u>is a vertex (edge) of</u> $\mathcal{G}$, <u>then so is</u> $-v$ $(-e)$ <u>and</u>

$$P_p\{v \text{ is occupied}\} = P_p\{-v \text{ is occupied}\}.$$

(<u>Of course</u> $-v = (-v(1),-v(2))$ <u>if</u> $v = (v(1),v(2))$ <u>and similarly for</u> $-e$).

<u>If</u>

$$\theta(p) > 0,$$

<u>then for every rectangle</u> B

$$(12.3) \qquad P_p\{\exists \text{ an occupied circuit on } \mathcal{G} \text{ surrounding } B\} = 1.$$

We do not prove this theorem. We merely give the easy deduction of (12.2) for the three-parameter bond-percolation problem on the triangular lattice from this theorem. Let $p_0 = (p_0(1),p_0(2),p_0(3)) >> 0$ satisfy (12.1). Assume that $\theta(p_0) > 0$. We derive a contradiction from this as follows. The three parameter bond-problem on the triangular lattice has the symmetry property (ii) above.

Thus, if $p_0 >> 0$ and $\theta(p_0) > 0$ then (12.3) holds for $p = p_0$ on $\mathcal{G}$, the covering graph of the triangular lattice. However, the proof of Condition A for Application 3.4(iii), or more precisely, the proof of (3.79), shows that then also

$$(12.4) \qquad P_{p_0}\{\exists \text{ vacant circuit on } \mathcal{G}^* \text{ surrounding } B\} = 1,$$

for each rectangle B. As we saw in the proof of Theorem 3.1(i) this implies $\theta(p_0) = 0$ (see the lines following (7.34)). It follows from this contradiction that $\theta(p_0) = 0$ for all $p_0 >> 0$ which satisfy (12.1), and a fortiori for all $p_0 >> 0$ with

$$p_0(1)+p_0(2)+p_0(3)-p_0(1)p_0(2)p_0(3) \leq 1.$$

Thus (12.2) holds.

To settle Problem 1 we would have to prove $\theta(p) > 0$ for any $0 << p << 1$ which satisfies

$$p(1)+p(2)+p(3)-p(1)p(2)p(3) > 1.$$

The present proof of Theorem 3.1 relies on Theorem 6.1, which we have been unable to prove so far without the symmetry condition (i) of Theorem 12.1. This leads as directly to the next question, which is more general than Problem 1.

Problem 2. Prove a version of Theorem 3.1 which does not require the symmetry property (i) of Theorem 12.1. ///

Perhaps even more disturbing than the symmetry restrictions in Theorems 3.1 and 3.2 is the fact that these results apply only to special graphs imbedded in the plane. No results seem to be known in dimension greater than two. This gives rise to the following questions.

Problem 3. Prove that

$$p_T = p_H$$

for a percolation problem on a periodic graph $\mathcal{G}$ imbedded in $\mathbb{R}^d$ with $d > 2$. ///

This problem is not even settled for $\mathcal{G} = \mathbb{Z}^3$.

Problem 4. Is it true that there can be at most one infinite occupied cluster on a periodic graph $\mathcal{G}$? ///

Newman and Schulman (1981) proved that if there can be more than one infinite occupied cluster under P_p, then

(12.5) $\qquad P_p\{\exists$ infinitely many infinite occupied clusters$\} = 1$.

It seems likely that if $\mathcal{G}$ is imbedded in $\mathbb{R}^2$, then (12.5) cannot occur. In fact we know this to be the case whenever Theorem 3.1 or 3.2 apply. However, if $\mathcal{G}$ is imbedded in $\mathbb{R}^d$ with $d > 2$ then very little is known about the impossibility of (12.5). For the site- or bond-percolation on $\mathbb{Z}^d$, $d \geq 2$, we can prove that

(12.6) $\qquad P_p\{\exists$ a unique infinite occupied cluster$\} = 1$

whenever $p > p_H^\infty$. Here p_H^∞ is the decreasing limit of p_H^k, $k \to \infty$, and $p_H^k = p_H(\mathcal{G}^k)$ is the critical probability for site-, respectively bond-percolation on the graph $\mathcal{G}^k := \mathbb{Z}^2 \times \{0,1,\dots,k\}$. $\mathcal{G}^k$ is the restriction of $\mathbb{Z}^3$ to $(k+1)$ copies of $\mathbb{Z}^2$ on top of each other; $\mathcal{G}^0$ is isomorphic to $\mathbb{Z}^2$. From Ex. 10.2(iii) we know that $p_H^\infty < p_H(\mathbb{Z}^2)$, and in particular for the bond-problem $p_H^\infty < \frac{1}{2}$. We conjecture (but have no proof) that

$$p_H^\infty = p_H(\mathbb{Z}^3),$$

both for site- and bond-percolation.

To Chapter 5.

The uniqueness of infinite clusters (see Problem 4) is related to continuity of the percolation probability $\theta(p)$. The relationship between the two problems was mainly one of similarity in methods of attack in the case of graphs imbedded in the plane. For both problems one tries to show that if $\theta(p) > 0$, then crossing probabilities of certain large rectangles are close to one and consequently arbitrarily large circuits exist. (cf. Russo (1981), Prop. 1 and the proof of Theorem 3.1 in Ch.7). However, recently M. Keane and J. van der Berg (private communication) have made the relationship between the problems far more explicit. They prove that in a one-parameter problem, if $p > p_H$ and (12.6) holds, then $\theta(\cdot)$ is continuous at p. Perhaps the converse also holds. In any case, the continuity properties of $\theta(\cdot)$ are of interest. Partial results about these are given in the Remark following Theorem 5.4, but in general the following question remains.

Problem 5. Is $\theta(p)$ a continuous function of p in every one-parameter percolation problem? In particular, is always

$$\theta(p_H) = 0? \qquad ///$$

One may also want to investigate further smoothness properties of $\theta(p,v)$, as a function of p, especially in one-parameter problems. For $\mathcal{G}$ one of a pair of matching graphs we already pointed out in Remark 5.2(iii) that under some symmetry condition $\theta(p,v)$ is infinitely differentiable for $p > p_H$.

Problem 6. Is $\theta(p,v)$ an analytic function of p for $p > p_H$? ///

For a directed site-percolation problem on the plane and p close to one Problem 6 was answered affirmatively by Vasil'ev (1970) (see Griffeath (1981), especially Sect. 9, for the relation between Vasil'ev's result and directed percolation).

To Chapter 6.

We already pointed out that Problem 1 would be solved if we could prove the Russo-Seymour-Welsh Theorem without symmetry assumptions. The same holds for Problems 2, and in dimension two also for Problems 4 and 5. Thus, one possible attack on these problems is to try and settle the following more specific problem.

Problem 7. Can one prove Theorem 6.1 without symmetry assumption? ///

To Chapter 8.

Problem 8. Prove any of the power laws (8.1)-(8.3) and get good estimates (or the precise values) of β and $\gamma_\pm$. ///

It is believed that (8.1) holds for a $0 < \beta < 1$. We do not even know for any graph whether

$$\frac{d}{dp}\,\theta(p) \to \infty \quad \text{as} \quad p \downarrow p_H \,. \tag{12.7}$$

Grimmett (private communication) suggested that Russo's formula (Prop. 4.2) might be helpful, since $\theta(p,v_0)$ is the P_p-probability of the increasing event $\{\#W(v_0) = \infty\}$. It does seem very difficult though to estimate the number of pivotal sites for this event.

To Chapter 9.

Problem 9. Does the function $p \to \Delta(p,\mathcal{G})$ introduced in Ch. 9 (cf. (9.12)) have a singularity at $p_H(\mathcal{G})$ for suitable $\mathcal{G}$? If yes, is there a power law of the form

$$\Delta(p,\mathcal{G}) \sim C_0|p-p_H|^{\nu\pm} \quad \text{as} \quad p \downarrow p_H$$
$$\text{or} \quad p \uparrow p_H, \text{ respectively?}$$ ///

The first part of the above problem is of historical interest, because Sykes and Essam (1964) wanted to base their arguments on $\Delta(p,\mathcal{G})$ having a unique singularity at $p = p_H$, at least for certain nice $\mathcal{G}$. Theorem 9.3 shows that for "nice" $\mathcal{G}$ $\Delta(p,\mathcal{G})$ can only have a singularity at $p = p_H$, but we could not establish that there really is a singularity at p_H (see also Remark 9.3 (iv)).

To Chapter 10.

Problem 10. Prove that $p_H(\mathcal{H}) > p_H(\mathcal{G})$ in cases where $\mathcal{H}$ is a subgraph of $\mathcal{G}$ formed by removing edges of $\mathcal{G}$ (see Remark 10.2(ii)).

Problem 11. Find a quantitative estimate for $p_H(\mathcal{H})-p_H(\mathcal{G})$ in the cases where this quantity is known to be strictly positive by Theorem 10.3.

To Chapter 11.

As in Chapter 11, let R_n be the resistance of the restriction of $\mathbb{Z}^d$ to $[0,n]^d$ between the faces $A_n^0 := \{0\} \times [0,n]^{d-1}$ and $A_n^1 := \{n\} \times [0,n]^{d-1}$, when the resistances of the individual edges are independent random variables.

Problem 12. Does

$$\lim_{n\to\infty} n^{d-2} R_n$$

exist in probability or with probability one? (See also Remark 11.1(vi).)

Problem 13. If the distribution of the individual resistances $R(e)$ are given by (11.6)-(11.8), is

$$\limsup_{n\to\infty} n^{d-2} R_n < \infty$$

whenever $1-p(\infty) = P\{R(e) < \infty\} >$ critical probability for bond-percolation on $\mathbb{Z}^d$? ///

Of course Theorem 11.2 answers Problem 13 affirmatively for $d = 2$.

We only discussed in Ch. 11 the resistance between two opposite faces of a cube. It is also interesting to look at the resistance, r_n say, between the origin and the boundary of the cube $[-n,n]^d$. If all edges of $\mathbb{Z}^d$ have resistance 1 ohm, then r_n is bounded as $n \to \infty$ for $d \geq 3$.

Problem 14. If $d \geq 3$, $P\{R(e) = 1\} = p$, $P\{R(e) = \infty\} = q = 1-p$ and $p >$ percolation probability for bond-percolation on $\mathbb{Z}^d$, does it follow that

$$P_p\{\limsup r_n < \infty \mid \text{the origin is connected to infinity by a conducting path}\} = 1?$$