

## CATEGORIAL SET THEORY

*“... the mathematics of the future, like that of the past, will include developments which are relevant to the philosophy of mathematics.... They may occur in the theory of categories where we see, once again, a largely successful attempt to reduce all of pure mathematics to a single discipline”.*

Abraham Robinson

While a topos is in general to be understood as a “generalised universe of sets”, there are, as we have seen, many topoi whose structure is markedly different from that of **Set**, the domain of classical set theory. Even within a topos that has classical logic (is Boolean) there may be an infinity of truth-values, non-initial objects that lack elements, distinct arrows not distinguished by elements of their domain etc. So in order to identify those topoi that “look the same” as **Set** we will certainly impose conditions like well-pointedness and (hence) bivalence.

However, in order to say precisely which topoi look like **Set** we have to know precisely what **Set** looks like. Thus far we have talked blithely about *the* category of all sets without even acknowledging that there might be some doubt as to whether, or why, such a unique thing may exist at all. We resolve (sidestep?) this matter by introducing a formal first-order language for set-theory, in which we write down precise versions of set-theoretic principles. Instead of referring to “the universe **Set**”, we confine ourselves to discussion of interpretations of this language. The notion of a topos is also amenable to a first-order description, as indicated in the last chapter, and so the relationship between topos theory and set theory can be rigorously analysed in terms of the relationship between models of two elementary theories.

Before looking at the details of this program we need to develop two more fundamental aspects of the category of sets.

## 12.1 Axioms of choice

Let  $f: A \twoheadrightarrow I$  be an epic (onto) set function. Then, construing  $f$  as a bundle over  $I$ , we may construct a *section* of  $f$ , i.e. a function  $s: I \rightarrow A$  having  $f \circ s = \text{id}_I$ . The point here is that for each  $i \in I$  the stalk  $A_i$  over  $i$  is non-empty (since  $f$  is onto) and so we may choose some element of  $A_i$  and take it as  $s(i)$  (unless  $I = \emptyset$ , in which case  $A = \emptyset$  so we take  $s$  as the empty map  $! : \emptyset \rightarrow \emptyset$ ). The section  $s$  is sometimes said to “split” the epic  $f$ . In sum then we have produced an argument to the effect that in **Set**, all epics split. We lift this now to the categorial statement

ES:         $\text{Each epic } a \xrightarrow{f} b \text{ has a section } b \xrightarrow{s} a \text{ with } f \circ s = 1_b.$

EXERCISE 1. Show that a section is always monic. □

The principle ES is a variant of what is known as the *axiom of choice*. The name relates to our making an arbitrary choice of the element  $s(i)$  of  $A_i$ . The function  $s$ , in selecting an element from each  $A_i$  is called a *choice function*. Informally, the axiom of choice asserts that it is permissible to make an unlimited number of arbitrary choices. It was first isolated as a principle of mathematical reasoning by Zermelo in 1904 and subsequently has been shown to be implied by, indeed equivalent to, many substantial “theorems” of classical mathematics. To many classically minded mathematicians the axiom of choice is a perfectly acceptable principle. It is difficult for someone so minded to see what could be wrong with the above argument that purports to show that ES is true of **Set**.

Nonetheless the status of the axiom of choice remained in doubt until Paul Cohen [66] proved that it was not derivable from the Zermelo–Fraenkel axioms for set theory (Gödel [40] had earlier shown that it was not refutable by this system). The point would seem to be that the choice function  $s$  cannot be explicitly defined in terms of any set-theoretic operations involving  $f: A \rightarrow I$ . In general we are unable to formulate a rule for  $s$  of the form “let  $s(i)$  be the element of  $A_i$  such that  $\varphi$ ”, where  $\varphi$  is some property that demonstrably is possessed by only one element of  $A_i$ . So if we wish to include ES in our account of what **Set** looks like we will simply have to take it as an axiom (unless of course we adopt some equally “unprovable” axiom that implies it).

Now if  $f: A \rightarrow I$  is a function that is not onto, then  $f$  will not have a section. This, as explained in §11.8, is why the **Bn**( $I$ )-object  $a = (A, f)$  is empty, i.e. has no elements  $1 \rightarrow a$ . However  $f$  will have a “partial

section"  $s: I \rightsquigarrow A$ . For, taking the epi-monic factorisation

$$\begin{array}{ccc}
 A & \xrightarrow{f} & I \\
 \searrow f^* & & \nearrow \hookrightarrow \\
 & f(A) &
 \end{array}$$

of  $f$  we find that a section of the epic  $f^*$  is a partial function from  $I$  to  $A$ .

Now the image  $f(A)$  is sometimes known as the *support* of the bundle  $a$ . It is the subset of  $I$

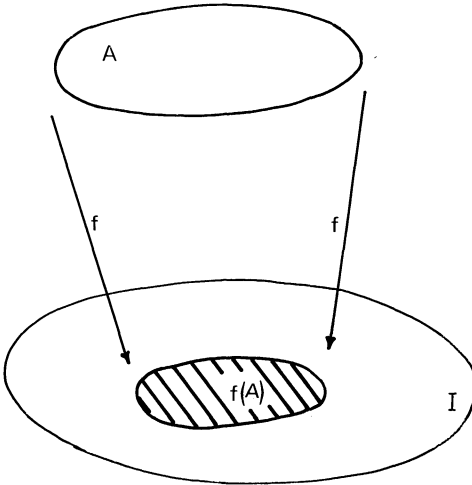


Fig. 12.1.

over which the stalks actually “sit”. As a subset of  $I$ ,  $f(A)$  is identifiable with a subobject of  $1 = (I, \text{id}_I)$  in  $\mathbf{Bn}(I)$ . Indeed since

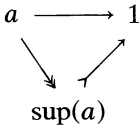
$$\begin{array}{ccc}
 A & \xrightarrow{f} & I \\
 \searrow f^* & \searrow f & \nearrow \hookrightarrow \\
 & f(A) & \\
 & \downarrow \text{id}_I & \\
 & I &
 \end{array}$$

commutes in  $\mathbf{Set}$ , so does

$$\begin{array}{ccc}
 a & \xrightarrow{f} & 1 \\
 \searrow f^* & & \nearrow \hookrightarrow \\
 & \text{sup}(a) &
 \end{array}$$

where the object  $\text{sup}(a)$  is the function (bundle)  $f(A) \hookrightarrow I$ .

Lifting this to a general topos  $\mathcal{E}$  we define the *support* of an  $\mathcal{E}$ -object  $a$  to be the subobject  $\text{sup}(a) \twoheadrightarrow 1$  of  $1$  given by the epi-monic factorisation



of the unique arrow  $! : a \rightarrow 1$ .

We may now formulate axiom

**SS (supports split):** *The epic part  $a \twoheadrightarrow \text{sup}(a)$  of the epi-monic factorisation of  $a \rightarrow 1$  has a section  $s : \text{sup}(a) \rightarrow a$ .*

Notice that a splitting  $s$  of the support of  $a$  yields a partial element  $s : 1 \rightsquigarrow a$  of  $a$ , so the principle SS is closely related to the question of (non) emptiness of objects. To pursue this we need axiom

**NE:** *For every non-initial  $a$  there exists an arrow  $x : 1 \rightarrow a$ .*

**LEMMA.** *In any  $\mathcal{E}$ , if  $g : a \twoheadrightarrow 1$  is a subobject of  $1$ , then there exists an element  $x : 1 \rightarrow a$  of  $a$  iff  $g \simeq 1_1$  iff  $g : a \cong 1$ .*

**PROOF.** This is the essence of Case 2 in the proof of Theorem 5.4.2.  $\square$

**CONVENTION.**  $\mathcal{E}$  is always non-degenerate, i.e.  $0 \neq 1$ .

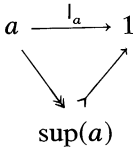
**NOTATION.** We write  $\mathcal{E} \models \text{NE}$ ,  $\mathcal{E} \models \text{SS}$  etc. to mean that NE (SS etc.) holds for  $\mathcal{E}$ .

**THEOREM 1.** *For any topos  $\mathcal{E}$ ,*

$$\mathcal{E} \models \text{NE} \text{ iff } \mathcal{E} \text{ is bivalent and } \mathcal{E} \models \text{SS}.$$

**PROOF.** Suppose  $\mathcal{E} \models \text{NE}$ , and let  $t : 1 \rightarrow \Omega$  be a truth-value. Pull  $t$  back along  $\top$  to get  $g : a \twoheadrightarrow 1$  with  $\chi_g = t$ . Then if  $t \neq \perp$ ,  $a$  is non-initial, so by NE there exists  $x : 1 \rightarrow a$ . But then by the Lemma,  $g \simeq 1_1$ , so  $\chi_g = \chi_{1_1}$ , i.e.  $t = \top$ . Hence  $\mathcal{E}$  is bivalent.

To see why supports split, consider



If  $\text{sup}(a) \cong 0$ , then  $a \cong 0$  (Theorem 3.16.1, (2)) and so the unique arrow  $\text{sup}(a) \rightarrow a$  will split the unique  $a \rightarrow \text{sup}(a)$ . If not  $\text{sup}(a) \cong 0$ , then by NE there is an element  $1 \rightarrow \text{sup}(a)$ , from which by the Lemma,  $\text{sup}(a)$  is terminal,  $\text{sup}(a) \cong 1$ , and hence  $l_a$  is epic. Then if  $a \cong 0$ ,  $l_a$  would be monic (Theorem 3.16.1, (4)), hence altogether iso, making  $0 \cong a \cong 1$ , and thus  $\mathcal{E}$  degenerate. So we may invoke NE again to get an element  $x : 1 \rightarrow a$ . Since  $\text{sup}(a) \cong 1$  this yields an arrow  $\text{sup}(a) \rightarrow a$  which must be a section of the unique  $! : a \rightarrow \text{sup}(a)$ .

Conversely if  $\mathcal{E}$  is bivalent then in Sub(1),  $\text{sup}(a) \twoheadrightarrow 1$  can only be  $0_1$  or  $1_1$ . But if  $a \neq 0$ , then  $\text{sup}(a) \neq 0$  (as above), so it cannot be  $0_1$ . We must then have  $\text{sup}(a) \twoheadrightarrow 1 = 1_1$ , so  $\text{sup}(a) \cong 1$ . Then if  $\mathcal{E} \models \text{SS}$ , there is an arrow  $\text{sup}(a) \rightarrow a$ , hence an arrow  $1 \rightarrow a$ . This establishes NE.  $\square$

**COROLLARY.**  $\mathcal{E}$  is well-pointed iff  $\mathcal{E}$  is Boolean (classical), bivalent, and has splitting supports.

**PROOF.** Theorem 5.4.5 (proven in §7.6) gives  $\mathcal{E}$  well-pointed iff  $\mathcal{E}$  is classical and  $\mathcal{E} \models \text{NE}$ .  $\square$

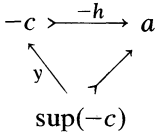
Even when there are more than two truth-values, the splitting of epics in a Boolean topos has implications for extensionality. We will say that  $\mathcal{E}$  is *weakly extensional* if for every pair  $f, g : a \rightrightarrows b$  with  $f \neq g$  there is a partial element  $x : 1 \dashv\vdash a$  such that  $f \circ x \neq g \circ x$ . Recall that  $x : 1 \dashv\vdash a$  means that  $\text{cod } x = a$  and there is a monic  $\text{dom } x \rightarrow 1$  (hence  $x$  could not be  $! : 0 \rightarrow a$  if  $f \circ x \neq g \circ x$ ).

Category theorists will recognise “ $\mathcal{E}$  is weakly extensional” as “Sub(1) is a set of generators for  $\mathcal{E}$ ”.

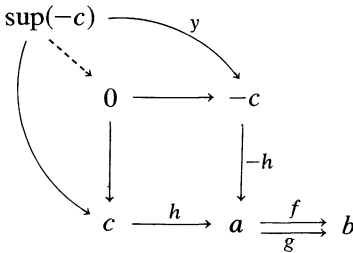
**THEOREM 2.** If  $\mathcal{E}$  is Boolean and  $\mathcal{E} \models \text{SS}$ , then  $\mathcal{E}$  is weakly extensional.

**PROOF.** Let  $h : c \twoheadrightarrow a$  equalise  $f, g : a \rightrightarrows b$ , and let  $-h : -c \twoheadrightarrow a$  be the

complement of  $h$  in  $\text{Sub}(a)$ . Then as in §7.6, if  $f \neq g$ ,  $-c \neq 0$ . Now if  $y : \text{sup}(-c) \rightarrow -c$  is a section of  $-c \rightarrow \text{sup}(-c)$ ,



then putting  $x = -h \circ y$  gives  $x : 1 \rightsquigarrow a$ . If  $f \circ x = g \circ x$ , then reasoning as in §7.6,



$x$  would factor through  $h$ , ultimately making  $\text{sup}(-c) \cong 0$  and hence  $-c \cong 0$ . Therefore  $x$  distinguishes  $f$  and  $g$ . □

**EXAMPLE.** In general  $\mathbf{Bn}(I)$ , though Boolean, is not extensional (well-pointed), since NE fails. However  $\mathbf{Bn}(I)$  is weakly extensional. Given bundles  $a = (A, h)$ ,  $b = (B, k)$  and distinct arrows  $f, g : a \rightrightarrows b$ , then the distinguishing  $x : 1 \rightsquigarrow a$ , as in Theorem 2, is a local section of  $a$ , defined on a subset  $-C$  of the support  $h(A)$  of  $a$ . For each  $i \in -C$  (hence  $A_i \neq \emptyset$ ),  $x$  selects an element  $x_i$  of the stalk  $A_i$  that distinguishes  $f$  and  $g$ , i.e.  $f(x_i) \neq g(x_i)$ . □

Returning to **Set** once more, let  $f : A \rightarrow I$  be any function and, invoking ES, let  $s : f(A) \rightarrow A$  be a section of  $f^* : A \rightarrow f(A)$ . Then if  $A \neq \emptyset$ , by choosing a particular  $x_0 \in A$  we can obtain a function  $g : I \rightarrow A$  by the rule

$$g(y) = \begin{cases} s(y) & \text{if } y \in f(A) \\ x_0 & \text{otherwise.} \end{cases}$$

Of course if there exists  $y \notin f(A)$ ,  $g$  will not be a section of  $f$ , since  $f(g(y)) \notin f(A)$ . However, starting with  $x \in A$  we find that  $g(f(x)) = s(f(x))$  lies in the stalk over  $f(x)$  so  $f$  simply takes  $g(f(x))$  to  $f(x)$ , i.e.  $f \circ g \circ f(x) = f(x)$ . This yields another version of the axiom of choice, due to Maclane,

that has the categorial formulation

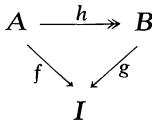
AC: If  $a \neq 0$  then for any arrow  $a \xrightarrow{f} b$  there exists  $b \xrightarrow{g} a$  with  $f \circ g \circ f = f$ .

THEOREM 3. If  $\mathcal{E} \models \text{AC}$ , then  $\mathcal{E} \models \text{NE}$ ,  $\mathcal{E} \models \text{ES}$ , and  $\mathcal{E}$  is bivalent.

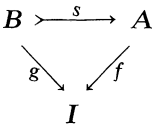
PROOF. If  $a \neq 0$ , apply AC to  $! : a \rightarrow 1$  to get  $g : 1 \rightarrow a$ . Hence NE holds. To derive ES, observe that if  $f : a \rightarrow b$  is epic, and  $a \cong 0$ , then  $f$  is monic, (Theorem 3.16.1), hence altogether iso, so is split by its inverse. If  $a \neq 0$ , apply AC to get  $g : b \rightarrow a$ , with  $f \circ g \circ f = f = 1_b \circ f$ . Since  $f$  is right cancellable, we get  $f \circ g = 1_b$ , making  $g$  a section of  $f$ .

For bivalence, observe that if  $g : a \rightarrow 1$  has  $a \neq 0$ , then by AC there is an arrow  $1 \rightarrow a$ . Hence, as in Theorem 1,  $g \approx 1_1$ . Thus  $\text{Sub}(1)$  has only the two elements  $0_1$  and  $1_1$ . □

The argument that yields AC from ES in **Set** will lift to a topos only if that topos is sufficiently “**Set-like**”. To see this, consider a set  $I$  with at least two elements. Then  $\mathbf{Bn}(I)$  has at least four truth-values (subsets of  $I$ ) so by the last result AC fails (alternatively observe that NE fails). But if epics split in **Set**, they will in  $\mathbf{Bn}(I)$  also. For  $h : (A, f) \twoheadrightarrow (B, g)$  means that  $h$  is an onto function with



$g \circ h = f$ . But then if  $s : B \rightarrow A$  is a section of  $h$ ,

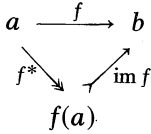


will commute, making  $s$  a splitting of  $h$  in  $\mathbf{Bn}(I)$ .

Rather than rely on the assumption that ES holds in **Set**, we can use the result of Gödel that there exist models of formal set theory in which the axiom of choice is true. We may then construct a category of bundles of “sets” from such a model to obtain a topos in which ES holds but AC fails.

THEOREM 4. If  $\mathcal{E} \models \text{ES}$ , and  $\mathcal{E}$  is well-pointed, then  $\mathcal{E} \models \text{AC}$ .

PROOF. Take  $f : a \rightarrow b$  and perform the factorisation

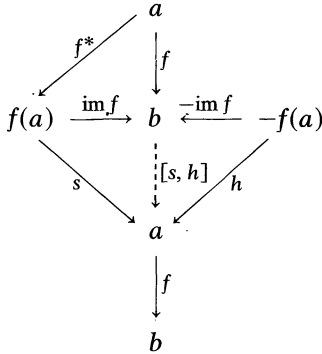


Since  $\mathcal{E}$  is well-pointed, it is Boolean, so  $\text{im } f$  has a complement  $-\text{im } f : -f(a) \rightarrow b$ , with  $\text{im } f \cup -\text{im } f \simeq 1_b$  in  $\text{Sub}(b)$ . But  $\text{im } f$  and  $-\text{im } f$  are disjoint monics (Theorem 7.2.3), and so  $[\text{im } f, -\text{im } f] : f(a) + -f(a) \rightarrow b$  is monic (Lemma, §5.4). But then

$$[\text{im } f, -\text{im } f] \simeq \text{im } f \cup -\text{im } f \simeq 1_b,$$

and so this co-product arrow is iso. This allows us to use  $b$  as a co-product object for  $f(a)$  and  $-f(a)$ , with  $\text{im } f$  and  $-\text{im } f$  serving as the associated injections.

Now suppose  $a \neq 0$ . Then as well-pointed topoi satisfy NE, we take some  $x : 1 \rightarrow a$  and let  $h : -f(a) \rightarrow a$  be the composite  $x \circ ! : -f(a) \rightarrow 1 \rightarrow a$ . Since  $\mathcal{E} \models \text{ES}$ , we have also a section  $s : f(a) \rightarrow a$  of  $f^*$ . Then



$$\begin{aligned}
 f \circ [s, h] \circ f &= \text{im } f \circ f^* \circ [s, h] \circ \text{im } f \circ f^* \\
 &= \text{im } f \circ f^* \circ s \circ f^* && (\text{im } f \text{ as injection}) \\
 &= \text{im } f \circ f^* && (f^* \circ s = 1_{f(a)}) \\
 &= f.
 \end{aligned}$$

Thus  $g = [s, h]$  gives the required arrow for AC. □

The hypothesis of Theorem 4, as stated, assumes more than it need do. We know that “well-pointed” = “NE plus Boolean”. But in the presence of ES, the last of these conditions can be derived! We have the remarkable fact, discovered by Radu Diaconescu [75], that the axiom of choice implies that the logic of a topos must be classical.



**THEOREM 5.** *If  $\mathcal{E}$  satisfies ES, then  $\mathcal{E}$  is Boolean.*

The basis of Diaconescu's result is that if epics with domain  $d + d$  have sections, then each subobject  $f: a \rightarrow d$  of  $d$  has a complement in  $\text{Sub}(d)$ . The construction, as described in Boileau [75], is best illustrated in **Set**, where we can see how it produces a categorical characterisation of the complement  $-A$  in  $D$  of a subset  $A \subseteq D$ .

(1) Form the co-product  $i_1, i_2: d \rightarrow d + d$ , with injections,  $i_1, i_2$ .

In **Set** we take  $D_1$  and  $D_2$  as two disjoint "copies" of  $D$ , containing copies  $A_1$  and  $A_2$  respectively of  $A$ .  $D + D$  is  $D_1 \cup D_2$ .

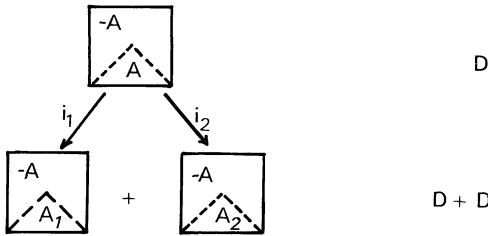


Fig. 12.2.

(2) Let  $g: d + d \rightarrow b$  be the co-equaliser (hence an epic) of  $i_1 \circ f: a \rightarrow d + d$  and  $i_2 \circ f: a \rightarrow d + d$ .

In **Set**  $f$  is the inclusion  $A \hookrightarrow D$ . The effect of  $g$  is to amalgamate the two copies  $A_1$  and  $A_2$  of  $A$  into a single copy  $A' \cong A$ , and to leave  $-A_1$  and  $-A_2$  as they are

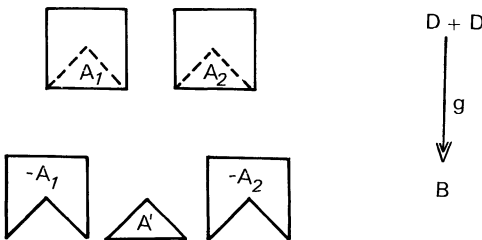


Fig. 12.3.

(3) Let  $s: b \rightarrow d + d$  be a section of  $g$ .

In **Set**,  $s$  acts to literally split  $A'$  into two pieces, part going into  $D_1$  and part into  $D_2$

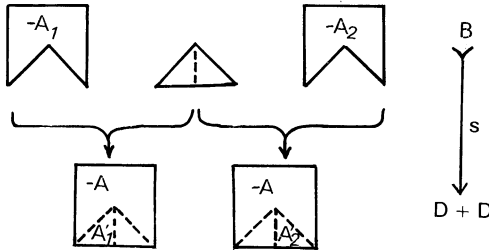


Fig. 12.4.

$A'_1$  is the  $s$ -image of  $A'$  in  $D_1$ ,  $A'_2$  the  $s$ -image in  $D_2$ .

(4) Form the pullbacks of  $i_1$  and  $i_2$  along  $s$

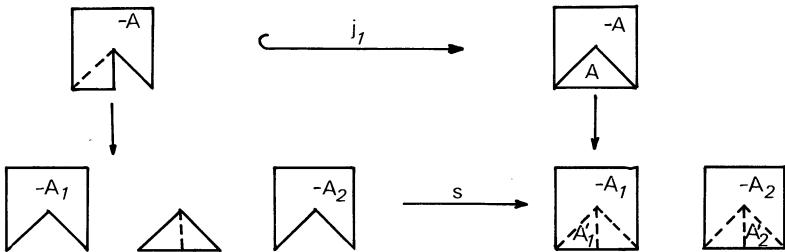


Fig. 12.5.

In **Set** the pullback of  $i_1$  produces the subobject (inclusion) of  $D$  whose domain is obtained by removing from  $D$  the part isomorphic to  $A'_2$ .

Similarly the pullback of  $i_2$  along  $s$  yields



Fig. 12.6.

(5) Form the intersection (pullback) of  $j_1$  and  $j_2$ .

In **Set** this gives the intersection

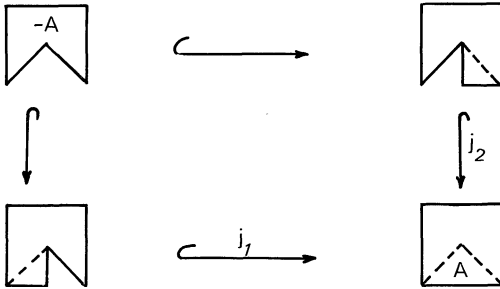


Fig. 12.7.

of the domains of  $j_1$  and  $j_2$ , i.e. the subset  $-A$ .

The five steps of this construction can be carried out in any topos to show that the intersection of the pullbacks of  $i_1$  and  $i_2$  along a section of the co-equaliser of the diagram

$$a \xrightarrow{f} d \begin{matrix} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{matrix} d+d$$

is a complement of  $f$  in  $\text{Sub}(d)$ . Thus all elements of  $\text{Sub}(d)$  have complements if  $\mathcal{E} \models \text{ES}$ , and since  $\text{Sub}(d)$  is a distributive lattice, it must therefore be a Boolean algebra. A detailed proof of Theorem 5, using a modification of this construction, and due to G. M. Kelly, is given by Brook [74]. There is also a proof given in Johnstone [77], Chapter 5.

Note that, by §7.3, for  $\mathcal{E}$  to be Boolean it suffices to have a complement for  $\text{true} : 1 \rightarrow \Omega$  in  $\text{Sub}(\Omega)$ . Thus a sufficient condition for Booleanness is that the co-equaliser of

$$1 \xrightarrow{\text{true}} \Omega \begin{matrix} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{matrix} \Omega + \Omega$$

splits.

**THEOREM 6.**  $\mathcal{E} \models \text{AC}$  iff  $\mathcal{E} \models \text{ES}$  and  $\mathcal{E} \models \text{NE}$ . □

We have already noted that topoi, e.g.  $\mathbf{Bn}(I)$ , can have splitting epics but not be fully extensional (well-pointed). However in view of Theorem 5, we see from Theorem 2 that if  $\mathcal{E} \models \text{ES}$ , then  $\mathcal{E}$  is at least weakly extensional, since then  $\mathcal{E} \models \text{SS}$  and  $\mathcal{E}$  is Boolean. Extensionality on the other hand does not imply ES or AC. By Cohen's work [66] there are

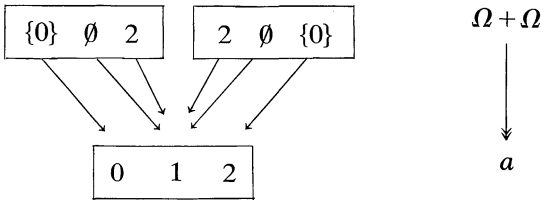
models of set theory, hence well-pointed topoi, in which the axiom of choice fails.

It follows from the foregoing results that AC implies Booleanness for any topos. An independent proof of this is given by Anna Michaelides Penk [75], who also considers a formalisation of the version of the choice principle that reads

“for each set  $X \neq \emptyset$  there is a function  $\sigma : \mathcal{P}(X) \rightarrow X$  such that whenever  $B$  is a non-empty subset of  $X$ ,  $\sigma(B) \in B$ ”.

This leads to a categorial statement that is implied by AC, independent of ES, and equivalent to AC (and ES) in well-pointed topoi.

We end this section with an illustration of a



“non-splitting” epic arrow  $\Omega + \Omega \twoheadrightarrow a$  in the topos  $\mathbf{M}_2$ . Here  $a = (\{0, 1, 2\}, \lambda)$  has  $\lambda(1, x) = x$ , and  $\lambda(0, x) = 1$ , all  $x \in \{0, 1, 2\}$ .

EXERCISE 2. Show that  $\lambda$  as defined is an action on  $\{0, 1, 2\}$  and that the displayed epic is an  $\mathbf{M}_2$ -arrow (equivariant). Explain why it has no section.

EXERCISE 3. Make a similar display of the co-equaliser of

$$1 \begin{array}{c} \xrightarrow{i_1 \circ \tau} \\ \xrightarrow{i_2 \circ \tau} \end{array} \Omega + \Omega$$

in  $\mathbf{M}_2$  and explain why it has no section.

EXERCISE 4. Show that SS holds in  $\mathbf{M}_2$ , and (hence?) that NE does as well.

EXERCISE 5. Show that SS and (hence?) NE fail in  $\mathbf{Z}_2\text{-Set}$  where  $\mathbf{Z}_2$  is the group

|   |  |   |   |
|---|--|---|---|
| + |  | 0 | 1 |
| 0 |  | 0 | 1 |
| 1 |  | 1 | 0 |

of the integers mod 2 under addition. Explain why the situation is typical, i.e. why SS and NE always fail in **M-Set** when **M** is a (non-trivial) group.

EXERCISE 6. Carry out Exercise 3 for the topos **Set**<sup>→</sup>. □

### 12.2. Natural numbers objects

An obvious difference between **Set**, and the topoi **Finset** and **Finord** is that in the latter, all objects are finite. Various definitions of “finite object” in a topos are explored by Brook [74], and Kock, Lecouturier, and Mikkelsen [75]. Our concern now is with the existence in set theory of infinite objects, the primary example being the set  $\omega = \{0, 1, 2, \dots\}$  of all finite ordinals, whose members are the set-theoretic representatives of the intuitively conceived natural numbers.

$\omega$  can be thought of as being generated by starting with 0 and “repeatedly adding 1”, to produce the series  $1 = 0 + 1, 2 = 1 + 1, 3 = 2 + 1, \dots$ . The process of “adding 1” yields the *successor* function  $s : \omega \rightarrow \omega$  which for each input  $n \in \omega$  gives output  $n + 1$ . That is,  $s(n) = n + 1$ .

(Notice that  $n = \{0, \dots, n - 1\}$  and  $n + 1 = \{0, \dots, n\}$  so that an explicit set-theoretic definition of  $s$  is available:–  $s(n) = n + 1 = n \cup \{n\}$ .)

Now the initial ordinal 0 may be identified with an arrow  $O : 1 \rightarrow \omega$  in the usual way (indeed the arrow is the inclusion  $\{0\} \hookrightarrow \omega$ ). Then we have a diagram

$$1 \xrightarrow{O} \omega \xrightarrow{s} \omega$$

which was observed by Lawvere [64] to enjoy a kind of co-universal property that characterises the natural numbers uniquely up to isomorphism in **Set**. The property that the diagram has is that all diagrams of its type, i.e. of the type

$$1 \xrightarrow{x} A \xrightarrow{f} A$$

factor uniquely through it. For, given functions  $x$  and  $f$  as shown we may use  $f$  and the element  $x(0)$  of  $A$  to generate a sequence

$$x(0), f(x(0)), f(f(x(0))), f(f(f(x(0)))) , \dots$$

in  $A$  by “repeatedly applying  $f$ ”. Now this sequence can itself be

described as a function  $h : \omega \rightarrow A$  from  $\omega$  to  $A$ , displayed as

$$h(0), h(1), h(2), h(3), \dots$$

$h$  is defined inductively, or *recursively* in two parts.

(1) We let  $h(0)$  be the first term  $x(0)$  in the sequence, i.e.

$$(*) \quad h(0) = x(0),$$

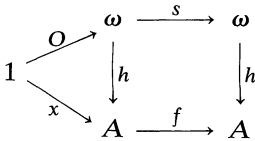
(2) Having defined the  $n$ -th term  $h(n)$ , apply  $f$  to it to get the next term  $h(n+1)$ , i.e.

$$h(n+1) = f(h(n)).$$

Since  $n+1 = s(n)$ , this equation becomes

$$(**) \quad h \circ s(n) = f \circ h(n).$$

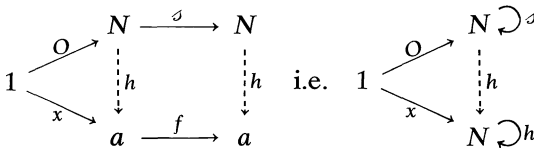
(\*) and (\*\*) mean that the diagram



commutes, giving the “factoring” mentioned above. But also we see that the only way for this diagram to commute is for  $h$  to obey the equations (\*) and (\*\*), so  $h$  can only be the function generated in the way we did it.  $h$  is said to be *defined recursively* from the data  $x$  and  $f$ .

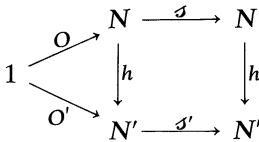
Inductive definitions of this type are called definitions by *simple recursion* and would seem to originate with Dedekind [88]. They lead us to the following axiom, which we have seen to be true of **Set**.

**NNO:** *There exists a natural numbers object (nno), i.e. an object  $N$  with arrows  $1 \xrightarrow{0} N \xrightarrow{s} N$  such that for any object  $a$ , and arrows  $1 \xrightarrow{x} a \xrightarrow{f} a$  there is exactly one arrow  $h : N \rightarrow a$  making*



*commute.*

EXERCISE 1. If  $1 \xrightarrow{O} N \xrightarrow{\sigma} N$  and  $1 \xrightarrow{O'} N' \xrightarrow{\sigma'} N'$  are nno's, then the unique  $h$  in



is iso. □

This exercise establishes that natural numbers objects are unique up to isomorphism in any category. Arrows  $h : N \rightarrow a$  with  $\text{dom} = N$  will on occasion be called *sequences*.

A multiplicity of examples of nno's is provided by

THEOREM 1. For any (small) category  $\mathcal{C}$ ,  $\mathbf{Set}^{\mathcal{C}} \models \mathbf{NNO}$ .

CONSTRUCTION FOR PROOF. Let  $N : \mathcal{C} \rightarrow \mathbf{Set}$  be the constant functor having

$$\begin{aligned}
 N(a) &= \omega, \quad \text{all } \mathcal{C}\text{-objects } a \\
 N(f) &= \text{id}_{\omega}, \quad \text{all } \mathcal{C}\text{-arrows } f.
 \end{aligned}$$

$\sigma : N \rightarrow N$  is the constant natural transformation with component  $\sigma_a : N(a) \rightarrow N(a)$  being the successor function  $s : \omega \rightarrow \omega$  for each  $a$ .

$O : 1 \rightarrow N$  is the constant transformation with each component  $O_a : 1(a) \rightarrow N(a)$  being  $\{0\} \hookrightarrow \omega$ . That this construction satisfies the axiom NNO is left for the reader to establish (the definition of the unique  $h$  is obvious, that it is a natural transformation is not). □

EXERCISE 2. Describe the natural numbers objects in  $\mathbf{Set}^2$ ,  $\mathbf{Set}^{\rightarrow}$ , and  $\mathbf{M-Set}$ , in terms appropriate to the way these topoi were originally defined. □

In  $\mathbf{Bn}(I)$  as one would expect,  $N$  is a bundle of copies of  $\omega$ . Formally  $N$  is  $\text{pr}_I : I \times \omega \rightarrow I$ , so that the stalk  $N_i$  over  $i$  is

$$\{i\} \times \omega \cong \omega.$$

$\sigma : N \rightarrow N$  has  $\sigma(\langle i, n \rangle) = \langle i, n + 1 \rangle$ , i.e.  $\sigma$  acts as the successor function on

each stalk.  $O : 1 \rightarrow N$  has  $O(i) = \langle i, 0 \rangle$ , so that

$$\begin{array}{ccccc}
 I & \xrightarrow{O} & I \times \omega & \xrightarrow{\sigma} & I \times \omega \\
 & \searrow \text{id}_I & \downarrow \text{pr}_I & \swarrow \text{pr}_I & \\
 & & I & & 
 \end{array}$$

commutes, making  $O$  and  $\sigma$  arrows in  $\mathbf{Bn}(I)$ .

Given a bundle  $a = (A, g)$  and arrows  $x : 1 \rightarrow a$ ,  $f : a \rightarrow a$ , then

$$\begin{array}{ccccc}
 & & I \times \omega & \xrightarrow{\sigma} & I \times \omega \\
 & \nearrow O & \downarrow h & & \downarrow h \\
 I & & A & \xrightarrow{f} & A \\
 & \searrow x & & & 
 \end{array}$$

a unique arrow  $h : I \times \omega \rightarrow A$  may be defined to make the last diagram commute. Fixing attention on the stalk over  $i$ , we recursively define  $h$  on that stalk by

$$\begin{aligned}
 h(i, 0) &= x(i) \\
 h(i, n + 1) &= f(h(n, i)).
 \end{aligned}$$

This is evidently the only way to make the diagram commute and so  $h$  provides the unique arrow from  $N$  to  $a$  in  $\mathbf{Bn}(I)$  defined recursively from the data  $x$  and  $f$ .

EXERCISE 3. Verify (inductively) that  $h : N \rightarrow a$ , i.e. that  $g \circ h = \text{pr}_I$ .

EXERCISE 4. Show that  $\sigma$  is the product map  $\text{id}_I \times s$ , and  $O = \langle \text{id}_I, O_I \rangle$ , where  $O_I : I \rightarrow \omega$  has  $O_I(i) = 0$ , all  $i \in I$ . □

The spatial topos  $\mathbf{Top}(I)$  of sheaves of sets of germs over a topological space  $I$  also has a natural numbers object – the same one as  $\mathbf{Bn}(I)$ . We take the product topology on the stalk space  $I \times \omega$ , assuming the discrete topology on  $\omega$ . Thus the basic sets are all those of the form  $U \times A$ , with  $U$  open in  $I$  and  $A$  any subset of  $\omega$ . For each point  $\langle i, n \rangle$ , if  $U$  is any open neighbourhood of  $i$  in  $I$  (e.g.  $U = I$ ), then  $U \times \{n\}$  will be an open



neighbourhood of  $\langle i, n \rangle$  in  $I \times \omega$  that projects homeomorphically

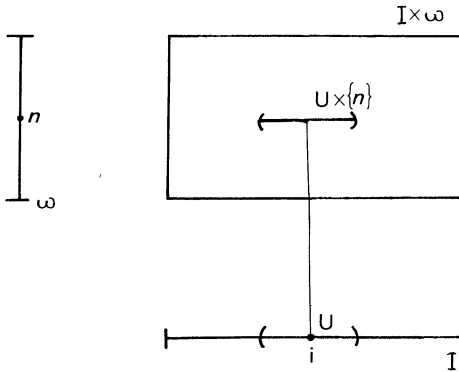


Fig. 12.8.

onto  $U$ . Thus  $pr_I$  is a local homeomorphism. Moreover each of  $\sigma = id_I \times s$  and  $O = \langle id_I, O_I \rangle$  is a product of continuous maps, hence is continuous, i.e. a **Top**( $I$ )-arrow.

**EXERCISE 5.** If  $x : 1 \rightarrow a$  and  $f : a \rightarrow a$  are **Top**( $I$ )-arrows, so that  $x$  and  $f$  are continuous, prove (inductively) that the unique  $h$  defined recursively from  $x$  and  $f$  in **Bn**( $I$ ) is also continuous, hence a **Top**( $I$ )-arrow.  $\square$

We shall reconsider the structure of nno's in **Top**( $I$ ) again in Chapter 14, in relation to "locally constant natural-number-valued functions on  $I$ ".

In any topos satisfying NNO a good deal of the arithmetic of the natural numbers can be developed. This will be considered in the next chapter.

The co-universal property of a natural numbers object will be fully elucidated in Chapter 15.

### 12.3. Formal set theory

The first-order language  $\mathcal{L}$  that we shall use for set-theory has a single binary predicate  $\epsilon$ , and no function symbols, or individual constants. Thus  $\mathcal{L} = \{\epsilon\}$ .

The definition of  $\mathcal{L}$ -model that we shall adopt is a little wider than that of §11.2. A model is a structure  $\mathfrak{A} = \langle A, E, \simeq \rangle$ , where  $E$  and  $\simeq$  are binary

relations on  $A$ , such that the identity axioms  $I1$  and  $I2$  are valid in  $\mathfrak{A}$  when  $\epsilon$  is interpreted as  $E$  and  $\approx$  as  $\simeq$ . Thus we are giving up the requirement that the identity predicate be always interpreted as the “diagonal” relation  $\Delta = \{\langle x, y \rangle : x = y\}$  on  $A$ . If  $I1$  and  $I2$  are valid then  $\simeq$  will be an equivalence relation, and we could, by replacing elements of  $A$  by their  $\simeq$ -equivalence classes, obtain a *normal* model in which  $\simeq$  is interpreted as the diagonal and which is semantically indistinguishable from  $\mathfrak{A}$ . However it is convenient for expository purposes to allow the wider interpretation of identity (note the parallel with the way we have treated equality of subobjects in a category).

Using the language  $\mathcal{L}$ , we are able to write out sentences (strings of symbols) that formally express properties of sets. By considering sentences that our intuitions may incline us to believe to correctly codify ways that sets actually do behave, and by using the precise and rigorous machinery of deduction in elementary logic, we are able to examine the consequences of our intuitively based assumptions about sets. Thus if  $\Sigma$  is a collection of sentences expressing what we take to be truths of set theory, and  $\varphi$  holds in all  $\mathcal{L}$ -models of  $\Sigma$ , then we would regard  $\varphi$  as a truth of set theory, whatever “the universe of sets” looks like.

Our intention then is to regard an  $\mathcal{L}$ -structure  $\mathfrak{A} = \langle A, E, \simeq \rangle$  as a formal, abstract, model or representation of the intuitively-conceived universe of all sets, from which we developed the idea of the category **Set**. There is a conceptual barrier to this that seems to belong uniquely to the study of set theory. While we have no difficulty in thinking of, say, a Boolean algebra as being any model of a certain group of axioms, since a Boolean algebra is *conceived of* as an abstract set satisfying appropriate laws, it is difficult not to think of a model for set theory as consisting of very particular sorts of things, namely sets. We regard the variables  $v_1, v_2, \dots$  as referring to collections, whereas the individuals in  $\mathfrak{A}$  are just that—individuals with no particular presupposed structure. We give the atomic formula  $v_1 \epsilon v_2$  its *intended* reading “ $v_1$  is a *member* of  $v_2$ ”, whereas all we mean is  $\mathfrak{A} \models v_1 \epsilon v_2[x_1, x_2]$ , i.e.  $x_1 E x_2$ .

Having taken pains to spell this out, we should recognise it as being, not a source of pedantry, but rather the very essence of the enterprise itself. By forcing ourselves to regard  $\epsilon$  as being an abstract relation between indeterminate things, we force ourselves to stand back from our presuppositions about what “membership” means, and thereby to identify those assumptions and determine what they commit us to.

We must also be careful to distinguish between *metalanguage* and *object-language*, between the language *in* which we speak and the language *about* which we speak. The object language is the first-order

language  $\mathcal{L}$ . The metalanguage is the language we use to talk about  $\mathcal{L}$  and about the meanings of  $\mathcal{L}$ -sentences (interpretations, models). It is the language in which we make statements like “ $\varphi$  is satisfied by every valuation in  $\mathfrak{A}$ ”. This metalanguage consists basically of sentences of English and unformalised, intuitive, set theory, which is concerned with actual collections. Thus the  $\mathcal{L}$ -formulas form a collection, a model  $\mathfrak{A}$  is based on a collection  $A$  of individuals, the relation  $E$  is a collection of ordered pairs, and so on. These collections are described by the metalanguage. They are “metasets”, and we continue to use the symbol  $\in$  to denote membership of such collections. The individuals in  $A$  on the other hand might be called “sets in the sense of  $\mathfrak{A}$ ”, or simply “ $\mathfrak{A}$ -sets”.

The distinction between these two levels can perhaps be made, somewhat colloquially, by contrasting our perspective, as we look at  $\mathfrak{A}$  “from outside”, with that of an imaginary person who lives “inside”  $\mathfrak{A}$  and is aware only of the existence of the individuals in  $A$ , i.e. of the  $\mathfrak{A}$ -sets. While to us,  $A$  is a set – an individual in our metauniverse of metasets – the  $\mathfrak{A}$ -person does not see  $A$  at all as an individual in his world. Rather,  $A$  represents the whole universe for the  $\mathfrak{A}$ -person. Similarly if  $B$  is a subset of  $A$  (i.e.  $B \subseteq A$ ), the metaset  $B$  may not be an  $\mathfrak{A}$ -set (if  $B \notin A$ ). However it is possible in some cases that  $B$  corresponds to an  $\mathfrak{A}$ -set. This occurs when there is an  $\mathfrak{A}$ -set  $b$  (i.e.  $b \in A$ ) whose  $E$ -members are just the  $\in$ -members of  $B$ , i.e.  $B = \{x: x \in A \text{ and } xEb\}$ . We shall return to this point shortly.

Now if  $a$  and  $b$  are members of  $A$  ( $a, b \in A$ ), then the statement “ $a$  is a member of  $b$ ” when interpreted on the metalevel means  $a \in b$ . However when uttered by the  $\mathfrak{A}$ -person it means  $aEb$ . In some models, the *standard* ones, these two interpretations are the same. Thus a model is standard if  $E$  is simply the meta-membership relation restricted to  $A$ , i.e. the relation

$$\in \upharpoonright A = \{\langle x, y \rangle: x \in A, y \in A, \text{ and } x \in y\}.$$

In a standard model, the metalevel/object-level distinction can be very delicate. If  $y$  is an  $\mathfrak{A}$ -set, and  $x \in y$ , we cannot then assume that the statement “ $x \in y$ ” makes any sense inside  $\mathfrak{A}$ . Unless  $x \in A$  as well, which is not necessary, the  $\mathfrak{A}$ -person will be unaware of the existence of  $x$ . Thus he may not recognise all the  $y$ -members that we do.

We recall now the expression  $\varphi \equiv \psi$  as an abbreviation for the  $\mathcal{L}$ -formula  $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$ .

AXIOM OF EXTENSIONALITY. This is the  $\mathcal{L}$ -formula

$$\text{Ext: } (\forall t)(t\epsilon u \equiv t\epsilon v) \supset u \approx v,$$

which formalises the principle that sets with the same members are equal.

In a model  $\mathfrak{A}$ , if  $x \in A$ , let

$$E_x = \{z : z \in A \text{ and } zEx\}.$$

Then  $\mathfrak{A} \models \text{Ext}$  iff  $E_x = E_y$  implies  $x \approx y$ , for all  $x, y \in A$ .

NULL SET:

$$(\exists t)(\forall u)(\sim(u\epsilon t))$$

“there exists a set with no members”. In  $\mathfrak{A}$  this is true when there is some  $x \in A$  such that  $E_x$  is the empty metaset.

PAIRS:

$$\forall u \forall v \exists t [\forall w (w\epsilon t \equiv w \approx u \vee w \approx v)]$$

“given sets  $x$  and  $y$  there exists a set having just  $x$  and  $y$  as members”, i.e. “ $\{x, y\}$  exists”.

POWERSETS: Let “ $v \subseteq u$ ” abbreviate the formula  $\forall w (w\epsilon v \supset w\epsilon u)$ , i.e. “ $v$  is a subset of  $u$ ”.

The axiom of powersets is the sentence

$$\forall u \exists t [\forall v (v\epsilon t \equiv v \subseteq u)]$$

formalising the statement “for any  $x$ , there is a set whose members are just the subsets of  $x$ ”.

UNIONS:

$$\forall u \exists t [\forall v (v\epsilon t \equiv \exists w (w\epsilon u \wedge v\epsilon w))]$$

Intuitively, all individuals in the universe are sets, so the members of  $x$  are themselves collections. This axiom states the existence of the union of all the members of  $x$ .

SEPARATION: If  $\varphi(v)$  is a formula with free  $v$ , the following is an instance of the Separation axiom schema

$$\text{Sep}_\varphi: \quad \forall u \exists t [\forall v (v\epsilon t \equiv v\epsilon u \wedge \varphi(v))]$$

i.e. “given  $x$ , there exists a set consisting just of the members of  $x$  satisfying  $\varphi$ ”. Or, “given  $x$ ,  $\{y : y \in x \ \& \ \varphi(y)\}$  exists”. This is a formal statement of the separation principle discussed in Chapter 1.

**BOUNDED SEPARATION:** A formula  $\varphi$  is *bounded* if all occurrences of  $\forall$  in  $\varphi$  are at the front of a subformula of  $\varphi$  of the form  $\forall v(v \in t \supset \psi)$ , and all occurrences of  $\exists$  are of the form  $\exists v(v \in t \wedge \psi)$ . Thus quantifiers in bounded formulae have readings of the form “for all  $v$  in  $t$ ” and “there exists a  $v$  in  $t$ ”. The *bounded separation* ( $\Delta_0$ -separation) schema takes as axioms all the formulae  $\text{Sep}_\varphi$  for *bounded*  $\varphi$ . It allows us to “separate out” a subset of  $x$  defined by a formula, provided that the quantifiers of that formula are restricted to range over sets.

The system  $Z_0$  of axiomatic set theory has, in addition to the classical axioms for first-order logic with identity (§11.3), the axioms of Extensionality, Null Set, Pairs, Powersets, Unions, and Bounded Separation. From  $\text{Sep}_\varphi$  and Ext one can derive in  $Z_0$  the sentence

$$\forall u \exists ! t [\forall v (v \in t \equiv v \in u \wedge \varphi(v))]$$

that asserts the existence of a unique set having the property that its members are precisely those members of  $x$  for which  $\varphi$  holds. Because of this we introduce expressions of the form  $\{u : \varphi\}$ , called *class abstracts*, as abbreviations for certain  $\mathcal{L}$ -formulae. The use of class abstracts is determined by stipulating that we write

$$\begin{aligned} v \in \{u : \varphi\} & \text{ for } \varphi[u/v] \\ v \approx \{u : \varphi\} & \text{ for } \forall t (t \in v \equiv \varphi[u/t]) \\ \{u : \varphi\} \in v & \text{ for } \exists t (t \in v \wedge t \approx \{u : \varphi\}) \end{aligned}$$

Class abstracts play the same sort of role in  $\mathcal{L}$  as do the corresponding expressions in the metalanguage. If  $\varphi$  has only the variable  $u$  free, then intuitively  $\{u : \varphi\}$  denotes a collection, the collection of all sets (individuals in the universe) having the property  $\varphi$ . For a model  $\mathfrak{A}$ ,  $\{u : \varphi\}$  will determine a metasubset of  $A$ , viz the collection

$$\mathfrak{A}_\varphi = \{x : x \in A \text{ and } \mathfrak{A} \models \varphi[x]\}.$$

In some cases, the metaset  $\mathfrak{A}_\varphi$  will correspond to an  $\mathfrak{A}$ -set, as above. This occurs when there is some  $y \in A$  such that  $\mathfrak{A}_\varphi = E_y = \{x : x \in A \text{ and } x E y\}$ . Thus if  $\varphi$  is  $\sim(u \approx u)$ , we find that  $\mathfrak{A}_\varphi = \emptyset$  (the empty metaset), and  $\mathfrak{A}_\varphi$  corresponds to an  $\mathfrak{A}$ -set iff the Null Set axiom is true in  $\mathfrak{A}$ .

The formula  $\text{Sep}_\varphi$  can now be given in the form

$$\forall u \exists t (t \approx \{v : v \in u \wedge \varphi(v)\}).$$

This is true in  $\mathfrak{A}$  when for each  $x \in A$  there is some  $y \in A$  such that  $E_y = E_x \cap \mathfrak{A}_\varphi$ .

Some familiar abstracts, and their abbreviations are

$$\begin{aligned}
 \mathbf{0} & \text{ for } \{u: \sim(u \approx u)\} \\
 \{u, v\} & \text{ for } \{t: t \approx u \vee t \approx v\} \\
 \{u\} & \text{ for } \{u, u\} \\
 u \cap v & \text{ for } \{t: t \varepsilon u \wedge t \varepsilon v\} \\
 u \cup v & \text{ for } \{t: t \varepsilon u \vee t \varepsilon v\} \\
 u - v & \text{ for } \{t: t \varepsilon u \wedge \sim(t \varepsilon v)\} \\
 \cup u & \text{ for } \{z: \exists t(t \varepsilon u \wedge z \varepsilon t)\} \\
 \cap u & \text{ for } \{z: \forall t(t \varepsilon u \supset z \varepsilon t)\} \\
 \mathbf{1} & \text{ for } \{\mathbf{0}\} \\
 u + 1 & \text{ for } u \cup \{u\} \\
 \mathcal{P}(u) & \text{ for } \{z: z \subseteq u\}
 \end{aligned}$$

EXERCISE 1. Let  $\varphi(v)$  be the formula  $v \approx \{u: u \varepsilon v\}$ . Explain why, for any  $x \in A$ ,  $\mathfrak{A} \models \varphi[x]$ . Show that  $\varphi(v)$  is a theorem of first-order logic.

EXERCISE 2. Let  $\varphi(t, u, v)$  be the formula  $t \approx \{u, v\}$ . Show that  $\mathfrak{A} \models \varphi[x, y, z]$  iff  $E_x = \{y, z\}$ .

EXERCISE 3. Show that the Pairs axiom can be written as

$$\forall u \forall v \exists t (t \approx \{u, v\}).$$

EXERCISE 4. Rewrite the other axioms of  $Z_0$  using class abstracts.  $\square$

To formalise the notions of *relation* and *function* we denote by  $\langle u, v \rangle$  the abstract  $\{\{u\}, \{u, v\}\}$ . The point of this definition is simply that it works, i.e. that we can derive in  $Z_0$  the sentence

$$\langle \langle u, v \rangle \approx \langle t, w \rangle \rangle \equiv (u \approx t \wedge v \approx w)$$

which captures the essential property of ordered pairs. Then we put

$$\begin{aligned}
 \{\langle u, v \rangle: \varphi\} & \text{ for } \{t: \exists u \exists v (t \approx \langle u, v \rangle \wedge \varphi)\} \\
 t \times w & \text{ for } \{\langle u, v \rangle: u \varepsilon t \wedge v \varepsilon w\} \\
 \mathbf{OP}(u) & \text{ for } \exists t \exists v (u \approx \langle t, v \rangle) \\
 \mathbf{Rel}(u) & \text{ for } \forall v (v \varepsilon u \supset \mathbf{OP}(v)) \\
 \mathbf{Fn}(u) & \text{ for } \mathbf{Rel}(u) \wedge \forall v \forall t \forall w (\langle v, t \rangle \varepsilon u \wedge \langle v, w \rangle \varepsilon u \supset t \approx w) \\
 \mathbf{Dom}(u) & \text{ for } \{t: \exists v (\langle t, v \rangle \varepsilon u)\}
 \end{aligned}$$

$$\begin{aligned} \mathbf{Im}(u) & \text{ for } \{t: \exists v(\langle v, t \rangle \varepsilon u)\} \\ \mathbf{\Delta}(u) & \text{ for } \{\langle v, v \rangle: v \varepsilon u\} \\ v \circ u & \text{ for } \{\langle t, w \rangle: \exists s(\langle t, s \rangle \varepsilon u \wedge \langle s, w \rangle \varepsilon v)\} \end{aligned}$$

Using these definitions we can construct from any  $Z_0$ -model  $\mathfrak{A} = \langle A, E, \approx \rangle$  a category  $\mathcal{E}(\mathfrak{A})$  by formalising our definition of the category **Set**. The  $\mathcal{E}(\mathfrak{A})$ -objects are the  $\mathfrak{A}$ -sets, i.e. the elements  $a \in A$ . The  $\mathcal{E}(\mathfrak{A})$ -arrows are the triples  $f = \langle a, k, b \rangle$ , where  $a, k$ , and  $b$  are  $\mathfrak{A}$ -sets, such that

$$\mathfrak{A} \models \varphi[a, k, b]$$

where  $\varphi(t, u, v)$  is the formula

$$\mathbf{Fn}(u) \wedge \mathbf{Dom}(u) \approx t \wedge \mathbf{Im}(u) \subseteq v.$$

We take the domain of arrow  $f$  to be  $a$ , and the codomain to be  $b$ . The composite of  $f = \langle a, k, b \rangle$  and  $g = \langle b, l, c \rangle$ , where  $\text{cod } f = \text{dom } g$ , is  $g \circ f = \langle a, h, c \rangle$ , where  $h \in A$  has

$$\mathfrak{A} \models \psi[h, k, l],$$

$\psi(t, u, v)$  being the formula  $t \approx v \circ u$ .

The identity arrow for  $a$  is  $\text{id}_a = \langle a, k, a \rangle$ , where, for  $\varphi(t, u)$  the formula  $t \approx \mathbf{\Delta}(u)$ , we have

$$\mathfrak{A} \models \varphi[k, a].$$

**THEOREM 1.** *If  $\mathfrak{A}$  is a model of all the  $Z_0$ -axioms, then  $\mathcal{E}(\mathfrak{A})$  is a well-pointed topos.*

**EXERCISE 5.** Verify in detail that Theorem 1 holds, by formalising in  $\mathcal{L}$ , and interpreting in  $\mathfrak{A}$ , the descriptions of pullbacks, terminal object, exponentials, and subobject classifier given for **Set**. □

**AXIOM OF INFINITY:** Let  $\mathbf{inf}(u)$  be the formula

$$\mathbf{0} \varepsilon u \wedge \forall v(v \varepsilon u \supset v \cup \{v\} \varepsilon u).$$

Intuitively  $\mathbf{inf}(u)$  asserts of a set  $x$  that the initial ordinal  $\emptyset$  is an element of  $x$ , and  $x$  is closed under the successor function (recall  $n + 1 = n \cup \{n\}$  in **Set**). Hence  $\omega \subseteq x$ , and  $x$  has infinitely many members. The axiom of infinity is

$$\mathbf{Inf}: \quad \exists u(\mathbf{inf}(u)).$$

In  $Z_0 + \mathbf{Inf}$  one can derive

$$\exists t(\mathbf{inf}(t) \wedge t \approx \cap \{u: \mathbf{inf}(u)\})$$

and so in any  $Z_0$  model  $\mathfrak{A}$  such that  $\mathfrak{A} \models \text{Inf}$ , there will be an  $\mathfrak{A}$ -set that the  $\mathfrak{A}$ -person thinks is the set of all finite ordinals. By formalising the discussion of §12.2 we can then show that this  $\mathfrak{A}$ -set produces a natural numbers object for  $\mathcal{E}(\mathfrak{A})$ , i.e.  $\mathcal{E}(\mathfrak{A}) \models \text{NNO}$ .

**AXIOM OF CHOICE:** There is some choice about which sentence we use to formalise the choice principle in classical set theory. Perhaps the simplest is

$$\forall u \forall v (\text{Fn}(u) \wedge \sim(\text{Dom}(u) \approx \mathbf{0}) \wedge \text{Im}(u) \subseteq v \supset \exists t (\text{Fn}(t) \wedge \text{Dom}(t) \approx v \wedge \text{Im}(t) \subseteq \text{Dom}(u) \wedge u \circ t \circ u \approx u))$$

which formalises the statement AC of §12.1. For a  $Z_0$ -model of this sentence we will have  $\mathcal{E}(\mathfrak{A}) \models \text{AC}$ .

**AXIOM OF REGULARITY:**

$$\text{Reg: } \forall u (\sim(u \approx \mathbf{0}) \supset \exists v (v \in u \wedge v \cap u \approx \mathbf{0}))$$

Intuitively, Reg asserts that if  $x \neq \emptyset$  then  $x$  has a member  $y \in x$  such that  $y$  and  $x$  have no members in common. The basic viewpoint of set theories of the type that we are developing is that sets are built up “from below” by operations such as union, powerset, separation etc. Reg asserts that if  $x$  exists, then its construction must have started somewhere, i.e. we cannot have all members of  $x$  consisting of members of  $x$ . This axiom proscribes relationships like  $x \in x$ ,  $x \in y \in x$ ,  $x \in y \in z \in x$ , etc., as well as “infinitely descending” membership chains  $x_1 \ni x_2 \ni x_3 \ni \dots$

**AXIOM OF REPLACEMENT:** Intuitively, the replacement axiom schema asserts that if the domain of a function is a set (individual in the universe) then so is its range, or image. The type of function it deals with is the functional relation defined by a formula  $\varphi$  with two free variables.

$$\text{Rep}_\varphi \quad \forall u \forall v \forall w (\varphi(u, v) \wedge \varphi(u, w) \supset v \approx w) \supset \forall t \exists s (s \approx \{v : \exists u (u \in t \wedge \varphi(u, v))\})$$

This asserts that if the ordered pairs satisfying  $\varphi$  form a relation with the “unique output” property of functions, and if for each  $u \in t$ ,  $f(u)$  is the unique individual such that  $\langle u, f(u) \rangle$  satisfies  $\varphi$ , then the collection  $\{f(u) : u \in t\}$  is a set.

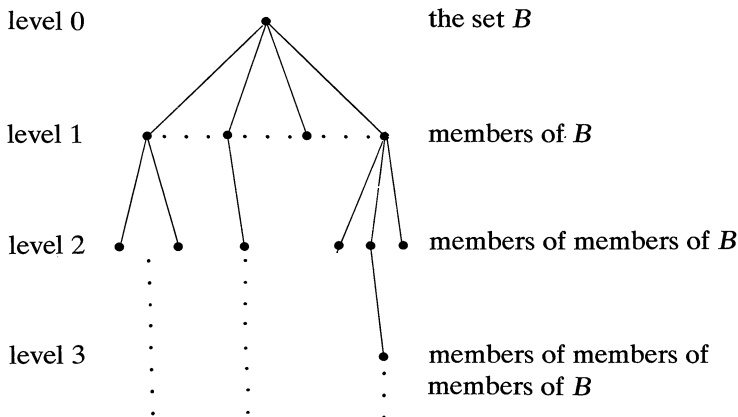
The Zermelo–Fraenkel system of set-theory, ZF, can be defined as  $Z_0 + \text{Inf} + \text{Reg} + \text{Replacement}$ . We see then that ZF is a much more powerful system than is needed to construct topoi. The description of **Set**,



when formalised, turns any model of the weaker system  $Z_0$  into a well-pointed topos. In order to reverse the procedure, and construct models of set theory from topoi, we have to analyse further the arrow-theoretic account of the membership relation.

### 12.4. Transitive sets

A set  $B$  determines a metamembership structure that can be displayed as:



This diagram is called the *membership tree* of  $B$ . The tree is in fact upside down – from each point there is a unique path upward towards the root (top point) of the tree. The collection  $T_B$  of all points in the tree except the top point  $B$  has a special property called *transitivity*. In general a set  $A$  is *transitive* if it satisfies the condition

$$x \in A \text{ implies } x \subseteq A,$$

i.e. if  $x$  is a member of  $A$  then all members of  $x$  are themselves members of  $A$ . (Notice that if a model  $\mathcal{U}$  is standard, and is based on a transitive  $A$ , then for each  $\mathcal{U}$ -set  $x$  all the metamembers of  $x$  will be  $\mathcal{U}$ -sets. Thus the  $\mathcal{U}$ -person will see the same members of  $x$  that we do.)

Now if  $x$  appears in  $T_B$  at say level  $n$ , then all the members of  $x$  appear in  $T_B$  at level  $n + 1$ . So  $T_B$  is transitive. But if  $A$  is any transitive set that contains  $B$ , it follows that  $T_B \subseteq A$ . The assumption that  $B \subseteq A$  means that all level 1 points of  $T_B$  are in  $A$ . Then if all level  $n$  points are in  $A$ , transitivity of  $A$  puts all level  $n + 1$  points in  $A$ . Thus by an inductive proof we show that  $T_B$  is contained in all transitive sets containing  $B$ . It is

the “smallest” transitive set containing  $B$ , and so is called the *transitive closure* of  $B$ .

**AXIOM OF TRANSITIVITY:** We write  $\mathbf{Tr}(u)$  for the formula  $\forall v(v\in u \supset v \subseteq u)$ . The axiom of transitivity is

$$\text{TA: } \quad \forall t \exists u (t \subseteq u \wedge \mathbf{Tr}(u))$$

In  $Z_0 + \text{TA}$  we can derive

$$\forall t \exists ! u (t \subseteq u \wedge \mathbf{Tr}(u) \wedge \forall v (t \subseteq v \wedge \mathbf{Tr}(v) \supset u \subseteq v))$$

which, under interpretation, states that the transitive closure of any set exists as an individual in the universe.

**EXERCISE 1.** Derive, in  $Z_0 + \text{TA}$ ,

$$\forall t \exists u (u \approx \bigcap \{v : t \subseteq v \wedge \mathbf{Tr}(v)\}) \quad \square$$

The role of trees in describing membership is this:  $A \in B$  iff the membership tree of  $A$  is isomorphic to the tree of all points below a particular level 1 point of the  $B$ -tree. This observation was lifted to the topos setting by William Mitchell [72] and Julian Cole [73] to define the notion of “ $\mathcal{E}$ -tree” and thereby construct models of set-theory from Boolean topoi.

An alternative approach to a topos-theoretic reconstruction of set theory was subsequently developed by Gerhard Osius [74], based on a characterisation of those **Set**-objects that are transitive as sets. Transitivity of  $A$  simply means that if  $x \in A$  then  $x \in \mathcal{P}(A)$ , i.e.  $A$  is transitive iff  $A \subseteq \mathcal{P}(A)$ . This property gives transitive sets a tractability not enjoyed by sets that are not “closed under  $\in$ ”. The relations  $E \subseteq A \times A$  on a set  $A$  are in bijective correspondence with the functions  $r_E : A \rightarrow \mathcal{P}(A)$ . Given  $E$ , then  $r_E$  assigns to  $y \in A$  the subset

$$r_E(y) = \{x : x \in A \text{ and } xEy\} = E_y \text{ of } A.$$

In the case that  $E$  is the membership relation

$$\in \upharpoonright A = \{(x, y) : x \in A, y \in A \text{ and } x \in y\},$$

we find that

$$r_\in(y) = \{x : x \in A \text{ and } x \in y\}.$$

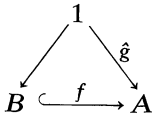
But if  $A$  is *transitive*, this simplifies:  $x \in y$  implies  $x \in A$  for  $y \in A$ , and so

$$r_\in(y) = \{x : x \in y\} = y.$$

Thus we see that for transitive  $A$ , the membership relation  $\in \upharpoonright A$  on  $A$  gives rise to the inclusion  $A \hookrightarrow \mathcal{P}(A)$  as  $r_\in$ , making  $A$  a subobject of  $\mathcal{P}(A)$ .

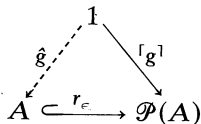
Now let us consider the problem of defining “membership” in a topos  $\mathcal{E}$ . We already know what  $x \in f$  means if  $x$  is an “element”  $1 \rightarrow a$  of an  $\mathcal{E}$ -object  $a$ , and  $f: b \rightarrow a$  is a subobject of  $a$  (§4.8). But what about  $g \in f$ , where  $g: c \rightarrow a$  is some other subobject of  $a$ ?

Returning to **Set**, we see that if  $g: C \hookrightarrow A$  and  $f: B \hookrightarrow A$  are subsets of  $A$ , then if  $C$  is going to be an element of  $B$ ,  $C \in B$ , then since  $B \subseteq A$  we will have  $C \in A$ , so there will be an arrow  $\hat{g}: \{0\} \rightarrow A$  with  $\hat{g}(0) = C$ . But then, knowing that  $\hat{g}$  exists, i.e.  $C \in A$ , deciding whether  $C \in B$  is equivalent to deciding whether  $\hat{g} \in f$ , i.e. whether



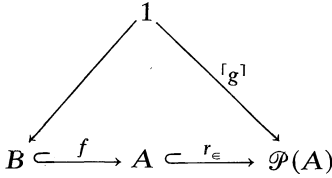
$\hat{g}$  factors through  $f$ .

Thus the question of membership of  $C$  in  $B$  can be resolved in the language of arrows once we know, categorially, whether  $\hat{g}$  exists. In the event that  $A$  is transitive, the problem can be transferred into  $\mathcal{P}(A)$  and restated. In general,  $g: C \hookrightarrow A$ , as a subset of  $A$ , corresponds to an “element”  $\lceil g \rceil: 1 \rightarrow \mathcal{P}(A)$  of the powerset of  $A$ , where  $\lceil g \rceil(0) = C$ . Identifying  $\mathcal{P}(A)$  with  $2^A$ , we see that  $\lceil g \rceil$  becomes  $\lceil \chi_g \rceil$ , the *name* of  $\chi_g: A \rightarrow 2$  as defined in §4.1. Then if there is an inclusion  $r_\in: A \hookrightarrow \mathcal{P}(A)$ , we have that  $C \in A$ , i.e.  $\hat{g}$  as defined is an arrow from  $1$  to  $A$ , iff  $\lceil g \rceil \in r_\in$ , that is,  $C \in A$  iff  $\hat{g}$  exists to make



$\lceil g \rceil$  factor (uniquely) through  $r_\in$ .

Altogether then, for transitive  $A$ , we can characterise the “local set theory” of subsets of  $A$ . For  $f: B \hookrightarrow A$  and  $g: C \hookrightarrow A$ , we have  $g \in f$  iff  $C \in B$  iff the *name* of  $g$  factors through  $r_\in \circ f$ ,



i.e. iff  $\lceil g \rceil \in r_\in \circ f$ .

Characterising the local set theory of an object (set) is, as Osius notes, sufficient for the needs of the “working mathematician”, who tends to deal with any given problem within the context of some fixed “universal” set  $A$ . But the “global” question of membership for  $\mathcal{E}$  can be reduced to the local one. First we need to deal with equality of subobjects. If  $f: b \twoheadrightarrow a$  and  $g: c \twoheadrightarrow a$  have the same codomain, we know what it means for  $f$  and  $g$  to represent the same “subset” – it means that  $f \approx g$  in  $\text{Sub}(a)$ . But  $f: b \twoheadrightarrow a$  and  $g: c \twoheadrightarrow d$  may still represent the same set, even if they have distinct codomains. In **Set**, the codomains of  $f: B \twoheadrightarrow A$  and  $g: C \twoheadrightarrow D$  may overlap, and indeed we may have  $f(B) = g(C) \subseteq A \cap D$ , in which case we would want to put  $f \approx g$ . But it is clear in this situation that if  $T$  is any set that includes both  $A$  and  $D$  (e.g.  $T = A \cup D$ ), so that there are inclusions  $i: A \hookrightarrow T$  and  $j: D \hookrightarrow T$ , then  $f(B) = g(C)$  iff  $i(f(B)) = j(g(C))$ . Thus  $f \approx g$  iff in  $\text{Sub}(T)$ ,  $i \circ f \approx j \circ g$ .

So the identification of subobjects – the general definition of  $f \approx g$  – is resolved by localising to the set-theory of any object that includes the co-domains of both  $f$  and  $g$ . The global membership for **Set** can now be described as follows. For  $f: B \twoheadrightarrow A$  and  $g: C \twoheadrightarrow D$  we put

$$g \in f \quad \text{iff} \quad \text{for some transitive } T \text{ including both } A \text{ and } D, \text{ in } \mathcal{P}(T) \text{ we have } [j(g(C) \hookrightarrow T)] \in [i(f(B)) \hookrightarrow T].$$

Here  $i$  and  $j$  are the inclusion as above. For a suitable  $T$  we may use the transitive closure of  $A \cup D$ . Although the arrows  $f$  and  $i(f(B)) \hookrightarrow T$  are not the same thing, the definition of membership is justified precisely because they are equal as subobjects, i.e. they bear the relation “ $\approx$ ” to each other. Similarly the arrows  $g$  and  $j(g(C)) \hookrightarrow T$  represent the same set.

**EXERCISE 2.** Verify this last statement.

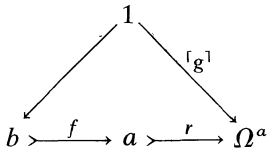
EXERCISE 3. Show that the definition of  $g \in f$  does not depend on the choice of appropriate  $T$ .

EXERCISE 4. For any sets  $A, B$ , show  $A \in B$  iff  $\text{id}_A \in \text{id}_B$ .

EXERCISE 5. Let  $T_A$  be the transitive closure of  $A$ , so that  $A \hookrightarrow T_A$ . Show that  $g \in f$  iff for some  $h: Y \hookrightarrow T_A$ ,  $g \simeq h$  and in  $\mathcal{P}(T_A)$ ,  $h \in (f(B) \hookrightarrow T_A)$ . Thus  $g \in f$  iff  $g$  is “equal” to a member of  $f(B)$  in  $T_A$ .  $\square$

In lifting these considerations to a topos  $\mathcal{E}$ , we take an  $\mathcal{E}$ -object  $a$  that is the domain of a subobject  $r: a \rightarrow \Omega^a$  of its own power object. Then a “membership” relation  $\in_r$  can be defined on  $\text{Sub}(a)$  by putting, for  $f: b \rightarrow a$  and  $g: c \rightarrow a$ ,

$$g \in_r f \text{ iff } \lceil g \rceil \in r \circ f$$



i.e. iff  $\lceil g \rceil$  factors through  $r \circ f$ , where  $\lceil g \rceil = \lceil \chi_g \rceil$  is the exponential adjoint of  $\chi_g \circ \text{pr}_a: 1 \times a \rightarrow \Omega$ .

Although this definition can be made for any  $r$  of this form, the simple requirement that  $r$  be monic does not capture the essence of transitivity. Indeed, it does not even capture the fact that for transitive  $A$ ,  $A \hookrightarrow \mathcal{P}(A)$  arises from the *metamembership* relation  $\in \upharpoonright A$ . For if  $\mathfrak{A} = \langle A, E, \Delta \rangle$  is any normal  $\mathcal{L}$ -model, then since  $r_E(y) = \{x: x \in A \text{ and } xEy\} = E_y$ ,  $r_E: A \rightarrow \mathcal{P}(A)$  will be monic if (and only if)  $\mathfrak{A} \models \text{Ext}$ .

So the problem remains of determining when  $r: A \rightarrow \mathcal{P}(A)$  represents the membership relation of a *transitive set*.

COLLAPSING LEMMA (Mostowski [49]). *Let  $E \subseteq A \times A$  be a relation on  $A$ . Then there exists a transitive set  $B$  such that*

$$\langle A, E \rangle \cong \langle B, \in \upharpoonright B \rangle$$

iff

- (1)  $E$  is extensional, and
- (2)  $E$  is well-founded.

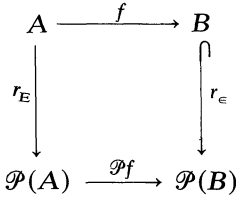
$\square$

Here, (1) means that  $r_E : A \rightarrow \mathcal{P}(A)$  is monic. Well-foundedness means that every non-empty subset of  $A$  has an  $E$ -minimal element. That is, if  $C \subseteq A$  and  $C \neq \emptyset$ , there exists  $x \in C$  such that  $E_x \cap C = \emptyset$ , so that if  $yEx$ , then  $y \notin C$ .

The sense of isomorphism in  $\langle A, E \rangle \cong \langle B, \in \upharpoonright B \rangle$  is that “ $E$ -membership” within  $A$  looks exactly like “ $\in$ -membership” within  $B$ . This requires that there be a bijective map  $f: A \cong B$  such that

$$(*) \quad xEy \text{ iff } f(x) \in f(y), \text{ all } x, y \text{ in } A.$$

For such an  $f$ , the diagram



commutes, where  $\mathcal{P}f$  assigns to  $C \in \mathcal{P}(A)$  (i.e.  $C \subseteq A$ ) its  $f$ -image  $f[C] = \{f(y) : y \in C\} \in \mathcal{P}(B)$ . The diagram requires, for  $x \in A$ , that

$$f[E_x] = f(x)$$

i.e.

$$\{f(y) : yEx\} = \{z : z \in f(x)\},$$

which for bijective  $f$  is equivalent to (\*).

Mostowski’s lemma has been stated as a fact about our metaset-theory. It can be expressed as a sentence of the formal language  $\mathcal{L}$ . “ $E$  is a relation on  $A$ ” would be replaced by “**Rel**( $u$ )  $\wedge u \subseteq v \times v$ ”,  $\in \upharpoonright B$  would be replaced by an abstract of the form  $\epsilon \upharpoonright t = \{\langle u, v \rangle : u \epsilon t \wedge v \epsilon t \wedge u \epsilon v\}$ , and so on. The resulting formal sentence can then be derived only if we assume the full strength of the ZF axioms. Thus Mostowski’s “theorem” is a theorem only if our metaset-theory satisfies all the ZF-axioms.

Note that the lemma implies in particular that  $\in \upharpoonright B$  is well-founded on  $B$ . This in fact can be deduced if we assume our metaset-theory satisfies the Regularity axiom. For then if  $C \subseteq B$  is non-empty there will be some  $x \in C$  with  $x \cap C = \emptyset$ , so that if  $y \in x$ ,  $y \notin C$ , making  $x$   $\in$ -minimal in  $B$ .

Now a well-founded relation  $E$  on  $A$  can be used to define functions with domain  $A$  by “recursion” in a similar manner to the operation of nno’s. The intuitive idea is that in order to define  $f(x)$ , where  $f: A \rightarrow B$ ,

we make the inductive assumption that  $f(y)$  has been defined for all  $y \in E_x$ , i.e.  $f$  is defined for all “ $E$ -members” of  $x$ . We then input the collection  $\{f(y): y \in E_x\}$  to some other function  $g$  and let  $f(x)$  be defined to be the resulting output. Thus

$$f(x) = g(\{f(y): y \in E_x\}) = g(f[E_x])$$

i.e.

$$(**) \quad f(x) = g(\mathcal{P}f(E_x))$$

Since we want  $f(x) \in B$ , and since  $\mathcal{P}f(E_x) \in \mathcal{P}(B)$ ,  $g$  has to be a function from  $\mathcal{P}(B)$  to  $B$ . Equation  $(**)$  states that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow r_E & & \uparrow g \\ \mathcal{P}(A) & \xrightarrow{\mathcal{P}f} & \mathcal{P}(B) \end{array}$$

commutes. But, given  $g$ , if  $f$  exists to make this diagram commute then it is uniquely determined by the equation  $(**)$ .

**THEOREM 1.**  *$E$  is well-founded on  $A$  iff for any set  $B$  and function  $g: \mathcal{P}(B) \rightarrow B$  there exists exactly one function  $f: A \rightarrow B$  making the last diagram commute.*

A proof of this result is given by Osius in [74]. Again the statement can be expressed as an  $\mathcal{L}$ -sentence, but this time it can be derived just using  $Z_0$ -axioms. Thus we see that in ZF, transitive sets are essentially extensional (monic) well-founded relations, and that well-foundedness can be characterised, even in  $Z_0$ , by an arrow-theoretic property.

This will lead us to a definition of “transitive sets” in a topos, for which we will also appeal to the following description of inclusions between transitive sets.

**THEOREM 2.** *If  $A$  and  $B$  are transitive then*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow r_E & & \downarrow r_B \\ \mathcal{P}(A) & \xrightarrow{\mathcal{P}f} & \mathcal{P}(B) \end{array}$$

commutes iff  $A \subseteq B$  and  $f$  is the inclusion  $A \hookrightarrow B$ .

PROOF. If  $f$  is the inclusion, it is clear, for  $x \in A$  (hence  $x \subseteq A$ ) that  $f[x] = \{y : y \in x\} = x$ , so the diagram commutes. On the other hand, if the diagram does commute, then  $f(x) = f[x]$ , for all  $x \in A$ . To show that  $f$  is the inclusion we have to show that  $f(x) = x$ , all  $x$  in  $A$ , or that

$$C = \{x : x \in A \text{ and } f(x) \neq x\} = \emptyset$$

To do this we need to assume  $\in \upharpoonright A$  is well-founded.

Then if  $C$  were a non-empty subset of  $A$  it would have an element  $x_0$  that is  $\in$ -minimal in  $C$ . Thus  $x_0 \neq f(x_0)$ , but (using transitivity)

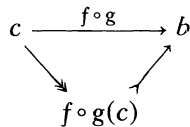
$$y \in x_0 \text{ implies } y \notin C, \text{ and so } f(y) = y.$$

But then  $f(x_0) = f[x_0] = \{f(y) : y \in x_0\} = \{y : y \in x_0\} = x_0$ , a contradiction.  $\square$

Theorem 2 can be expressed as an  $\mathcal{L}$ -sentence derivable in  $Z_0 + \text{Reg}$  (Regularity being used to give well-foundedness of  $\in \upharpoonright A$ ). The proof of the theorem indicates what lies behind Theorem 1, i.e. how inductive definitions and constructions depend on the property of well-foundedness for their validity.

### 12.5. Set-objects

IMAGES: If  $f : a \rightarrow b$  is an arrow in topos  $\mathcal{E}$ , then for each subobject  $g : c \rightarrow a$  of  $a$  we define the image  $f[g] : f(g(c)) \rightarrow b$  of  $g$  under  $f$  to be the monic part of the epi-monic factorisation



Thus  $f[g] = \text{im}(f \circ g)$ .

This construction establishes a map from  $\text{Sub}(a)$  to  $\text{Sub}(b)$ , that in fact has an internal version  $\Omega^f : \Omega^a \rightarrow \Omega^b$ . In **Set**  $\Omega^f$  is the function  $\mathcal{P}f : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$  used in the last section.

Now by the identification of subobjects with their characters, the image construction assigns to each  $h : a \rightarrow \Omega$  an arrow  $f[h] : b \rightarrow \Omega$ . Then, starting with  $f : a \rightarrow b$  we form  $1 \times f : \Omega^a \times a \rightarrow \Omega^a \times b$  and then take the image  $1_{\Omega^a} \times f[ev_a]$  of  $ev_a : \Omega^a \times a \rightarrow \Omega$  under  $1_{\Omega^a} \times f$ .



$\Omega^f$  is then defined as the unique arrow making

$$\begin{array}{ccc}
 \Omega^b \times b & \xrightarrow{ev_b} & \Omega \\
 \Omega^f \times 1_b \uparrow & \nearrow & \\
 \Omega^a \times b & \xrightarrow{1_{\Omega^a} \times f[ev_a]} & \Omega
 \end{array}$$

commute, i.e.  $\Omega^f$  is the exponential adjoint of  $1_{\Omega^a} \times f[ev_a]$ .

EXERCISE 1. If  $f: a \twoheadrightarrow b$  is monic, then  $f[g] = f \circ g$ .

EXERCISE 2. Verify that the definition of  $\Omega^f$ -characterises  $\mathcal{P}f$  in **Set**.

EXERCISE 3. Show that  $\Omega^{1_a} = 1_{\Omega^a}$ , and that if

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 & \searrow h & \downarrow g \\
 & & c
 \end{array}$$

commutes, then so does

$$\begin{array}{ccc}
 \Omega^a & \xrightarrow{\Omega^f} & \Omega^b \\
 & \searrow \Omega^h & \downarrow \Omega^g \\
 & & \Omega^c
 \end{array}$$

i.e.  $\Omega^{g \circ f} = \Omega^g \circ \Omega^f$ .

EXERCISE 4. Given  $c \twoheadrightarrow^g a \xrightarrow{f} b$ , show

$$\begin{array}{ccc}
 & \xrightarrow{g^1} & \Omega^a \\
 1 & \nearrow & \downarrow \Omega^f \\
 & \xrightarrow{[f[g]]^1} & \Omega^b
 \end{array}$$

commutes. □

DEFINITION. A *transitive set object* (tso) is an  $\mathcal{E}$ -arrow  $r: a \twoheadrightarrow \Omega^a$  that is

- (1) *extensional*, i.e. monic, and
- (2) *recursive*, i.e. for any  $\mathcal{E}$ -arrow of the form  $g: \Omega^b \rightarrow b$  there is

exactly one  $\mathcal{E}$ -arrow  $f: a \rightarrow b$  making

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow r & & \uparrow g \\ \Omega^a & \xrightarrow{\Omega^f} & \Omega^b \end{array}$$

commute. ( $f$  is said to be defined recursively from  $g$  over  $r$ :—  $f = \text{rec}_r(g)$ ).

EXERCISE 5.  $0 \rightarrow \Omega^0$  is a tso.

EXERCISE 6.  $\perp : 1 \rightarrow \Omega^1$  is a tso (why is this so in **Set**) □

If  $r: a \rightarrow \Omega^a$  and  $s: b \rightarrow \Omega^b$  are “relations” then  $h: a \rightarrow b$  is an *inclusion* from  $r$  to  $s$ , written  $h: r \hookrightarrow s$ , iff

$$\begin{array}{ccc} a & \xrightarrow{h} & b \\ \downarrow r & & \downarrow s \\ \Omega^a & \xrightarrow{\Omega^h} & \Omega^b \end{array}$$

commutes. We write  $r \subseteq s$  if there exists an inclusion  $h: r \hookrightarrow s$ .

EXERCISE 7. Show that  $(0 \rightarrow \Omega^0) \subseteq (r: a \twoheadrightarrow \Omega^a)$ , for any tso  $r$ .

EXERCISE 8.  $r \subseteq r$ .

EXERCISE 9.  $r \subseteq s \subseteq t$  implies  $r \subseteq t$ . (cf. Exercise 3) □

An inclusion between transitive set-objects, if it exists, is unique. To see this, we introduce a construction that assigns to each monic  $s: b \twoheadrightarrow \Omega^b$  a unique arrow  $\hat{s}: \Omega^{\tilde{b}} \rightarrow \tilde{b}$ , where  $\tilde{b}$  is the codomain of the partial arrow classifier  $\eta_b: b \rightarrow \tilde{b}$  described in §11.8. The arrow

$$\Omega^{\eta_b}: \Omega^b \rightarrow \Omega^{\tilde{b}}$$

will in fact be monic, since  $\eta_b$  is (Osius [74], Proposition 5.8(a)).  $\hat{s}$  is then defined as the unique arrow making

$$\begin{array}{ccc} b & \xrightarrow{\Omega^{\eta_b} \circ s} & \Omega^{\tilde{b}} \\ \downarrow 1_b & & \downarrow \hat{s} \\ b & \xrightarrow{\eta_b} & \tilde{b} \end{array}$$

a pullback (note that  $\Omega^{n_b} \circ s$  is monic iff  $s$  is monic).

**THEOREM 1.** *If  $r : a \rightarrow \Omega^a$  is recursive, and  $s : b \rightarrow \Omega^b$  extensional, then*

(1)  *$f : a \rightarrow b$  is an inclusion iff*

$$\begin{array}{ccccc}
 a & \xrightarrow{f} & b & \xrightarrow{\eta_b} & \tilde{b} \\
 \downarrow r & & & & \uparrow \hat{s} \\
 \Omega^a & \xrightarrow{\Omega^f} & \Omega^b & \xrightarrow{\Omega^{\eta_b}} & \Omega^{\tilde{b}}
 \end{array}$$

*commutes, iff  $\eta_b \circ f = \text{rec}_r(\hat{s})$ .*

(2) *If  $r \subseteq s$  then there is a unique inclusion  $r \hookrightarrow s$  of  $r$  into  $s$ .*

**PROOF.** (1) Consider

$$\begin{array}{ccccc}
 a & \xrightarrow{f} & b & \xrightarrow{\eta_b} & \tilde{b} \\
 \downarrow r & & \downarrow s & & \uparrow \hat{s} \\
 \Omega^a & \xrightarrow{\Omega^f} & \Omega^b & \xrightarrow{\Omega^{\eta_b}} & \Omega^{\tilde{b}}
 \end{array}$$

The right hand square always commutes, by the definition of  $\hat{s}$ . Then if  $f$  is an inclusion, the left hand square commutes, hence the whole diagram does. Conversely, if the perimeter of the diagram commutes then this means precisely that the perimeter of the diagram

$$\begin{array}{ccccc}
 & & & \Omega^{\eta_b} \circ \Omega^f \circ r & \\
 & \curvearrowright & & \searrow & \\
 a & & & & \Omega^{\tilde{b}} \\
 \downarrow f & \dashrightarrow k & \downarrow \Omega^{\eta_b} \circ s & & \downarrow \hat{s} \\
 & b & & & \Omega^b \\
 & \downarrow 1_b & & & \downarrow \eta_b \\
 & b & & & \tilde{b}
 \end{array}$$

commutes, and so by the universal property of the inner square as pullback, the unique  $k$  exists as shown to make the whole diagram commute. Then  $1_b \circ k = f$ , and so  $k = f$ . Hence from the upper triangle

$$\Omega^{\eta_b} \circ s \circ f = \Omega^{\eta_b} \circ \Omega^f \circ r$$

Since  $\Omega^{\eta_b}$  is monic, this gives  $s \circ f = \Omega^f \circ r$ , i.e. the left hand square of the previous diagram commutes, making  $f$  an inclusion.

To complete part (1), note that since  $\Omega^{(\eta_b \circ f)} = \Omega^{\eta_b} \circ \Omega^f$ , recursiveness of  $r$  implies the diagram commutes precisely when  $\eta_b \circ f$  is the unique arrow defined recursively from  $\hat{s}$  over  $r$ .

(2) If  $f_1: r \hookrightarrow s$  and  $f_2: r \hookrightarrow s$ , then by (1),  $\eta_b \circ f_1 = \eta_b \circ f_2 = \text{rec}_r(\hat{s})$ . Since  $\eta_b$  is monic we get  $f_1 = f_2$ . □

**THEOREM 2.** *If  $r$  and  $s$  are tso's, then*

- (1) *If  $r \subseteq s$ , the (unique) inclusion  $r \hookrightarrow s$  is monic.*
- (2) *If  $r \subseteq s \subseteq r$ , then  $r \cong s$ , i.e. the inclusions  $r \hookrightarrow s$  and  $s \hookrightarrow r$  are iso.*

**PROOF.** (1) Consider

$$\begin{array}{ccccc}
 a & \xrightarrow{f} & b & \xrightarrow{g} & \tilde{a} \\
 \downarrow r & & \downarrow s & & \uparrow \hat{r} \\
 \Omega^a & \xrightarrow{\Omega^f} & \Omega^b & \xrightarrow{\Omega^g} & \Omega^{\tilde{a}}
 \end{array}$$

Here  $\hat{r}$  is defined by the construction prior to Theorem 1, so  $\hat{r} \circ \Omega^{\eta_a} \circ r = \eta_a \circ 1_a = \eta_a$ . Hence

$$\begin{array}{ccc}
 a & \xrightarrow{\eta_a} & \tilde{a} \\
 \downarrow r & & \uparrow \hat{r} \\
 \Omega^a & \xrightarrow{\Omega^{\eta_a}} & \Omega^{\tilde{a}}
 \end{array}$$

commutes, showing that  $\eta_a$  is the arrow  $\text{rec}_r(\hat{r})$ .

In the previous diagram,  $f$  is the inclusion  $r \hookrightarrow s$ , so the left hand diagram commutes.  $g$  is defined to be the arrow  $\text{rec}_s(\hat{r})$  given by recursion from  $\hat{r}$  over  $s$ . But then the whole diagram commutes, and so  $g \circ f = \text{rec}_r(\hat{r}) = \eta_a$ . Thus  $g \circ f$  is monic, so  $f$  itself must be monic (Exercise 3.1.2).

(2) If  $r \subseteq s \subseteq r$ , then from

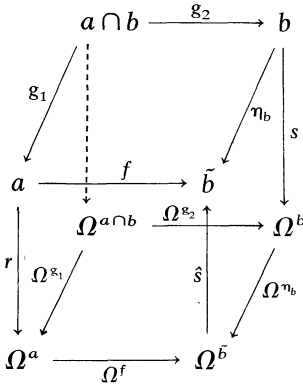
$$\begin{array}{ccccc}
 a & \xleftarrow{f} & b & \xleftarrow{g} & a \\
 \downarrow r & & \downarrow s & & \downarrow r \\
 \Omega^a & \xrightarrow{\Omega^f} & \Omega^b & \xrightarrow{\Omega^g} & \Omega^a
 \end{array}$$

we see that  $g \circ f: r \hookrightarrow r$ . But obviously  $1_a: r \hookrightarrow r$ , so by Theorem 1 (2),  $g \circ f = 1_a$ . Similarly  $f \circ g = 1_b$ , hence  $f: a \cong b$ , with  $f$  and  $g$  inverse to each other. □

Thus, defining  $r \approx s$  iff  $r \subseteq s$  and  $s \subseteq r$  leads to a definition of equality of (isomorphism classes of) transitive set-objects, with respect to which the inclusion relation becomes a partial ordering. Osius then gives constructions for

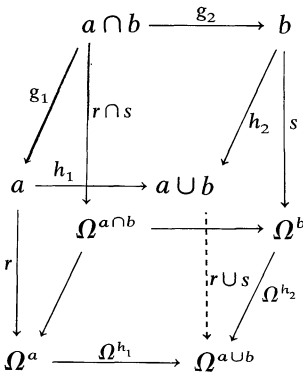
- (i) the intersection  $r \cap s : a \cap b \rightarrow \Omega^{a \cap b}$ , which proves to be the greatest lower bound of  $r$  and  $s$  in the inclusion ordering of tso's; and
- (ii) the union  $r \cup s : a \cup b \rightarrow \Omega^{a \cup b}$ , which is the least upper bound of  $r$  and  $s$ .

For (i), the cube



is formed by first defining  $f$  to be  $\text{rec}_*(\hat{s})$ , and obtaining the top face as the pullback of  $f$  along  $\eta_b$ . Thus the right-hand face is the square defining  $\hat{s}$ , the front face the square defining  $f$ . The bottom square then proves to be a pullback whose universal property yields the unique arrow  $a \cap b \rightarrow \Omega^{a \cap b}$  making the whole diagram commute. This arrow is  $r \cap s$ .

For (ii),  $a \cup b$  comes from the pushout



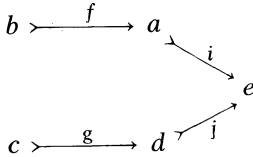
of  $g_1$  and  $g_2$ , with  $r \cup s$  arising from the co-universal property of pushouts.

DEFINITION. A *set-object* in a topos  $\mathcal{E}$  is a pair  $(f, r)$  of  $\mathcal{E}$ -arrows of the form

$$b \xrightarrow{f} a \xrightarrow{r} \Omega^a,$$

where  $r$  is a transitive set-object.

Equality of set-objects is defined as follows:  $(f, r) \simeq_{\mathcal{E}} (g, s)$  iff for some tso  $t: e \rightarrow \Omega^e$  such that  $r \subseteq t$  and  $s \subseteq t$ ,



we have  $i[f] \simeq j[g]$  in  $\text{Sub}(e)$ , (i.e.  $i \circ f \simeq j \circ g$ , since  $i$  and  $j$  are monic) where  $i$  and  $j$  are the inclusions  $i: r \hookrightarrow t$  and  $j: s \hookrightarrow t$ .

Osius establishes that the definition is independent of the choice of the tso  $t$  containing  $r$  and  $s$ : the condition holds for some such  $t$  iff it holds for all such  $t$  (hence iff it holds when  $t = r \cup s$ ).

EXERCISE 10.  $(f, r) \simeq_{\mathcal{E}} (g, r)$  iff in  $\text{Sub}(a)$ ,  $f \simeq g$ .

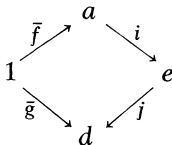
EXERCISE 11. Suppose that  $\lceil f \rceil \in r$  and  $\lceil g \rceil \in s$ , i.e. there are commutative diagrams



for certain elements  $\bar{f}$  and  $\bar{g}$ . Show that  $f \in_r 1_a$ ,  $g \in_s 1_d$ . For  $t$  such that  $r \subseteq t$  and  $s \subseteq t$ , show

$$(f, r) \simeq_{\mathcal{E}} (g, s) \quad \text{iff} \quad i \circ \bar{f} = j \circ \bar{g},$$

i.e.

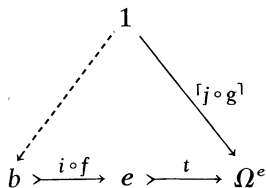


commutes. □

“Membership” for set objects is defined by

$$(g, s)E_{\mathcal{E}}(f, r)$$

iff for some toso  $t: e \rightarrow \Omega^e$  such that  $r \subseteq t$  and  $s \subseteq t, j \circ g \in_i i \circ f,$



i.e.  $j \circ g$  factors through  $t \circ i \circ f$ .

Again the definition is independent of the choice of  $t$ , and can be given with  $t = r \cup s$ .

Equivalent definitions of  $(g, s)E_{\mathcal{E}}(f, r)$  are

(i) There exist set objects  $(g', t)$  and  $(f', t)$  with

$$(g, s) \simeq_{\mathcal{E}}(g', t), \quad (f, r) \simeq_{\mathcal{E}}(f', t)$$

and

$$g' \in_t f',$$

and

(ii) There exists  $g': c' \rightarrow a$  such that

$$(g, s) \simeq_{\mathcal{E}}(g', r)$$

and

$$g' \in_r f.$$

EXERCISE 12. For set objects  $(g, r), (f, r)$ ,

$$(g, r)E_{\mathcal{E}}(f, r) \text{ iff } g \in_r f.$$

□

We now have a definition of an  $\mathcal{L}$ -model

$$\mathfrak{A}(\mathcal{E}) = \langle A_{\mathcal{E}}, E_{\mathcal{E}}, \simeq_{\mathcal{E}} \rangle,$$

where  $A_{\mathcal{E}}$  is the collection of all set objects in  $\mathcal{E}$ . Notice that the definition has been given for any topos  $\mathcal{E}$ . Osius proves

**THEOREM 3.** *If  $\mathcal{E}$  is well-pointed, then  $\mathfrak{A}(\mathcal{E})$  is a model of all of the  $Z_0$ -axioms, together with the axiom of Regularity and the Transitivity axiom (TA). If NNO (respectively ES) holds in  $\mathcal{E}$  then the Axiom of Infinity (respectively Axiom of Choice) holds in  $\mathfrak{A}(\mathcal{E})$ .* □

It is also shown that for each tso  $r: a \twoheadrightarrow \Omega^a$ , the set object  $(1_a, r)$  is a “transitive set” in the sense of  $\mathfrak{A}(\mathcal{E})$ , i.e. the  $\mathcal{L}$ -formula  $\mathbf{Tr}(u)$  is satisfied in  $\mathfrak{A}(\mathcal{E})$  when  $u$  is interpreted as the  $\mathfrak{A}(\mathcal{E})$ -set  $(1_a, r)$ .

### 12.6 Equivalence of models

We now have two construction processes

$$\begin{aligned} \mathfrak{A} &\mapsto \mathcal{E}(\mathfrak{A}) \\ \mathcal{E} &\mapsto \mathfrak{A}(\mathcal{E}) \end{aligned}$$

of well-pointed topoi from models of  $Z_0$  and conversely. It remains to determine the extent to which these constructions are inverse to each other.

To do this we will need to assume that Mostowski’s lemma is true in  $\mathfrak{A}$ . Rather than confine ourselves to ZF-models, we take the statement of the lemma as a further axiom.

**AXIOM OF TRANSITIVE REPRESENTATION:** This is the  $\mathcal{L}$ -sentence that formally expresses the statement

**ATR:** Any extensional, well-founded relation  $r: A \rightarrow \mathcal{P}(A)$  is isomorphic to the membership relation  $r_\epsilon: B \hookrightarrow \mathcal{P}(B)$  of some transitive set  $B$ .

$B$  is called the *transitive representative* of  $r$ .

The system  $Z$  is  $Z_0 + \text{Reg} + \text{TA} + \text{ATR}$ .

Now let us assume  $\mathfrak{A} = \langle A, E, = \rangle$  is a  $Z$ -model. If  $b \in A$  is an  $\mathfrak{A}$ -set, then, working “inside”  $\mathfrak{A}$ , from  $Z_0 + \text{TA}$  there will be an  $\mathfrak{A}$ -set  $a$  that is the transitive closure of  $b$  in the sense of  $\mathfrak{A}$ , and so there will be an  $\mathfrak{A}$ -inclusion  $f: b \hookrightarrow a$ . Moreover by Ext and Reg the  $\mathfrak{A}$ -membership relation  $r_a: a \rightarrow \mathcal{P}(a)$  on the  $\mathfrak{A}$ -transitive object  $a$  will be  $\mathfrak{A}$ -monic and  $\mathfrak{A}$ -well-founded, hence  $\mathfrak{A}$ -recursive. But the  $\mathfrak{A}$ -functions  $f$  and  $r_a$  will be arrows in the topos  $\mathcal{E}(\mathfrak{A})$ , and so  $\langle f, r_a \rangle$  will be a set-object in  $\mathcal{E}(\mathfrak{A})$ , i.e. an individual (“set”) in the  $\mathcal{L}$ -model  $\mathfrak{A}(\mathcal{E}(\mathfrak{A}))$ . Putting  $Ob(a) = \langle f, r_a \rangle$  gives a transformation from  $\mathcal{L}$ -model  $\mathfrak{A}$  to  $\mathcal{L}$ -model  $\mathfrak{A}(\mathcal{E}(\mathfrak{A}))$  that satisfies

$$a \simeq c \quad \text{iff} \quad Ob(a) \simeq_{\mathcal{E}(\mathfrak{A})} Ob(c)$$

and

$$aEc \quad \text{iff} \quad Ob(a) E_{\mathcal{E}(\mathfrak{A})} Ob(c).$$



In the opposite direction, given a set-object  $X = (f : b \multimap a, r : a \multimap \Omega^a)$  in  $\mathfrak{A}(\mathcal{E}(\mathfrak{A}))$ , then  $r$  is a monic, recursive arrow in  $\mathcal{E}(\mathfrak{A})$ , i.e. an extensional, well-founded relation in  $\mathfrak{A}$ . Since ATR holds in  $\mathfrak{A}$  there is some  $\mathfrak{A}$ -transitive set  $c \in A$ , and an  $\mathfrak{A}$ -bijection  $g : a \rightarrow c$  that makes  $r$   $\mathfrak{A}$ -isomorphic to the  $\mathfrak{A}$ -membership relation on  $c$ . We let  $St(X)$  be the  $\mathfrak{A}$ -set “ $g(f(a))$ ”, i.e. the  $\mathfrak{A}$ -image of  $b$  in  $c$  under the  $\mathfrak{A}$ -function  $g \circ f$ .

In view of Theorem 2 of §12.4, transitive representatives are unique (in  $Z$ ) and so this gives us a map  $St$  from  $\mathfrak{A}(\mathcal{E}(\mathfrak{A}))$  to  $\mathfrak{A}$  that can be shown to satisfy

$$X \simeq_{\mathcal{E}(\mathfrak{A})} Y \quad \text{iff} \quad St(X) \simeq St(Y)$$

and

$$XE_{\mathcal{E}(\mathfrak{A})} Y \quad \text{iff} \quad St(X)ESt(Y).$$

Moreover  $Ob$ , and  $St$  are “almost inverse” in the sense that we have

$$a \simeq St(Ob(a))$$

and

$$X \simeq_{\mathcal{E}(\mathfrak{A})} Ob(St(X))$$

Were we to “normalise”  $\mathfrak{A}$  and  $\mathfrak{A}(\mathcal{E}(\mathfrak{A}))$  by replacing individuals by their  $\simeq$ -equivalence classes we would obtain two fully isomorphic  $\mathcal{L}$ -models.

EXERCISE 1. Show, for any  $\mathcal{L}$ -formula  $\varphi$ , that

$$\mathfrak{A} \models \varphi[a] \quad \text{iff} \quad \mathfrak{A}(\mathcal{E}(\mathfrak{A})) \models \varphi[Ob(a)]$$

and

$$\mathfrak{A}(\mathcal{E}(\mathfrak{A})) \models \varphi[X] \quad \text{iff} \quad \mathfrak{A} \models \varphi[St(X)].$$

EXERCISE 2. Show

$$\mathfrak{A} \models \varphi[a] \quad \text{iff} \quad \mathfrak{A} \models \varphi[St(Ob(a))]$$

and

$$\mathfrak{A}(\mathcal{E}(\mathfrak{A})) \models \varphi[X] \quad \text{iff} \quad \mathfrak{A}(\mathcal{E}(\mathfrak{A})) \models \varphi[Ob(St(X))]. \quad \square$$

Beginning now with a well-pointed topos  $\mathcal{E}$ , a transformation  $F : \mathcal{E}(\mathfrak{A}(\mathcal{E})) \rightarrow \mathcal{E}$  is defined as follows. If  $X$  is an  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$  object then  $X$  is an  $\mathfrak{A}(\mathcal{E})$ -set, i.e. a set-object  $(f, r)$ , where  $f : b \multimap a$  and  $r : a \rightarrow \Omega^a$  are  $\mathcal{E}$ -arrows. We put  $F(X) = \text{dom } f = b$ .

Osius shows how to define  $F$  on  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$ -arrows so that it becomes a functor from  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$  to  $\mathcal{E}$ . The image of  $F$  in  $\mathcal{E}$  proves to be a full subcategory of  $\mathcal{E}$  containing those  $\mathcal{E}$ -objects  $b$  that are *partially transitive*. Partial transitivity of  $b$  means that there exists a tso  $r : a \rightarrow \Omega^a$  in  $\mathcal{E}$ , and an  $\mathcal{E}$ -monic  $f : b \rightarrow a$  from  $b$  to  $a$ . This makes  $(f, r)$  a set-object, i.e. an object in  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$ , with  $F(f, r) = b$ .

AXIOM OF PARTIAL TRANSITIVITY:

APT: Every object is partially transitive.

Notice that if  $\mathfrak{A}$  is any  $\mathbf{Z}$ -model, then the topos  $\mathcal{E}(\mathfrak{A})$  of  $\mathfrak{A}$ -sets and  $\mathfrak{A}$ -functions always satisfies APT. The definition of  $Ob(b)$  shows that every  $b$  is partially transitive.

Now if  $\mathcal{E} \models \text{APT}$ , then the functor  $F$  described above will be “onto” – its image is the whole of  $\mathcal{E}$ . Moreover  $F$  will then be an equivalence of categories, as defined in Chapter 9. Thus  $\mathcal{E}$  and  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$  are equivalent categories. They are “isomorphic up to isomorphism”. By identifying isomorphic objects in each we obtain two (skeletal) categories that are isomorphic in the category **Cat** of all small categories. Furthermore if  $\mathcal{E}$  is partially transitive (i.e.  $\mathcal{E} \models \text{APT}$ ) then the functor  $F$  can be used to show that the axiom ATR of transitive representation holds in  $\mathfrak{A}(\mathcal{E})$ , and so  $\mathfrak{A}(\mathcal{E})$  is a  $\mathbf{Z}$ -model. For, if  $R$  is an extensional well-founded relation on  $X$  inside  $\mathfrak{A}(\mathcal{E})$  then  $R$  corresponds to an  $\mathfrak{A}(\mathcal{E})$ -function  $r : X \rightarrow \mathcal{P}(X)$  which becomes a tso in  $\mathcal{E}(\mathfrak{A}(\mathcal{E}))$ .  $F$  transfers this to a tso  $t : a \rightarrow \Omega^a$  in  $\mathcal{E}$ . The set object  $\langle 1_a, t \rangle$  then proves to be the transitive representative of  $X$  in  $\mathfrak{A}(\mathcal{E})$ .

In summary then, there is an exact correspondence between models of the set theory  $\mathbf{Z}$  and well-pointed, partially transitive, topoi. The concept of a “well-pointed partially transitive topos” can be expressed in the first-order language of categories, and so we have an exact correspondence between models of two first-order theories. Indeed the whole exercise can be treated as a syntactic one, the set-theoretic definition of “function (arrow)” and the categorial definition of “set-object” providing theorem-preserving interpretations of two formal systems in each other.

The theory as developed may be extended to stronger set theories. A categorial version of the Replacement schema can be defined to characterise those topoi that correspond to models of ZF. Further results of this nature are given in Section 9 of Osius. In the event that epics split in well-pointed  $\mathcal{E}$ , the axiom APT is redundant. By lifting to  $\mathcal{E}$  the set-theoretic proof that any object  $A$  has a well-ordering (and hence yields a tso  $A \rightarrow \mathcal{P}(A)$ ), it can be shown from ES that all objects are partially

transitive. Thus well-pointed topoi satisfying ES correspond exactly to models of ZC (Z+axiom of choice).

A fuller account of the technical details of the theory just described, including proofs of the main results, is to be found in Chapter 9 of Johnstone [77].