10. INEQUALITIES FOR CRITICAL PROBABILITIES.

We first give a theorem of Hammersley's (1961) stating that for any connected graph $\mathcal{G}$ the critical probability in a one-parameter problem for site-percolation on $\mathcal{G}(= p_H(\mathcal{G})$ in our notation) is at least as large as the critical probability for bond-percolation on $\mathcal{G}(= p_H(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{G}}$ is the covering graph of $\mathcal{G}$; see Sect. 2.5). Actually, the result is obtained by comparing the probabilities that a fixed vertex z_0 is connected to some set of vertices V via a path with all vertices occupied, and via a path with all edges open, respectively. The proof given below is from Oxley and Welsh (1979). Hammersley (1980) has generalized this further to mixed bond and site problems (see Remark 10.1(i) below).

Special cases of the above mentioned inequality

$$(10.1) \qquad p_H(\mathcal{G}) \geq p_H(\tilde{\mathcal{G}})$$

are

$$(10.2) \qquad p_H(\mathcal{G}_0) = \text{critical probability for site-percolation on } \mathbb{Z}^2 \geq p_H(\mathcal{G}_1) = \frac{1}{2} \quad .$$

(see Ex. 2.1(i), 2.1(ii) and Application 3.4(ii)) and

$$(10.3) \qquad p_H(\mathcal{T}) = \frac{1}{2} \geq \text{critical probability for bond percolation on the triangular lattice} = 2\sin\frac{\pi}{18} \quad .$$

(see Ex. 2.1(iii) and Applications 3.4(i) and (iii)). In (10.3) we clearly have a strict inequality, and various data (Essam (1972)) indicate that $p_H(\mathcal{G}_0) \approx .59$ so that one long expected (10.2) to be a strict inequality as well. Higuchi (1982) recently gave the first proof of this strict inequality. Intuitively, the most important basis for a comparison of $p_H(\mathcal{G}_0)$ and $p_H(\mathcal{G}_1)$ is the fact that $\mathcal{G}_0$ can be realized as a subgraph of $\mathcal{G}_1$; one obtains (an isomorphic copy of) $\mathcal{G}_0$ by deleting certain edges from $\mathcal{G}_1$, see Fig. 2.1 and 2.2. and Fisher (1961). The principal result of this chapter implies that for many pairs of periodic graphs $\mathcal{H}$, $\mathcal{G}$ with $\mathcal{H}$ a subgraph of $\mathcal{G}$ one has

$$(10.4) \qquad p_H(\mathcal{H}) \quad > \quad p_H(\mathcal{G}).$$

Of course one always has $p_H(\mathcal{H}) \geq p_H(\mathcal{G})$ whenever $\mathcal{H}$ is a subgraph of $\mathcal{G}$. The strength of Theorems 10.2 and 10.3 is that they give a strict inequality in many examples such as (10.2) and (10.3) (see Ex. 10.2(i), (ii)). Theorem 10.2 is actually much more general, and also gives strict inclusions for the percolative regions in some multiparameter percolation problems (see Ex. 10.2(i) below). The price for the generality is a very involved combinatorial argument in Sect. 10.3. The reader is advised to look first at the simple special case treated in Higuchi (1982).

10.1 Comparison of bond and site problems.

Let $\mathcal{G}$ be any graph with vertex (edge) set $\mathcal{V}(\mathcal{E})$, and let P_p be the one-parameter probability measure on the occupancy configurations of its sites, given by

$$P_p = \Pi \ \mu_v$$

with (3.61), as in Sect. 3.4. For a vertex z_0 of $\mathcal{G}$ and a set of vertices V of $\mathcal{G}$ set

$$\sigma_p(z_0,V) = \sigma_p(z_0,V,\mathcal{G}) = P_p\{\ \exists \text{ path } (v_0,e_1,\dots,e_\nu,v_\nu) \text{ with } v_0 = z_0,\ v_\nu \in V \text{ and all its vertices occupied } | \ z_0 \text{ is occupied}\} \ .$$

Analogously, we define $\tilde{P}_p$ as a measure on the configurations of passable and blocked edges of $\mathcal{G}$. As in Sect. 3.1 we take

$$\tilde{P}_p = \Pi \ \mu_e$$

and

$$\mu_e\{\omega(e) = 1\} = 1 - \mu_e\{\omega(e) = -1\} = p \ .$$

Also, with z_0 and V as above we set

$$\beta_p(z_0,V) = \beta_p(z_0,V,\mathcal{G}) = \tilde{P}_p\{\ \exists\ \text{path } (v_0,e_1,\ldots,e_\nu,v_\nu)$$
$$\text{with } z_0 = v_0,\ v_\nu \in V \text{ and all its edges passable}\}$$

Lastly we remind the reader that $\theta(p,z_0)$ was defined in (3.25), and define here its analogue

$$\tilde{\theta}(p,z_0) := \tilde{P}_p\{\ \exists\ \text{infinitely many vertices connected to } z_0$$
$$\text{by a path with all its edges passable}\}\ .$$

<u>Theorem 10.1. Let</u> $\mathcal{G}$ <u>be any connected graph,</u> z_0 <u>a fixed vertex of</u> $\mathcal{G}$<u>, and</u> V <u>a collection of vertices of</u> $\mathcal{G}$. <u>Then</u>

$$\sigma_p(z_0,V) \le \beta_p(z_0,V)\ ,\quad 0 \le p \le 1\ . \tag{10.5}$$

<u>Moreover</u>,

$$\theta(p,z_0) \le p\tilde{\theta}(p,z_0)\ , \tag{10.6}$$

<u>and consequently</u>

$$p_H(\mathcal{G}) \ge p_H(\tilde{\mathcal{G}})\ , \tag{10.7}$$

<u>where</u> $\tilde{\mathcal{G}}$ <u>is the covering graph of</u> $\mathcal{G}$.

<u>Proof</u>: We only have to prove (10.5). One then obtains (10.6) by taking for V the set

$$V_n := \{v:\ v \text{ a vertex of } \mathcal{G} \text{ such that all paths from}$$
$$z_0 \text{ to } v \text{ contain at least } n \text{ vertices}\}\ .$$

and letting $n \to \infty$. Indeed one has the simple relations

$$\theta(p,z_0) = \lim_{n\to\infty} P_p\{z_0 \text{ is connected by an occupied path}$$
$$\text{to } V_n\} = \lim_{n\to\infty} p\,\sigma_p(z_0,V_n),$$

$$\tilde{\theta}(p,z_0) = \lim_{n\to\infty} \beta_p(z_0,V_n)\ .$$

(10.7) in turn follows from (10.6), the definition (3.62) of $p_H(\mathcal{G})$ and the corresponding formula

$$p_H(\tilde{\mathcal{G}}) = \sup\{p \in [0,1] : \tilde{\theta}(p,z_0) = 0\} \quad .$$

(Here we use the fact that bond percolation on $\mathcal{G}$ is equivalent to site percolation on $\tilde{\mathcal{G}}$, as proved in Prop. 3.1.)

For proving (10.5) we shall drop the restriction that $\mathcal{G}$ is connected. It suffices then to consider only finite graphs $\mathcal{G}$, by virtue of the following simple limit relation. Let $\mathcal{G}_n$ be the graph obtained from $\mathcal{G}$ by deleting all vertices in V_n and all edges incident to some vertex in V_n. Then clearly

$$\sigma_p(z_0,V,\mathcal{G}) = \lim_{n \to \infty} \sigma_p(z_0,V \cap \mathcal{G}_n,\mathcal{G}_n) \ .$$

We now prove (10.5) for a finite graph $\mathcal{G}$ by induction on the number of edges in $\mathcal{G}$. First assume $\mathcal{G}$ has one edge e only. If $z_0 \in V$ then $\sigma_p(z_0,V,\mathcal{G}) \geq P_p\{z_0$ is occupied $| \, z_0$ is occupied$\} = 1$. Thus $\sigma_p(z_0,V) = 1$ and similarly $\beta(z_0,V) = 1$. If $z_0 \notin V$ and e is not incident to z_0, then both sides of (10.5) are zero. If e connects z_0 with a vertex z_1, then both sides of (10.5) are still zero if $z_1 \notin V$. If, however, $z_1 \in V$, then (10.5) follows from

$$\sigma_p(z_0,V) = P_p\{z_0 \text{ and } z_1 \text{ are occupied} \mid z_0 \text{ is occupied}\}$$

$$= p = \tilde{P}_p\ \{e \text{ is passable}\} = \beta_p(z_0,V)$$

(since z_0 can be connected only to z_1). Now assume that (10.5) has been proven for all graphs with m or fewer edges, and let $\mathcal{G}$ have $(m+1)$ edges. As before the case with $z_0 \in V$ is trivial. Assume $z_0 \notin V$. If there is no edge incident to z_0, then again $\sigma_p(z_0,V) = \beta_p(z_0,V) = 0$. Otherwise let e be an edge with endpoints z_0 and some other vertex, z_1 say. Introduce the following two graphs:

$\mathcal{G}^d$ = graph obtained by deleting $\mathring{e} = e \setminus \{z_0,z_1\}$ from $\mathcal{G}$,

$\mathcal{G}^c$ = graph obtained by contracting e,i.e, deleting $\mathring{e} = e \setminus \{z_0,z_1\}$, but identitying z_1 with z_0 .

$\mathcal{G}^c$ has as vertex set the vertex set of $\mathcal{G}$ minus z_1, and has as many edges from z_0 to v as there are edges in $\mathcal{G}$ from z_0 or z_1 to v. Both $\mathcal{G}^d$ and $\mathcal{G}^c$ have at most m edges. Next, denote by $B(z_0,V,\mathcal{G})$ $(S(z_0,V,\mathcal{G}))$ the event that there exists a path $(v_0,e_1,\dots,e_\nu,v_\nu)$ with $v_0 = z_0$, $v_\nu \in V$ and all its bonds or edges passable (all its sites or vertices occupied). One easily sees that if e is blocked, then $B(z_0,V,\mathcal{G})$ occurs if and only if $B(z_0,V,\mathcal{G}^d)$ occurs, since any passable path from z_0 to V does not contain e. Therefore

$$\tilde{P}_p\{B(z_0,V,\mathcal{G}) \text{ and } e \text{ blocked}\} = (1-p)\tilde{P}_p\{B(z_0,V,\mathcal{G}^d)\} = (1-p)\beta_p(z_0,V,\mathcal{G}^d)$$

Similarly, if z_1 is vacant and $S(z_0,V,\mathcal{G})$ occurs, then there is an occupied path on $\mathcal{G}$ from z_0 to V, which does not go through e, because any path which does not go through z, cannot contain e either. In other words z_1 must be vacant and on the graph $\mathcal{G}^d$ minus the vertex z_1(and the edges incident to z_1 on $\mathcal{G}$ there must exist an occupied path from z_0 to V. Since this occupied path is automatically a path on $\mathcal{G}^d$ we have

$$\begin{aligned} &P_p\{S(z_0,V,\mathcal{G}) \text{ and } z_1 \text{ is vacant} \mid z_0 \text{ is occupied}\} , \\ &\le (1-p)\sigma_p(z_0,V,\mathcal{G}^d) . \end{aligned}$$

Next consider the case in which e is passable. Then, if $B(z_0,V,\mathcal{G}^c)$ occurs, also $B(z_0,V,\mathcal{G})$ occurs. Indeed, if $(z_0,e_1,v,\dots,e_\nu,v_\nu)$ is a passable path on $\mathcal{G}^c$ from z_0 to $v_\nu \in V$, then either $(z_0,e_1,v_1,\dots,e_\nu,v_\nu)$ or $(z_0,e,z_1,e_1,v_1,\dots,e_\nu,v_\nu)$ is a passable path on $\mathcal{G}$ from z_0 to v_ν. (We abuse notation somewhat here by using the same symbol for an edge or vertex on $\mathcal{G}$ and the corresponding edge or vertex, respectively on $\mathcal{G}^c$. Also if $z_1 \in V$ on $\mathcal{G}$, then on $\mathcal{G}^c$ the vertex z_0, resulting from identifying z_0 and z_1 on $\mathcal{G}$, belongs to V). Conversely it is just as easy to go from a passable path on $\mathcal{G}$ to a passable path with possible double points on $\mathcal{G}^c$ by removal of the edge e and identifying z_0 and z_1. Therefore

$$P_p\{B(z_0,V,\mathcal{G}) \text{ and } e \text{ passable}\} = p\,\beta_p(z_0,V,\mathcal{G}^c) .$$

Finally, if z_1 is occupied, then $S(z_0,V,\mathcal{G})$ implies that there exists an occupied path on $\mathcal{G}^c$ from z_0 to V. By considering separately the cases $z_1 \in V$ and $z_1 \notin V$ one obtains

$$P_p\{S(z_0,V,\mathcal{G}) \text{ and } z_1 \text{ occupied} | z_0 \text{ is occupied}\} = p\, \sigma_p(z_0,V,\mathcal{G}^c) .$$

Finally, by the induction hypothesis

$$\beta_p(z_0,V,\mathcal{G}^c) \geq \sigma_p(z_0,V,\mathcal{G}^c) \quad \text{and}$$

$$\beta_p(z_0,V,\mathcal{G}^d) \geq \sigma_p(z_0,V,\mathcal{G}^d) .$$

Putting all these inequalities together we obtain

$$\begin{aligned}
&\beta_p(z_0,V,\mathcal{G}) = \tilde{P}_p\{B(z_0,V,\mathcal{G}) \text{ and } e \text{ is blocked}\}\\
&+ \tilde{P}_p\{B(z_0,V,\mathcal{G}) \text{ and } e \text{ is passable}\}\\
&= (1-p)\, \beta_p(z_0,V,\mathcal{G}^d) + p\, \beta_p(z_0,V,\mathcal{G}^c)\\
&\geq (1-p)\, \sigma_p(z_0,V,\mathcal{G}^d) + p\, \sigma_p(z_0,V,\mathcal{G}^c)\\
&\geq P_p\{S(z_0,V,\mathcal{G}) \text{ and } z_1 \text{ vacant} \mid z_0 \text{ is occupied}\}\\
&+ P_p\{S(z_0,V,\mathcal{G}) \text{ and } z_1 \text{ occupied} \mid z_0 \text{ is occupied}\\
&= \sigma_p(z_0,V,\mathcal{G}).
\end{aligned}$$

□

Remark .

(i) We can also ask for the probability

$\gamma(p,p',z_0,V) := P\{\ \exists$ path $(v_0,e_1,\ldots,e_\nu,v_\nu)$ with

$v_0 = z_0$, $v_\nu \in V$ and all its edges passable and all its vertices occupied} ,

when each vertex is occupied with probability p and each edge is passable with probability p' (all edges and all vertices independent) . Hammersley (1980) gives the following generalization of a result of McDiarmid (1980).

(10.8) $\quad \gamma(\delta p,p',z_0,V) \leq \gamma(p,\delta p',z_0,V),\ 0 \leq \delta,p,p' \leq 1$.

Here is Hammerersley's quick proof of (10.8). Let $\mathcal{H}$ be the random graph obtained by deleting each site other than z_0 of $\mathcal{G}$ with probability $1-p$ and each edge of $\mathcal{G}$ with probability $1-p'$. $\mathcal{H}$ may have some edges for which only one or no endpoint is a vertex of

$\mathcal{H}$. Despite this slight generalization (10.5) remains valid for $\mathcal{H}$ since one can simple ignore all edges which do not have a vertex of $\mathcal{H}$ for both of their endpoints. Now take the expectation over $\mathcal{H}$ of the inequality

$$\sigma_\delta(z_0,V;\mathcal{H}) \leq \beta_\delta(z_0,V;\mathcal{H}) \quad .$$

This gives (10.8). E.g. in the left hand side one can pass through an edge only if it remained in $\mathcal{H}$; this event has probability p'. One can go through a vertex only if it stayed in $\mathcal{H}$ and is now occupied in $\mathcal{H}$; this event has probability δp. Thus

$$E\ \sigma_\delta(z_0,V,\mathcal{H}) = \gamma(\delta p,p',z_0,V) \ .$$

Similarly

$$E\ \beta_\delta(z_0,V;\mathcal{H}) = \gamma(p,\delta p',z_0,V).$$

(10.5) can be recovered from (10.8) by taking $p = p' = 1$, since

$$\sigma_\delta(z_0,V) = \gamma(\delta,1,z_0,V) \quad \text{and} \quad \beta_\delta(z_0,V) = \gamma(1,\delta,z_0,V) \ .$$

10.2 Strict inequalities for a graph and a subgraph .

The set-up in this section will be the following.

(10.9) $(\mathcal{G},\mathcal{G}^*)$ is a matching pair of periodic graphs in $\mathbb{R}^2$, based on $(\mathcal{M},\mathcal{F})$,

(10.10) $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ is a periodic partition of the vertices of $\mathcal{M}$,

and P_p is the λ-parameter probability measure defined as in (3.22), (3.23). We further assume that

(10.11) one of the coordinate axes, call it L, is an axis of symmetry for $\mathcal{G},\mathcal{G}^*$ and the partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$.

We shall later be interested in subgraphs $\mathcal{H}$ of $\mathcal{G}$ and the inequality (10.4). For the time being, though, we concentrate on comparing the percolation probabilities on $\mathcal{G}$ (or rather $\mathcal{G}_{p\ell}$) under two different probability measures. We shall show after Theorem 10.2 how the case of a subgraph $\mathcal{H}$ of $\mathcal{G}$ fits into our framework. For a little while our attention will be on $\mathcal{G}_{p\ell}$. $\mathfrak{w}$ will be a periodic subclass of the vertices of $\mathcal{G}_{p\ell}$. Unfortunately we have to impose an ugly and complicated looking technical condition. It is a purely combinatorial condition, whose purpose is to guarantee that sufficiently many sites in $\mathfrak{w}$ can be pivotal for the occurrence of occupied horizontal and vertical crossings on $\mathcal{G}_{p\ell}$ of large rectangles. Despite its forbidding appearance the condition is rather mild, as the examples after Theorem 10.2 will show. We shall also show by example that some condition of this form is needed to obtain the inequality (10.4). Before formulating the condition we remind the reader of some of the constants Λ, Λ_i introduced earlier. These depend on $\mathcal{M}, \mathcal{G}, \mathcal{G}^*$, $\mathcal{G}_{p\ell}$ and $\mathcal{G}^*_{p\ell}$ only.

(10.12) $\quad \Lambda \geq$ diameter of any edge of $\mathcal{G}, \mathcal{G}^*$, $\mathcal{G}_{p\ell}$ or $\mathcal{G}^*_{p\ell}$.

Λ_3 and $\Lambda_5 \geq 1$ are such that each horizontal (vertical) strip of height Λ_3(width Λ_3) posseses a horizontal (vertical) crossing on $\mathcal{M}$ (and hence also on $\mathcal{G}$ as well as on $\mathcal{G}^*$) with the property that for any two points y_1, y_2 on the crossing the diameter of the segment of the crossing between y_1 and y_2 is at most

$$\Lambda_5(|y_1 - y_2| + 1) .$$

Such Λ_3, Λ_5 exist by Lemma A.3 (Note that this lemma allows us to construct crossings which consist of translates of a fixed path independent of the length of the strip.) As before $\Lambda_4 = \lceil \Lambda_3 + \Lambda \rceil + 1$. We also choose Λ_6 such that any two vertices of $\mathcal{G}_{p\ell}(\mathcal{G}^*_{p\ell})$ within distance $\Lambda_3 + 10\Lambda$ of each other can be connected by a path on $\mathcal{G}_{p\ell}(\mathcal{G}^*_{p\ell})$ of diameter $\leq \Lambda_6$. Further we use the following abbreviations

$$\Lambda_7 = \Lambda_3 + 4\Lambda \quad ,$$

$$\Lambda_8 = (3\Lambda_5 + 1)(2\Lambda_6 + 4\Lambda_3 + 10\Lambda + 1) .$$

Lastly we make the following definitions.

Def. 10.1. A path $(v_0, e_1, \ldots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ is called minimal if for any $i < j$ for which v_i and v_j are adjacent on $\mathcal{G}_{p\ell}$ one has $j = i + 1$.

Def. 10.2. A shortcut of one edge of the path $(v_0, e_1, \dots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ is an edge e of $\mathcal{G}_{p\ell}$ between two vertices v_i and v_j on the path with $j \geq i + 2$.

Comment.

(i) A path is minimal exactly when it has no shortcuts of one edge. ///

Now let $\mathfrak{w}$ be a periodic subclass of the vertices of $\mathcal{G}_{p\ell}$.

Condition D. For some vertex $x = (x(1), x(2)) \in \mathfrak{w}$ there exists a constant $\Delta \geq 2\Lambda_8$, a minimal path $U = (u_0, e_1, \dots, e_\rho, u_\rho)$ on $\mathcal{G}_{p\ell}$ and a path $V^* = (v_0, e_1, \dots, e^*_\sigma, v^*_\sigma)$ on $\mathcal{G}^*_{p\ell}$ such that the following conditions are satisfied:

a) $x = u_{i_0}$ for some i_0, i.e., U goes through x.

b) If i and $j \geq i + 2$ are such that u_i and u_j lie on the perimeter of a single face $F \in \mathfrak{F}$, whose central vertex does not belong to $\mathfrak{w}$, then either $i + 2 \leq j \leq i_0$ or $i_0 \leq i \leq j - 2$,

c) U is a horizontal crossing of

$$B = B(x) := [x(1) - \Delta, x(1) + \Delta] \times [x(2) - \Delta, x(2) + \Delta].$$

U lies below the horizontal line $\mathbb{R} \times \{x(2) + \Delta - \Lambda_8\}$. Moreover, $(u_{i_0} = x, e_{i_0+1}, \dots, e_\rho, u_\rho)$ lies to the right of the vertical line $\{x(1) - \Delta + \Lambda_8\} \times \mathbb{R}$, while $(u_0, e_1, \dots, e_{i_0}, u_{i_0} = x)$ lies to left of the vertical line $\{x(1) + \Delta - \Lambda_8\} \times \mathbb{R}$,

d) V^* connects x to the top edge of B inside the strip $[x(1) - \Delta + \Lambda_8, x(1) + \Delta - \Lambda_8] \times \mathbb{R}$, i.e., $(v^*_0, e^*_1, \dots, e^*_\sigma, v^*_\sigma)$ are contained in this strip, $(v^*_0, e^*_1, \dots, e^*_{\sigma-1}, v^*_{\sigma-1}) \subset B(x)$, but e^*_σ intersects $[x(1) - \Delta + \Lambda_8, x(1) + \Delta - \Lambda_8] \times \{x(2) + \Delta\}$. Moreover, v^*_0 and x are adjacent on $\mathcal{M}_{p\ell}$.

e) U and V^* have no vertex in common.

Comments.

(ii) Basically a),c),d) and e) state that there exists a horizontal crossing U of $B(x)$ on $\mathcal{G}$ through x, and a connection V^* from

x to the top edge of B above U. There are some restrictions on the location of U and V^*, and U has to be minimal. However, condition b) may put a crucial restriction of another kind on U. Basically it requires that the pieces of U before and after x should not come too close to each other in a certain sense. On the other hand, condition b) is vacuous if $\mathfrak{I} = \emptyset$ or if all central vertices of $\mathcal{G}_{p\ell}$ belong to $\mathfrak{w}$. This happens in several of the examples below. The reader is urged to look at these examples to get a feeling for Condition D. Example v) also illustrates that some restriction is necessary to obtain (10.4).

(iii) In condition c) and d) there is an asymmetry between the roles of the horizontal and vertical direction, and between the roles of the positive and negative vertical direction. This was merely done not to complicate the conditions still further. One can always interchange the positive and negative direction of an axis, or the first and second coordinate axis by rotating the graph over 180° or 90°. ///

We now turn to a discussion of the probability measures to be considered. We assume that $p_0 \in \mathcal{P}_\lambda$ is such that

(10.13) $$\overline{0} \ll p_0 \ll \overline{1}$$

and that P_{p_0} is given by (3.22), (3.23) with $p = p_0$. Further

(10.14) Condition A or B of Sect. 3.3 is satisfied for p_0.

As usual we extend P_{p_0} to a probability measure on the occupancy configurations of $\mathcal{M}_{p\ell}$ by means of (7.2) and (7.3). The extended measure P_{p_0} is still a product measure of the form (3.22), (3.23) with $\mathfrak{v}$ = vertex set of $\mathcal{M}_{p\ell}$. We shall also consider another probability measure, $P_{p'}$, on the occupancy configurations of $\mathcal{M}_{p\ell}$. $P_{p'}$ too will be a product measure:

(10.15) $$P_{p'} = \prod_{\substack{v \text{ a vertex} \\ \text{of } \mathcal{M}_{p\ell}}} \nu_v$$

with ν_v a probability measure on $\{-1, +1\}$. We assume that

(10.16) $$\nu_v = \mu_v \quad \text{for } v \notin \mathfrak{w}$$

but

(10.17) $\nu_v\{\omega(v) = 1\} < \mu_v\{\omega(v) = 1\}$, $v \in \mathfrak{w}$,

where $\mathfrak{w}$ is a periodic subset of vertices of $\mathcal{G}_{p\ell}$. $P_{p'}$ is also assumed periodic, i.e.,

(10.18) $\nu_v = \nu_w$ if $w = v + k_1\xi_1 + k_2\xi_2$ for some

$$k_1, k_2 \in \mathbb{Z} \quad .$$

Thus, $P_{p'}$ {v is occupied} takes still only a finite number of values, on periodic subclasses of the vertices. We think of these values as the components of a vector p', thereby justifying the notation $P_{p'}$. Note, however, that p' can have more (or fewer) components than p; $P_{p'}$ does not have to be a λ-parameter probability measure. Also, for a central vertex v of $\mathcal{G}_{p\ell}$ which belongs to $\mathfrak{w}$ (10.17) and (7.2) imply

(10.19) $\nu_v\{\omega(v) = 1\} < 1 = \mu_v\{\omega(v) = 1\}$.

We are therefore no longer restricting ourselves to measures in which all central vertices of $\mathcal{G}_{p\ell}$ are occupied with probability one. However, by (10.16) and (7.3) we still have

(10.20) $\nu_v\{\omega(v) = -1\} = \mu_v\{\omega(v) = -1\} = 1$ for every central vertex of $\mathcal{G}^*_{p\ell}$.

It is also worth pointing out that (10.15) - (10.17) imply

$$P_{p'}\ \{v \text{ is occupied}\} \leq P_{p_0}\ \{v \text{ is occupied}\}$$

for all vertices v of $\mathcal{M}_{p\ell}$.

<u>Theorem 10.2. Assume $\mathcal{G}, \mathcal{G}^*, \mathfrak{v}_1, \dots, \mathfrak{v}_\lambda$ satisfy (10.9) - (10.11) and that $\mathfrak{w}$ is a periodic subset of the vertices of $\mathcal{G}_{p\ell}$ such that Condition D holds. Further let p_0 be such that (10.13) and (10.14) hold, and assume that P_{p_0} is extended such that (7.2) and (7.3) hold for $p = p_0$. Let $P_{p'}$ be defined by (10.15) and satisfy (10.16) and (10.17). Then, for any vertex z_0 of $\mathcal{G}_{p\ell}$</u>

(10.21) $\quad E_{p'}\{\#W_{p\ell}(z_0)\} < \infty$ and $P_{p'}\{\#W_{p\ell}(z_0) = \infty\} = 0$,

where $W_{p\ell}(z_0)$ is the occupied cluster on $\mathcal{G}_{p\ell}$ of z_0.

We now explain how this result can be applied to deal with subgraphs $\mathcal{H}$ of $\mathcal{G}$. We will consider subgraphs $\mathcal{H}$ of $\mathcal{G}$ formed by one or both of the following two procedures in succession:

(10.22) Remove all vertices of $\mathcal{G}$ in some periodic subclass $\mathcal{V}_0$ of the vertices of $\mathcal{G}$. Also remove all edges incident to any vertex of $\mathcal{V}_0$.

(10.23) Remove the close-packing in all faces of $\mathcal{F}_0$, where $\mathcal{F}_0$ is a periodic subset of $\mathcal{F}$.

Note that we do not make any symmetry requirements for $\mathcal{H}$ with respect to any line. The periodicity requirement in (10.22) for $\mathcal{V}_0$ means of course that (3.18) holds for $\mathcal{V}_0$, while for $\mathcal{F}_0$ in (10.23) it means that if $F \in \mathcal{F}_0$ then also $F + k_1\xi_1 + k_2\xi_2 \in \mathcal{F}_0$ for any integers k_1, k_2. To remove the close packing of F means to remove all edges which run through the interior of F and connect two vertices on the perimeter of F. Recall that these edges where inserted to manufacture $\mathcal{G}$ from $\mathcal{M}$ (see Sect. 2.2).

Now let p_0 satisfy (10.13) and (10.14) ((10.14) is a condition on p_0 and $\mathcal{G}$). P_{p_0} also induces a probability measure on the occupancy configurations of $\mathcal{H}$(we merely have to restrict P_{p_0} to the vertices of $\mathcal{H}$, i.e., to $\bigcup_i^{\lambda} \mathcal{V}_i \setminus \mathcal{V}_0$). To define $P_{p'}$ in the present situation we take

(10.24) $\mathcal{W} = \mathcal{V}_0 \cup \{$the central vertices of faces $F \in \mathcal{F}_0\}$ ($\mathcal{W}$ is a subset of the vertex set of $\mathcal{G}_{p\ell}$. $\mathcal{V}_0 = \emptyset$ if only (10.23) is applied to form $\mathcal{H}$; also $\mathcal{F}_0 = \emptyset$ if only (10.22) is applied to form $\mathcal{H}$).

Next, we take for v a vertex of $\mathcal{M}_{p\ell}$

(10.25) $P_{p'}\{v$ is occupied$\} = P_{p_0}\{v$ is occupied$\}$ if $v \notin \mathcal{W}$,

and

(10.26) $\qquad P_{p'}\ \{v \text{ is occupied}\} = 0 \quad \text{if} \quad v \in \mathfrak{w}$.

Later on we shall show the easy fact that percolation on $\mathcal{H}$ under P_{p_0} is equivalent to percolation on $\mathcal{G}_{p\ell}$ under $P_{p'}$, and that (10.15) - (10.18) hold for the above $\mathfrak{w}$ and $P_{p'}$. This then leads to the following result for subgraphs $\mathcal{H}$.

Theorem 10.3. Assume $\mathcal{G},\mathcal{G}^*, \mathfrak{b}_1,\dots,\mathfrak{b}_\lambda$ satisfy (10.9) - (10.11) and p_0 satisfies (10.13) and (10.14). Let $\mathcal{H}$ be a subgraph of $\mathcal{G}$ formed by one or both of the procedures (10.22), (10.23) and assume Condition D holds with $\mathfrak{w}$ as in (10.24). Then, for any p_1 in some open neighborhood (in $\mathcal{P}_\lambda$) of p_0 and any vertex z_0 of $\mathcal{H}$:

$$E_{p_1}\{\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{H})\} < \infty \quad ,$$

$$P_{p_1}\{\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{H}) = \infty\} = 0 \ .$$

Special case. In a one-parameter problem (i.e., $\lambda = 1$) with $\mathcal{G}$ and $\mathcal{H}$ as in Theorem 10.3 one obtains

$$p_H(\mathcal{H}) > p_H(\mathcal{G}) \ .$$

Examples .

Before turning to the proofs we illustrate the use of Theorems 10.2 and 10.3 and the verifiability of condition D with a few examples.

(i) Let $\mathcal{G} = \mathcal{G}_1$, the graph corresponding to bond percolation on $\mathbb{Z}^2$, imbedded as in Fig. 2.3 (see Ex. 2.1(ii); the vertices are located at $(i + \frac{1}{2}, i_2)$ and $(i_1,i_2, + \frac{1}{2})$, $i_1,i_2 \in \mathbb{Z})$. $\mathcal{G}_{1,p\ell}$ has in addition vertices at (i_1,i_2), $i_1,i_2 \in \mathbb{Z}$. (see Ex. 2.3(ii) where the same graph is discussed, but rotated over 45°). $\mathcal{G}^*_{1,p\ell}$ is shown in Fig. 10.1 below. It has vertices at $(i_1,i_2 + \frac{1}{2})$, $(i_1 + \frac{1}{2}, i_2)$, $(i_1 + \frac{1}{2}, i_2 + \frac{1}{2})$, i_1, $i_2 \in \mathbb{Z}$. For $\mathfrak{w}$ we take the vertices of $\mathcal{G}_{1,p\ell}$ on $\mathbb{Z}^2$,i.e.,

$$\mathfrak{w} = \{(i_1,i_2) : i_1,i_2 \in \mathbb{Z}\} \quad .$$

We easily see that condition D holds in this example with x = the origin. For U we take a path from $(-\Delta,0)$ to $(\Delta,0)$ along the first

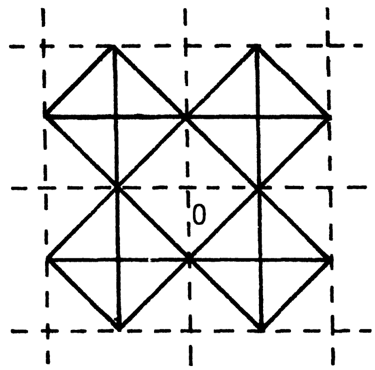

Figure 10.1 $\mathcal{G}^*_{1,p\ell}$. The solid segments are the edges of $\mathcal{G}^*_{1,p\ell}$. The dashed lines are the lines $x(1) = k_1$ or $x(2) = k_2$, $k_i \in \mathbb{Z}$.

coordinate axis. For V^* we take the path from $v_0^* = (0, \frac{1}{2})$ along the 45° line to $(\frac{1}{2}, 1)$ and then upwards along the vertical line $x(1) = \frac{1}{2}$ to the point $(\frac{1}{2}, \Delta)$ (see Fig. 10.2). b) is automatically fulfilled since $\mathfrak{w}$ contains all central vertices of $\mathcal{G}_{p\ell}$.

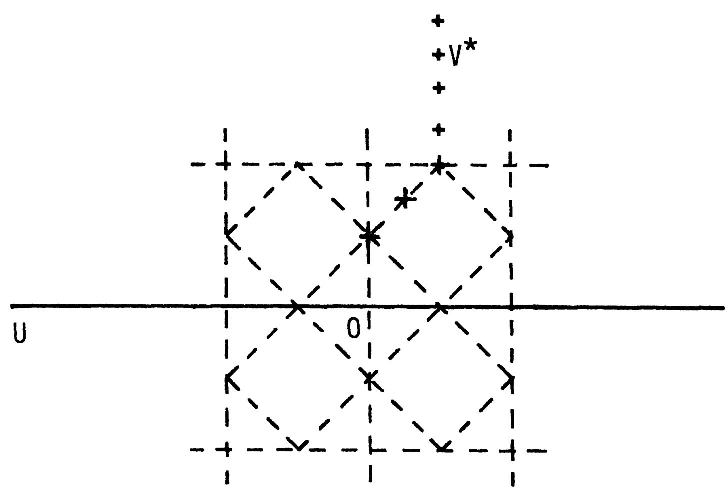

Figure 10.2 The dashed lines represent edges of $\mathcal{G}_{1,p\ell}$. The path U is drawn solidly. The path V^* is indicated by +++ ; it runs on $\mathcal{G}^*_{1,p\ell}$.

Now as in Application 3.4(ii), let

$$\mathcal{V}_1 = \{i_1 + \frac{1}{2}, i_2) : i_1, i_2 \in \mathbb{Z}\},$$

$$\mathcal{V}_2 = \{(i_1, i_2 + \frac{1}{2}) : i_1, i_2 \in \mathbb{Z}\}.$$

We consider the corresponding two-parameter problem, as defined in (3.20) - (3.23). Take $p_0 = (p_0(1), p_0(2))$ such that

$$p_0(1) + p_0(2) = 1,\ 0 < p_0(i) < 1,\ i = 1,2 .$$

By Application 3.4(ii) condition A holds for such a p_0. By Theorem 10.2 we therefore have

(10.27) $$E_{p'} \{\#W_{p\ell}(z_0)\} < \infty$$

for any $P_{p'}$ of the form (10.15) with

(10.28) $$P_{p'} \{v \text{ is occupied}\} = p_0(i),\ v \in \mathcal{V}_i ,\ i = 1,2, .$$

(10.29) $$P_{p'} \{(i_1,i_2) \text{ is occupied}\} < 1,\ i_1,i_2 \in \mathbb{Z} .$$

Actually the set of $p >> 0$ in parameter space where (10.27) holds is open, by Cor. 5.1. Thus (10.27) continues to hold when p_0 in (10.28) and is replaced by p_1 sufficiently close to p_0, even when $p_1(1) + p_1(2) > 1$. The best illustration for this is provided by Theorem 10.3. We now define $\mathcal{H}$ as the subgraph of $\mathcal{G}$, obtained by removing the close packing of all the faces which contain a point (i_1, i_2), $i_1,i_2 \in \mathbb{Z})$. (Thus, if we call this last collection of faces $\mathcal{F}_0$, then we only apply (10.23) with this $\mathcal{F}_0$). The resulting $\mathcal{H}$ is clearly isomorphic to $\mathcal{G}_0$, the simple quadratic lattice, and $\mathcal{V}_1$ and $\mathcal{V}_2$ are such that the resulting two-parameter problem on $\mathcal{H}$ is precisely the two-parameter problem for site-percolation on $\mathbb{Z}^2$ considered in Application 3.4(iv). We conclude from Theorem 10.3 that no percolation occurs under P_{p_1} for $p_1 = (p_1(1), p_1(2)$, in some neighborhood of p_0. In particular, <u>the non-percolative region for two-parameter site-percolation on $\mathbb{Z}^2$ contains the (anti-) diagonal</u>

$$\{p: 0 \le p(i) \le 1,\ p(1) + p(2) = 1\}$$

<u>strictly in its interior</u>. Strictly speaking we only obtain this conclusion from Theorem 10.3 for $0 << p << 1$. However, we already know from Application 3.4(iv) that no percolation occurs for $0 \le p(1) \le p_H(\mathcal{G}_0^*)$, $p(2) = 1$, and hence by monotonicity (Lemma 4.1) no percolation occurs for $0 \le p(1) \le p_H(\mathcal{G}_0^*)$, $p(2) \ge 1 - p_H(\mathcal{G}_0^*)$ (see Fig. 3.8). Similarly no percolation occurs for $1-p_H(\mathcal{G}_1^*) \le p(1) \le 1$, $0 \le p(2) \le p_H(\mathcal{G}_0^*)$.

When restricted to $p(1) = p(2)$ the above shows that there is no percolation in a neighborhood of $p(1) = p(2) = \frac{1}{2}$. This shows that (10.2) is really a strict inequality .

(ii) This time let $\mathcal{G}$ be the triangular lattice. In order to obtain the familiar picture we imbed this lattice in such a way that its faces are equilateral triangles (i.e., we use the imbedding of Fig. 2.4 rather than the one for $\mathcal{T}$ described in Ex. 2.1(iii).) $\mathcal{G}_{p\ell} = \mathcal{G}$ in this case. Let the vertices be located at

$$(k_1 + \frac{k_2}{2}, \frac{k_2}{2}\sqrt{3}), \; k_1, k_2 \in \mathbb{Z},$$

and take

$$\mathfrak{w} = \{2k_1 + k_2, k_2\sqrt{3}\}, \; k_1, k_2 \in \mathbb{Z}.$$

In a way $\mathfrak{w}$ consists of every other point; see Fig. 10.3.

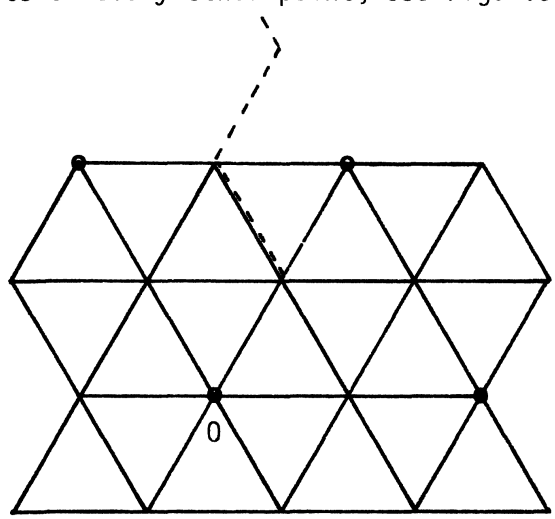

Figure 10.3 The triangular lattice with the points of $\mathfrak{w}$ indicated by circles. V^* is the dashed path.

Again condition D is easily seen to hold with x = the origin. For U we take again a path from $(-\Delta,0)$ to $(\Delta,0)$ along the first coordinate axis. For V^* we take a path with "zig-zags" upward from the point $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ alternatingly through points $(\frac{1}{2},(j - \frac{1}{2})\sqrt{3})$ and $(0,j\sqrt{3})$, $j = 1,2,\dots, \Delta$.

We may therefore apply Theorem 10.3 to the one-parameter problem on $\mathcal{G}$. We know from application 3.4(i) that $p_0 = \frac{1}{2}$ = critical probability for site-percolation on $\mathcal{G}$ satisfies condition A. Let $\mathcal{H}$ be the graph obtained by removing the vertices in $\mathfrak{w}$ from $\mathcal{G}$.(Thus we apply only (10.22) with $\mathcal{V}_0 = \mathfrak{w}$.) We conclude that $p_H(\mathcal{H}) > \frac{1}{2}$.

However, one easily sees that removing the sites in $\mathfrak{w}$ from $\mathcal{G}$ yields the Kagome lattice of Ex. 2.5(i) for $\mathcal{H}$. This is the covering graph of the hexagonal lattice, so that $p_H(\mathcal{H})$ = critical probability for bond percolation on the hexagonal lattice $= 1 - 2\sin\frac{\pi}{18}$ (see Prop. 3.1 and Application 3.4(iii).). Thus, we obtained the obvious inequality $1-2\sin\frac{\pi}{18} = p_H(\mathcal{H}) > \frac{1}{2}$.

Since by Application 3.4(iii) the critical probability for bond-percolation on the triangular lattice equals one minus the critical probability for bond percolation on the hexagonal lattice, we also have

$$p_H(\text{bond percolation on triangular lattice}) < \frac{1}{2} .$$

This is precisely (10.3) with a strict inequality.

(iii) In this example we compare $\mathbb{Z}^3$ with $\mathbb{Z}^2$. We concentrate on site-percolation, but practically the same argument works for bond-percolation on $\mathbb{Z}^3$, or even the restriction of $\mathbb{Z}^3$ to $\mathbb{Z}^2 \times \{0,1\}$ (i.e., two layers of $\mathbb{Z}^2$). The latter graph contains the following graph $\mathcal{G}$, which is obtained by decorating one out of nine faces of $\mathcal{G}_0$ (see Ex. 2.1(i) for $\mathcal{G}_0$). Each face $(i_1,i_1+1) \times (i_2,i_2+1)$ with both $i_1 \equiv 1 (\mathrm{mod}\ 3)$ and $i_2 \equiv 1(\mathrm{mod}\ 3)$ is decorated as shown in Fig. 10.4.

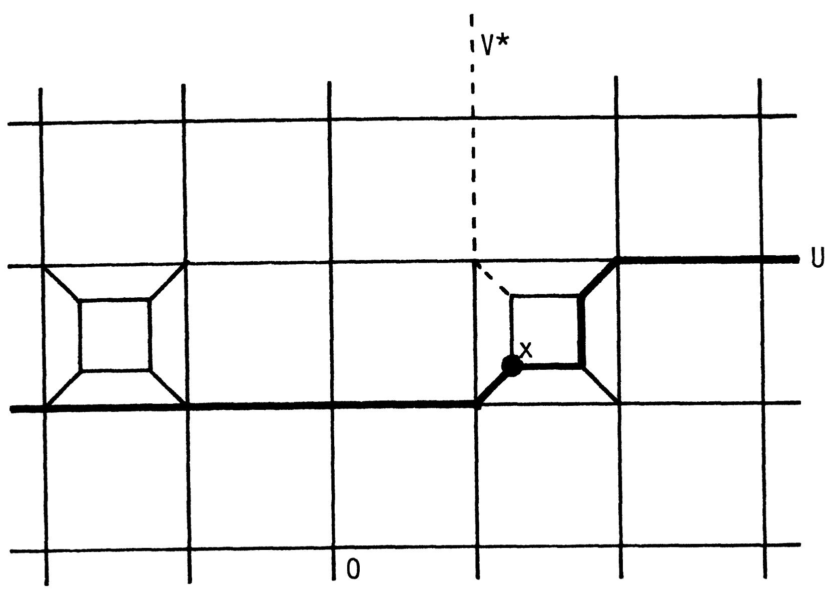

Figure 10.4 The graph $\mathcal{G}$, obtained from $\mathcal{G}_0$ by the indicated decorations. The blackened circle is the vertex x. The boldly drawn path is U. The path V^* is dashed.

$\mathcal{G}$ is a mosaic so that we can view $\mathcal{G}$ as one of a matching pair based on $(\mathcal{G},\emptyset)$. In this case $\mathcal{G} = \mathcal{G}_{p\ell}$. Both coordinate axes are axes of symmetry for $\mathcal{G}$. For the subgraph $\mathcal{H}$ we take $\mathcal{G}_0$ = the simple square lattice. This corresponds to applying (10.22) only, with $\mathcal{V}_0$ = the collection of vertices used in decorating $\mathcal{G}_0$ when forming $\mathcal{G}$. Condition D b) is vacuous since $\mathcal{F} = \emptyset$, and Fig. 10.4 illustrates that the other parts of Condition D can also easily be satisfied for any choice of $x \in \mathcal{V}_0$. Since $\mathcal{G}$ is invariant under a rotation over 90° around the origin, it is immediate that (3.52) - (3.55) hold for the one-parameter problem on $\mathcal{G}$; compare Applications 3.4(iv) and (v). As in those Applications it follows from Theorem 3.2 that Condition B of Sect. 3.3 holds for $p_0 = p_H(\mathcal{G})$. We therefore conclude from the one-parameter case of Theorem 10.3 that

(10.30) $$\begin{aligned} &p_H(\text{site-percolation on } \mathbb{Z}^3) \\ &\le p_H(\text{site-percolation on two layers of } \mathbb{Z}^2) \le p_H(\mathcal{G}) \\ &< p_H(\text{site-percolation on } \mathbb{Z}^2) = p_H(\mathcal{G}_0). \end{aligned}$$

To obtain a similar conclusion for bond-percolation on $\mathbb{Z}^3$ we compare the covering graph $\tilde{\mathcal{G}}$ of $\mathcal{G}$ with the covering graph of $\mathbb{Z}^2$, (see Ex. 2.5(ii)). We draw some faces of $\tilde{\mathcal{G}}$ in Fig. 10.5. The central square in this figure corresponds to one of the decorated faces in $\mathcal{G}$. $\mathcal{H}$ is now formed from $\tilde{\mathcal{G}}$ by removing all vertices of $\mathcal{H}$ which correspond to edges of the decorations. These vertices are marked by solid circles in Figure 10.5. We leave it to the reader to verify that $\mathcal{H}$ is nothing but $\mathcal{G}_1$. We therefore conclude in the same way as in (10.30) that

$$p_H(\text{bond-percolation on } \mathbb{Z}^3) < \frac{1}{2} = p_H(\mathcal{G}_1)$$

(see Application 3.4(ii) for the last equality).

(iv) The graph $\mathcal{G}$ in this example will be $\mathcal{D}^*$, the matching graph of the diced lattice $\mathcal{D}$. $\mathcal{D}$ was introduced in Ex. 2.1(v); $\mathcal{D}^*$ is illustrated in Fig. 10.6. One can think of $\mathcal{D}^*$ as a "decoration" of the hexagonal lattice. Note that $\mathcal{D}^*$ is not identical with the matching graph of the hexagonal graph, because $\mathcal{D}^*$ has a vertex in the center of each hexagon (the solid circles in Fig. 10.6). $\mathcal{D}^*_{p\ell}$ is also drawn in Fig. 10.6. It has a central vertex in each face of $\mathcal{D}$ (see Fig. 2.7; these central vertices are indicated by the open

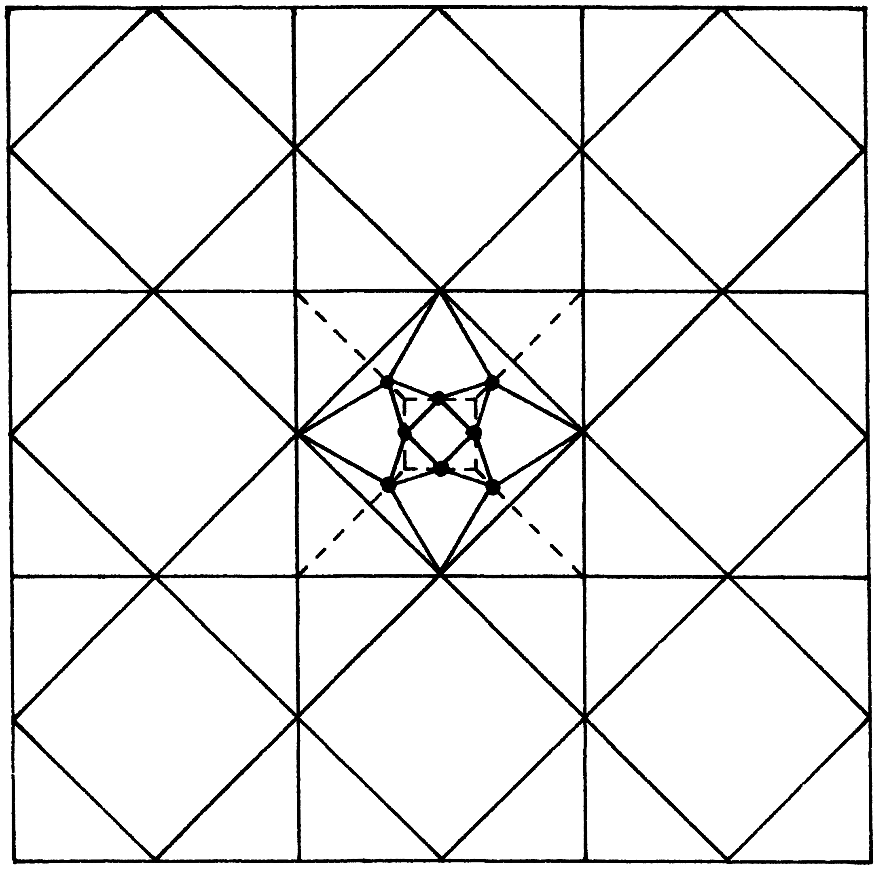

Figure 10.5 The covering graph $\tilde{\mathcal{G}}$ of $\mathcal{G}$. The dashed edges form the decoration of one face of $\mathcal{G}_0$.

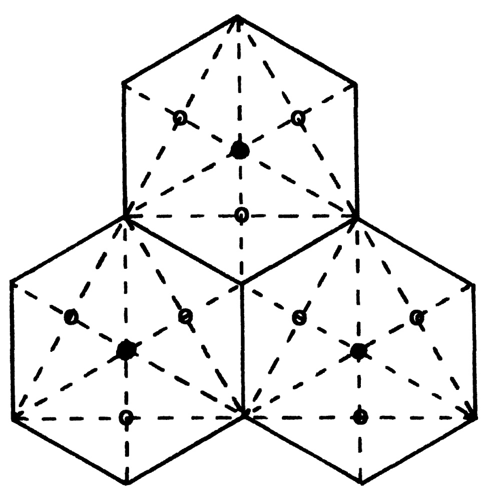

Figure 10.6 $\mathcal{D}^*$ drawn as a "decoration" of a hexagonal lattice. The "decoration" is indicated by dashed lines. There is a vertex of $\mathcal{D}^*$ at each center of the hexagons (drawn as a solid circle). There is no vertex of $\mathcal{D}^*$ at the open circles; however, there is a vertex of $\mathcal{D}^*_{p\ell}$ at each open circle.

circles in Fig. 10.6). For $\mathfrak{w}$ we take the collection of these central vertices. Condition D again holds. We content ourselves with a picture of a possible choice for U and V^* in Fig. 10.7 for x an arbitrary vertex in $\mathfrak{w}$. Note that Condition D b) is again vacuous since $\mathfrak{w}$ contains all central vertices of $\mathcal{G}_{p\ell} = \mathcal{S}^*_{p\ell}$. Also $(\mathcal{S}^*)^* = \mathcal{S}$ and $\mathcal{S}_{p\ell} = \mathcal{S}$.

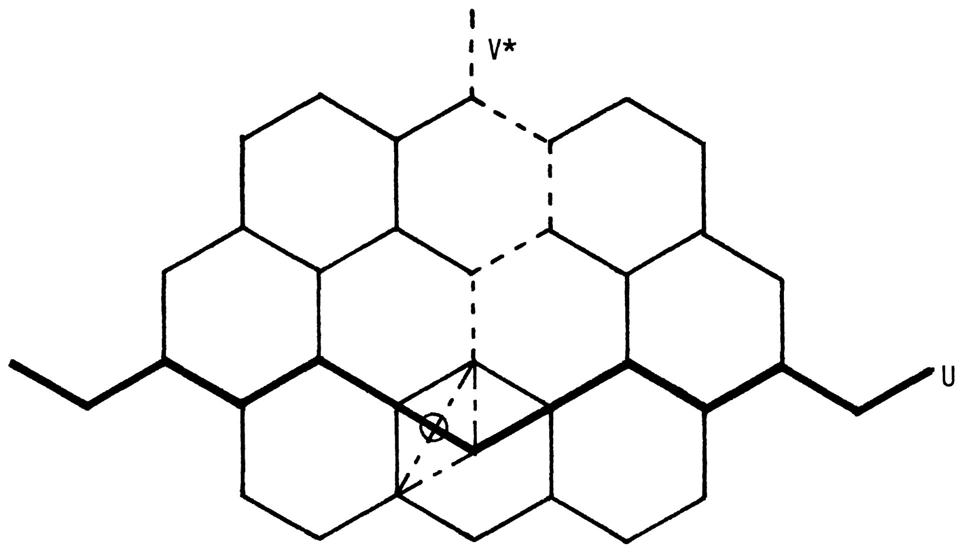

Figure 10.7 The open circle is the vertex x . The boldly drawn path is U . The dashed path is V^* . The edges indicated by — - — belong to $\mathcal{S}_{p\ell}$.

Once more we apply Theorem 10.3. This time we take for $\mathcal{H}$ the graph obtained by removing the close packing in all faces of $\mathcal{S}^*$, i.e., we apply only (10.23) with $\mathfrak{F}_0$ all faces of $\mathcal{S}$. The resulting $\mathcal{H}$ is just $\mathcal{S}$ itself. $p_0 = p_H(\mathcal{S}^*)$ satisfies condition B when $\mathcal{G}$ is taken $\mathcal{S}^*$ (by Application 3.3(v) and Theorem 3.2). (Actually we checked (3.52) - (3.55) for $\mathcal{G} = \mathcal{S}$. However, (3.52) - (3.55) remain unchanged when $\mathcal{G}$ is replaced by $\mathcal{G}^*$ and p by $1-p$. Thus (3.52)-(3.55) hold when $\mathcal{G} = \mathcal{S}^*$.) Theorem 3.2 then shows that Condition B holds for $p_0 = p_H(\mathcal{S}^*)$. The conclusion of the one-parameter case of Theorem 10.3 is now

$$p_H(\mathcal{S}) > p_H(\mathcal{S}^*) .$$

But by Theorems 3.2 and 3.1 $p_H(\mathcal{S}) = 1-p_H(\mathcal{S}^*)$ so that we find

$$p_H(\mathcal{S}) > \frac{1}{2} > p_H(\mathcal{S}^*).$$

From Fig. 10.7 one also sees that Condition D is fulfilled if we take $\mathcal{G} = \mathcal{D}$, and $\mathfrak{w}$ the collection of the centers of the hexagons. If $\mathcal{D}$ is imbedded as described in Ex. 2.1(v) these are the points

$$((k_1 + \frac{\ell}{2})\sqrt{3}\ ,\ 3(k_2 + \frac{1}{2}\ell)),\ k_i \in \mathbb{Z}\ ,\ \ell = 0 \text{ or } 1\ ;$$

in Fig. 10.6 this means that we remove the solid circles. If we apply (10.22) with $\mathfrak{v}_0 = \mathfrak{w}$ the resulting graph is the hexagonal lattice and we obtain

$$p_H(\text{hexagonal lattice}) > p_H(\text{diced lattice}) > \frac{1}{2}\ .$$

(these are critical probabilities for site-percolation).

<u>Remark</u> .

i) The procedure illustrated in this example will work in many examples of matching pairs $(\mathcal{G},\mathcal{G}^*)$ based on $(\mathcal{M},\emptyset)$ to yield

$$p_H(\mathcal{G}) = p_H(\mathcal{M}) > \frac{1}{2} > p_H(\mathcal{G}^*)\quad .$$

Indeed apply Theorem 10.3 with $\mathcal{G}$ replaced by $\mathcal{G}^*$, and $\mathcal{F}_0$ the collection of all faces of $\mathcal{M}$. When removing the close-packing from $\mathcal{G}^*$ in all faces of $\mathcal{M}$ as in (10.23), the resulting subgraph $\mathcal{H}$ is just $\mathcal{M}$, or $\mathcal{G}$. Lastly one uses $p_H(\mathcal{G}) + p_H(\mathcal{G}^*) = 1$, assuming Theorem 3.1 or 3.2 applies. One could have obtained $p_H(\mathcal{G}_0) > \frac{1}{2}$ in Ex. 10.2(i) above in this way. ///

v) This "negative" example shows that some kind of condition like Condition D has to be imposed. We take for $\mathcal{H}$ a mosaic, and for $\mathcal{G}$ a graph obtained by decorating a periodic subclass of faces of $\mathcal{H}$. Choose the decoration in a face F such that it is attached to only one vertex v , or two adjacent vertices v' , v'', of $\mathcal{H}$ on the

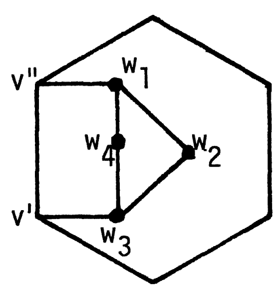

Figure 10.8. F is the interior of the hexagon, which is a face of $\mathcal{H}$. The vertices w_1-w_4 and the edges in F have been used to "decorate" F.

perimeter of F, e.g. as in Fig. 10.8. Even though $\mathcal{H}$ is a subgraph of $\mathcal{G}$ one always has $p_H(\mathcal{H}) = p_H(\mathcal{G})$. Indeed in site-percolation, in the situation of Fig. 10.8, the decoration could be of help in forming an infinite occupied cluster only if v' and v" are both occupied. But in this case v' and v" belong to that same cluster even if no decoration is present. Condition D fails because there exists no minimal path through any of the added vertices in F which starts and ends ouside F.

Remark .

ii) The theorems of this section give no strict inequality if $\mathcal{H}$ is obtained from $\mathcal{G}$ by removing edges of $\mathcal{M}$ or partial close-packings only. If we remove one of the edges introduced when close-packing a face of $\mathcal{M}$, then we usually cannot find a subgraph $\mathcal{H}_{p\ell}$ which serves as the planar modification of $\mathcal{H}$, and we therefore have trouble in defining a "lowest" horizontal occupied crossing. On the other hand, if an edge e of $\mathcal{M}$ (and it translates by integral vectors) is removed to form $\mathcal{H}$, then one can artifically turn this into a situation where one removes a vertex. One introduces a new kind of vertex for $\mathcal{M}$, situated somewhere on $\overset{\circ}{e}$, and connected only to the endpoints of e. The new vertex should be occupied with probability one on $\mathcal{G}$, and it is this vertex which is removed to form $\mathcal{H}$. However, this introduces a new vertex on the perimeter of some faces of $\mathcal{M}$, and therefore $\mathcal{G}$ may no longer be obtained from the modified $\mathcal{M}$ by close-packing faces. Nevertheless we believe a more complicated proof may work when only edges of $\mathcal{M}$ are removed from $\mathcal{G}$ to form $\mathcal{H}$. ///

Proof of Theorem 10.2. The proof consists of two parts. First a combinatorial, or topological, part which derives another ugly condition - Condition E stated in Step (ii) - from Condition D. We begin with a probabilistic part, and defer the derivation of Condition E from Condition D to a separate section (to make it easier to skip the unpleasant and not very interesting part of the argument).

The probabilistic part begins like the proof of Lemma 7.4. By virtue of Theorem 5.1 it suffices for (10.21) to prove

$$\lim_{\ell\to\infty} \tau(2\overline{M}_\ell;\ i,p',\mathcal{G}_{p\ell}) = 0\ ,\ i = 1,2,$$

for some sequence $\overline{M}_\ell = (M_{\ell 1},M_{\ell 2})$ with

$$M_{\ell i} \to \infty\ (\ell \to \infty)\ ,\ i = 1,2,$$

In this whole proof we restrict ourselves to horizontal crossings, i.e., we prove only

$$(10.31) \qquad \lim_{\ell \to \infty} \tau(2\overline{M}_\ell;\ 1, p', \mathcal{G}_{p\ell}) = 0$$

for suitable $\overline{M}_\ell$. The same proof can be used to show

$$\lim_{\ell \to \infty} \tau(2M_\ell; 2, p', \mathcal{G}_{p\ell}) = 0 ;$$

the asymmetry between the horizontal and vertical direction in Condition D discussed in Comment 10. 2(iii) will play no role in the proof of (10.31).

Let $\tilde{E}$ be the event that there exists an occupied horizontal crossing on $\mathcal{G}_{p\ell}$ of a certain large rectangle. We want to show that $P_{p'}\{\tilde{E}\}$ is small. As in Lemma 7.4 this is essentially done by showing $\frac{d}{dt} P_{p(t)}\{\tilde{E}\}$ is large with $p(t) = tp_0 + (1-t)p'$. Russo's formula (4.22) reduces this to proving that the number of pivotal sites in $\mathfrak{w}$ for $\tilde{E}$ is large. This is really the content of (10.62), which is our principal new estimate here. From Lemma 7.4 and Remark 7(ii) we know that with high probability there are many pivotal sites for $\tilde{E}$ on the lowest occupied horizontal crossing of the large rectangle. (10.62) claims that many of these have to belong to $\mathfrak{w}$. The proof of this is based on the idea that if few of the pivotal sites belong to $\mathfrak{w}$, then one can make local modifications in the occupancy configuration so as to obtain many pivotal sites in $\mathfrak{w}$. To obtain (10.62) one has to make the modifications in such a way that one can more or less go back, i.e., reconstruct the original occupancy configuration from the modified one. For this one first has to locate the sites whose occupancy has been modified. To achieve this we must have good control over the changes in the lowest occupied crossing under our modifications of the occupancy configuration. The various parts of Condition E give the necessary control.

Before we can even formulate Condition E we need a preparatory step.

Step (i). Since p_0 satisfies (10.13) and (10.14) the conclusion of Lemma 7.2 holds. For the remainder of this chapter we choose $\overline{M}_\ell$ and $\delta_k > 0$ such that (7.17), (7.19) and (7.21) hold. For

large ℓ we construct a Jordan curve J_ℓ on $\mathcal{M}$ close to the perimeter of $[0,2M_{\ell 1}] \times [0,12M_{\ell 2}]$ by the method of Lemma 7.4. Specifically, we find simple curves ϕ_1 and ϕ_3 on $\mathcal{M}$ which connect the top and bottom edges of the strips
$[0,\Lambda_3] \times [-\Lambda_4,12M_{\ell 2} + \Lambda_4]$ and $[2M_{\ell 1} - \Lambda_3, 2M_{\ell 1}] \times [-\Lambda_4,12M_{\ell 2} + \Lambda_4]$, respectively. Such curves can be found as parts of

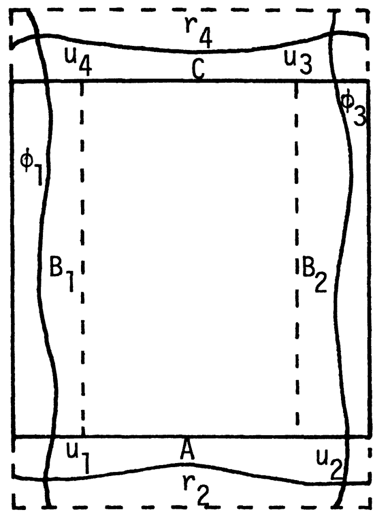

Figure 10.9 The solid rectangle is $[0,2M_{\ell 1}] \times [0,12M_{\ell 2}]$; the outer dashed rectangle is $[0,2M_{\ell 1}] \times [-\Lambda_4,12M_{\ell 2}+\Lambda_4]$; the inner dashed rectangle is $[\Lambda_3,2M_{\ell 1}-\Lambda_3] \times [0,12M_{\ell 2}]$.

vertical crossings of these strips. Also we take self-avoiding horizontal crossings r_2 and r_4 on $\mathcal{M}$ of the strips
$[0, 2M_{\ell 1}] \times [-\Lambda_4, - 1]$ and $[0,2M_{\ell 1}] \times [12M_{\ell 2} + 1, 12M_{\ell 2} + \Lambda_4]$, respectively. Starting from the left endpoint of $r_2(r_4)$ let $u_1(u_4)$ be the last intersection of $r_2(r_4)$ with ϕ_1; and $u_2(u_3)$ the first intersection of $r_2(r_4)$ with ϕ_3(see Fig. 10.9). As in Lemma 7.4 we denote the closed segment of ϕ_1 from u_4 to u_1 by B_1, the closed segment of ϕ_3 from u_2 to u_3 by B_2, the closed segment of r_2 from u_1 to u_2 by A and the closed segment of r_4 from u_3 to u_4 by C(again see Fig. 10.9). J_ℓ is the Jordan curve consisting of B_1,A,B_2 and C. We shall be considering paths $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ on $\mathcal{G}_{p\ell}$ with the properties (7.39)- (7.41) (with J_ℓ for J in these). For brevity we shall refer to such paths simply as <u>crosscuts of int</u>(J_ℓ) in this chapter. For any such

path we define, as in Def. 2.11 $J_\ell^-(r)(J_\ell^+(r))$ as the component of $\mathrm{int}(J_\ell)\setminus r$ with $A(C)$ in its boundary.

With any crosscut r which in addition satisfies

$$r \subset [0,2M_{\ell 1}] \times [-\infty, 6M_{\ell 2}], \tag{10.32}$$

(roughly speaking this means that r lies in the lower half of $\bar{J}$) we shall associate a crosscut $r^{\#}$ which also satisfies (7.34) -(7.41) and which lies "above" r. This associated $r^{\#}$ is found by means of a specially chosen circuit K on $\mathcal{G}_{p\ell}$ and surrounding the origin. To choose K recall that Δ is chosen in Condition D and set

$$\begin{aligned} \Lambda_9 &= 20(\Lambda_5 + \Lambda_6 + \Lambda_7 + \Lambda_8 + \Lambda + \Delta + 1), \\ \Theta &= (6\Lambda_5 + 1)\,(3\Lambda_9 + \Lambda_6 + 4\Lambda_3 + 7\Lambda + 1). \end{aligned} \tag{10.33}$$

Next take for K a circuit on $\mathcal{G}_{p\ell}$ surrounding the origin in the annulus

$$[-2\Theta - \Lambda_3,\ 2\Theta + \Lambda_3] \times [-2\Theta - \Lambda_3,\ 2\Theta + \Lambda_3] \setminus (-2\Theta, 2\Theta) \times (-2\Theta, 2\Theta). \tag{10.34}$$

Such a circuit can be constructed in the manner of J_ℓ above from two vertical crossings s_1 and s_2 on $\mathcal{G}_{p\ell}$ of $[-2\Theta - \Lambda_3, -2\Theta] \times [-2\Theta - \Lambda_3, 2\Theta + \Lambda_3]$ and $[2\Theta, 2\Theta + \Lambda_3] \times [-2\Theta - \Lambda_3, 2\Theta + \Lambda_3]$, together with two horizontal crossings s_2 and s_4 on $\mathcal{G}_{p\ell}$ of $[-2\Theta - \Lambda_3, 2\Theta + \Lambda_3] \times [-2\Theta - \Lambda_3, -2\Theta]$ and $[-2\Theta - \Lambda_3, 2\Theta + \Lambda_3] \times [2\Theta, 2\Theta + \Lambda_3]$, respectively. By our choice of the constant Λ_5 (just after (10.12)) we can take the s_i such that for any two points y_1, y_2 on one s_i, there is a segment of s_i connecting y_1 and y_2 with diameter $\le \Lambda_5(|y_1 - y_2| + 1)$. We claim that any pair of points y_1, y_2 on K is then connected by an arc of K of diameter at most

$$3\Lambda_5(|y_1 - y_2| + 2\Lambda_3 + 1)\ . \tag{10.35}$$

This is obvious of y_1, y_2 lie on one s_i. When y_1 lies on s_1, y_2 on s_2 and u is the intersection of s_1 and s_2 on K, then u lies in $[-2\Theta - \Lambda_3, -2\Theta] \times [-2\Theta - \Lambda_3, -2\Theta]$, y_1 to the left of $\mathbb{R} \times \{-2\Theta\}$ and y_2 below $\{-2\Theta\} \times \mathbb{R}$. From this it is not hard to see that

$$|y_i - u| \leq |y_1 - y_2| + 2\Lambda_3 \quad , \; i = 1,2,$$

One therefore obtains the estimate (10.35) for the arc which goes from y_1 to u along s_1 and then from u to y_2 along s_2. When y_1 lies on s_1 and y_2 on s_3, then y_1 lies to the left of the vertical line $x(1) = -2\Theta$ and y_2 to the right of $x(2) = 2\Theta$. In this case

$$3(|y_1 - y_2| + 2\Lambda_3) \geq 12\Theta + 6\Lambda_3 \geq \text{diameter of the annulus (10.34)}$$

Since K is contained in the annulus, (10.35) is obvious in this case too (Λ_5 has to be ≥ 1 by its definition). Thus we showed (10.35) in all typical cases.

We also want to arrange matters such that

$$(10.36) \qquad K \text{ is minimal,}$$

in the sense that if v_1 and v_2 are two vertices of $\mathcal{G}_{p\ell}$ on K which are adjacent on $\mathcal{G}_{p\ell}$, then K contains an edge of $\mathcal{G}_{p\ell}$ from v_1 to v_2. (This is the obvious extension of Def. 10.1 to a circuit). If K is not minimal, then we can make it minimal by inserting a number of suitable shortcuts of one edge. E.g., if v_1 and v_2 are adjacent and e is an edge of $\mathcal{G}_{p\ell}$ between them, but K itself does not contain such an edge, then we can replace K by one of the arcs of K between v_1 and v_2 and the edge e. Since diameter $(e) \leq \Lambda$, the new circuit will still surround the square

$$(10.37) \qquad (-2\Theta + \Lambda, 2\Theta - \Lambda) \times (-2\Theta + \Lambda, 2\Theta + \Lambda),$$

and lie inside the square

$$(10.38) \quad [-2\Theta - \Lambda_3 - \Lambda, 2\Theta + \Lambda_3 + \Lambda] \times [-2\Theta - \Lambda_3 - \Lambda, 2\Theta + \Lambda_3 + \Lambda].$$

(Of course this holds only if we combine e with one of the two arcs of K between v_1 and v_2; it fails for the other arc). Also the estimate (10.35) changes only a little. Any two points y_1, y_2 on the new circuit are now connected by an arc of the new circuit with diameter at most

$$(10.39) \qquad 3\Lambda_5(|y_1-y_2| + 2\Lambda_3 + 3\Lambda + 1).$$

These observations remain valid even if we replace several arcs of K by shortcuts. Indeed, denote for the time being the circuit obtained after the insertion of shortcuts by K'. Then any $y \in K'$ lies within Λ of some vertex $z \in K \cap K'$. In particular, if $y_1, y_2 \in K'$, then there exist $z_1, z_2 \in K \cap K'$, $|z_1-z_2| \le |y_1-y_2| + 2\Lambda$. Also some arc of K between z_1 and z_2 has diameter $\le 3\Lambda_5(|z_1-z_2| + 2\Lambda_3 + 1)$. One can now find an arc of K' from y_1 to y_2 which is within distance Λ from the arc of K from z_1 to z_2. (10.39) is immediate from this, as well as the fact that K' lies outside (10.37) and inside (10.38). We drop the prime in K' and for the remainder we assume that K is a fixed circuit inside (10.38), which surrounds (10.37), satisfies (10.36) and the estimate (10.39).

For any vertex $v = (v(1), v(2))$ of $\mathcal{G}_{p\ell}$ we set

$$K(v) = K + \lfloor v(1) \rfloor \xi_1 + \lfloor v(2) \rfloor \xi_2 \quad .$$

$K(v)$ is the translate of K by $(\lfloor v(1) \rfloor, \lfloor v(2) \rfloor)$ and therefore $v \in \mathrm{int}(K(v))$. For any crosscut $r = (v_0, e_1, \ldots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ of J_ℓ which satisfies (10.32) (in addition to (7.39) - (7.41)) we set

$$(10.40) \qquad \mathcal{E}(r) = \bar{J}_\ell^-(r) \cup \bigcup_w \bar{K}(w),$$

where the union runs over the vertices $w = (w(1), w(2))$ of $\mathcal{G}_{p\ell}$ on r which satisfy

$$(10.41) \qquad \frac{1}{2}M_{\ell 1} - \Theta - 2\Lambda \le w(1) \le \frac{3}{2}M_{\ell 1} + \Theta + 2\Lambda \quad ,$$

and $\bar{K} = \mathrm{int}(K) \cup K$. Also $\mathfrak{F}(r)$ denotes the component of $\mathrm{int}(J_\ell) \setminus \mathcal{E}(r)$ with C in its boundary. Note that $\mathcal{E}(r)$ is a somewhat fattened up (near r) version of $\bar{J}_\ell^-(r)$. $\mathcal{E}(r)$ still lies below the horizontal line $x(2) = 6M_{\ell 2} + 2\Theta + \Lambda_3 + \Lambda$ (by (10.32) and (10.38)) so that for all large ℓ $\mathfrak{F}(r)$ is well defined and even contains a whole strip of $\mathrm{int}(J)$ near its upper edge C(C lies above $x(2) = 12M_{\ell 2}$). We claim that for sufficiently large ℓ there exists a crosscut $r^\#$ on $\mathcal{G}_{p\ell}$ which satisfies (7.39) - (7.41) and

(10.42) $\mathfrak{I}(r) = J_\ell^+(r^\#)$,

(10.43) $\overline{J}_\ell^-(r) \subset \overline{J}_\ell^-(r^\#)$, $J_\ell^+(r^\#) \subset J_\ell^+(r)$,

and

(10.44) $$r^\# \subset r \cup \bigcup_w K(w) ,$$

where the union in (10.44) runs over the same w as in (10.40). Of course $r^\#$ will simply be the "lower part" of the boundary of $\mathfrak{I}(r)$. A formal proof of the existence of $r^\#$ proceeds by induction. Assume the vertices which enter in the union in (10.40) are $w_1,\dots,w_m$. Let

$$\mathcal{E}_k = \overline{J}_\ell^-(r) \cup \bigcup_{i=1}^k \overline{K}(w_i),$$

and $\mathfrak{I}_k$ the component of $\text{int}(J_\ell) \setminus \mathcal{E}_k$ with C in its boundary. Assume we already proved that $\mathfrak{I}_k = J_\ell^+(r_k)$ for some r_k on $\mathcal{G}_{p\ell}$ satisfying (7.39) - (7.41) and

(10.45) $$\overline{J}_\ell^-(r) \subset \overline{J}_\ell^-(r_k),\ r_k \subset r \cup \bigcup_1^k K(w_i) .$$

This statement is true for $k = 0$ if we take $\mathcal{E}_0 = \overline{J}_\ell^-(r)$, $\mathfrak{I}_0 = J_\ell^+(r)$, $r_0 = r$. We now show that the statement is then also true for k replaced by $k+1$. We shall find r_{k+1} by a method similar to the construction of r from r_1 and r_2 in the beginning of the proof of Prop. 2.3 (see the Appendix). $\mathcal{E}_{k+1} = \mathcal{E}_k \cup \overline{K}(w_{k+1})$. For large enough ℓ $K(w_{k+1})$ does not intersect the left and right pieces B_1 and B_2 of J by virtue of (10.41). If α is an arc of $K(w_{k+1})$ which lies in $J^+(r_k)$ except for its endpoints, v_1 and v_2, which lie on r_k (see Fig. 10.10) then replace the piece of r_k between v_1 and v_2 by α. This gives a new crosscut, $\tilde{r}_k$ say, of $\text{int}(J_\ell)$ such that

$$\tilde{r}_k \subset \overline{J}_\ell^+(r_k) \quad\text{and}\quad r_k \subset \overline{J}_\ell^-(\tilde{r}_k) .$$

The proof of this statement is the same as for (A.38) - (A.40). If $K(w_{k+1})$ still contains a point above $\tilde{r}_k$, and hence an arc above $\tilde{r}_k$,

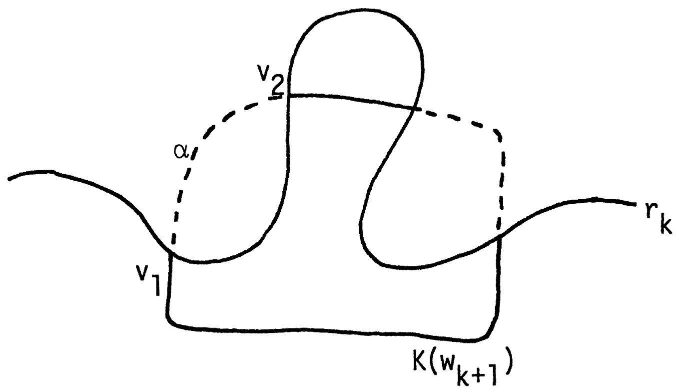

Figure 10.10 The two dashed pieces of the circuit $K(w_{k+1})$ lie in $J_\ell^+(r_k)$.

i.e., in $J^+(\tilde{r}_k)$, we repeat the procedure until we arrive at a crosscut r_{k+1}, made up from pieces of r_k and $K(w_{k+1})$ such that $K(w_{k+1})$ contains no more points of $J_\ell^+(r_{k+1})$. It is clear from the construction and the induction hypothesis (10.45) that

$$r_{k+1} \subset r_k \cup K(w_{k+1}) \subset r \cup \bigcup_1^{k+1} K(w_i), \tag{10.46}$$

Also, as in (A.38)

$$r_{k+1} \subset \bar{J}^+(r_k)$$

and this implies, just as (A.38) implies (A.39),

$$\bar{J}_\ell^-(r) \subset \bar{J}_\ell^-(r_k) \subset \bar{J}_\ell^-(r_{k+1}) \tag{10.47}$$

(of course (10.45) is used for the first inclusion). Finally, we must show that $\mathfrak{J}_{k+1} = J_\ell^+(r_{k+1})$. But, by (10.46) $r_{k+1} \subset \mathcal{E}_{k+1}$. Therefore, the connected subset $\mathfrak{J}_{k+1}$ of $\text{int}(J_\ell) \setminus \mathcal{E}_{k+1} \subset \text{int}(J_\ell) \setminus r_{k+1}$ with C in its boundary is contained in $J_\ell^+(r_{k+1})$. To prove the inclusion in the other direction, let $y \in J_\ell^+(r_{k+1})$. Then y can be connected by a continuous curve, ϕ say, to C, such that ϕ minus its endpoint on C lies in $J_\ell^+(r_{k+1}) \subset J_\ell^+(r_k)$ (by (10.47); compare (A.40)). Thus $y \in J_\ell^+(r_k)$ and $y \notin \bar{J}^-(r_k)$. But neither can y lie in $\bar{K}(w_{k+1})$. Indeed, the endpoint of ϕ on C lies in $\text{ext}(K(w_{k+1}))$ (recall that $\mathcal{E}(r)$ lies below $x(2) = GM_{\ell 2} + 2\Theta + \Lambda_3 + \Lambda$). If $y \in \bar{K}(w_{k+1})$, then ϕ would intersect $K(w_{k+1})$, and since ϕ minus its endpoint on C lies in $J_\ell^+(r_{k+1})$, this would imply that $K(w_{k+1})$ still contains a point of $J_\ell^+(r_{k+1})$, contrary to our construction of r_{k+1}. Thus

$$y \notin \bar{J}_\ell^-(r_k) \cup \bar{K}(w_{k+1}) \ .$$

Since y was arbitrary in $J^+(r_{k+1})$, and $\mathcal{E}_{k+1} \subset \bar{J}_\ell^-(r_k) \cup \bar{K}(w_{k+1})$ by the induction hypothesis, and $\bar{K}(w_i) \cap \mathfrak{F}_k = \emptyset$ for $i \leq k$ we now have

$$J_\ell^+(r_{k+1}) \subset \mathrm{int}(J_\ell) \setminus \mathcal{E}_{k+1} \ .$$

Further, since $J_\ell^+(r_{k+1})$ is connected and contains C in its boundary we also have

$$J_\ell^+(r_{k+1}) \subset \mathfrak{F}_{k+1} \ ,$$

and therefore

$$J_\ell^+ (r_{k+1}) = \mathfrak{F}_{k+1} \ . \tag{10.48}$$

as desired.

Now that we have shown how to obtain r_{k+1} from r_k we can take $r^\# = r_m$, the crosscut obtained after the last $K(w)$ has been added. (10.42) - (10.44) are then just (10.48) with $k+1 = m$ and (10.45) with $k = m$ (plus the simple observation

$$J_\ell^+(r^\#) = \mathrm{int}(J_\ell) \setminus \bar{J}_\ell^-(r^\#) \subset \mathrm{int}(J_\ell) \setminus \bar{J}_\ell^-(r) \subset J_\ell^+(r)$$

for the second part of (10.43)).

Step (ii). In this step we formulate Condition E, by means of the path $r^\#$. Throughout we shall assume $M_{\ell 1}$, $M_{\ell 2}$ large enough so that the construction of Step (i) can be carried out. The specific properties of $r^\#$ will only be used later; for the time being we only use the fact that to each r with properties (7.39) - (7.41) and (10.32) we have assigned an $r^\#$ in a specific way. Now assume that ω is an occupancy configuration on $\mathcal{M}_{p\ell}$ with all central vertices of faces $F \in \mathfrak{F}$ ($F \notin \mathfrak{F}$) occupied (vacant), and such that there exists an occupied crosscut of $\mathrm{int}(J_\ell)$ which also satisfies (10.32). Analogously to Lemma 7.4 we shall say that a vertex a on r has a vacant connection to $\mathcal{C}$ above r inside a set Γ if there exists a vacant path $s^* = (w_0^*, f_1^*, \dots, f_\tau^*, w_\tau^*)$ on $\mathcal{G}_{p\ell}^*$ which satisfies the following conditions (10.49) - (10.51) :

(10.49) there exists an edge f^* of $\mathcal{M}_{p\ell}$ between a and w_0^* such that $\overset{\circ}{f}{}^* \subset J_\ell^+(r) \cap \Gamma$,

(10.50) $$w_\tau^* \in \overset{\circ}{C} \quad ,$$

(10.51) $$(w_0^*, f_1^*, \dots, w_{\tau-1}^*, f_\tau^* \setminus \{w_\tau^*\}) = s^* \setminus \{w_\tau^*\}) \subset J_\ell^+(r) \cap \Gamma \ .$$

When $\Gamma = \mathbb{R}^2$ (so that the restrictions due to Γ are vacuous) we simply talk about a vacant connection from a to $\overset{\circ}{C}$ above r.

Once there exists some occupied r which satisfies (7.39)-(7.41) and (10.32) we know from Prop. 2.3 that there then also exists a unique such r with minimal $J_\ell^-(r)$. We denote this path by $R = (v_0, e_1, \dots, e_\nu, v_\nu)$. Associated with R is a path $R^\#$ as in Step (i). Now assume $a^\#$ is a vertex of $R^\#$ which has a vacant connection to $\overset{\circ}{C}$ above $R^\#$ inside

$$\Gamma_\ell : = [\tfrac{1}{2}M_{\ell 1}, \tfrac{3}{2}M_{\ell 1}] \times \mathbb{R} \ .$$

Finally assume that $x \in \mathfrak{w}$ is such that Condition D holds for this x, and set

$$\mathfrak{w}_0 = \{x + k_1\xi_1 + k_2\xi_2 : k_i \in \mathbb{Z}, i = 1,2,\} \ .$$

Since $\mathfrak{w}$ was assumed periodic, $\mathfrak{w}_0 \subset \mathfrak{w}$. Moreover, by the periodicity assumptions in (10.9) Condition D remains valid when x is replaced by any element of $\mathfrak{w}_0$.

We now formulate Condition E. It requires that for suitable constants κ_i we can find a configuration $\tilde{\omega}$ which satisfies (10.53) - (10.57). The specific values of the κ_i are unimportant. We only need $0 < \kappa_i < \infty$ and that the κ_i depend only on $\mathcal{G}_{p\ell}, \mathcal{G}_{p\ell}^*$ and Δ but not on $\ell, \omega, R, a^\#$, p_0 or p'.

<u>Condition E</u>. Let $\ell \geq \kappa_0$ and let ω be an occupancy configuration on $\mathcal{M}_{p\ell}$ which has an occupied crosscut of $\mathrm{int}(J_\ell)$ and is such that

(10.52) all central vertices of $\mathcal{G}_{p\ell}$ outside $\mathfrak{w}$ are occupied, while all central vertices of $\mathcal{G}_{p\ell}^*$ are vacant.

Let R be the occupied crosscut of $\mathrm{int}(J_\ell)$ with $J_\ell^-(R)$ minimal and let $R^{\#}$ be associated to R as in Step (i). Then for every vertex $a^{\#}$ on $R^{\#}$ which has a vacant connection to $\mathring{C}$ above $R^{\#}$ inside Γ_{ℓ} there exists an occupancy configuration $\tilde{\omega} = \tilde{\omega}(\omega, a^{\#})$ on $\mathcal{M}_{p\ell}$ with the following properties (10.53) - (10.57).

(10.53) $\tilde{\omega}(v) = \omega(v)$ for all vertices v of $\mathcal{M}_{p\ell}$ with $|v(i) - a^{\#}(i)| > \kappa_1$ for $i \geq 1$ or 2.

(Recall that $\tilde{\omega}(v)$ is the value of $\tilde{\omega}$ at the vertex v of $\mathcal{M}_{p\ell}$, and similarly for $\omega(v)$; the more explicit notation $\tilde{\omega}(\omega, a^{\#})(v)$ for $\tilde{\omega}(v)$ should not be necessary.)

(10.54) If v is a central vertex of $\mathcal{G}_{p\ell}$ which does not belong to $\mathfrak{w}$, and hence $\omega(v) = 1$, then $\tilde{\omega}(v) = 1$.

(10.55) If v is a central vertex of $\mathcal{G}^*_{p\ell}$, and hence $\omega(v) = -1$, then $\tilde{\omega}(v) = -1$.

(10.56) In the configuration $\tilde{\omega}$ there exists an occupied crosscut $\tilde{R}$ of $\text{int}(J_{\ell})$ satisfying (7.39) - (7.41) and with $J_{\ell}^{-}(\tilde{R})$ minimal among all such crosscuts. Moreover on $\tilde{R}$ there exists a vertex $\tilde{x}$ from $\mathfrak{w}_0$ with a vacant (in the configuration $\tilde{\omega}$) connection Y^* to $\mathring{C}$ above $\tilde{R}$, and such that $|\tilde{x}-a^{\#}| \leq \kappa_2$.

(10.57) Any vertex y from $\mathfrak{w}_0$ which lies on the $\tilde{R}$ of (10.56) and which has a vacant connection to $\mathring{C}$ above $\tilde{R}$ in the configuration $\tilde{\omega}$ satisfies (a) or (b) below.

a) y lies on R and has a vacant connection to $\mathring{C}$ above R in the configuration ω.

b) $|y-a^{\#}| \leq \kappa_3$. ///

We merely add one explanatory comment. The requirements (10.54) and (10.55) just guarantee that $\tilde{\omega}$ also satisfies (10.52). By (7.2), (7.3), and (10.16) the condition (10.52) has to be satisfied with P_{p_0}-probability one as well as with $P_{p'}$-probability one. If we did not have (10.52) for $\tilde{\omega}$, then the simple estimate (10.64) would fail. Unfortunately, (10.54) necessitates much extra work; "strong minimality" and "shortcuts of two edges" are used in Steps (iv) - (ix) only for (10.54). The purpose of the other requirements in Condition E should become evident in the next step.

Step (iii). In this step we derive (10.21) from Condition E. As we observed above it suffices to prove

$$\lim_{\ell \to \infty} \tau(2\overline{M}_\ell; i, p', \mathcal{G}_{p\ell}) = 0 \ , \ i = 1,2,$$

and we shall only deal with (10.31). The proof of (10.31) mimicks the proof of (7.35), at least initially. We restrict ourselves to $i = 1$. Analogously to Lemma 7.4 we shall drop the subscript ℓ for the time being, and set

$$E = \{ \exists \text{ occupied path } r = (v_0, e_1, \ldots, e_\nu, v_\nu) \text{ on } \mathcal{G}_{p\ell} \text{ with the properties (7.39) - (7.41) and (10.32)} \} .$$

Note that we have added the requirement (10.32); this was absent in the definition of E in Lemma 7.4. Moreover, J as defined in Step (i) differs somewhat from the J in Lemma 7.4. Nevertheless the argument used in Lemma 7.4 still shows that

$$\tau(2\overline{M}_\ell; 1, p', \mathcal{G}_{p\ell}) \le P_{p'}\{E\} ,$$

so that it suffices to prove

$$\lim_{\ell \to \infty} P_{p'}\{E\} = 0 \ . \tag{10.58}$$

In addition to E we also introduce the event in which the restriction (10.32) is dropped. We denote this by E_1:

$$E_1 = \{ \exists \text{ occupied path } r = (v_0, e_1, \ldots, e_\nu, v_\nu) \text{ on } \mathcal{G}_{p\ell} \text{ with the properties (7.39) - (7.41)} \} .$$

Analogously to Lemma 7.4 we write for any r which satisfies (7.39)-(7.41)

$N(r) = N(r,\omega) = \#$ of vertices of $\mathcal{G}_{p\ell}$ in $\mathbb{L}_{0_\circ}$ and on $r \cap \text{int}(J)$ which have a vacant connection to $\mathring{C}$ above r .

Again note the slight differences with (7.50); the fact that we only count vertices in $\mathfrak{w}_0$ is crucial. If E occurs then all the vertices of R counted in $N(R,\omega)$ are pivotal for (E_1,ω), by Ex. 4.2(iii). Set

$$p(t) = t\, p_0 + (1-t)p'$$

and for x as in the last step

$$\alpha = P_{p_0}\{x \text{ is occupied}\} - P_{p'}\{x \text{ is occupied}\} .$$

Then, since $\mathfrak{w}_0$ consists of the translates of x by integral vectors we have

$$\alpha = \min_{v \in \mathfrak{w}_0} \{P_{p_0}\{v \text{ is occupied}\} - P_{p'}\{v \text{ is occupied}\}\}$$

and by assumption (10.17) $\alpha > 0$. By Russo's formula (Prop. 4.2) we have as in (7.42), and (7.51).

$$(10.59) \qquad \frac{d}{dt} P_{p(t)}\{E_1\} \geq \alpha E_{p(t)}\{\# \text{ of pivotal sites in } \mathfrak{w}_0 \text{ for } E_1\} \geq \alpha E_{p(t)}\{N(R);\ E \text{ occurs}\} .$$

We must now find a lower bound for the right hand side of (10.59) Assume that E occurs, and that $R = r$ for a path r satisfying (7.39) - (7.41) and (10.32). If $r^{\#}$ be the path associated to r as in Step (i), set

$M(r^{\#})$ = number of vertices $a^{\#}$ on $r^{\#}$ which have a vacant connection to $\mathring{C}$ above $r^{\#}$ inside Γ.

Our first estimate is that for each m we can choose $\ell_0 = \ell_0(m)$ such that for all $\ell \geq \ell_0$ and all $0 \leq t \leq 1$.

$$(10.60) \qquad P_{p(t)}\{ E \text{ occurs and } M(R^{\#}) \geq m\} \geq P_{p'}\{E\} \frac{1}{2} \delta_{27} ,$$

where δ_{27} is as in (7.19). To see this we observe that as in (7.46), (7.51).

$$P_{p(t)}\{E \text{ occurs and } M(R^{\#}) \geq m\} .$$

$$\geq \sum_r P_{p(t)}\{R = r, R^{\#} = r^{\#}\} P_{p(t)}\{M(r^{\#}) \geq m | R = r,\ R^{\#} = r^{\#}\}$$

where the sum is over all paths r which satisfy (7.39) - (7.41) and (10.32) and $r^{\#}$ is the path associated to r by Step (i). By definition of $M(R^{\#})$ and vacant connections. (see (10.49)- (10.51)) $M(r^{\#})$ depends only on the occupancies of vertices outside $\bar{J}^{-}(r^{\#})$. On the other hand the event $\{R = r, R^{\#} = r^{\#}\} = \{R = r\}$, since $r^{\#}$ and $R^{\#}$ are the paths which are associated uniquely to r and R, respectively. Further, by Prop. 2.3 $\{R = r\}$ depends only on occupancies of the vertices in $\bar{J}^{-}(r) \subset \bar{J}^{-}(r^{\#})$ (by (10.43)). Therefore, for fixed $r^{\#}$ satisfying (7.39)-(7.41).

$$P_{p(t)}\{M(r^{\#}) \geq m | R = r, R^{\#} = r^{\#}\} = P_{p(t)}\{M(r^{\#}) \geq m\} .$$

Now as in Remark 7(ii) to Lemma 7.4 (especially (7.76)) we have for all sufficiently large ℓ

(10.61) $$P_{p(t)}\{M(r^{\#}) \geq m\} \geq \frac{1}{2} P_{p_0}\{ \exists \text{ at least one vertex } a^{\#} \text{ on } r^{\#} \text{ with a vacant connection to } \mathring{C} \text{ above } r^{\#} \text{ inside } \Gamma'\},$$

where

$$\Gamma' = \Gamma'_{\ell} = [\tfrac{3}{4} M_{\ell 1}, \tfrac{5}{4} M_{\ell 1}] \times \mathbb{R}$$

Moreover, exactly as in (7.61), the probability in the right hand side of (10.61) is at least

$$P_{p_0}\{ \exists \text{ vacant vertical crossing on } \mathcal{G}^{*}_{p\ell} \text{ of } [\tfrac{3}{4} M_{\ell 1}, \tfrac{5}{4} M_{\ell 1}] \times [-\Lambda_4, 12M_{\ell 2} + \Lambda_4]$$

$$\geq \sigma^{*}((\tfrac{1}{2}M_{\ell 1} - 1), 13M_{\ell 2}); 2, p_0, \mathcal{G}_{p\ell}) \geq \delta_{27} .$$

(10.60) follows by combining these observations with the facts

$$E = \bigcup_{r} \{R = r, R^{\#} = r^{\#}\} ,$$

where the union runs over all r which satisfy (7.39)-(7.41) and (10.32), and

$$P_{p(t)}\{E\} \geq P_{p'}\{E\} \quad , 0 \leq t \leq 1 ,$$

which follows from Lemma 4.1 and the fact that E is an increasing event.

The second important estimate for our proof concerns the event

$$G(m,\eta):= \{E \text{ occurs}, \ M(R^{\#}) \geq m, \text{ but } N(R) \leq \eta m\} .$$

We shall show that for some τ_1,τ_2 independent of η , m and ℓ(but dependent on p_0 and p').

$$(10.62) \qquad P_{p(t)} \{G(m,\eta)\} \leq \tau_1(\eta + \frac{\tau_2}{m}) , \quad \frac{1}{4} \leq t \leq \frac{3}{4} ,$$

for all sufficiently large ℓ . Before proving (10.62) we show that it quickly implies (10.58). Indeed, on the event

$$\{E \text{ occurs}, \ M(R^{\#}) \geq m, \text{ but } G(m,\eta) \text{ fails}\}$$

one has $N(R) > \eta m$, so that by (10.60) and (10.62)

$$\begin{aligned} &E_{p(t)}\{N(R);\ E \text{ occurs}\} \\ &\geq \eta m(P_{p(t)}\{E \text{ occurs}, M(R^{\#}) \geq m\} - P_{p(t)}\{G(m,\eta)\}) \\ &\geq \eta m(\tfrac{1}{2}\delta_{27}\ P_{p'}\{E\} - \tau_1\eta - \frac{\tau_1\tau_2}{m}), \quad \frac{1}{4} \leq t \leq \frac{3}{4} . \end{aligned}$$

Thus, by (10.59), for large ℓ

$$\begin{aligned} 1 &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{d}{dt} P_{p(t)} \{E_1\}\, dt \\ &\geq \alpha\eta m \int_{\frac{1}{4}}^{\frac{3}{4}} (\tfrac{1}{2}\, \delta_{27}\ P_{p'}\{E\} - \tau_1\eta - \frac{\tau_1\tau_2}{m})\, dt \\ &= \frac{1}{2}\, \alpha\eta m\ (\tfrac{1}{2}\, \delta_{27}\ P_{p'}\{E\} - \tau_1\eta - \frac{\tau_1\tau_2}{m}) . \end{aligned}$$

Consequently, for all η,m

$$\limsup_{\ell\to\infty} \{\tfrac{1}{2}\, \delta_{27}\ P_{p'}\{E\} - \tau_1\eta - \frac{\tau_1\tau_2}{m})\} \leq \frac{2}{\alpha\eta m} ,$$

or equivalently

$$\limsup_{\ell\to\infty} P_{p'}\{E\} \leq \frac{2}{\delta_{27}} (\frac{2}{\alpha\eta m} + \tau_1\eta + \frac{\tau_1\tau_2}{m})$$

By first choosing η small, then m large we obtain the desired (10.58). As we saw above this implies (10.31) and (10.21).

Theorem 10.2 has been reduced to (10.62) which we now prove by means of Condition E. Let

$$H(\lambda,\eta) = \{E \text{ occurs, } M(R\#) = \lambda, \text{ but } N(R) \leq \eta m\} .$$

Then

$$(10.63) \qquad G(m,\lambda) = \bigcup_{\lambda \geq m} H(\lambda,\eta) .$$

Let ω be a configuration in $H(\lambda,\eta)$ which satisfies (10.52). Then by definition of $M(.)$, in the configuration ω there are λ vertices on $R^{\#}$ which have a vacant connection to $\mathring{C}$ above $R^{\#}$ inside Γ. Denote these by $a_1^{\#},\dots,a_\lambda^{\#}$ in an arbitrary order. To each one of these there is assigned by Condition E a configuration $\tilde{\omega}(\omega,a_j^{\#})$ with the properties (10.53)-(10.57). Let S be any square. Denote by ω_S the set of all configurations which agree with ω at all sites in S. We show first that there exists a constant $\tau_3 > 0$ (which depends on p' and p_0, but not on ℓ, S,ω, R or $a^{\#}$) such that

$$(10.64) \qquad P_{p(t)}\{\tilde{\omega}_S(\omega, a_j^{\#})\} \geq \tau_3 \, P_{p(t)}\{\omega_S\}, \quad \frac{1}{4} \leq t \leq \frac{3}{4} .$$

This is easy to see, since $\tilde{\omega}(\omega, a_j^{\#})$ is obtained from ω by changing at most κ_4 sites for some κ_4 depending on the graph only, by (10.53). Moreover, if v is a site with $\omega(v) = +1$, $\tilde{\omega}(v) = -1$, then either v is not a central site of $\mathcal{G}_{p\ell}$, or it is a central site of $\mathcal{G}_{p\ell}$ which belongs to $\mathfrak{w}$ (by (10.54)). In the former case, for $t \geq 1/4$

$$P_{p(t)}\{v \text{ is vacant}\} \geq t\, P_{p_0}\{v \text{ is vacant}\}$$
$$\geq \frac{1}{4} P_{p_0}\{v \text{ is vacant}\} > 0,$$

since $p_0 \ll \overline{1}$ (see (10.13)). In the latter case, for $t \leq 3/4$

$$P_{p(t)}\{v \text{ is vacant}\} \geq (1-t)\, P_{p'}\{v \text{ is vacant}\}$$
$$\geq \frac{1}{4} \, \nu_v\{\omega(v) = -1\} > 0$$

by (10.17). On the other hand, if $\omega(v) = -1$ and $\tilde{\omega}(v) = +1$, then by (10.55) v is not a central vertex of $\mathcal{G}^*_{p\ell}$. Therefore, for

$t \geq 1/4$

$$P_{p(t)}\{v \text{ is occupied}\} \geq \frac{1}{4} P_{p_0}\{v \text{ is occupied}\} > 0,$$

this time by $p_0 >> \overline{0}$ (see (10.13)). Therefore, in all cases, if v has a different state in $\tilde{\omega}$ then in ω, then

$$P_{p(t)}\{v \text{ is in the state prescribed by } \tilde{\omega}\}$$
$$\geq \delta P_{p(t)}\{v \text{ is in the state prescribed by } \omega\}$$

for some $\delta = \delta(p_0, p') > 0$. Consequently (10.64) holds with $\tau_3 = \delta^{\kappa_4}$.

Next we note that for fixed ℓ we can choose S so large that all events which we consider only depend on the configuration in S. Indeed we are only interested in $\omega(v)$ for v in $\overline{J} = J \cup \text{int}(J)$, and $\tilde{\omega}(v) = \omega(v)$ except possibly for v with $|v(i) - a^{\#}(i)| \leq \kappa_1$ for some $a^{\#} \in J$ (see (10.53)). The last property also allows us to choose $\tilde{\omega}_S(\tilde{\omega}, a^{\#})$ as a function of ω_S and $a^{\#}$ only (when S is large enough). Accordingly we denote it by $\tilde{\omega}_S(\omega_S, a^{\#})$ below. We also repeat the observation that by (7.2), (7.3) and (10.16) the condition (10.52) holds with P_{p_0}-probability one as well as with $P_{p'}$-probability one. Consequently it also holds with $P_{p(t)}$-probability one for all $0 \leq t \leq 1$. We therefore conclude from (10.64) that

$$(10.65) \qquad P_{p(t)}\{H(\lambda,\eta)\} = \sum_{\omega_S} P_{p(t)}\{\omega_S\}$$
$$\leq \frac{1}{\tau_3{}^{\lambda}} \sum_{\omega_S} \sum_{j=1}^{\lambda} P_{p(t)}\{\tilde{\omega}_S(\omega_S, a_j^{\#})\} \quad ,$$

where $\sum_{\omega_S}$ is the sum over all configurations ω_S in S for which $H(\lambda,\eta)$ occurs, and (10.52) holds inside S. We now rearrange the double sum in the last member of (10.65); on the outside we sum over the possible "values" of $\tilde{\omega}_S(\omega_S, a_j^{\#})$, and inside we sum over the ω_S and j for which $\tilde{\omega}_S(\omega_S, a_j^{\#})$ equals a specified configuration. This yields

$$P_{p(t)}\{H(\lambda,\eta)\} \le \frac{1}{\tau_3\lambda} \sum_{\overline{\omega}_S} P_{p(t)}\{\overline{\omega}_S\} \text{ . (number of}$$

pairs ω_S and $a^\#$ on $R^\#(\omega_S)$ with $\tilde{\omega}_S(\omega_S,a^\#) = \overline{\omega}_S$ and ω_S such that $H(\lambda,\eta)$ occurs) .

The sum over $\overline{\omega}_S$ runs over all possible configurations in S, and we have written $R^\#(\omega_S)$ for $R^\#(\omega)$, again because $R^\#$ depends on ω_S only for large S. If we sum the last inequality over $\lambda \ge m$, then we obtain, by virtue of (10.63),

$$P_{p(t)}\{G(m,\eta)\} \le \frac{1}{\tau_3 m} \sum_{\overline{\omega}_S} P_{p(t)}\{\overline{\omega}_S\}\text{. (number of pairs}$$

ω_S and $a^\#$ on $R^\#(\omega_S)$ with $\tilde{\omega}_S(\omega_S,a^\#) = \overline{\omega}_S$ and ω_S such that $G(m,\eta)$ occurs).

Finally we shall prove that for any given $\overline{\omega}_S$ there are at most $\kappa_5(\eta m + \kappa_6)$ pairs ω_S and $a^\#$ on $R^\#(\omega_S)$ with $\tilde{\omega}_S(\omega_S,a^\#) = \overline{\omega}_S$ and such that $G(m,\eta)$ occurs in ω_S. This will imply

$$P_{p(t)}\{G(m,\eta)\} \le \frac{1}{\tau_3 m}\kappa_5(\eta m + \kappa_6),$$

which is the desired (10.62) (κ_5 and κ_6 depend only on κ_1-κ_3 and $\mathcal{G}_{p\ell}$).

Now fix a configuration $\overline{\omega}_S$ in S and let ω_S be a configuration such that $G(m,\eta)$ occurs and let $a^\#$ lie on $R^\#(\omega_S)$ such that $\tilde{\omega}_S(\omega_S,a^\#) = \overline{\omega}_S$. Then $a^\#$ has to be a vertex with a vacant connection to $\mathring{\mathcal{C}}$ above $R^\#(\omega_S)$ (these were the only $a^\#$ for which we ever considered $\tilde{\omega}(\omega, a^\#)$). By (10.56) $\tilde{\omega}_S(\omega_S,a^\#) = \overline{\omega}_S$ must then be such that it has a lowest crosscut $\tilde{R}$ of $\underset{\sim}{J}$ and a vertex $\tilde{x}$ from $\mathfrak{w}_0$ with a vacant connection to $\mathring{\mathcal{C}}$ above $\tilde{R}$ in configuration $\overline{\omega}_S$ and such that $|\tilde{x}(i) - a^\#(i)| \le \kappa_2$. Now we are only given $\overline{\omega}_S$, and know neither $R,R^\#$ nor $a^\#$. However $\tilde{R}$ is the lowest crosscut in configuration $\overline{\omega}_S$, and hence there is at most one possibility for $\tilde{R}$ for a given $\overline{\omega}_S$. Next we must check how many possibilities there are for $\tilde{x}$. By (10.57), if $\overline{\omega}_S$ arose as $\tilde{\omega}_S(\omega_S,a^\#)$, then the number of vertices from $\mathfrak{w}_0$ on $\underset{\sim}{\tilde{R}}$ with a vacant connection above $\tilde{R}$ to $\mathring{\mathcal{C}}$ in $\overline{\omega}_S$ is limited. It either is of the type described in (10.57)(a) or (10.57)(b). There are at most ηm vertices of type

(10.57)(a) in int(J) if ω_S is such that $G(m,\eta)$ occurs (because by definition $N(R,\omega_S) \leq \eta m$ in this case). Also, there are at most κ_6 vertices of type (10.57)(b) or on $\tilde{R} \cap J$. Thus, any $\bar{\omega}_S$ which can arise from an ω_S for which $G(m,\eta)$ occurs has at most $\eta m + \kappa_6$ vertices in $\mathfrak{w}_0$ with a vacant connection above $\tilde{R}(\bar{\omega}_S)$ to $\mathcal{C}$ in configuration $\bar{\omega}_S$. Thus, there are at most $\eta m + \kappa_6$ choices for $\tilde{x}$ for any $\bar{\omega}_S$ which can arise at all. But once we picked $\tilde{x}$, we have at most κ_7 choices for $a^{\#}$ by (10.56). Finally, if we know $\bar{\omega}_S = \tilde{\omega}_S(\omega_S,a^{\#})$ and $a^{\#}$, then there are at most κ_8 possibilities for ω_S, because (by (10.53)) ω_S differs from $\tilde{\omega}_S(\omega_S,a^{\#}) = \bar{\omega}_S$ only in a fixed neighborhood of $a^{\#}$. In total, starting with $\bar{\omega}_S$ we can make at most $(\eta m + \kappa_6)\kappa_7\kappa_8$ choices for $\tilde{x}$, $a^{\#}$ and ω_S. This bound completes the proof of (10.62) and Theorem 10.2 (modulo the derivation of Condition E from Condition D in the next section). □

<u>Proof of Theorem 10.3</u>. The principal idea was already explained before the statement of the theorem. Let $\mathcal{K}$ be the graph obtained from $\mathcal{M}$ by close-packing only the faces F in $\mathcal{F}_1 := \mathcal{F} \setminus \mathcal{F}_0$, where $\mathcal{F}_0$ is as in (10.23). ($\mathcal{F}_0 = \emptyset$ if $\mathcal{H}$ is obtained by applying only (10.22)). Clearly $\mathcal{K}$ is one of a matching pair of graphs, based on $(\mathcal{M},\mathcal{F}_1)$, and $\mathcal{K}$ is a subgraph of $\mathcal{G}$, while $\mathcal{H}$ is the subgraph of $\mathcal{K}$ obtained by removing all vertices in $\mathcal{V}_0$($\mathcal{V}_0$ as in (10.22); again $\mathcal{V}_0 = \emptyset$ if only (10.23) is applied to construct $\mathcal{H}$). An occupied cluster on $\mathcal{H}$ is an occupied cluster on $\mathcal{K}$ which does not contain any vertices of $\mathcal{V}_0$, and hence remains unchanged if all vertices in $\mathcal{V}_0$ are made vacant with probability one. Moreover Cor. 2.1 applied to $\mathcal{K}$ shows that for any vertex z_0 of $\mathcal{K}$

$$\begin{aligned}&\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{K})\\ &\leq (\#\text{occupied cluster of } z_0 \text{ on } \mathcal{K}_{p\ell}).\end{aligned}$$

Therefore

$$\begin{aligned}(10.66)\qquad &E_{p_0}(\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{H}))\\ &\leq E(\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{K}_{p\ell})) ,\end{aligned}$$

where in the right hand side we make vertices in $\mathcal{V}_0$ vacant with probability one, and for other vertices of $\mathcal{K}_{p\ell}$ we use the measure

P_{p_0}. However, $\mathcal{K}_{p\ell}$ is just $\mathcal{G}_{p\ell}$ with the central vertices of faces in $\mathfrak{F}_0$ (and the edges incident to these vertices removed). The right hand side of (10.66) therefore equals

(10.67) $\quad E_{p'}(\#(\text{occupied cluster of } z_0 \text{ on } \mathcal{G}_{p\ell})$,

where

$P_{p'}\{v$ is occupied$\} = 0$ if $v \in \mathcal{V}_0$ or if v is a central vertex of a face $F \in \mathfrak{F}_0$,

while

$P_{p'}\{v$ is occupied$\} = P_{p_0}\{v$ is occupied$\}$ for all other vertices v of $\mathcal{G}_{p\ell}$.

With $\mathfrak{w}$ as in (10.24), these are just the relations (10.25) and (10.26), which in turn say that $P_{p'}$ is of the form (10.15) and satisfies (10.16) and (10.17). Indeed, for $v \in \mathfrak{w}$ we now have

$$\nu_v\{\omega(v) = 1\} = P_{p'}\{\omega(v) = 1\} = 0 < \mu_v\{\omega(v) = 1\}$$
$$= P_{p_0}\{\omega(v) = 1\},$$

because of $p_0 >> \overline{0}$ and (7.2). Thus Theorem 10.2 applies and (10.67) is finite. But then also the left hand side of (10.66) is finite. Theorem 10.3 now follows from Cor. 5.1 applied to the graph $\mathcal{H}$. □

10.3 Derivation of Condition E from Condition D.

In this section we fill the gap left in the proof of Theorem 10.2. The proof is broken down into six steps, numbered (iv)-(ix) (because we already had Steps (i)-(iii) of the proof of Theorem 10.2). Condition E says that one can make a local modification in the occupancy configuration around a site $a^\#$ on $R^\#$ with a vacant connection in $\mathring{C}$. The modified configuration is to have a site from $\mathfrak{w}_0$ (defined in step (ii)) with a vacant connection above the lowest horizontal crossing in the new configuration. Basically this is obtained by translating the point x together with the paths U and V^* of condition D and "splicing in" the translate of U into the lowest crossing R and connecting the translate of V^* to the vacant connection from $a^{\#}$ to $\mathring{C}$. A good part of the construction takes place in $\mathrm{int}(K(a))$ (see Step (i) for an a with $a^{\#}$ on $K(a)$. We begin with a method for making well controlled connections between (endpoints of) paths.

Step (iv). By a corridor $\mathcal{K}$ of width Λ_7 we mean the union of a finite sequence of rectangles $D_0,\ldots,D_\lambda$ or $D_1,\ldots,D_\lambda$ of the form

(10.68) $$D_{2i} = [a_{2i},a_{2i}+\Lambda_7] \times [b_{2i},b_{2i}+k_{2i}],$$

(10.69) $$D_{2i+1} = [a_{2i+1},a_{2i+1}+k_{2i+1}] \times [b_{2i},b_{2i}+\Lambda_7]$$

with $k_{2i},k_{2i+1} \geq 2\Lambda_7$ and arbitrary a_j, b_j, and satisfying the connectivity condition that D_j and D_{j+1} have a corner in common and intersect in a square of size $\Lambda_7 \times \Lambda_7$. However, D_{j-1} and D_{j+1} must have disjoint interiors; see Fig. 10.11. The first edge of the corridor

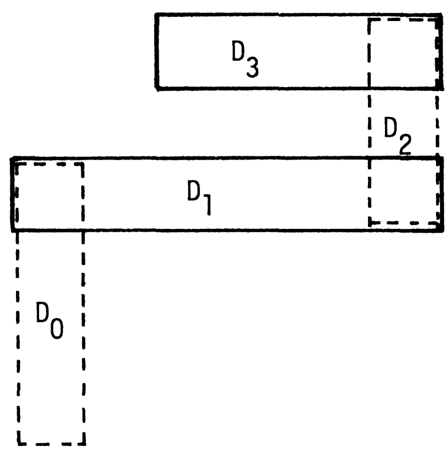

Figure 10.11 A typical corridor. The solid rectangles have odd indices, the dashed rectangles have even indices.

$\mathcal{K} = \bigcup_{i=0}^{\lambda} D_i$ will be the short edge of D_0 which does not belong to D_1, i.e., $[a_0,a_0+\Lambda_7] \times \{b_0\}$ or $[a_0,a_0+\Lambda_7] \times \{b_0+k_0\}$, whichever one is disjoint from D_1. The last edge of $\mathcal{K}$ is that short edge of D_λ which does not belong to $D_{\lambda-1}$. A similar definition holds if $\mathcal{K} = \bigcup_{i=1}^{\lambda} D_i$ (which starts with a rectangle of odd index). For the duration of this proof only we shall call a path $r = (v_0, e_1, \dots, e_\nu, e_\nu)$ on $\mathcal{G}_{p\ell}$ strongly minimal if it is minimal (see Def. 10.1), and if in addition for any $i < j$ such that v_i and v_j are vertices of $\mathcal{M}$ which are not adjacent on $\mathcal{M}$, but lie on the perimeter of one face $F \in \mathfrak{F}$ whose central vertex u does not belong to $\mathfrak{w}$, one has $j = i+2$ and v_{i+1} is a central vertex of $\mathcal{G}_{p\ell}$ which does not belong to $\mathfrak{w}$. In a strongly minimal path two vertices on the perimeter of a single face $F \in \mathfrak{F}$ whose central vertex does not belong to $\mathfrak{w}$ are always connected in one of two ways: either by a single edge of the path which belongs to the perimeter of F, or by two successive edges of the path which go through a central vertex of $\mathcal{G}_{p\ell}$ not in $\mathfrak{w}$. Note that two vertices v_i and v_j may be simultaneously on the perimeter of several faces and that there may be several central vertices which are adjacent to both v_i and v_j; for this reason we did not require $v_{i+1} = u$ in the above definition. In analogy with Def. 10.2 we shall call a shortcut of two edges of the path $(v_0, e_1, \dots, e_\nu, v_\nu)$ a string e, u, f of an edge, vertex and edge of $\mathcal{G}_{p\ell}$ such that for some $i < j$, v_i and v_j are not adjacent on $\mathcal{G}_{p\ell}$, v_i and u (u and v_j) are the endpoints of $e(f)$ and u is a central vertex of $\mathcal{G}_{p\ell}$ which does not belong to $\mathfrak{w}$, and is different from all the v_i, $0 \le i \le \nu$. (Since a central vertex has only non-central neighbors (Comment 2.3(iv)) v_i and v_j have to lie on the perimeter of some face $F \in \mathfrak{F}$ of $\mathcal{M}$ if there is a shortcut of two edges between them.)

A minimal path for which there do not exist shortcuts of two edges is strongly minimal. However, the converse is not quite true. A strongly minimal path $(v_0, e_1, \dots, e_\nu, v_\nu)$ can have a shortcut of two edges e, u, f between two vertices v_i and v_j, but this can happen only if $j = i+2$, v_i and v_j lie on the perimeter of a face $F_1 \in \mathfrak{F}$ of $\mathcal{M}$ and v_{i+1} is the central vertex of F_1, but does not belong to $\mathfrak{w}$. In this case u has to be the central vertex of another face $F_2 \in \mathfrak{F}$ of $\mathcal{M}$, u must be outside $\mathfrak{w}$ and v_i, v_{i+2} must lie on the perimeter of F_2, as well as on the perimeter of F_1.

In this step we prove that for every corridor $\mathcal{K}$ of width Λ_7 there exists a strongly minimal path $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ such that

(10.70) $\quad r \subset \mathcal{K}$ and $v_0(v_\nu)$ are within distance 3Λ from the first (last) edge of $\mathcal{K}$.

This statement remains true if $\mathcal{G}_{p\ell}$ is replaced by $\mathcal{G}^*_{p\ell}$. Note that no statements about the occupancy of r are made. The proof is carried out only for $\mathcal{G}_{p\ell}$ and only by means of a single case illustration. Assume $\mathcal{K} = \bigcup_{i=0}^{2\nu} D_i$ and that a corner on the top edge of D_0, $[a_0,a_0+\Lambda_7] \times \{b_0+k_0\}$, is also a corner of D_1. Then the first edge of $\mathcal{K}$ is the bottom edge of D_0, $[a_0,a_0+\Lambda_7] \times \{b_0\}$. Assume also that $D_{2\nu-1}$ and $D_{2\nu}$ have a corner in common which lies on the bottom edge of $D_{2\nu}$. Then the last edge of $\mathcal{K}$ is the top edge of $D_{2\nu}$, $[a_{2\nu},a_{2\nu}+\Lambda_7] \times \{b_{2\nu}+k_{2\nu}\}$. To find a strongly minimal r satisfying (10.70) let s_{2i} be a vertical crossing on $\mathcal{G}_{p\ell}$ of

$$\tilde{D}_{2i} := [a_{2i}+2\Lambda,a_{2i}+\Lambda_7-2\Lambda] \times [b_{2i}+2\Lambda,b_{2i}+k_{2i}-2\Lambda]$$

and s_{2i+1} a horizontal crossing on $\mathcal{G}_{p\ell}$ of

$$\tilde{D}_{2i+1} := [a_{2i+1}+2\Lambda,a_{2i+1}+k_{2i+1}-2\Lambda] \times [b_{2i+1}+2\Lambda,b_{2i+1}+\Lambda_7-2\Lambda].$$

All these crossings exist by our choice of Λ_3 and $\Lambda_7 = \Lambda_3+4\Lambda$. Now, since D_j and D_{j+1} intersect in a $\Lambda_7 \times \Lambda_7$ square, $\tilde{D}_j$ and $\tilde{D}_{j+1}$ intersect in a $(\Lambda_7-4\Lambda) \times (\Lambda_7-4\Lambda) = \Lambda_3 \times \Lambda_3$ square. The latter square is crossed horizontally by s_j and vertically by s_{j+1}, if j is odd. Thus s_j and s_{j+1} intersect, necessarily in a vertex of $\mathcal{G}_{p\ell}$. A similar argument works for even j. We can therefore put together pieces of $s_0,\dots,s_{2\nu}$ to obtain a path $\tilde{s} = (\tilde{u}_0,\tilde{f}_1,\dots,\tilde{f}_\sigma,\tilde{u}_\sigma)$ with possible double points, which satisfies

$$(10.71) \qquad (\tilde{u}_1,\tilde{f}_2,\dots,\tilde{f}_{\sigma-1},\tilde{u}_{\sigma-1}) \subset \tilde{\mathcal{K}} := \bigcup_{i=0}^{2\nu} \tilde{D}_i$$

and

$$(10.72) \qquad \tilde{f}_1 \text{ intersects } [a_0+2\Lambda,a_0+\Lambda_7-2\Lambda] \times \{b_0+2\Lambda\}, \quad \text{while } \tilde{f}_\sigma \text{ intersects } [a_{2\nu}+2\Lambda,a_{2\nu}+\Lambda_7-2\Lambda] \times \{b_{2\nu}+k_{2\nu}-2\Lambda\}$$

(see Fig. 10.12 for $\nu = 1$). By loop removal, as described in Sect. 2.1 we can make $\tilde{s}$ into a self-avoiding path, without changing its initial or endpoint. Since loop removal only takes away pieces of a path, we obtain after loop removal a self-avoiding path, which we shall denote by $s = (u_0,f_1,\dots,f_\tau,u_\tau)$, which satisfies the analogue of (10.71), i.e.

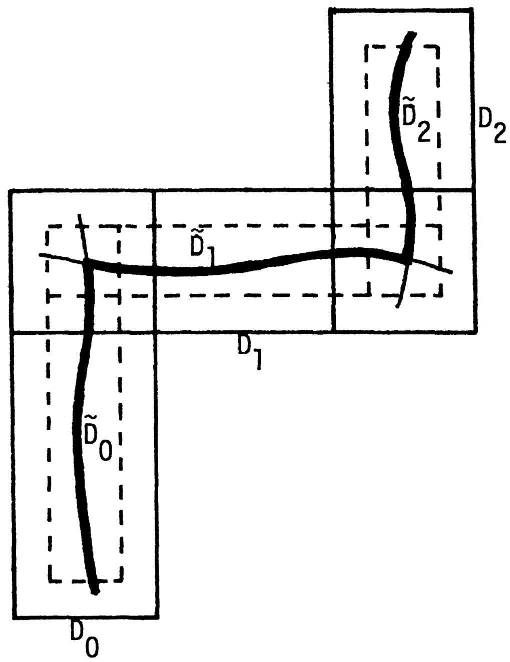

Figure 10.12 An illustration of $\mathcal{K}$, $\tilde{\mathcal{K}}$ and $\tilde{s}$ for $\nu = 1$. The solid rectangles are the D_i, the dashed ones the $\tilde{D}_i$. The boldly drawn path is $\tilde{s}$.

$$(10.73) \qquad (u_1,f_1,\ldots,u_{\tau-1},u_{\tau-1}) \subset \tilde{\mathcal{K}} .$$

However (10.72) need not be valid any longer. Nevertheless $u_0 = \tilde{u}_0$, $u_\tau = \tilde{u}_\sigma$ so that, by (10.72) and (10.12)

(10.74) u_0 is within distance Λ of $[a_0+2\Lambda, a_0+\Lambda_7-2\Lambda] \times \{b_0+2\Lambda\}$, and u_σ is within distance Λ of $[a_\nu+2\Lambda, a_\nu+\Lambda_7-2\Lambda] \times \{b_{2\nu}+k_{2\nu}-2\Lambda\}$.

We shall now replace s by a minimal path, by introducing shortcuts of one edge, whenever necessary. Specifically, assume s is not minimal. Let u_i be the first vertex which is adjacent on $\mathcal{G}_{p\ell}$ to a u_j with $j \geq i+2$. Take the highest j with this property and replace the piece $f_{i+1},u_{i+1},\ldots,f_{j-1}$ of s by a single edge of $\mathcal{G}_{p\ell}$ from u_i to u_j. By repeated application of this procedure we obtain a minimal path from u_0 to u_τ, which we still denote by $s = (u_0,f_1,\ldots,f_\tau,u_\tau)$. Since its vertices form a subset of the vertices of the original s we have (see (10.73))

$$(10.75) \qquad \{u_1,\ldots,u_{\tau-1}\} \subset \tilde{\mathcal{K}} .$$

Of course (10.74) remains valid.

If s is not strongly minimal we also introduce shortcuts of two edges. This time we take the smallest i for which there exists a $j \geq i+2$ such that u_i and u_j lie on the perimeter of a face $F \in \mathcal{F}$, whose central vertex does not belong to $\mathfrak{w}$, but not such that $j = i+2$ and u_{i+1} a central vertex of $\mathcal{G}_{p\ell}$ outside $\mathfrak{w}$. Again we take the maximal j with this property, and replace the piece $f_{i+1},u_{i+1},\ldots,f_{j-1}$ of s by a piece of two edges and a vertex in between, e, u, f say, with u the central vertex of F and $e(f)$ the edge from u_i to u (from u to u_j). The insertion of this piece of two edges neither introduces double points, nor destroys the minimality of s. Indeed if u were equal to u_k for some $k < i$, then by the minimality of s this would require $i = k+1$ and $j = k+1$ (since u_k would then be adjacent to u_i and u_j). This is clearly impossible, as is $u = u_i$. A similar argument excludes $u = u_k$ with $k \geq j$. Thus the new path has no double points. Also, if u is adjacent to some u_k with $k < i$ then u_k has to lie on the perimeter of F (the central vertex of F is adjacent only to vertices on the perimeter of F). Then u_k and u_j with $j > k+2$ lie on the perimeter of F whose central vertex u is outside $\mathfrak{w}$. This contradicts the choice of u_i as the first vertex with such a property. Thus u is not adjacent to u_k with $k < i$ and a similar argument works for $k > i$. Consequently the new path is minimal, as claimed. After a finite number of insertions of shortcuts of two edges we arrive at a strongly minimal path $r = (v_0,e_1,\ldots,e_\nu,v_\nu)$ with $v_0 = u_0$, $v_\nu = u_\tau$. We claim that this path r satisfies (10.70). r satisfies the last part of (10.70) by virtue of (10.74). But also $r \subset \mathcal{K}$ follows. Indeed any edge or shortcut of two edges has diameter at most 2Λ, by virtue of (10.12). Therefore r contains only points within distance 2Λ from some vertex $u_1,\ldots,u_{\tau-1}$, i.e.,

$$r \subset (2\Lambda)\text{-neighborhood of } \tilde{\mathcal{K}} \subset \mathcal{K} \quad \text{(see (10.75))}.$$

Thus r has the properties claimed in (10.70). It is clear that the whole argument goes through unchanged on $\mathcal{G}^*_{p\ell}$.

We shall use the above procedure for making a path strongly minimal a few more times. We draw the readers attention to two aspects of the procedure. Firstly, we do not insert a shortcut of two edges between any pair of vertices u_i and u_{i+2} if u_{i+1} is already a central vertex of some face $F \in \mathfrak{F}$ of $\mathcal{M}$, with $u_{i+1} \notin \mathfrak{U}$. Secondly, the procedure is carried out in a specific order, first loop removal, then insertion of shortcuts of one edge and finally insertion of shortcuts of two edges. In all three of these subprocedures we work from the initial vertex of the path to the final one.

Step (v). In this step we make a remark about combining strongly minimal paths. Let $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ and $s = (u_0, f_1, \dots, f_\sigma, u_\sigma)$ be strongly minimal paths on $\mathcal{G}_{p\ell}$ such that

$$|v_\nu - u_0| \leq \Lambda_7 + 6\Lambda .$$

By definition of Λ_6 (see the lines following (10.12)) there then exists a path t on $\mathcal{G}_{p\ell}$ from v_ν to u_0 with diameter $(t) \leq \Lambda_6$. Now consider the path (with possible double points) consisting of r, t and s (in this order) and make it into a strongly minimal path from v_0 to u_σ. The procedure for making the path strongly minimal consists of loop removal and insertion of shortcuts of one or two edges as described for $\tilde{s}$ and s in the last step. Denote the resulting strongly minimal path from v_0 to u_σ by $\langle r,t,s\rangle$. Then the following holds.

(10.76) $\langle r,t,s\rangle$ contains all vertices u_i of s for which (distance from u_j to r) $> \Lambda_6 + 2\Lambda$ for all $j > i$, $j \leq \sigma$.

To prove (10.76) observe that u_i can be removed from s during loop removal only if u_i belongs to a loop which starts on $r \cup t$ and ends with a u_j, $j > i$, because s itself is self-avoiding. But this means that u_j equals some vertex on $r \cup t$. In this case the distance from u_j to r is $\leq \Lambda_6$, since any point of t is within distance Λ_6 from the initial point of t, which equals the endpoint v_ν of r. Next assume u_i is removed when a shortcut of one or two edges is inserted. One endpoint of the shortcut has to be a vertex of the combination of r, t and s following u_i. This has to be a u_j with $j > i$. If the shortcut has any point in common with $r \cup t$ then the above argument again gives us (10.76), in view of the fact that the diameter of the shortcut is at most 2Λ. Finally any shortcut disjoint from $r \cup t$ would be a shortcut for s itself, and no such shortcuts are inserted because s was already strongly minimal. Thus (10.76) always holds.

Assume now that s lies within distance 2Λ from some rectangle B, and that r lies outside B (in addition to the assumptions on r and s already made in the beginning of this step). Then $<r,t,s>$ also has the following property:

(10.77) $<r,t,s>$ contains only points of $r \cup s$ plus points within distance $\Lambda_6+4\Lambda$ from each of r, s and $Fr(B)$.

The proof of (10.77) is essentially contained in the proof of (10.76). Certainly t lies within distance Λ_6 from each of its endpoints, v_ν (which lies on r) and u_0 (which lies on s). Moreover u_0 lies inside B or within 2Λ from $Fr(B)$. In the former case t runs from the outside of B to a point inside B and hence intersects $Fr(B)$. In both cases t lies within $\Lambda_6+2\Lambda$ from $Fr(B)$. The only points on $<r,t,s>$ which do not belong to $r \cup t \cup s$ are points of certain shortcuts. If the shortcut contains a point of t or runs from a point of r to a point of s, then the above argument again shows that all points of the shortcut are within distance $\Lambda_6+4\Lambda$ from r, s and from $Fr(B)$. Finally, as we saw in the proof of (10.76) no shortcuts from a point of s to a point of s are inserted, and for the same reason no shortcuts from a point of r to a point of r are inserted. This takes care of all possible cases and proves (10.77).

Step (vi). This very long step gives a number of preparatory steps for the description of the local modifications of occupancy configurations which figure in Condition E. The basic objective is to construct a path $\tilde{R}$ which is a crosscut of $\text{int}(J_\ell)$ and which differs only slightly from the "lowest occupied crosscut" R of $\text{int}(J_\ell)$ and, most importantly, contains a translate $\tilde{x}$ of the vertex x in Condition D, such that $\tilde{x}$ has (almost) a vacant connection to $\mathring{C}$ above $\tilde{R}$. We choose for $\tilde{x}$ a translate of x, such that $\tilde{x}$ is not too far away from R and is near a point $a^\#$ which has a vacant connection s^* to $\mathring{C}$ above R (actually above $R^\#$). To obtain $\tilde{R}$ we replace a piece of R by a curve on $\mathcal{G}_{p\ell}$ which contains $\tilde{x}$. To construct the vacant connection from $\tilde{x}$ to $\mathring{C}$ we construct a connection on $\mathcal{G}^*_{p\ell}$ from $\tilde{x}$ to the initial point of s^*, near $a^\#$, and then continue along s^* to $\mathring{C}$. Unfortunately, the details are complicated and the reader is advised to refer frequently to Figure 10.13-10.17 to try and see what is going on.

Now for the details. Let ω be an occupancy configuration in which the event E occurs (see Step (iii) for E). Let $R = (v_0,e_1,\dots,e_\nu,v_\nu)$ be the occupied crosscut of $\mathrm{int}(J_\ell)$ with minimal $J_\ell^-(R)$ among all occupied crosscuts which satisfy (7.39)-(7.41) and (10.32). Associated with it is a crosscut $R^\#$ satisfying (7.39)-(7.41) and (10.42)-(10.44) (with r, $r^\#$ replaced by R, $R^\#$) as in Step (i). Assume further that $a^\# \in R^\#$ has a vacant connection $s^* = (w_0^*,f_1^*,\dots,f_\tau^*,w_\tau^*)$ to $\mathring{C}$ above $R^\#$ in Γ_ℓ .

We shall now use the specific properties of $R^\#$ to prove that the following relations hold (K is the special circuit of Step (i) and $K(a) = K + \lfloor a(1) \rfloor \xi_1 + \lfloor a(2) \rfloor \xi_2$ as before):

(10.78) $$(\text{distance from } a^\# \text{ to } R) \geq \Theta \quad ,$$

(10.79) $a^\# \in K(a)$ for some vertex a on R with
$$\frac{1}{2} M_{\ell 1} - \Theta - 2\Lambda \leq a(1) \leq \frac{3}{2} M_{\ell 1} + \Theta + 2\Lambda \text{ , and}$$

(10.80) $$s^* \subset \mathrm{ext}\, K(a).$$

Assume that (10.78) fails. Then we can find some point b on R with $|a^\#-b| < \Theta$ and hence for some vertex w of $\mathcal{G}_{p\ell}$ on R (w can be taken as an endpoint of the edge containing b)

$$\begin{aligned} |w_0^*-w| &\leq |w_0^*-a^\#| + |a^\#-b| + |b-w| \\ &< \Theta + 2\Lambda \quad . \end{aligned}$$

Since K surrounds the square (10.37) this means that $w_0^* \in \mathrm{int}(K(w))$. Further, from $w_0^* \in \Gamma_\ell$ we obtain

$$\frac{1}{2} M_{\ell 1} - \Theta - 2\Lambda \leq w(1) \leq \frac{3}{2} M_{\ell 1} + \Theta + 2\Lambda \quad .$$

In other words $\overline{K}(w) \subset \mathcal{E}(R)$ (see (10.40) and (10.41)). This, however, is impossible since $\mathcal{E}(R)$ is disjoint from $\mathfrak{F}(R)$ (by definition of $\mathfrak{F}(R)$), while by (10.42) $\mathfrak{F}(R) = J_\ell^+(R^\#)$. Thus w_0^* , which is a point of $J_\ell^+(R^\#)$ (see (10.51)) cannot lie in $\mathcal{E}(R)$. This contradiction implies that (10.78) holds.

(10.79) is now easy. By virtue of (10.44) $a^\# \in R^\#$ lies on R or on some $K(a)$ for which (10.79) holds. $a^\# \in R$ is excluded by (10.78). Also (10.80) follows, since $s^*\setminus\{w_\tau^*\} \subset J_\ell^+(R^\#)$ (see (10.51)), and as we saw above $J_\ell^+(R^\#) = \mathfrak{J}(R)$ is disjoint from all $\overline{K}(a)$ which can arise in (10.79). Moreover $w_\tau^* \in \overset{\circ}{C}$ (by (10.50)) lies above the line $x(2) = 12M_{\ell 2}$ (see Step (i)) and outside $\overline{K}(a)$ since R satisfies (10.32).

For the remainder fix a vertex a of $\mathcal{G}_{p\ell}$ on R such that (10.78)-(10.80) hold. For the sake of argument assume that $a^\#$ lies on the "left half of the lower edge of $K(a)$", i.e.,

$$a^\#(1) \le a(1), \quad a^\#(2) \le a(2) - 2\Theta + \Lambda \quad ; \tag{10.81}$$

see Fig. 10.13. Similar arguments will apply in the other cases. Let $x \in \mathfrak{w}$ have the properties listed in Condition D and choose $k_1, k_2 \in \mathbb{Z}$

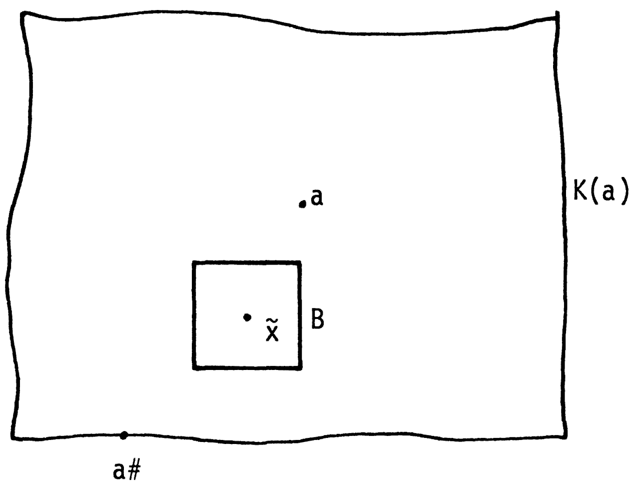

Figure 10.13

such that $\tilde{x} := x+(k_1,k_2)$ lies in the closed unit square centered at

$$a^\#+(5\Lambda_7+2\Lambda_8+3\Lambda+\Delta+1,\Lambda_6+\Lambda_5+\Lambda_3+10\Lambda+\Delta+1). \tag{10.82}$$

Then, by the periodicity $\tilde{x}$ also has the properties listed in Condition D. We can therefore find $B = B(\tilde{x})$ and paths U on $\mathcal{G}_{p\ell}$, V^* on $\mathcal{G}^*_{p\ell}$ such that a)-e) of Condition D (with $\tilde{x}$ for x) hold. We note that by (10.37) and (10.38) $K(a)$ lies in the annulus

$$\mathfrak{a} := [\lfloor a(1) \rfloor -2\Theta-\Lambda_3-\Lambda, \lfloor a(1) \rfloor +2\Theta+\Lambda_3+\Lambda]$$

$$\times [\lfloor a(2) \rfloor -2\Theta-\Lambda_3-\Lambda, \lfloor a(2) \rfloor +2\Theta+\Lambda_3+\Lambda]$$

$$\setminus (\lfloor a(1) \rfloor -2\Theta+\Lambda, \lfloor a(1) \rfloor +2\Theta-\Lambda)$$

$$\times (\lfloor a(2) \rfloor -2\Theta+\Lambda, \lfloor a(2) \rfloor +2\Theta-\Lambda).$$

By (10.81), (10.82) and (10.33) $B = B(\tilde{x})$ lies in the interior of the inner boundary of $\mathfrak{a}$. In fact

(10.83) $$\text{distance } (B,\mathfrak{a}) > \Lambda_6+6\Lambda \quad .$$

We now want to "splice U into R" and connect V^* to s^*. We first connect those endpoints of U and V^* near the perimeter of B to $K(a) \subset \mathfrak{a}$, by paths which run to the outside of $\mathfrak{a}$. These paths should not interfere with each other, nor should they be too far away from B (for purposes of the construction to follow). We put these paths inside three corridors $\mathcal{K}_\ell$, $\mathcal{K}_r$ and $\mathcal{K}^*$ of width Λ_7. A typical illustration of these corridors is shown in Fig. 10.14. Formally, we require that they have the properties (10.84)-(10.92) below.

(10.84) The corridors are disjoint from $\overset{\circ}{B}$ (= interior of $B(\tilde{x})$).

(10.85) The first edge of $\mathcal{K}_\ell(\mathcal{K}_r)$ is on the left (right) edge of B, i.e., on $\{\tilde{x}(1)-\Delta\} \times [\tilde{x}(2)-\Delta,\tilde{x}(2)+\Delta]$ $(\{\tilde{x}(1)+\Delta\} \times [\tilde{x}(1)-\Delta,x(2)+\Delta])$. Moreover the first edge of $\mathcal{K}_\ell$ $(\mathcal{K}_r)$ intersects the edge e_1 (e_ρ) of U (cf. Condition Dc). Finally the distance between $\mathcal{K}_\ell$ $(\mathcal{K}_r)$ and $\{u_{i_0},\dots,u_\rho\}$ $(\{u_0,\dots,u_{i_0}\})$ is at least $\Lambda_6+9\Lambda$, while the distance between $\mathcal{K}_\ell \cup \mathcal{K}_r$ and V^* is at least $\Lambda_6+5\Lambda$.

(10.86) The first edge of $\mathcal{K}^*$ is on the top edge of B, in the segment $[\tilde{x}(1)-\Delta+\Lambda_8,\tilde{x}(1)+\Delta-\Lambda_8] \times \{\tilde{x}(2)+\Delta\}$ and intersects the edge e^*_σ of V^*. The distance between $\mathcal{K}^*$ and U is at least Λ_8.

(10.87) Let D_ℓ be the last rectangle in the corridor $\mathcal{K}_\ell$. It is of the even-indexed type (10.68) and intersects $\mathfrak{a}$ only in the latter's bottom strip

$[\lfloor a(1)\rfloor-2\Theta-\Lambda_3-\Lambda, \lfloor a(1)\rfloor+2\Theta+\Lambda_3+\Lambda]\times[\lfloor a(2)\rfloor-2\Theta-\Lambda_3-\Lambda, \lfloor a(2)\rfloor-2\Theta+\Lambda])$. The intersection of D_ℓ and this bottom strip is a rectangle of size $\Lambda_7\times(\Lambda_3+2\Lambda)$. The last edge of $\mathcal{K}_\ell$ lies in the exterior of $\mathcal{G}$, at a distance $\geq 3\Lambda$ from $\mathcal{G}$. D_ℓ lies to the right of the vertical line $\{\lfloor a(1)\rfloor-2\Theta+3\Lambda\}\times\mathbb{R}$, i.e., more than 2Λ units to the right of the left strip of $\mathcal{G}$. Lastly, all points of $\mathcal{K}_\ell$ within distance 2Λ from $\mathcal{G}$ lie in D_ℓ.

(10.88) Either (10.87) also holds with $\mathcal{K}_\ell$ replaced by $\mathcal{K}_r$ and D_ℓ by the last rectangle D_r of $\mathcal{K}_r$, or D_r is of the odd-indexed type (10.69) and intersects $\mathcal{G}$ only in the latter's left strip, $[\lfloor a(1)\rfloor-2\Theta-\Lambda_3-\Lambda, \lfloor a(1)\rfloor-2\Theta+\Lambda]\times[\lfloor a(2)\rfloor-2\Theta-\Lambda_3-\Lambda, \lfloor a(2)\rfloor+2\Theta+\Lambda_3+\Lambda]$. In this case the intersection of D_r and the left strip is a rectangle of size $(\Lambda_3+2\Lambda)\times\Lambda_7$, the last edge of $\mathcal{K}_r$ lies in the exterior of $\mathcal{G}$ at a distance $\geq 3\Lambda$ from $\mathcal{G}$. Also all points of $\mathcal{K}_r$ within distance 2Λ of $\mathcal{G}$ lie in D_r, and D_r lies above the horizontal line $\mathbb{R}\times\{\lfloor a(2)\rfloor-2\Theta+3\Lambda\}$ (i.e., more than 2Λ units above the bottom strip of $\mathcal{G}$).

(10.89) $a^\# \in \mathcal{K}^*\cap K(a)\subset\mathcal{K}^*\cap\mathcal{G}$; $\mathcal{K}^*\cap\mathcal{G}$ lies below the horizontal line $\mathbb{R}\times\{\lfloor a(2)\rfloor-\Theta\}$.

(10.90) The distance between any pair of the corridors $\mathcal{K}_\ell$, $\mathcal{K}_r$ and $\mathcal{K}^*$ is at least Λ_8 (Λ_8 is defined before Condition D).

(10.91) All three corridors $\mathcal{K}_\ell$, $\mathcal{K}_r$ and $\mathcal{K}^*$ lie within distance Λ_9 of $a^\#$ (Λ_9 is defined in (10.33)).

(10.92) $\mathcal{K}^*\cap K(a)$ lies "between $\mathcal{K}_\ell\cap\mathcal{G}$ and $\mathcal{K}_r\cap\mathcal{G}$". More precisely, if b is any point of $\mathcal{K}^*\cap K(a)$, then any continuous curve from $\mathcal{K}_\ell$ to $\mathcal{K}_r$ inside $\mathcal{G}$ of diameter $\leq\Theta$ intersects the line segment $b+t(1,1)$, $-2\Lambda_3-4\Lambda\leq t\leq 2\Lambda_3+4\Lambda$.

These horrendous conditions are actually not difficult to satisfy as illustrated in Fig. 10.14 for the case where $a^\#$ is sufficiently far away from the left edge of $\mathcal{G}$ so that (10.87) can be satisfied for $\mathcal{K}_\ell$ as well as $\mathcal{K}_r$. We content ourselves with this figure and a few minor comments indicating why (10.84)-(10.92) can be satisfied. For

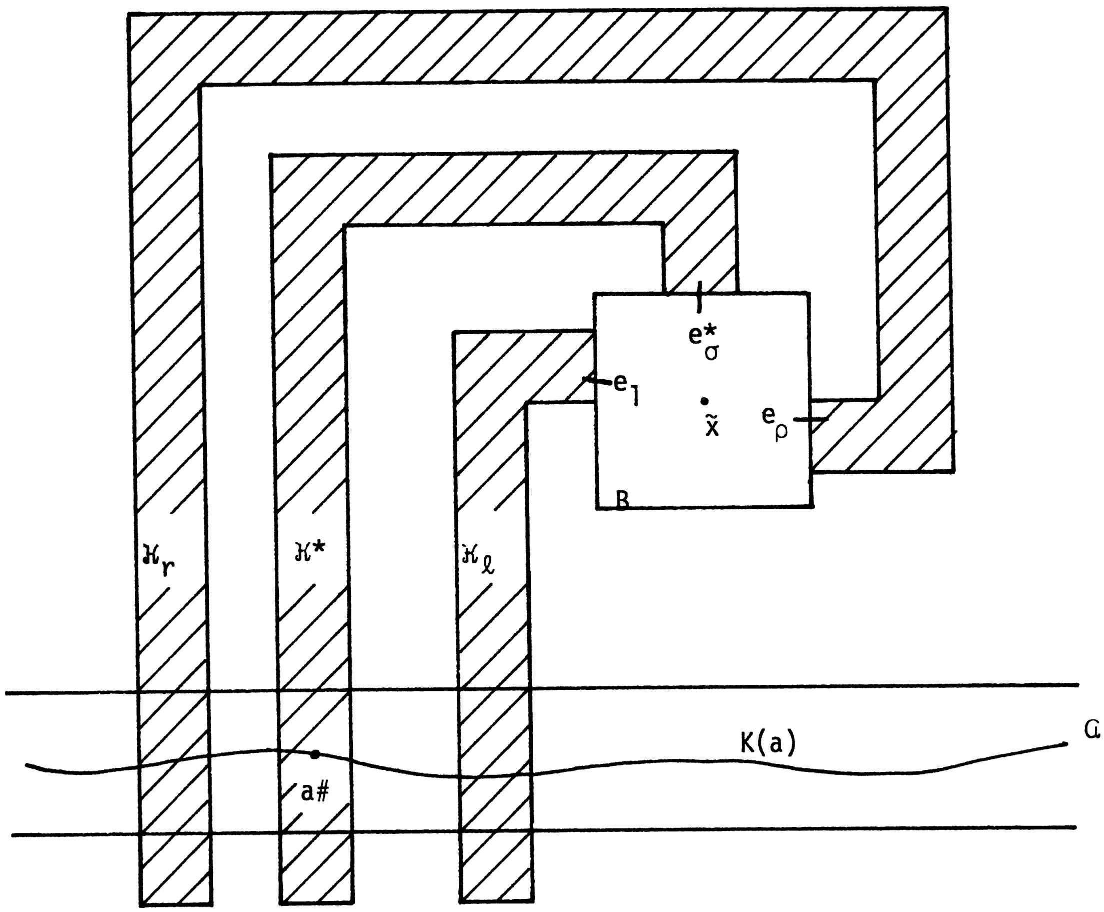

Figure 10.14 The hatched regions are the corridors $\mathcal{K}_r$, $\mathcal{K}_\ell$ and $\mathcal{K}^*$.

(10.85) and (10.86) we remind the reader that e_1, e_ρ and e^*_σ intersect the left, right and top edge of B, respectively, by Condition D. Moreover, $U = (u_0, e_1, \dots, e_\rho, u_\rho)$ lies below the horizontal line $\mathbb{R} \times \{\tilde{x}(2) + \Delta - \Lambda_8\}$, while V^* lies in the vertical strip $[\tilde{x}(1) - \Delta + \Lambda_8, \tilde{x}(1) + \Delta - \Lambda_8] \times \mathbb{R}$. Lastly $(u_{i_0}, e_{i_0+1}, \dots, u_\rho)$ lies to the right of $\{x(1) - \Delta + \Lambda_8\} \times \mathbb{R}$ and $(u_0, e_1, \dots, u_{i_0})$ lies to the left of $\{x(1) + \Delta - \Lambda_8\} \times \mathbb{R}$. (10.91) can be satisfied by (10.33) and because

$$|a^{\#}(i) - \tilde{x}(i)| \leq 5(\Lambda_5 + \Lambda_6 + \Lambda_7 + \Lambda_8 + \Lambda + \Delta + 1) \tag{10.93}$$

(see (10.82)). Lastly, with regard to (10.92) we remark that the segment $b + t(1,1)$, $|t| \leq 2\Lambda_3 + 4\Lambda$, is on a 45° line through b and cuts $\mathcal{G}$ "close to" the lower strip of $\mathcal{G}$. Also $\mathcal{K}_\ell \cap \mathcal{G}$ and $\mathcal{K}_r \cap \mathcal{G}$ lie close to the lower edge of $\mathcal{G}$. A path from $\mathcal{K}_\ell \cap \mathcal{G}$ to $\mathcal{K}_r \cap \mathcal{G}$ of diameter $\le \Theta$, has to remain below the horizontal line $x(2) = \lfloor a(2) \rfloor - \Theta + \Lambda$ (by (10.87)). Therefore such a path cannot intersect the segment $\{\lfloor a(1) \rfloor\} \times [\lfloor a(2) \rfloor + 2\Theta - \Lambda, \lfloor a(2) \rfloor + 2\Theta + \Lambda_3 + \Lambda\}$ which cuts the top strip of $\mathcal{G}$. This segment together with the segment $b+t(1,1), |t| \le 2\Lambda_3 + 4\Lambda$ divide $\mathcal{G}$ into two components, whenever $b \in \mathcal{K}^* \cap \mathcal{G}$ (by (10.89)). (10.92) basically says that $\mathcal{K}_\ell \cap \mathcal{G}$ and $\mathcal{K}_r \cap \mathcal{G}$ do not lie in the same component of $\mathcal{G}$ when $\mathcal{G}$ is cut by these two segments. This is obviously the case when $\mathcal{K}_\ell$, $\mathcal{K}_r$ and $\mathcal{K}^*$ are located as in Fig. 10.14.

It should be obvious that the precise values of the various constants Λ_i and Θ are without significance.

Once the corridors $\mathcal{K}_\ell$, $\mathcal{K}_r$ and $\mathcal{K}^*$ have been chosen we choose strongly minimal paths r_i on $\mathcal{G}_{p\ell}$ inside $\mathcal{K}_i$, $i = \ell$ or r, which start within distance 3Λ from its first edge and end within distance 3Λ from its last edge, by the method of Step (iv) (see (10.70)). Since the last edge of $\mathcal{K}_i$ is at least 3Λ units outside $\mathcal{G}$ (by (10.87), (10.88)), the endpoint of r_i lies in the exterior or on the exterior boundary of $\mathcal{G}$. The first point of r_i lies within 3Λ from B and therefore inside the inner boundary of $\mathcal{G}$ and at distance $> 3\Lambda$ from this inner boundary (by (10.83)). Hence r_i intersects $K(a)$. A fortiori there exists a first vertex of r_i, b_i say, which can be connected to a vertex of $K(a)$, $c_{\alpha(i)}$ say, by a path of two edges on $\mathcal{G}_{p\ell}$. We connect b_i to $c_{\alpha(i)}$ by such a path of two edges. If possible we take for the intermediate vertex between b_i and $c_{\alpha(i)}$ a central vertex of $\mathcal{G}_{p\ell}$ which does not belong to $\mathfrak{w}$. Also we connect the initial point of r_ℓ (r_r) to u_1 $(u_{\rho-1})$ by a path t_ℓ (t_r) on $\mathcal{G}_{p\ell}$ of diameter $\le \Lambda_6$. This can be done by one choice of Λ_6 since the initial point of r_ℓ is within 3Λ from the first edge of $\mathcal{K}_\ell$, which intersects e_1 by (10.85). Thus the distance between the initial point of r_ℓ and u_1 is at most $4\Lambda+\Lambda_7$. A similar statement holds for r_r and $u_{\rho-1}$. Next we make the piece of U from u_1 to $u_{\rho-1}$ into a strongly minimal path, $\tilde{U}$ say, which still runs from u_1 to $u_{\rho-1}$, by insertion of shortcuts of two edges if necessary (see the method used for the path s in Step (iv); recall that U is minimal by Condition D). Now consider the following path on $\mathcal{G}_{p\ell}$ (with possible double points) from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$: From $c_{\alpha(\ell)}$ go via two edges to the vertex b_ℓ of r_ℓ, traverse r_ℓ backwards, then go along t_ℓ to u_1, along $\tilde{U}$ from u_1 to $u_{\rho-1}$, along t_r to the initial point of r_r, then along r_r to the vertex b_r of r_r, and finally via two edges to $c_{\alpha(r)}$ (see

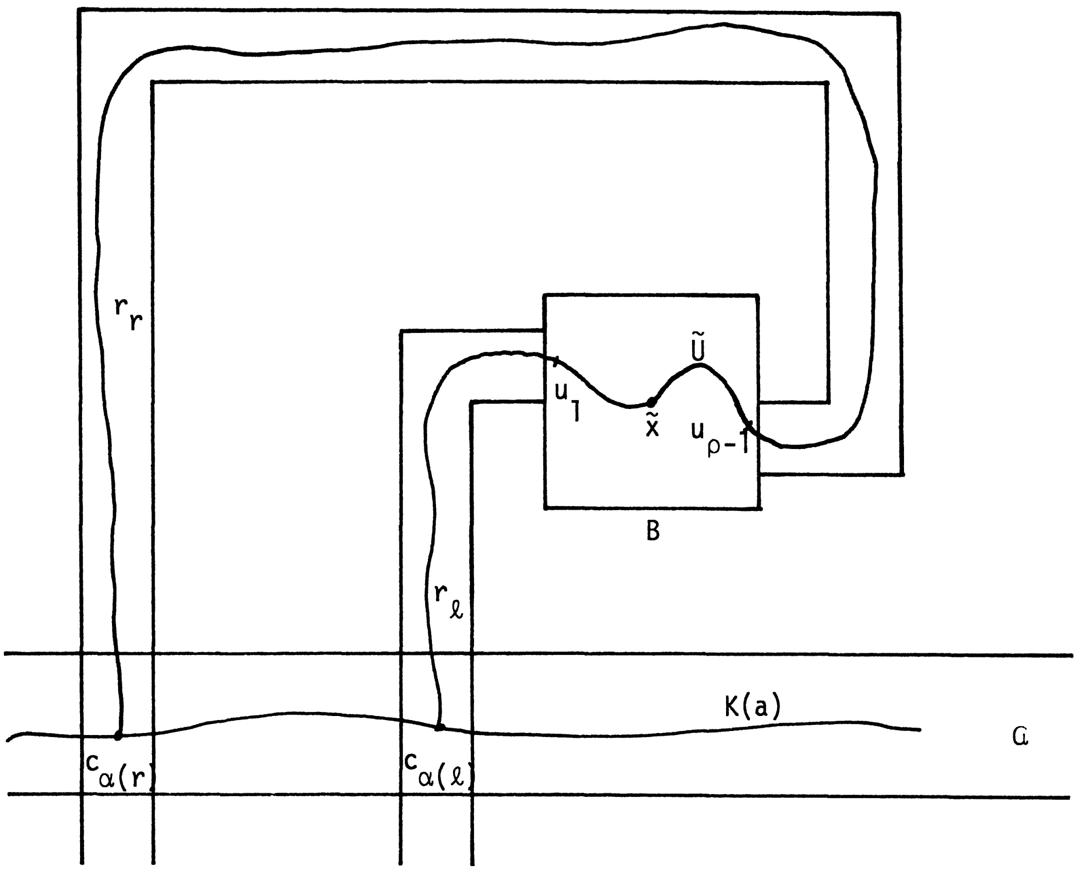

Figure 10.15

Fig. 10.15). This whole path is made into a strongly minimal path $\tilde{X}$ in the following way. First make r_i till b_i plus the two-edge connection from b_i to $c_{\alpha(i)}$ into a strongly minimal path, $\tilde{r}_i$ say, by the method applied to the path s in Step (iv). Since r_i itself was already strongly minimal, and since b_i is the first point on r_i which can be connected by two edges to $K(a)$, one easily sees that no loops have to be removed, nor shortcuts of one edge have to be inserted during the formation of $\tilde{r}_i$. Moreover, at most one shortcut of two edges has to be inserted to obtain a strongly minimal $\tilde{r}_i$. Indeed, if the connection from b_i to $c_{\alpha(i)}$ goes through the vertex y_i of $\mathcal{G}_{p\ell}$, then the only shortcut which may have to be inserted is from some vertex on the piece of r_i between its initial point and b_i to y_i. Note that such a shortcut lies within 3Λ from $K(a)$ and hence further than 2Λ away from U (by virtue of (10.83)). Now that $\tilde{r}_i$ and $\tilde{U}$ have been formed we first combine $\tilde{r}_\ell$, t_ℓ and $\tilde{U}$ into the strongly minimal path $<\tilde{r}_\ell,t_\ell,\tilde{U}>$ as in Step (v) (see (10.76)). Finally we obtain the strongly minimal path $\tilde{X}$ as the combination $<<\tilde{r}_\ell,t_\ell,\tilde{U}>,t_t,\tilde{r}_r>$ of this last path with t_r and $\tilde{r}_r$.

It will be very important that one has

(10.94) $\qquad \tilde{x}$ is a vertex on $\tilde{X}$,

as we now prove. Firstly $\tilde{x}$ cannot be removed when U is turned into the strongly minimal path $\tilde{U}$. This is so because U is already minimal by hypothesis (see Condition D), and $\tilde{x} = u_{i_0}$ could be removed only by insertion of a shortcut of two edges, and only if such a shortcut runs from u_i to u_j with $i < i_0 < j$. By Condition D b) no such shortcuts exist. Secondly, when we form $<\tilde{r}_\ell,t_\ell,\tilde{U}>$ from $\tilde{r}_\ell$, t_ℓ and $\tilde{U}$, by the method of Step (v), then $\tilde{x} = u_{i_0}$ is not removed, on account of (10.76) and (10.85). Indeed, all points $u_{i_0+1},\dots,u_\rho$ have a distance of at least $\Lambda_6+9\Lambda$ to $r_\ell \subset \mathcal{K}_\ell$. Thus, also any shortcuts introduced in the formation of $\tilde{U}$ and ending at one of $u_{i_0+1},\dots,u_\rho$ have distance at least $\Lambda_6+5\Lambda$ to $\tilde{r}_i$ (which lies within 2Λ from r_i). Thirdly, when $\tilde{r}_r$, t_r and $<\tilde{r}_\ell,t_\ell,\tilde{U}>$ are combined to $<<\tilde{r}_\ell,t_\ell,\tilde{U}>,t_r,\tilde{r}_r>$ $= \tilde{X}$, then $\tilde{x}$ is still maintained. This is so because no intersections or shortcuts between $\tilde{r}_\ell \cup t_\ell$ and $\tilde{r}_r \cup t_r$ exist, the distance between these two sets being at least

$$\Lambda_8-2\Lambda_6-4\Lambda > 6\Lambda ,$$

by virtue of (10.90). Also the distance between $u_1,\dots,u_{i_0-1}$, or any shortcuts ending at one of these points, and $\tilde{r}_r$ is at least $\Lambda_6+5\Lambda$, by (10.85) again. As in the proof of (10.76) one obtains from this that $\tilde{x}$ will not be removed when forming $\tilde{X}$. This proves (10.94).

We set

(10.95) $\qquad \tilde{X}_i(\tilde{x})$ = closed segment of $\tilde{X}$ between $c_{\alpha(i)}$ on $K(a)$ and $\tilde{x}$, $i = \ell$ or r.

The proof of (10.94) just completed also shows that

(10.96) $\qquad$ There exist no shortcuts of two edges for $\tilde{X}$ with one endpoint each on of $\tilde{X}_\ell(\tilde{x})\setminus\{\tilde{x}\}$ and $\tilde{X}_r(\tilde{x})\setminus\{\tilde{x}\}$.

We also leave it to the reader to use (10.83), (10.77) and the description of $\tilde{X}$ - especially the statements about $\tilde{r}_i$ before the proof of (10.94) - to verify that

(10.97) any vertex on $\tilde{X}$ which can be connected to $K(a)$ by one or two edges of $\mathcal{G}_{p\ell}$ lies within distance 2Λ of $\{c_{\alpha(\ell)}, c_{\alpha(r)}\}$.

For later purposes it is also useful to know that

$$\tilde{X} \setminus \{c_{\alpha(\ell)}, c_{\alpha(r)}\} \subset \mathrm{int}(K(a)). \tag{10.98}$$

To prove this we go back to the construction of $\tilde{r}_i$. This is made from the piece of r_i from its initial point to b_i, a two-edge connection from b_i via the vertex y_i to $c_{\alpha(i)}$, and possibly a shortcut from y_i via a central vertex, y_i' say, to a vertex, y_i'' say, on the piece of r_i between its initial point and b_i. Since r_i from its initial point to b_i lies in $\mathrm{int}(K(a))$, we see that also $\tilde{r}_i \setminus c_{\alpha(i)} \subset \mathrm{int}(K(a))$, unless y_i or y_i' belongs to $K(a)$. However y_i cannot lie on $K(a)$ by the minimality properties of b_i, for if $y_i \in K(a)$, then b_i would be connectable to $K(a)$ by a single edge. Similarly $y_i \notin K(a)$, because y_i'' cannot be connected to $K(a)$ by a single edge. Thus

$$\tilde{r}_\ell \cup \tilde{r}_r \setminus \{c_{\alpha(\ell)}, c_{\alpha(r)}\} \subset \mathrm{int}(K(a))$$

and also

$$t_\ell \cup \tilde{U} \cup t_r \text{ lie in } \mathrm{int}(K(a)), \text{ even at a distance } > 4\Lambda \text{ from } K(a) .$$

(Again recall (10.83) and the fact that t_ℓ (t_r) has one endpoint at u_1 $(u_{\rho-1})$.) Finally any shortcuts inserted while making $\tilde{X}$ from $\tilde{r}_\ell$, t_ℓ, $\tilde{U}$, t_r, $\tilde{r}_r$ lie in $\mathrm{int}(K(a))$ by (10.83) and (10.77) with its proof (recall that there are no shortcuts between $\tilde{r}_0 \cup t_0$ and $\tilde{r}_\nu \cup t_\nu$).

It is our objective to make (most of) $\tilde{X}$, including $\tilde{x}$, part of the "lowest" occupied horizontal crosscut of J_ℓ in the modified occupancy configuration. Before we can do this we also have to describe part of the path which will form the vacant connection from $\tilde{x}$ to $\mathring{C}$ in the modified configuration. Specifically we construct a path on $\mathcal{G}^*_{p\ell}$ from v_0^* (= the initial point of V^*) to w_0^* (= the initial point of s^*). We first take a path r^* on $\mathcal{G}^*_{p\ell}$ in $\mathcal{H}^*$ which begins within distance 3Λ from the first edge of $\mathcal{H}^*$ and ends within $\Lambda_7+3\Lambda$ from

$a^{\#}$. This can be done by virtue of (10.89) and the fact that $\mathcal{H}^*$ has width Λ_7. We connect v_σ^* to the first point of r^* by a path on $\mathcal{G}_{p\ell}^*$ of diameter $\leq \Lambda_6$. We also connect the final point of r^* to w_0^* by a path on $\mathcal{G}_{p\ell}^*$ of diameter Λ_6. This can be done since $|a^{\#}-w_0^*| \leq \Lambda$ (cf. (10.49)) so that the distance from the last point of r^* to w_0^* is at most $\Lambda_7+4\Lambda$. Now take the path (with possible double points) from v_0^* to w_0^* which proceeds via V^*, the connection between v_σ^* and the first point of r^*, r^* and finally the connection from the last point of r^* to w_0^*. Make it self-avoiding by loop-removal (it is not important that it become (strongly) minimal). The resulting path on $\mathcal{G}_{p\ell}^*$, will still run from v_0^* to w_0^*. Call it X^*. We shall need the fact that

(10.99) $\qquad X^*$ is disjoint from $\tilde{X}$.

This follows from the following remarks, Firstly $\tilde{U}$ and V^* have no point in common by virtue of Condition De) and the fact that the only vertices which can lie on $\tilde{U}\setminus U$ are central vertices of $\mathcal{G}_{p\ell}$, and hence are not on the path V^* on $\mathcal{G}_{p\ell}^*$. Secondly, all points of $\tilde{X}(X^*)$ further away than $\Lambda_6+4\Lambda$ from $\mathcal{H}_\ell \cup \mathcal{H}_r$ $(\mathcal{H}^*)$ must belong to $\tilde{U}(V^*)$. Finally points within $\Lambda_6+4\Lambda$ from $\mathcal{H}_\ell \cup \mathcal{H}_r$ $(\mathcal{H}^*)$ cannot belong to $X^*(\tilde{X})$ by (10.90), (10.85) and (10.86).

We now start on making (most of) $\tilde{X}$ part of the lowest crossing. In order to achieve this we want to connect $\tilde{X}$ with R. Note first that

(10.100) $\qquad \tilde{X}$ is disjoint from R,

because by construction $\tilde{X}$ lies within $\Lambda_6+4\Lambda$ from $B \cup \mathcal{H}_\ell \cup \mathcal{H}_r$, hence within

$$\Lambda_6+4\Lambda+2\Lambda_9$$

from $a^{\#}$ (see (10.91) and (10.33)), which is less than the distance from R to $a^{\#}$ (by (10.78)). Despite (10.100) R is not too far away from $\tilde{X}$. Indeed R contains the vertex a in the interior of $K(a)$, while for large enough ℓ, the initial (final) point of R on $B_1(B_2)$ has first coordinate $\leq \Lambda_3$ $(\geq 2M_{\ell 1}-\Lambda_3)$, and therefore lies in $\text{ext}(K(a))$ for all sufficiently large ℓ. (See Step (i) for B_i and recall that a satisfies (10.79).) Thus R intersects $K(a)$ at least twice. We next derive some information about the location of these intersections. Let $K_{\#}$ be the arc of $K(a)$ from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ through $a^{\#}$. We claim that

(10.101) $\quad \text{diameter}(K_\#) \le 6\Lambda_5(2\Lambda_9+2\Lambda_3+5\Lambda+1) < \Theta$,

and

(10.102) $$\mathcal{K}^* \cap K(a) \subset K_\# .$$

To prove (10.101) and (10.102) let b be any point of $\mathcal{K}^* \cap K(a)$. Then, by (10.91)

$$|b-c_{\alpha(i)}| \le |b-b_i| + |b_i-c_{\alpha(i)}| \le 2\Lambda_9+2\Lambda, \ i = \ell,r,$$

since $b_i \in \mathcal{K}_i$ and b_i is connected to $c_{\alpha(i)}$ by two edges. Thus, by the construction of K - in particular by (10.39) - b and $c_{\alpha(\ell)}$ are connected by an arc, ϕ say, of $K(a)$ of diameter at most

(10.103) $$3\Lambda_5(2\Lambda_9+2\Lambda_3+5\Lambda+1).$$

First we must show that this arc does not contain $c_{\alpha(r)}$. Assume to the contrary that moving along ϕ from $c_{\alpha(\ell)}$ to b one passes $c_{\alpha(r)}$ before reaching b. Then the subarc ϕ' of ϕ from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ does not contain b. However, $b_i \in \mathcal{K}_i$ and $|b_i-c_{\alpha(i)}| \le 2\Lambda$. Thus by (10.87), (10.88) b_i lies in the last rectangle D_i of $\mathcal{K}_i$. Since the location of D_ℓ is at least 2Λ units to the right of the left strip of $\mathcal{G}$ (see (10.87)) and within Λ_9 of $a^\#$ (see (10.91)) - which lies to the left of a (see (10.81)) - it follows that $c_{\alpha(\ell)}$ (which lies within 2Λ from $D_\ell \subset \mathcal{K}_\ell$) lies in the lower strip of $\mathcal{G}$. Hence, $c_{\alpha(\ell)}$ can be connected to some point of D_ℓ by a horizontal line segment in the lower strip of $\mathcal{G}$ and of length $\le 2\Lambda$. Similarly, $c_{\alpha(r)}$ can be connected to a point of $D_r \subset \mathcal{K}_r$ by a straight line segment in $\mathcal{G}$ (horizontal or vertical) of length $\le 2\Lambda$. ϕ' together with the two straight line segments from $c_{\alpha(i)}$ to $\mathcal{K}_i$ form a continuous curve in $\mathcal{G}$ from $\mathcal{K}_\ell$ to $\mathcal{K}_r$ of diameter $\le 4\Lambda$ plus the expression in (10.103). Since this diameter is at most Θ, (10.92) implies that the curve must intersect the segment $b+t(1,1)$, $|t| \le 2\Lambda_3+4\Lambda$. The two straight line segments which were added to ϕ' lie within 2Λ of $\mathcal{K}_\ell \cup \mathcal{K}_r$, and by virtue of (10.90) do not intersect the segment $b+t(1,1)$, $|t| \le 2\Lambda_3+4\Lambda$, which lies within $2\Lambda_3+4\Lambda$ from $\mathcal{K}^*$. Thus ϕ' already intersects the segment in some point b', whose distance from b is at most $2\Lambda_3+4\Lambda$. Again by the construction of K and the estimate (10.39), b' is connected to b by an arc, ψ say, of $K(a)$ of diameter at most

(10.104) $$3\Lambda_5(4\Lambda_3+7\Lambda+1).$$

Now by our assumption the curve ϕ starting at $c_{\alpha(\ell)}$ first passes through b', then through $c_{\alpha(r)}$ and then ends at b. ψ cannot be the piece of ϕ from b' through $c_{\alpha(r)}$ to b, in fact ψ cannot contain $c_{\alpha(r)}$, for then by (10.90) its diameter would be at least

$$|c_{\alpha(r)}-b| \geq \text{distance } (\mathcal{H}_r,\mathcal{H}^*)-2\Lambda \geq \Lambda_8-2\Lambda ,$$

which exceeds (10.104). Thus, the piece of ϕ from b' to b and ψ have to be two arcs of K(a) from b' to b, exactly one of which contains the point $c_{\alpha(r)}$ of K(a). This can only be if together these two arcs make up all of the Jordan curve K(a), and if at least one of these arcs has a diameter $\geq \frac{1}{2}$ diameter $(K(a)) \geq 2\Theta-\Lambda$ (see (10.37)). Since this is not the case we have derived a contradiction from the assumption that ϕ contains the point $c_{\alpha(r)}$. Thus the path ϕ from $c_{\alpha(\ell)}$ to b does not contain $c_{\alpha(r)}$. In the same way we find an arc θ of K(a) from b to $c_{\alpha(r)}$ of diameter at most equal to the expression in (10.103) and not containing $c_{\alpha(\ell)}$. ϕ followed by θ gives us an arc of K(a) from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ through b and of diameter at most equal to the right hand side of (10.101). This arc must be the same for all choices of b in $\mathcal{H}^* \cap K(a)$. Otherwise, as above, K(a) would be the union of two different arcs from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$, each with diameter at most equal to the right hand side of (10.101). This, however, contradicts the fact that diameter (K(a)) $\geq 4\Theta-2\Lambda$. But for $b = a^{\#}$ the arc from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ through $a^{\#}$ is just $K_{\#}$ so that (10.101) and (10.102) follow.

We shall use two consequences of (10.101) and (10.102). These are

$$R \cap K_{\#} = \emptyset \tag{10.105}$$

and

$$X^* \cap K(a) \subset K_{\#} \text{ and hence } X^* \cap K(a) \cap R = \emptyset . \tag{10.106}$$

(10.105) is immediate from (10.101) since $a^{\#} \in K_{\#}$ has distance at least Θ to R (see (10.78)). The second statement in (10.106) will follow from the first part and (10.105). As for the first part of (10.106), by (10.83) and the construction of X^*, any point c of $X^* \cap K(a)$ lies on $r^* \cap K(a) \subset \mathcal{H}^* \cap K(a)$ or lies on the connection of diameter $\leq \Lambda_6$ from the endpoint of r^* to w_0^*. Since $|w_0^*-a^{\#}| \leq \Lambda$, any point c of $X^* \cap K(a)$ lies within distance $\Lambda_6+\Lambda$ from some point b in $\mathcal{H}^* \cap K(a)$. Again by the estimate (10.39) b is then connected to c by an arc ζ of K(a) of diameter at most

$$3\Lambda_5(\Lambda_6+2\Lambda_3+4\Lambda+1) < \Lambda_8-2\Lambda \ .$$

On the other hand b is connected to $c_{\alpha(\ell)}$ and $c_{\alpha(r)}$ by two arcs of $K(a)$ of diameter at least

$$\lim_{i=\ell,r} |c_{\alpha(i)}-b| \geq \min_{i=\ell,r} \text{distance}(\mathcal{H}_i,\mathcal{H}^*)-2\Lambda \geq \Lambda_8-2\Lambda$$

(see (10.90)), and as we saw in the proof of (10.101) and (10.102) these arcs have only the point b in common and together make up $K_{\#}$. ζ must start out following one of these arcs, and the endpoint of ζ must come before the endpoint of this arc since

$$\min_{i=\ell,r} |c_{\alpha(i)}-b| > \text{diameter}(\zeta).$$

Consequently ζ is contained in one of the above arcs from b to $c_{\alpha(i)}$, $i = \ell$ or r, and a fortiori ζ is contained in $K_{\#}$. This proves (10.106).

We now know that R intersects $K(a)\setminus K_{\#}$ at least twice (see the lines immediately preceding (10.101), and (10.105)). Therefore if one moves along the arc of $K(a)$ from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ which is not $K_{\#}$, then one passes through at least two points of R. Let $R = (v_0,e_1,\ldots,e_\nu,v_\nu)$ and let $v_{\beta(\ell)}$ $(v_{\beta(r)})$ be the first (last) point of R one meets in going along $K(a)\setminus K_{\#}$ from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$. Denote by K_i, $i = \ell$ or r, the (closed) arc of $K(a)$ between $c_{\alpha(i)}$ and $v_{\beta(i)}$ which does not contain $K_{\#}$ (see Fig. 10.16). From the above description we see that $v_{\beta(\ell)} \neq v_{\beta(r)}$, and

(10.107) $$K_i \cap K_{\#} = \{c_{\alpha(i)}\} \quad K_\ell \cap K_r = \emptyset \ ,$$

$$K_i \cap R = \{v_{\beta(i)}\} \quad K_{\#} \cap R = \emptyset \ .$$

We can now define a new crosscut $\overline{R}$ of $\text{int}(J_\ell)$ which contains $\tilde{X}$ "spliced into R" (see Step (i) for J_ℓ). The path $\overline{R}$ on $\mathcal{G}_{p\ell}$ consists of several pieces. We start with the piece of R from v_0 to $v_{\beta(\ell)}$ or $v_{\beta(r)}$, whichever comes first. Let γ and δ be such that

$$\beta(\gamma) = \min(\beta(\ell),\beta(r)), \quad \beta(\delta) = \max(\beta(\ell),\beta(r)) \ .$$

Thus $\{\gamma,\delta\} = \{\ell,r\}$ and the first piece of $\overline{R}$ is the piece of R from v_0 to $v_{\beta(\gamma)}$. $v_{\beta(\gamma)}$ is an endpoint of K_γ. We now continue $\overline{R}$ along K_γ to its other endpoint $c_{\alpha(\gamma)}$. Next we move along $\tilde{X}$ to

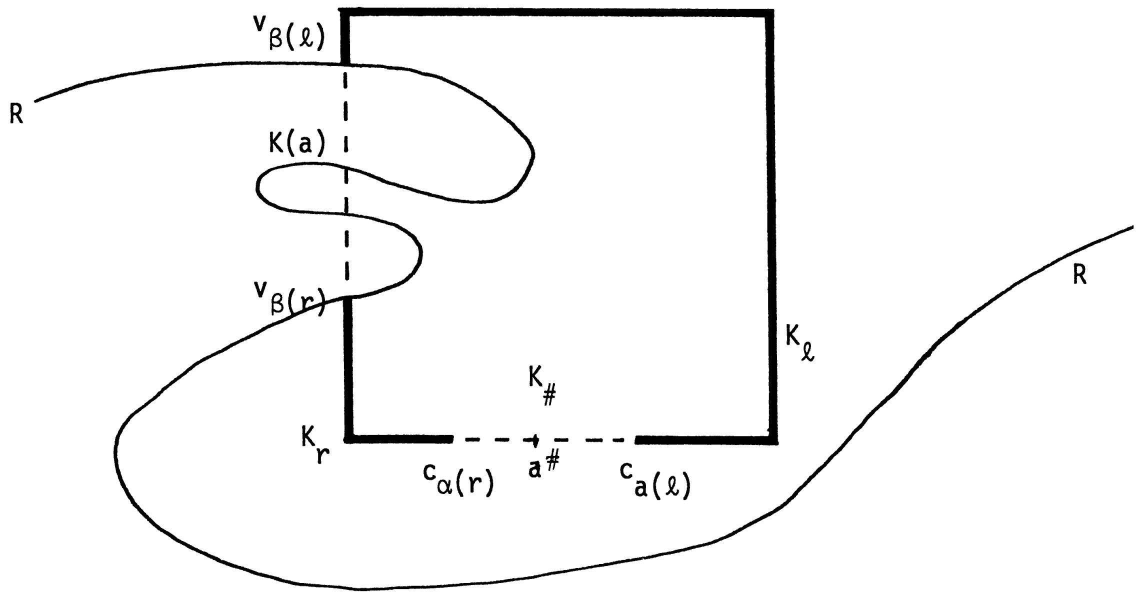

Figure 10.16 Schematic diagram. The dashed and boldly drawn curves together make up the circuit K(a) . The boldly drawn pieces of K(a) are the arcs K_ℓ and K_r , while $K_\#$ is the arc between K_ℓ and K_r which contains $a^\#$.

$c_{\alpha(\delta)}$ (recall that $\tilde{X}$ is a path with endpoints $c_{\alpha(\ell)}$ and $c_{\underline{\alpha}(r)}$). From $c_{\alpha(\delta)}$ we move along K_δ to $v_{\beta(\delta)}$. The last piece of $\overline{R}$ is the piece of R from $v_{\beta(\delta)}$ to v_ν. The curve traversed in this way from v_0 to v_ν is $\overline{R}$. It is made up of paths on $\mathcal{G}_{p\ell}$, and as we shall now show,

$$\overline{R} \text{ has no double points.}$$

Indeed, since R itself has no double points, and the same holds for the arcs K_ℓ and K_r of K(a) and for the path $\tilde{X}$, the only way $\overline{R}$ can have a double point is when $\tilde{X}$ intersects $R \cup K_\ell \cup K_r$ in a point distinct from its endpoints $c_{\alpha(\ell)}$ and $c_{\alpha(r)}$, or if K_i intersects R in a point other than $v_{\alpha(i)}$, $i = \ell$ or r. All these possibilities are ruled out though by (10.100), (10.98) and (10.107). Thus $\overline{R}$ is indeed a self-avoiding path on $\mathcal{G}_{p\ell}$ from v_0 to v_ν . We stress that $\overline{R}$ contains the vertex $\tilde{x}$ of $\tilde{X}$ (see (10.94)).

We want to show that $\overline{R}$ is a crosscut of J_ℓ, i.e., (10.108) $$\bar{R}\setminus\{v_0,v_\nu\} \subset \mathrm{int}(J_\ell),\ v_0 \in B_1, v_\nu \in B_2 .$$

In addition we want to know that $\bar{R}$ lies above R, i.e.,

(10.109) $$\bar{R} \subset \bar{J}_\ell^+(R) \quad \text{and} \quad J_\ell^+(\bar{R}) \subset J_\ell^+(R).$$

We begin with the first inclusion in (10.109). It is clear that the two pieces of R from v_0 to $v_{\beta(\gamma)}$ and from $v_{\beta(\delta)}$ to v_ν belong to $\bar{J}_\ell^+(R)$. Thus, for the first inclusion in (10.109) we only have to show that the connected curve consisting of K_ℓ, $\tilde{X}$ and K_r lies in $\bar{J}_\ell^+(R)$. As we just saw, (10.100), (10.98) and (10.107) imply that this curve only has its endpoints, $v_{\beta(\ell)}$ and $v_{\beta(r)}$, on R. It therefore suffices to show that $K_\ell \cup \tilde{X} \cup K_r\ \{v_{\beta(\ell)}, v_{\beta(r)}\}$ does not intersect $\mathrm{Fr}(J^+(R))$, but contains some point of $J_\ell^+(R)$. As a first step we show

$$a^\# \in J_\ell^+(R).$$

To see this note that $a^\# \in R^\# \setminus R \subset \bar{J}_\ell^+(R)\setminus R$, by virtue of (10.43), (10.78). But neither does $a^\#$ belong to the pieces B_1, B_2 or C of J_ℓ because $B_1(B_2)$ lies to the left (right) of the vertical line $\{\Lambda_3\} \times \mathbb{R}$ $(\{2M_{\ell 1}-\Lambda_3\} \times \mathbb{R})$ and C lies above the horizontal line $\mathbb{R} \times \{12M_{2\ell}\}$ (see Step (i)), while $a^\# \in K(a)$ with

$$\frac{1}{2} M_{\ell 1}-\Theta-2\Lambda \le a(1) \le \frac{3}{2} M_{\ell 1}+\Theta+2\Lambda$$

(see (10.79)), and $a \in R$, whence

$$a(2) \le 6M_{\ell 2}$$

(see (10.32) and beginning of this Step). Thus, for sufficiently large ℓ

(10.110) $$\text{distance } (a^\#, B_1 \cup B_2 \cup C) \ge \min\left(\tfrac{1}{2}M_{\ell 1}-\Theta-2\Lambda-\Lambda_3), 6M_{\ell 2}\right\} - \text{diameter } \bar{K}(a) > 2 \text{ diameter } \bar{K}(a) + \Lambda.$$

We have now shown that $a^\#$ does not belong to $\mathrm{Fr}(J_\ell^+(R)) \subset R \cup B_1 \cup B_2 \cup C$, so that indeed $a^\# \subset J_\ell^+(R)$.

Next, (10.110) shows that $\bar{K}(a) = K(a) \cup \mathrm{int}(K(a))$ does not intersect $B_1 \cup B_2 \cup C$. This and (10.105) imply that $K_\#$ does not intersect $Fr(J_\ell^+(R))$, and since $a^\# \varepsilon K_\#$ we see that

$$c_{\alpha(i)} \varepsilon K_\# \subset J_\ell^+(R),\ i = \ell, r.$$

By virtue of (10.107) we then obtain also

$$K_i \setminus \{v_{\beta(i)}\} \subset J_\ell^+(R),\ i = \ell, r.$$

Finally, we already saw in (10.98) and (10.100) that

$$\tilde{X} \subset \overline{K(a)} \setminus R,$$

which is disjoint from $Fr(J_\ell^+(R))$. Thus also

$$\tilde{X} \subset J_\ell^+(R).$$

This proves the first inclusion of (10.109). In the course of its proof we also saw that $K_\ell \cup \tilde{X} \cup K_r \subset \overline{K}(a)$ does not intersect $B_1 \cup B_2 \cup C$. But neither can $K_\ell \cup \tilde{X} \cup K_r$ intersect the arc A of J_ℓ since $A \subset \overline{J}_\ell^-(R) \cap J_\ell$, while

$$K_\ell \cup \tilde{X} \cup K_r \setminus \{v_{\beta(\ell)}, v_{\beta(r)}\} \subset J_\ell^+(R).$$

Also

$$v_{\beta(i)} \varepsilon R \setminus (B_1 \cup B_2) \subset int(J_\ell) \quad .$$

Thus $K_\ell \cup \tilde{X} \cup K_r \subset int(J_\ell)$. Since R is a crosscut of J_ℓ, (7.39) - (7.41) show that $R \setminus \{v_0, v_\nu\} \subset int(J_\ell)$, $v_0 \varepsilon B_1$, $v_\nu \varepsilon B_2$. (10.108) is now obvious. Finally, the second inclusion in (10.109) follows from the first one, in the same way as (A.40) follows from (A.38) in the Appendix.

As a final step before defining the modified configuration $\tilde{\omega}$ we construct a connection on $\mathcal{G}_{p\ell}^*$ to $\mathring{C}$ above $\overline{R}$. This connection, call it Y^*, will consist of X^* - which runs from v_0^* to w_0^* - followed by s^* - which runs from w_0^* to $w_\tau^* \varepsilon \mathring{C}$ (see beginning of this step). Actually X^* followed by s^* could still have double points; Y^* is the path obtained by loop-removal from the composition of X^* and s^*. To show that Y^* is a connection from $\tilde{x}$ to $\mathring{C}$ above $\overline{R}$ note first that Y^* ends at $w_\tau^* \varepsilon \mathring{C}$ and that $s^* \setminus \{w_\tau^*\} \subset J_\ell^+(R^\#)$, because by assumption s^* is a vacant connection of $a^\#$ to $\mathring{C}$ above $R^\#$. Thus, by (10.43) $s^* \setminus \{w_\tau^*\} \subset J_\ell^+(R)$ and a fortiori $s^* \setminus \{w_\tau^*\}$ does not intersect R. But neither does s^* intersect $K_\ell \cup \tilde{X} \cup K_r \subset \overline{K(a)}$ by virtue of (10.80). Thus, s^* does not intersect the crosscut $\overline{R}$ of J_ℓ and ends on $\mathring{C}$. Since some neighborhood of $\mathring{C}$ intersected with $int(J_\ell)$ belongs to $J^+(\overline{R})$ and $s^* \setminus \{w_\tau^*\} \subset int(J_\ell^-)$ we conclude

(10.111) $$s^* \setminus \{w^*_\tau\} \subset J^+_\ell(\bar{R}) .$$

Neither can X^* intersect $\bar{R}$. To see this, observe that we already know that X^* is disjoint from $\tilde{X}$ (see (10.99)) and that $X^* \cap K(a) \subset K_\#$ (see (10.106)). Also X^* does not contain the points $c_{\alpha(\ell)}$ and $c_{\alpha(r)}$ of $K(a)$ since by construction any point of X^* lies on V^* or is within distance Λ_6 from $\mathcal{H}^*$, while $c_{\alpha(i)}$ has a distance at least Λ_8-2Λ from $\mathcal{H}^*$ by (10.90), and $V^* \subset \mathrm{int}(K(a))$ by (10.83). This means that

$$X^* \cap (K_\ell \cup K_r) = X^* \cap K(a) \cap (K_\ell \cup K_r) \setminus \{c_{\alpha(\ell)}, c_{\alpha(r)}\}$$

$$\subset K_\# \cap (K_\ell \cup K_r) \setminus \{c_{\alpha(\ell)}, c_{\alpha(r)}\} = \emptyset \quad \text{(see (10.107))}.$$

Lastly, to show that X^* is disjoint from R we can copy the proof of (10.100) verbatim. X^* too lies within distance $\Lambda_6+4\Lambda+2\Lambda_9$ from $a^\#$. On the one hand, this together with (10.110) shows that X^* does not intersect $B_1 \cup B_2 \cup C$ for large ℓ. On the other hand, together with (10.78) this gives

(10.112) $$\mathrm{distance}(X^*,R) \geq \Theta-(\Lambda_6+4\Lambda+2\Lambda_9) > 2\Lambda .$$

Thus X^* is disjoint from $\bar{R} \cup B_1 \cup B_2 \cup C$ and a fortiori from $\mathrm{Fr}(J^+_\ell(\bar{R}))$. Since we already saw that the endpoint $w^*_0 \in s^*$ of X^* belongs to $J^+_\ell(\bar{R})$, it follows that all of X^* lies in $J^+_\ell(\bar{R})$. Combined with (10.111) this gives the desired conclusion

(10.113) $$Y^* \setminus \{w^*_\tau\} \subset J^+_\ell(\bar{R}).$$

We note also that the initial point of Y^* = initial point of $X^* = v^*_0$ which is adjacent on $\mathcal{M}_{p\ell}$ to $\tilde{x}$ by Condition Dd) (with x replaced by $\tilde{x}$). Thus Y^* is indeed a connection on $\mathcal{G}^*_{p\ell}$ from $\tilde{x}$ to $\mathring{C}$ above $\bar{R}$. Note that we do not claim Y^* to be vacant, though.

Figure 10.17 illustrates the end result of our construction of $\bar{R}$ and Y^*. In Fig. 10.17 we have more or less drawn the various pieces in the same relative location as in Fig. 10.13-10.16.

Step (vii). We are finally ready to describe the modification $\tilde{\omega}$ of the occupancy configuration ω. We remind the reader that ω satisfies (10.52). We form $\tilde{\omega}$ by means of the following steps:

(a) Make all sites on $\bar{R}$ which are vacant in ω occupied in $\tilde{\omega}$.

(b) Make vacant all sites of $\mathcal{G}_{p\ell}$ which lie in $\bar{J}^-_\ell(\bar{R}) \setminus \bar{R}$ and which

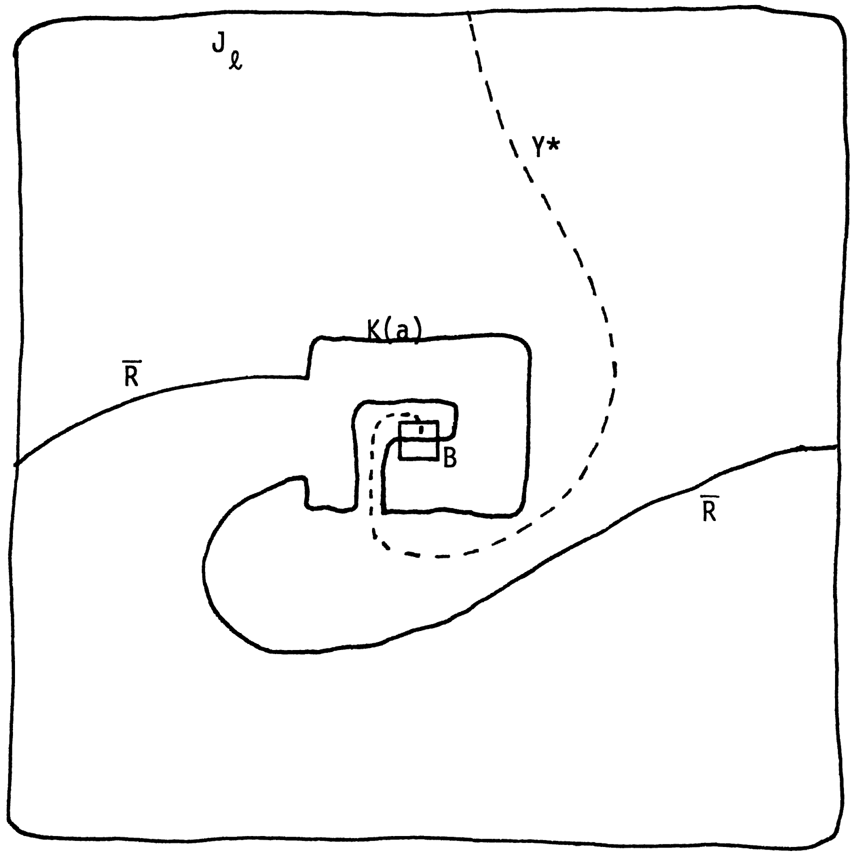

Figure 10.17 The outer circuit is J_ℓ . The solidly drawn crosscut is $\overline{R}$. The dashed path is Y^* . The small square near the center is $B = B(\tilde{x})$, which has $\tilde{x}$ as its center .

can be connected to a vertex on $K_\ell \cup \tilde{X} \cup K_r$ via one or two edges of $\mathcal{G}_{p\ell}$. Excluded from this change are central vertices of $\mathcal{G}_{p\ell}$ which do not belong to $\mathbb{W}$.

(c) Make all vertices on Y^* vacant.

(d) Make occupied all non-central vertices of $\mathcal{G}^*_{p\ell}$ which lie in $J^+_\ell(\overline{R}) \setminus Y^*$ and which are connected to a point of X^* via one or two edges of $\mathcal{G}^*_{p\ell}$.

No other changes than the ones listed in (a)-(d) are made in the configuration ω to obtain $\tilde{\omega}$.

Before we can start on the verification of Condition E we must show that the steps (a)-(d) are compatible, i.e., that they do not require a certain vertex to be made occupied as well as vacant. This is easy, however. Indeed (a) only involves vertices on $\overline{R}$, (b) only vertices in $\overline{J^-_\ell}(\overline{R}) \setminus \overline{R}$ and (c) and (d) only vertices in $J^+_\ell(\overline{R}) \cup \mathring{C}$ (by virtue of (10.113)). Thus, Steps (a), (b) and the pair (c) and (d) deal with disjoint sets of vertices. It is also clear that Steps (c) and (d) deal with disjoint sets of vertices. Therefore no conflict exists between any of the required modifications.

We denote by $\tilde{\omega}$ the occupancy configuration which results from ω by Steps (a)-(d). We check in this step that (10.53)-(10.55) hold for $\tilde{\omega}$. (10.54) is immediate from steps (a)-(d) and the fact that Y^* is a path on $\mathcal{G}^*_{p\ell}$, hence does not contain any central vertices of $\mathcal{G}_{p\ell}$. Thus, in none of the steps is an occupied central vertex of $\mathcal{G}_{p\ell}$ outside $\mathfrak{W}$ made vacant. Also (10.55) is immediate if we take into account that $\overline{R}$ is a path on $\mathcal{G}_{p\ell}$ and hence does not contain central vertices of $\mathcal{G}^*_{p\ell}$. Lastly, (10.53) follows from the fact that R is already occupied and s^* already vacant in the configuration ω (see beginning of Step (vi), where R and s^* are introduced). Therefore (a) requires only changes of vertices on $K_\ell \cup \tilde{X} \cup K_r \subset \overline{K}(a)$. Also (c) requires only changes of vertices on X^*, which by construction lies within distance Λ_6 from $\overline{K(a)} \cup \mathcal{K}^*$; in turn $\mathcal{K}^*$ lies within distance Λ_9 from $\overline{K(a)}$ by (10.91) and (10.79). The changes in (b) and (d) lie within 2Λ from the set $K_\ell \cup \tilde{X} \cup K_r$ or X^*. Consequently $\tilde{\omega}(v) \neq \omega(v)$ is only possible for a v within $\Lambda_6+\Lambda_9+2\Lambda$ from $\overline{K(a)}$, which contains $a^\#$, and which has diameter $\leq 8\Theta+4\Lambda_3+4\Lambda$ (see (10.38)). This proves (10.53).

<ins>Step (viii)</ins>. In this step we verify (10.56). The essential part is to show that in the configuration $\tilde{\omega}$ there exists a lowest occupied crosscut $\tilde{R}$ of $\text{int}(J_\ell)$ on $\mathcal{G}_{p\ell}$, which almost equals the path $\overline{R}$, and in particular contains $\tilde{x}$. The existence of a lowest occupied crosscut $\tilde{R}$ of $\text{int}(J_\ell)$ - i.e., an occupied path $\tilde{R}$ on $\mathcal{G}_{p\ell}$ which satisfies (7.39)-(7.41) and such that $J^-_\ell(\tilde{R})$ is minimal among all such paths - follows from Prop. 2.3, because $\overline{R}$ is an occupied crosscut on $\mathcal{G}_{p\ell}$ of J_ℓ in $\tilde{\omega}$ (by (10.108) and Step (viia)). Let $\tilde{R} = (y_0,h_1,\dots,h_\lambda,y_\lambda)$. We remind the reader that $R = (v_0,e_1,\dots,e_\nu,v_\nu)$ and that the pieces $(v_0,e_1,\dots,e_{\beta(\gamma)},v_{\beta(\gamma)})$ and $(v_{\beta(\delta)},e_{\beta(\delta)+1},\dots,e_\nu,v_\nu)$ of R are also the first and last piece of $\overline{R}$; between these pieces $\overline{R}$ consists of the composition of K_ℓ, $\tilde{X}$, and K_r (or this path in reverse). We shall now prove the following statements:

$$(10.114)\qquad (y_0,h_1,\dots,h_{\beta(\gamma)},y_{\beta(\gamma)}) = (v_0,e_1,\dots,e_{\beta(\gamma)},v_{\beta(\gamma)}),$$

$$(10.115)\qquad (y_{\lambda-\nu+\beta(\delta)},h_{\lambda-\nu+\beta(\delta)+1},\dots,h_\lambda,y_\lambda) = (v_{\beta(\delta)},e_{\beta(\delta)+1},\dots,e_\nu,v_\nu),$$

(10.116) Any y_i with $\beta(\gamma) < i < \lambda-\nu+\beta(\delta)$ lies within distance Λ of a vertex on $K_\ell \cup \tilde{X} \cup K_r$,

(10.117) $\tilde{x}$ is one of the y_i .

Of course (10.114)-(10.116) say that $\tilde{R}$ shares its beginning and last piece with R and $\overline{R}$ and in between deviates only little from $\overline{R}$.

To prove (10.114)-(10.117) we first must assemble some facts about the non-existence of certain shortcuts for $\tilde{R}$. It is convenient to use the following notation. $R(\gamma) = (v_0, e_1, \dots, e_{\beta(\gamma)}, v_{\beta(\gamma)})$, the beginning piece of R and $\overline{R}$; $R(\delta) = (v_{\beta(\delta)}, e_{\beta(\delta)+1}, \dots, e_\nu, v_\nu)$, the last piece of R and $\overline{R}$. Let z be an arbitrary point of $R(\gamma)\setminus v_{\beta(\gamma)}$. Since R and $\overline{R}$ are crosscuts of $\mathrm{int}(J_\ell)$ which have the piece $R(\gamma)$ in common, there exist arbitrarily small neighborhoods N of z such that

$$N \cap \mathrm{int}(J_\ell)\setminus R = N \cap \mathrm{int}(J_\ell)\setminus R(\gamma) = N \cap \mathrm{int}(J_\ell)\setminus \overline{R}$$

and such that $N \cap \mathrm{int}(J_\ell)\setminus R$ consists of two components, N^+ and N^- say, with

(10.118) $$N^+ \subset J_\ell^+(R), \quad N^- \subset J_\ell^-(R).$$

We claim that for any such N also

(10.119) $$N^+ \subset J_\ell^+(\overline{R}), \quad N^- \subset J_\ell^-(\overline{R}).$$

This is easy to see from (10.109). Indeed (10.109) implies

$$J_\ell^-(R) = \mathrm{int}(J_\ell)\setminus \overline{J_\ell^+}(R) \subset \mathrm{int}(J_\ell)\setminus \overline{J_\ell^+}(\overline{R}) = J_\ell^-(\overline{R})$$

and hence

$$N^- \subset J_\ell^-(\overline{R}).$$

But $N \cap \mathrm{int}(J_\ell)\setminus \overline{R}$ consists of the two connected sets N^- and N^+, and $N \cap \mathrm{int}(J_\ell)\setminus\overline{R}$ must intersect $J_\ell^+(\overline{R})$ as well as $J_\ell^-(\overline{R})$ (since N is a neighborhood of a point z on the crosscut $\overline{R}$ of $\mathrm{int}(J_\ell)$; see Newman (1951), Theorem V.11.7). Thus both inclusions in (10.119) must hold.

We use (10.119) to prove that if ω satisfies (10.52) then

(10.120) there does not exist a shortcut of one or two edges of $\overline{R}$ inside $\overline{J_\ell^-}(\overline{R})$, which has one endpoint among $v_0, v_1, \dots, v_{\beta(\gamma)-1}, v_{\beta(\delta)+1}, \dots, v_\nu$.

(See Def. 10.2 and Step (iv), for the definition of shortcuts.) Suppose first that the edge e of $\mathcal{G}_{p\ell}$ is a shortcut of $\overline{R}$ of one edge which runs from some v_i, $0 \le i \le \beta(\gamma)-1$ to a vertex u of $\overline{R}$, and is such that $e \subset \overline{J}_\ell^-(\overline{R})$. Then by Def. 10.2 e is not an edge of $\overline{R}$ itself, since u is a vertex of $\overline{R}$ which is not the immediate predecessor or successor of v_i on $\overline{R}$. But then $\mathring{e}$ is disjoint from $\overline{R}$. $\mathring{e}$ also cannot belong to J_ℓ because then both v_i and u must belong to $J_\ell \cap \overline{R} = \{v_0, v_\nu\}$ and the vertices v_0 and v_ν on B_1 and B_2 respectively (see (7.40), (7.41)) are too far apart to be connected by the single edge e. Thus, by the planarity of $\mathcal{G}_{p\ell}$, $\mathring{e}$ is also disjoint from J_ℓ. Since $e \subset \overline{J}_\ell^-(\overline{R})$ this implies $\mathring{e} \subset J_\ell^-(\overline{R})$. Therefore, if N is a neighborhood of v_i for which (10.118) and (10.119) hold, then $\mathring{e} \cap N \subset N^- \subset J_\ell^-(R)$. Consequently $\mathring{e} \subset J_\ell^-(R)$ entirely. On the other hand u is a vertex on $\overline{R} \subset \overline{J}_\ell^+(R)$ (by (10.109)) so that $u \in \overline{J}_\ell^-(R) \cap \overline{J}_\ell^+(R) = R$. This means that e connects v_i with u, two vertices of R, while $\mathring{e}$ lies strictly below R, i.e., in $J_\ell^-(R)$. Replacing the arc of R between v_i and u by e then gives an occupied crosscut of J_ℓ which lies in $\overline{J}_\ell^-(R)$ and which is not equal to R. This contradicts the choice of R as the occupied (in the configuration ω) crosscut of $\text{int}(J_\ell)$ with minimal $J_\ell^-(R)$; see Prop. 2.3. Thus, no shortcut of one edge for $\overline{R}$ exists which lies inside $\overline{J}_\ell^-(\overline{R})$ and has one endpoint among $v_0, \dots, v_{\beta(\gamma)-1}$.

Next suppose that e, u, f is a shortcut of two edges for $\overline{R}$ inside $\overline{J}_\ell^-(\overline{R})$ which starts at some v_i, $0 \le i \le v_{\beta(\gamma)-1}$. In this case u must be a central vertex of $\mathcal{G}_{p\ell}$ which neither belongs to $\mathfrak{w}$ nor is one of the vertices v_j, $0 \le j \le \beta(\gamma)$ of $\overline{R}$. This again excludes the possibility that e belongs to R or to J_ℓ (since J_ℓ lies on $\mathcal{M}$ and contains therefore no central vertices; see Step (i)). As above this implies $\mathring{e} \subset J_\ell^-(R)$. On the other hand the endpoint other than u of f lies on $\overline{R} \subset \overline{J}_\ell^+(R)$ (see (10.109)). Consequently $\mathring{e}, u, f$ intersects R, necessarily in a vertex, w say, of $\mathcal{G}_{p\ell}$. Thus e followed by f contains a path, t say, from v_i to w, t lies in $J_\ell^-(R)$, except for its endpoints v_i and w on R. t can contain at most one vertex not on R, to wit the vertex u. But as a central vertex of $\mathcal{G}_{p\ell}$ not in $\mathfrak{w}$,u is occupied in the configuration ω (by (10.52)). Thus we would have the occupied path t below R connecting the two vertices v_i and w on R. As above this contradicts the minimality of R. This proves the cases of (10.120) where the shortcut has one endpoint among v_i, $0 \le i \le \beta(\gamma)-1$. The same argument can be used for the v_i with

$\beta(\delta)+1 \le i \le \nu$.

We conclude from (10.120) that any shortcuts for $\overline{R}$ in $\overline{J}_\ell^-(\overline{R})$ have to have their endpoints on $K_\ell \cup \tilde{X} \cup K_r$ (this includes $v_{\beta(\gamma)}$ and $v_{\beta(\delta)}$). K_ℓ and K_r are pieces of $K(a)$, hence minimal paths (see (10.36)). $\tilde{X}$ was even taken strongly minimal in Step (vi). There can still be shortcuts for $\overline{R}$ between these three pieces. Some of these will be harmless but we have to rule out shortcuts between points on "opposite sides of $\tilde{x}$". Shortcuts from $\tilde{X}_\ell(\tilde{x})\setminus\{\tilde{x}\}$ to $\tilde{X}_r(\tilde{x})\setminus\{\tilde{x}\}$ are already ruled out by (10.96) (see (10.95) for the definition of $\tilde{X}_\ell$ and $\tilde{X}_r$). We now prove

(10.121) there do not exist shortcuts of one or two edges of $\overline{R}$ inside $\overline{J}_\ell$ with one endpoint on $\tilde{X}_\ell(\tilde{x})$ and the other on K_r or with one endpoint on $\tilde{X}_r(\tilde{x})$ and the other on K_ℓ.

Again we give an indirect proof of (10.121). Assume that e or (e,u,f) is a shortcut of $\overline{R}$ connecting a vertex z_1 on K_ℓ with a vertex z_2 on $\tilde{X}_r(\tilde{x})$. Then by (10.97)

$$|z_2-c_{\alpha(\ell)}| \le 2\Lambda \quad \text{or} \quad |z_2-c_{\alpha(r)}| \le 2\Lambda \quad .$$

Since by construction $c_{\alpha(\ell)}$ lies on $K(a)$ and within 2Λ from $b_\ell \in \mathcal{K}_\ell$ it has distance $> 2\Lambda$ from $\tilde{X}_r(\tilde{x})$, by (10.90) and (10.83). ($\tilde{X}_r$ contains only points within 2Λ from U or within $\Lambda_6+4\Lambda$ from $r_r \subset \mathcal{K}_r$, as in (10.77).) Thus

$$|z_2-c_{\alpha(r)}| \le 2\Lambda \quad \text{and} \quad |z_1-c_{\alpha(r)}| \le |z_1-z_2| + |z_2-c_{\alpha(r)}| \le 4\Lambda \quad .$$

By the estimate (10.39), there must then exist an arc K_1 of $K(a)$ from z_1 to $c_{\alpha(r)}$ with

$$\text{diameter}(K_1) \le 3\Lambda_5(7\Lambda+2\Lambda_3+1). \tag{10.122}$$

Now, since $z_1 \in K_\ell$ one arc between z_1 and $c_{\alpha(r)}$ contains $c_{\alpha(\ell)}$ and the arc $K_\#$ from $c_{\alpha(\ell)}$ to $c_{\alpha(r)}$ (see Fig. 10.18). Since the diameter of $K_\#$ is at least (see (10.90))

$$|c_{\alpha(\ell)}-c_{\alpha(r)}| \ge |b_\ell-b_r|-4\Lambda \ge \text{distance}(\mathcal{K}_\ell,\mathcal{K}_r)-4\Lambda \ge \Lambda_8-4\Lambda,$$

which exceeds the right hand side of (10.122). Thus K_1 must be the other arc of $K(a)$ from z_1 to $c_{a(r)}$. However, this second arc of $K(a)$ from z_1 to $c_{\alpha(r)}$ has to contain K_r from $v_{\beta(r)}$ to $c_{\alpha(r)}$

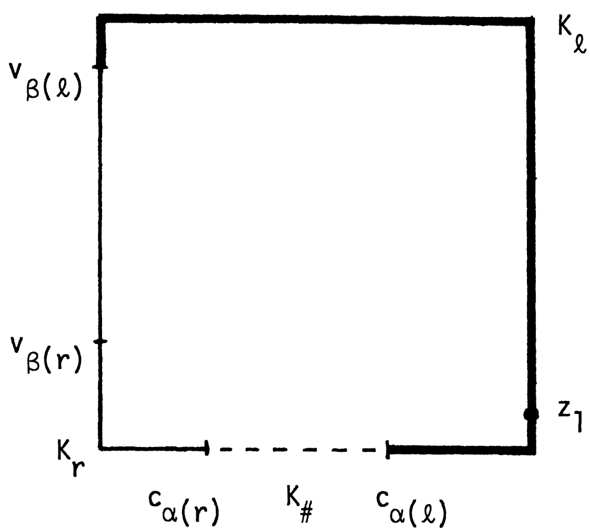

Figure 10.18 Schematic diagram of $K(a)$ indicating the relative location of various points. K_ℓ is boldly drawn; $K_\#$ is dashed.

and this has diameter at least

(10.123) $|v_{\beta(r)}-c_{\alpha(r)}| \geq |v_{\beta(r)}-a^\#| - |c_{\alpha(r)}-a^\#| \geq \Theta-\Lambda_9-2\Lambda$

since $|v_{\beta(r)}-a^\#| \geq \Theta$ (by (10.78)) and

$$|c_{\alpha(r)}-a^\#| \leq |c_{\alpha(r)}-b_r| + |b_r-a^\#| \leq 2\Lambda+\Lambda_9 ,$$

(by (10.91)). But the right hand side of (10.123) also exceeds the right hand side of (10.122), so that neither arc of $K(a)$ from z_1 to $c_{\alpha(r)}$ is possible for K_1. This contradiction proves that there is no shortcut of $\overline{R}$ from a vertex on K_ℓ to a vertex on $\tilde{X}_r(x)$. The same argument shows that there is no shortcut from K_r to $\tilde{X}_\ell(x)$ and therefore proves (10.121).

Our final claim about shortcuts is that if ω satisfies (10.52), then

(10.124) there do not exist shortcuts of one or two edges for $\overline{R}$ in $\overline{J}_\ell^-(\overline{R})$ with one endpoint on each of K_ℓ and K_r .

We prove (10.124) for shortcuts of two edges, the case of a shortcut of one edge being similar, but easier. Assume (e,u,f) is a shortcut of two edges for $\overline{R}$ in $\overline{J}_\ell^-(\overline{R})$, which starts at z_1 on K_ℓ and ends at z_2 on K_r. Let K_2 be the arc of $K(a)$ which connects z_1 to z_2 and is contained in $K_\ell \cup K_\# \cup K_r$ (see Fig. 10.19). As above

(10.125) $\quad \text{diameter}(K_2) \geq \text{diameter}(K_\#) \geq \Lambda_8 - 4\Lambda > 3\Lambda_5(5\Lambda+2\Lambda_3+1).$

On the other hand the estimate (10.39), together with the fact $|z_1-z_2| \leq 2\Lambda$, implies that there exists an arc K_3 of $K(a)$ from z_1 to z_2 with

$$\text{diameter}(K_3) \leq 3\Lambda_5(5\Lambda+2\Lambda_3+1).$$

Thus K_3 is not K_2, but K_3 must be the arc through $v_{\beta(\ell)}$ and $v_{\beta(r)}$ (see Fig. 10.19). Now consider the closed curve G consisting

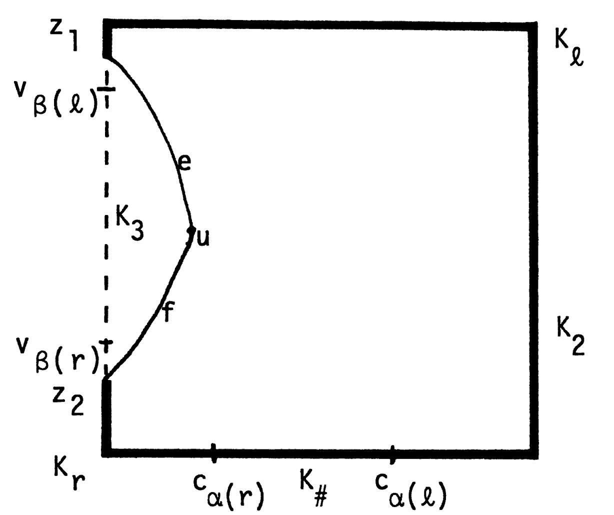

Figure 10.19 Schematic diagram of $K(a)$ indicating the relative location of various parts. K_2 is boldly drawn, K_3 is dashed.

of K_2 from z_1 to z_2 followed by f and e. G is a path with possible double points on $\mathcal{G}_{p\ell}$. We first show that

(10.126) $\quad G$ is a simple Jordan curve.

Since $K(a)$ is a simple Jordan curve, the only way G could have a double point is when $u \in K_2 \subset K_\ell \cup K_\# \cup K_r$. But $u \notin K_\ell \cup K_r \subset \overline{R}$ because if (e,u,f) is a shortcut of two edges for $\overline{R}$, then u is not a vertex of $\bar{R}$ (see Step (iv)). But also $u \in K_{\#}$ is impossible for then e and f are edges with both endpoints on $K(a)$, and therefore[1)] e followed by f would be an arc of $K(a)$ from z_1 to z_2 containing a point of $K_{\#}$. This could only happen if e followed by f constitutes the arc K_2 - which would go through the point u of $K_{\#}$ - and consequently diameter $(K_2) \leq 2\Lambda$. Since this is excluded by (10.125), it follows that (10.126) holds.

Next we observe that $a \in \text{int}(G)$. This must be so because by definition of $K(a)$ $a \in \text{int}(K(a))$ at a distance at least $2\Theta-\Lambda-1$ from any point of $K(a)$ (see (10.37)). On the other hand G is formed from $K(a)$ by replacing the arc K_3 between z_1 and z_2 by $e \cup f$. Since

$$\text{diameter}(K_3)+\text{diameter}(e)+\text{diameter}(f)$$

$$\leq 3\Lambda_5(5\Lambda+2\Lambda_3+1)+2\Lambda < \Theta < 2\Theta-\Lambda-1 \quad ,$$

this replacement cannot take a from $\text{int}(K(a))$ to $\text{ext}(G)$.

We now have the point a of R (see (10.79)) in $\text{int}(G)$, while $v_0 \in B_1$ is outside G, because B_1 is to the left of the vertical line $x(1) = \Lambda_3$ (see Step (i)) and

$$a(1) \geq \frac{1}{2} M_{\ell 1}-\Theta-2\Lambda > \text{diameter}(G)+\Lambda_3$$

for large ℓ (see (10.79)). Similarly, v_ν is outside G. Therefore R must intersect G in at least two distinct vertices of $\mathcal{G}_{p\ell}$, such that R intersects $\text{int}(G)$ in arbitrarily small neighborhoods of each of these vertices. By (10.105) R does not intersect $K_{\#}$, while by choice of K_ℓ and K_r R intersects $K_\ell \cup K_r$ only in $v_{\beta(\ell)}$ and $v_{\beta(r)}$ (see (10.107)). If $v_{\beta(\ell)}$ ($v_{\beta(r)}$) belongs to G at all, then it must equal z_1 (z_2), and in particular, belong to e (f) (see Fig. 10.19 and recall that $u \notin K_2$). It follows from this and from $G \subset K_\ell \cup K_{\#} \cup K_r \cup e \cup f$ that $R \cap G$ is contained in $e \cup f$. Starting at v_0 R must therefore enter $\text{int}(G)$ through a vertex on $e \cup f$ and exit again through another vertex of $e \cup f$ to reach v_ν. Since $e \cup f$ contains only the three vertices z_1, u and z_2, R must intersect $\text{int}(G)$ as well as $\text{ext}(G)$ in arbitrarily small neighborhoods of one of the z_i.

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1) A little care is needed here because we allowed multiple edges between a pair of vertices. However, in the present case u is a central vertex of $\mathcal{G}_{p\ell}$, and the construction of $\mathcal{G}_{p\ell}$ in Sect. 2.3 is such that there exists exactly one edge in $\mathcal{G}_{p\ell}$ between a central vertex and any of its neighbors.

For the sake of argument let this happen at z_1. Then z_1 belongs to $K_\ell \cap R = \{v_{\beta(\ell)}\}$, i.e., $z_1 = v_{\beta(\ell)}$ and one of the edges $e_{\beta(\ell)}$ or $e_{\beta(\ell)+1}$ of R has its interior in $\mathrm{int}(G)$ and the other in $\mathrm{ext}(G)$. This means that R intersects G transversally at $z_1 = v_{\beta(\ell)}$. But the arc K_ℓ from $v_{\beta(\ell)}$ to $c_{\alpha(\ell)}$ belongs to G, since $z_1 = v_{\beta(\ell)}$ and $z_2 \in K_r$ (see Fig. 10.19). As we saw in the proof of (10.109) this arc belongs to $J_\ell^+(R)$, or more precisely

$$K_\ell \setminus \{v_{\beta(\ell)}\} \subset J_\ell^+(R).$$

This, together with the transversality of R and G at $v_{\beta(\ell)}$ forces

$$\mathring{e} \subset J_\ell^-(R).$$

We are now in the same situation as in the proof (10.120). $\mathring{e}$ would have to be part of a path below R of one or two edges, and occupied in the configuration ω. No such path exists by the minimality of $J_\ell^-(R)$ and Prop. 2.3. (10.124) follows from this contradiction.

With (10.96), (10.120), (10.121) and (10.124) in hand, it is now relatively simple to prove (10.114)-(10.117). Assume first (10.114) fails and let ξ be the smallest index with $h_\xi \neq e_\xi$. Then $1 \leq \xi \leq \beta(\gamma)$. Since $\tilde{R}$ satisfies (7.39)-(7.41) $\mathring{h}_\xi \subset \mathrm{int}(J_\ell)$. Now consider first the case $\xi \geq 2$. Then by the minimality of ξ, $y_{\xi-1} = v_{\xi-1} \in R$ as well as $y_{\xi-1} = v_{\xi-1} \in \overline{R}$. Since $h_\xi \neq e_\xi$ and $\mathcal{G}_{p\ell}$ is planar, it follows that $\mathring{h}$ does not intersect $\overline{R}$, and for all sufficiently small neighborhoods N of $v_{\xi-1}$

$$\mathring{h}_\xi \cap N \subset (\tilde{R} \cap N) \setminus (J_\ell \cup \overline{R}). \tag{10.127}$$

On the other hand $\tilde{R}$ is the occupied crosscut of $\mathrm{int}(J_\ell)$ in the configuration $\tilde{\omega}$ with $J_\ell^-(\tilde{R})$ minimal. In particular

$$\tilde{R} \subset \overline{J}_\ell^-(\overline{R}) \ , \tag{10.128}$$

since also $\overline{R}$ is an occupied crosscut of $\mathrm{int}(J_\ell)$ in $\tilde{\omega}$, (by Step (viia); see also (2.27)). Thus, by (10.127) and (10.128),

$$\mathring{h}_\xi \cap N \subset N \cap (\overline{J}_\ell^-(\overline{R}) \setminus \mathrm{Fr}(J_\ell^-(\overline{R})) = N \cap J_\ell^-(\overline{R}).$$

In view of (10.118) and (10.119) this means also

$$\mathring{h}_\xi \cap N \subset N \cap J_\ell^-(R)$$

for suitable N. Since $h_\xi \neq e_\xi$ implies also that $\mathring{h}_\xi$ does not intersect R - again because $\mathcal{G}_{p\ell}$ is planar - it follows that

(10.129) $$\mathring{h}_\xi \subset J_\ell^-(R).$$

Exactly the same argument works if $\xi = 1$ and y_0 lies on $\overline{R}$, for then $y_0 \in \overline{R} \cap B_1 = \{v_0\}$, i.e., $y_0 = v_0$. On the other hand, if y_0 does not lie on $\overline{R}$, then automatically $\xi = 1$ and h_1 cannot reach $\overline{R}$ or R before y_1, i.e., $\mathring{h}_1 = \mathring{h}_\xi$ is again disjoint from $Fr(J_\ell^-(\overline{R}))$ and from $Fr(J_\ell^-(R))$ (use the analogue of (7.40) for $\tilde{R}$). Since also

$$h_1 \subset \tilde{R} \subset \overline{J}^-(\overline{R}),$$

the initial point y_0 of h belongs to the part of B_1 in $Fr(J_\ell^-(\overline{R}))\setminus \overline{R}$, which is the segment between u_1 and v_0 (apply (7.40) to $\overline{R}$ and see Fig. 10.9). But this segment of B_1 from u_1 to v_0 also equals $Fr(J_\ell^-(R))\setminus R$ so that $\mathring{h}_1$ belongs to $J_\ell^-(R)$ near y_0. (10.129) therefore holds in the case $\xi = 1$ as well.

To derive a contradication from (10.129) we consider the set

$$\Xi := \{\text{vertices of } \mathcal{G}_{p\ell} \text{ which are vacant in } \omega \text{ but occupied in } \tilde{\omega}\}.$$

If $\tilde{R}$ has no vertex in Ξ, then $\tilde{R}$ is also an occupied crosscut of $\text{int}(J_\ell)$ in the configuration ω. In this case (2.27) shows that $\tilde{R} \subset \overline{J}_\ell^+(R)$, and therefore $\tilde{R}$ cannot contain any part of any edge - such as $\mathring{h}_\xi$ - strictly below R. Thus, if (10.114) fails, and hence (10.129) holds, then $\tilde{R}$ must contain a vertex in Ξ. Let π be the smallest index with $y_\pi \in \Xi$. Now observe that by Steps (viia)-(viid) and (10.109)

$$\Xi \subset \overline{J}_\ell^+(\overline{R}) \subset \overline{J}_\ell^+(R).$$

Also, all vertices on R are occupied in ω, hence are outside Ξ, so that

(10.130) $$\Xi \subset \overline{J}_\ell^+(R)\setminus R, \text{ whence } \quad \Xi \cap \overline{J}_\ell^-(R) = \emptyset .$$

By definition of ξ and (10.129), $y_0,\dots,y_{\xi-1} \in \overline{J}_\ell^-(R)$ (even when $\xi = 1$) and hence by (10.130) $\pi \geq \xi$. We claim that

(10.131) $$y_i \notin R \quad \text{for} \quad \xi \leq i \leq \pi .$$

Indeed, if (10.131) would fail and j would be the smallest index $\geq \xi$ with $y_j \in R$, then $j \leq \pi$ and the path $(y_{\xi-1}, h_\xi, \dots, h_j, y_j)$ minus its endpoints $y_{\xi-1}, y_j$ would lie in $J_\ell^-(R)$ (by (10.129)) and have all its vertices $y_\xi,\dots,y_{j-1}$ occupied in ω since $j \le \pi$. This would again contradict the minimality of $J_\ell^-(R)$ in the configuration ω. Thus (10.131) holds. On the other hand, by (10.129), the path $(y_{\xi-1},h_\xi,\dots,h_\pi,y_\pi)$ starts with $\overset{\circ}{h}_\xi$ in $J_\ell^-(R)$, and by (10.130) cannot reach Ξ without intersecting R. This contradiction proves the impossibility of (10.129). Thus (10.114) must hold.

(10.115) must hold for the same reasons as (10.114). We merely have to interchange the roles of B_1 and v_0 with the roles of B_2 and v_ν.

Next we prove (10.116) and (10.117). Let $\bar{R} = (\bar{y}_0,\bar{h}_1,\dots,\bar{h}_\kappa,\bar{y}_\kappa))$. We already know from (10.114) and the definition of $\bar{R}$ that

$$y_i = \bar{y}_i = v_i,\ 0 \le i \le \beta(\gamma), \text{ and } h_j = \bar{h}_j = e_j,\ 1 \le j \le \beta(\gamma).$$

Now assume for a certain i

(10.132) $y_i = \bar{y}_j$ for some j with $y_i = \bar{y}_j \in K_\ell \cup \tilde{X} \cup K_r$.

By (10.114) this holds for $i = j = \beta(\gamma)$. If h_{i+1} is an edge of $\bar{R}$, then we can simply move along h_{i+1} to y_{i+1} and then (10.132) also holds with y_i replaced by y_{i+1} (unless $y_{i+1} \notin K_\ell \cup \tilde{X} \cup K_r$). The case of interest is the one where h_{i+1} is not an edge of $\bar{R}$. First consider the case where y_{i+1} again belongs to $\bar{R}$. Then the edge h_{i+1} forms a shortcut of one edge for $\bar{R}$. It lies necessarily below $\bar{R}$, i.e., in $\bar{J}_\ell^-(\bar{R})$ because of (10.128). By (10.120), (10.121), (10.124) the endpoints of h_{i+1}, $y_i = \bar{y}_j$ and $y_{i+1} = \bar{y}_k$ say, must in this case both belong to $K_\ell \cup \tilde{X}_\ell(\tilde{x})$, both to $K_r \cup \tilde{X}_r(\tilde{x})$, or both to $\tilde{X}$. The last case cannot occur because $\tilde{X}$ is a minimal path, by construction. In the other two cases $\tilde{x}$ does not occur between $\bar{y}_j$ and $\bar{y}_k$ on $\bar{R}$ since $\tilde{X}_i(\tilde{x})$ is the piece of $\tilde{X}$ between $\tilde{X} \cap K_i$ and $\tilde{x}$, $i = \ell$ or r, by the definition (10.95). Note that $\tilde{x} = \bar{y}_j$ or $\tilde{x} = \bar{y}_k$ is not excluded, though. In any case, if we replace the segment of $\bar{R}$ between $\bar{y}_j$ and $\bar{y}_k$ by h_{i+1} then $\tilde{x}$ still lies on the modified path. Moreover, $y_{i+1} = \bar{y}_k$ will again be a vertex of $\tilde{R}$ on $\bar{R}$, and as long as $y_{i+1} \in K_\ell \cup \tilde{X} \cup K_r$ we are back to (10.132) with y_i, $\bar{y}_j$ replaced by y_{i+1}, $\bar{y}_k$.

The other possibility allowed by (10.132) is that y_{i+1} does not belong to $\bar{R}$. Since $y_{i+1} \in \tilde{R} \subset \bar{J}_\ell^-(\bar{R})$ (by (10.128)) this implies $y_{i+1} \in \bar{J}_\ell^-(\bar{R}) \setminus \bar{R}$. Also, since $y_{i+1} \in \tilde{R}$ it must be occupied in the configuration $\tilde{\omega}$, and by Step (viib) this means that y_{i+1} has to be a central vertex of $\mathcal{G}_{p\ell}$ which does not belong to $\mathfrak{w}$. The neighbor y_{i+2} of y_{i+1} is then not a central vertex of $\mathcal{G}_{p\ell}$ (Comment 2.3(iv)). Again by Step (viib) it follows that y_{i+2} cannot lie in $\overline{J}_\ell^-(\overline{R})\setminus\overline{R}$. Since y_{i+2} is an endpoint of h_{i+1} - which starts at $y_{i+1} \varepsilon \overline{J}^-(\overline{R})\setminus\overline{R}$ - we have $y_{i+2} \varepsilon \overline{J}_\ell^-(\overline{R})$, and hence $y_{i+2} \varepsilon \overline{R}$, say $y_{i+2} = \overline{y}_k$. Thus, in this case either $\overline{y}_j$ and $\overline{y}_k$ are successive points of $\overline{R}$ or $(h_{i+1},y_{i+1},h_{i+2})$ is a shortcut of two edges for $\overline{R}$ in $\overline{J}_\ell^-(R)$. Again, if we replace the segment of $\overline{R}$ between $\overline{y}_j$ and $\overline{y}_k$ by $(h_{k+1},y_{i+1},h_{i+2})$, then we do not remove $\tilde{x}$. This is obvious if $\overline{y}_k$ and $\overline{y}_j$ are successive points on $\overline{R}$, while the argument is essentially as above in case $(h_{i+1},y_{i+1},h_{i+2})$ is a shortcut for $\overline{R}$. The only new case to consider this time is the one where the shortcut runs between two points of $\tilde{X}$. But then (10.96) guarantees that $\tilde{x}$ is not removed during the replacement. Once again, with $\overline{y}_k$ we are back at (10.132) with y_i, $\overline{y}_j$ replaced by y_{i+2}, $\overline{y}_k$. Starting with $y_{\beta(\gamma)}$, which satisfies (10.132) we use the above argument until we arrive at $y_{\lambda-\nu+\beta(\delta)} = \overline{y}_{\kappa+\nu+\beta(\delta)}$ after which

$$(y_{\lambda-\nu+\beta(\delta)},h_{\lambda-\nu+\beta(\delta)+1},\dots,h_\lambda,y_\lambda)$$
$$= (\overline{y}_{\kappa-\nu+\beta(\delta)},\overline{h}_{\kappa-\nu+\beta(\delta)+1},\dots,\overline{h}_\kappa,\overline{y}_\kappa)$$
$$= (v_{\beta(\delta)},e_{\beta(\delta)+1},\dots,e_\nu,v_\nu)$$

by (10.115) and the definition of $\overline{R}$. It follows from this that $\tilde{R}$ is formed from R by replacing a number of pieces of $\overline{R}$ between two vertices of $\overline{R}$ on $K_\ell \cup \tilde{X} \cup K_r$ by pieces of $\tilde{R}$ of one or two edges. None of these replacements results in the removal of $\tilde{x}$. This proves (10.116) and (10.117).

Finally we complete the proof of (10.56). The existence of the crosscut $\tilde{R}$ of $\mathrm{int}(J_\ell)$ with minimal $J_\ell^-(\tilde{R})$, and containing $\tilde{x}$ from $\mathfrak{w}_0$ we already proved (see especially (10.117)). We know from (10.93) and (10.33) that

$$|\tilde{x}-a^\#| \le \Theta \quad .$$

Also, we showed at the end of Step (vi) that Y^* is a connection on $\mathcal{G}^*_{p\ell}$ from $\tilde{x}$ to $\mathring{C}$ above $\overline{R}$. But $\tilde{R} \subset \overline{J}_\ell^-(\overline{R})$ (see (10.128)) and this implies

$$J_\ell^+(\overline{R}) \subset J_\ell^+(\tilde{R}) \tag{10.133}$$

as shown by the derivation of (A.41) from (A.38). Thus Y^* is also a connection from $\tilde{x}$ to $\mathring{C}$ above $\tilde{R}$. Moreover it is vacant in the configuration $\tilde{\omega}$ by Step (viic). Thus, everything claimed in (10.56) has been verified.

Step (ix). In this step we complete the deduction of Condition E by verifying (10.57). Let $y \in \mathfrak{w}_0$ be a vertex of $\tilde{R}$ and let $Z^* = (z_0^*, k_1^*, \dots, k_\theta^*, z_\theta^*)$ be a vacant connection on $\mathcal{G}^*_{p\ell}$ (in the configuration $\tilde{\omega}$) from y to $\mathring{C}$ above $\tilde{R}$ (cf. the definition (10.49)-(10.51) with $\Gamma = \mathbb{R}^2$). If y is not one of the v_i with $0 \le i < \le \beta(\gamma)-1$ or $\beta(\delta)+1 \le i \le \nu$, then by (10.114)-(10.116) y lies within distance Λ from $K_\ell \cup \tilde{X} \cup K_r \subset \overline{K}(a)$. Since $a^\# \in K(a)$, (b) of (10.57) holds for such y with $\kappa_3 = \text{diamter}(K)+\Lambda$. Thus we may restrict ourselves to $y = v_i \in \overline{R}$ with $0 \le i < \beta(\gamma)$; the case where $y = v_i$ with $\beta(\delta) < i \le \nu$ is similar.

We begin by showing that

(10.134) $\quad Z^*$ is a vacant connection (in $\tilde{\omega}$) from y to $\mathring{C}$ above $\overline{R}$.

The point of (10.134) is that Z^* is even above $\overline{R}$, not only above $\tilde{R}$. To see (10.134) we observe that (by requirement (10.49)) there exists an edge k^* of $\mathcal{M}_{p\ell}$ between y and z_0^* such that $\mathring{k}^* \subset J_\ell^+(\tilde{R})$. Thus for any small neighborhood N of y

$$\mathring{k}^* \cap N \subset N \cap J_\ell^+(\tilde{R}).$$

However, now that we have (10.114) we can use the argument which derives (10.119) from (10.118) - with (10.133) or (10.128) replacing (10.109) - to obtain also

$$N \cap J_\ell^+(\tilde{R}) = N \cap J_\ell^+(\overline{R})$$

for suitable small neighborhoods N of y. For such N one has

$$\mathring{k}^* \cap N \subset N \cap J_\ell^+(\overline{R}).$$

Thus, near y $\mathring{k}^*$ lies in $J_\ell^+(\overline{R})$, and then all of $\mathring{k}^*$ lies in $J_\ell^+(\overline{R})$. This is the analogue of (10.49) for Z^* and $\overline{R}$ instead of s^* and r. The analogue of (10.50) is $z_\theta^* \in \mathring{C}$, which is true because Z^* is a connection to $\mathring{C}$. To prove (10.134) it therefore suffices to show

$$Z^* \setminus \{z_\theta^*\} \subset J_\ell^+(\overline{R}). \tag{10.135}$$

Since $\overset{\circ}{k} \subset J_\ell^+(\overline{R})$ and z_0^* is an endpoint of k, hence in $\overline{J}_\ell^+(\overline{R})$, and since $Z^* \setminus \{z_\theta^*\} \subset \text{int}(J_\ell)$ (by (10.51) with r replaced by $\tilde{R}$), (10.135) can fail only if some point of Z^* lies on $\overline{R}$. The first intersection of Z^* and $\overline{R}$ has to be one of the vertices z_i^* in this case, say z_ξ^*. z_ξ^* also has to be a vertex of $\overline{R}$. This is not possible for then z_ξ^* has to vacant in the configuration $\tilde{\omega}$, being on Z^*, as well as occupied, being a vertex of $\overline{R}$, which is occupied in $\tilde{\omega}$, by Step (viia). Thus (10.135) and (10.134) hold.

We now introduce

$$\Xi^* := \{\text{vertices of } \mathcal{G}_{p\ell}^* \text{ which are occupied in } \omega \text{ but vacant in } \tilde{\omega}\} .$$

If Z^* has no vertex in Ξ^*, then Z^* is also vacant in the configuration ω. Moreover $J_\ell^+(\overline{R}) \subset J_\ell^+(R)$, as we saw in (10.109), so that in this case, by virtue of (10.134) Z^* is a vacant connection from y to $\overset{\circ}{C}$ above R in the configuration ω. Thus in this case (a) of (10.57) holds. There remains the case where Z^* has a vertex in Ξ^*. We shall now show that (a) of (10.57) must hold in this case as well. To prove this, let z_η^* be the first vertex of Z^* in Ξ^*. By Step (vii)(a)-(d),

$$\Xi^* \cap (\overline{J}_\ell^+(\overline{R}) \setminus \overline{R}) \subset Y^* .$$

Actually, we saw in the proof of (10.53) at the end of Step (vii) that Step (viic) requires only changes in the occupancies of vertices on X^*. Therefore

$$\Xi^* \cap (\overline{J}_\ell^+(\overline{R}) \setminus \overline{R}) \subset X^*. \tag{10.136}$$

In particular z_η^* is a vertex on $X^* \cap Y^*$ and we can define π as the smallest index $i \le \eta$ with $z_i^* \in X^* \cap Y^*$. Since z_0^* and z_1^* are within distance 2Λ of $y \in R$, and since (10.112) shows that distance $(X^*,R) > 2\Lambda$, z_0^* and z_1^* cannot lie on X^*. Therefore

$$2 \le \pi \le \eta \quad .$$

Now $z_{\pi-1}^*$ and $z_{\pi-2}^*$ both lie in $J_\ell^+(\overline{R})$ (by (10.135)) and they can be connected by one or two edges of $\mathcal{G}_{p\ell}^*$ to $z_\pi^* \in X^*$. Being vertices of Z^*, $z_{\pi-1}^*$ and $z_{\pi-2}^*$ have to be vacant in the configuration $\tilde{\omega}$. In view of Step (viid) this means that both $z_{\pi-1}^*$ and $z_{\pi-2}^*$ have to be central vertices of $\mathcal{G}_{p\ell}^*$ or belong to Y^*. If $z_{\pi-1}^* \in Y^*$, then

$$z_{\pi-1}^* \in Y^* \setminus X^* \subset S^*$$

(see the construction of Y^* towards the end of Step (vi)). If $z^*_{\pi-1}$ is the vertex w^*_ϕ of s^*, then

$$(10.137)\qquad (z^*_0, k^*_1, \dots, k^*_{\pi-1}, z^*_{\pi-1} = w^*_\phi, f^*_{\phi+1}, \dots, f^*_\tau, w^*_\tau)$$

is a path with possible double points on $\mathcal{G}^*_{p\ell}$, consisting of the beginning piece of Z^*, until an intersection of Z^* and s^*, and a final piece of s^* from this intersection of s^* with Z^* to $w^*_\tau \in \mathring{C}$. This path is vacant in the configuration ω, since $z^*_0, \dots, z^*_{\pi-1}$ do not lie in Ξ^* and are vacant in $\tilde{\omega}$, while s^* is a vacant connection in ω from $a^\#$ to $\mathring{C}$ above $R^\#$ (see beginning of Step (vi)). Also, as we saw above

$$\mathring{k}^* \subset J^+_\ell(\overline{R}) \subset J^+_\ell(R) \quad \text{(cf. (10.109))},$$

$$(z^*_0, k^*_1, \dots, k^*_{\pi-1}, z^*_{\pi-1}) \subset Z^* \setminus \{z^*_\theta\} \subset J^+_\ell(\overline{R}) \subset J^+_\ell(R)$$
$$\text{(cf. (10.135), (10.109))},$$

and finally, because s^* is a connection to $\mathring{C}$ above $R^\#$

$$(w^*_\phi, f^*_{\phi+1}, \dots, f^*_\tau \setminus \{w^*_\tau\}) \subset s^* \setminus \{w^*_\tau\} \subset J^+_\ell(R^\#) \subset J^+_\ell(R) \quad \text{(cf.(10.43))}.$$

It follows from this that the path in (10.137) after loop-removal, to make it self-avoiding, forms a vacant connection in ω from y to $\mathring{C}$ above R. Thus, if $z^*_{\pi-1} \in Y^*$, then (a) of (10.57) holds. The same argument works if $z^*_{\pi-2} \in Y^*$. This leaves only the case where neither $z^*_{\pi-1} \in Y^*$ nor $z^*_{\pi-2} \in Y^*$. This case, however, cannot arise, for as we saw above this would require both $z^*_{\pi-1}$ and $z^*_{\pi-2}$ to be central vertices of $\mathcal{G}^*_{p\ell}$. Since $z^*_{\pi-1}$ and $z^*_{\pi-2}$ are neighbors on $\mathcal{G}^*_{p\ell}$ this is impossible (Comment 2.3 (iv)). We have thus proved (10.57) in all cases and completed the proof of Condition E.