CHAPTER 10

SET CONCEPTS AND VALIDITY

"... a natural and useful generalisation of set theory to the consideration of 'sets which internally develop'"

F. W. Lawvere

10.1. Set concepts

We saw in Chapter 1 that a statement $\varphi(x)$, pertaining to individuals x, determines a set, viz the set $\{x: \varphi(x)\}$ of all things of which the statement is true. But according to the constructivist attitude outlined in Chapter 8, truth is not something ascribed to a statement *absolutely*, but rather is a "context-dependent" attribute. The truth-value of a sentence varies according to the state of knowledge existing at the time of assertion of the sentence. In these terms we might regard φ not as determining a set per se, but rather as determining, for each state p, the collection

$$\varphi_p = \{x: \varphi(x) \text{ is known at } p \text{ to be true}\}.$$

φ_p will be called the *extension* of φ at p.

Thus, given a frame **P** of states of knowledge, the assignment of φ_p to p determines a function $P \to \mathbf{Set}$. Moreover, if truth is taken to "persist in time", then if $x_0 \in \varphi_p$ and $p \sqsubseteq q$, we have $\varphi(x_0)$ true also at q, so $x_0 \in \varphi_q$. Thus

$$(*) \qquad p \sqsubseteq q \quad \text{implies} \quad \varphi_p \subseteq \varphi_q$$

This means that φ determines a functor $\mathbf{P} \to \mathbf{Set}$, which assigns the inclusion arrow $\varphi_p \hookrightarrow \varphi_q$ to each $p \to q$ in **P**.

EXAMPLE. Let $\varphi(x)$ be the statement "x is an integer greater than 2, and there are no non-zero integers a, b, c with $a^x + b^x = c^x$". Fermat's celebrated "last theorem" asserts that $\varphi(x)$ holds for every integer $x \geqslant 2$. At the present moment it is not known if this is correct, although it is known that φ is true for all $x \leqslant 25{,}000$. Until Fermat's "theorem" is decided either way we may expect the extension of φ to increase with time.

So, corresponding to an expression φ we have an object in the functor category $\mathbf{Set}^{\mathbf{P}}$. Such an object might be though of as a "variable set", as in Lawvere [75, 76]. We might also call it an *intensional set,* or a *set concept.* This terminology derives from semantic theories of the type set out by Rudolf Carnap [47]. In such theories the *extension* of an individual expression is taken to be the actual thing, or collection of things, to which it refers. The *intension* on the other hand is a somewhat more elusive entity, which is sometimes described as being the *meaning* of the expression. Carnap ([47], p. 41) defines the intension of an individual expression to be the "individual concept expressed by it". Thus if $\varphi(x)$ is the statement "x is a finite ordinal" then the intension of φ is the *concept* of a finite ordinal. This is represented by the functor that assigns to each p the set of things known at stage p to be finite ordinals. This functor can also be said to represent the concept of the set of finite ordinals. In this way we construe $\mathbf{Set}^{\mathbf{P}}$ as being a category of set concepts.

There are some difficulties with the theme just developed. Consider the expression "the smallest non-finite ordinal". This expresses quite a different concept to "the set of finite ordinals", and yet the two have the same extension, i.e. the set of finite ordinals *is* the smallest non-finite ordinal. Thus two different concepts might well be represented in $\mathbf{Set}^{\mathbf{P}}$ by the same object, i.e. $\mathbf{Set}^{\mathbf{P}}$ does not *faithfully* represent all concepts (for a more basic example consider the expressions "2 plus 2" and "2 times 2").

Another difficulty relates to the derivation of the principle $(*)$ above. The argument would seem to be simply fallacious in the event that x_0 is itself the extension of some set concept, i.e. $x_0 = \psi_p$ for some expression $\psi(x)$. Suppose for example that $\varphi(x)$ is the statement "$x = \{y: \psi(y)\}$". Then $\varphi_p = \{\psi_p\}$, the set whose only member is $\psi_p = x_0$, while $\varphi_q = \{\psi_q\}$. If $\psi_p \neq \psi_q$, then $x_0 \notin \varphi_q$. We do salvage from this however the fact that if $\psi_p \in \varphi_p$, then $\psi_q \in \varphi_q$. Perhaps we should then replace the inclusion function of $(*)$ by the map taking each element of φ_p to its counter-part in φ_q. In this way φ would still determine a functor. Unfortunately the notion of counterpart is ambiguous here – x_0 may also be the extension of some other expression $\theta(x)$ $(x_0 = \psi_p = \theta_p)$ whose extension at q differs from ψ $(\psi_q \neq \theta_q)$.

In spite of these problems, the notion of set concept would still seem appropriate to an understanding of the objects in $\mathbf{Set}^{\mathbf{P}}$, and to the viewpoint that $\mathbf{Set}^{\mathbf{P}}$ is the universe for a generalised "non-extensional" set theory. Indeed the study of $\mathbf{Set}^{\mathbf{P}}$ may help to clarify the philosophically difficult notions of "individual concept" and "intensional object" (for an indication of how intractable these ideas are, read Scott [70i]).

Certainly the notion of "variable structure" is a mathematically significant one. One thinks of the concept of "neighbourhood system" as represented by the assignment to each point in a topological space of its set of neighbourhoods – or the concept of "tangent space" as represented by the assignment to each point in a manifold of the space of vectors tangent to the manifold at that point.

In this chapter we propose to look in depth at the topos structure of $\mathbf{Set}^{\mathbf{P}}$, and in particular the nature of its truth arrows. The conclusion we will reach is that "the logic of variable sets is intuitionistic".

10.2. Heyting algebras in P

Let $\mathbf{P} = (P, \sqsubseteq)$ be a poset. For each $p \in P$, let

$$[p) = \{q : p \sqsubseteq q\}$$

be the set of P-elements "above" p in the ordering $\sqsubseteq$. If $q \in [p)$ and $q \sqsubseteq r$, then, by the transitivity of $\sqsubseteq$, $r \in [p)$. Thus $[p)$ is hereditary in $\mathbf{P}$ ($[p) \in \mathbf{P}^+$), and will be called the *principal* $\mathbf{P}$*-hereditary set generated by* p. Principal sets are very useful in describing the structure of the **HA** $\mathbf{P}^+$, as seen in the following

Exercises

Cf. §8.4 for notation.

EXERCISE 1. For any $S \subseteq P$, if $[p) \subseteq S$ then $p \in S$.

EXERCISE 2. $p \sqsubseteq q$ iff $[q) \subseteq [p)$.

EXERCISE 3. The following are equivalent, for any $S \subseteq P$:

(i) S is $\mathbf{P}$-hereditary;

(ii) for all $p \in P$, $p \in S$ iff $[p) \subseteq S$;

(iii) for all $p \in P$, $p \in S$ implies $[p) \subseteq S$.

EXERCISE 4. For any $S, T \in \mathbf{P}^+$,

$$S \Rightarrow T = \{p : S \cap [p) \subseteq T\}$$

$$\neg S = \{p : [p) \subseteq -S\} = \{p : [p) \cap S = \emptyset\}. \qquad \square$$

Now the relation $\sqsubseteq$ when restricted to the members of $[p)$ is still a partial ordering, and so we have a poset $([p), \sqsubseteq)$, and a collection $[p)^+$

consisting of all the sets that are hereditary in $[p)$. Now if $q \in [p)$, then the principal set generated in $[p)$ by q is

$$\begin{aligned}[q)_p &= \{r: r \in [p) \text{ and } q \sqsubseteq r\} \\ &= [p) \cap [q).\end{aligned}$$

But by Exercise 2, this is just $[q)$. In other words, the principal set of q in $\mathbf{P}$ is the same as the principal set of q with respect to $[p)$, $[q) = [q)_p$. From this we obtain a detailed account of the relationship between $\mathbf{P}^+$ and $[p)^+$.

If S is *any* subset of P, put

$$\begin{aligned}S_p &= S \cap [p) \\ &= \{q: q \in S \text{ and } p \sqsubseteq q\}.\end{aligned}$$

THEOREM 1. (1) *If* $S \subseteq [p)$, *then* $S = S_p$, *and* $S \in [p)^+$ *iff* $S \in \mathbf{P}^+$;

(2) *If* $S \in \mathbf{P}^+$ *then* $S_p \in [p)^+$;

(3) $T \in [p)^+$ *iff for some* $S \in \mathbf{P}^+$, $T = S_p$;

(4) *If* $S \in \mathbf{P}^+$, *then* $S = \bigcup\{S_p : p \in P\}$.

PROOF. (1) Clearly if $S \subseteq [p)$, then $S = S \cap [p)$. Moreover, by Exercise 3 (iii),

$$S \in [p)^+ \quad \text{iff} \quad q \in S \quad \text{implies} \quad [q)_p \subseteq S$$

while

$$S \in \mathbf{P}^+ \quad \text{iff} \quad q \in S \quad \text{implies} \quad [q) \subseteq S.$$

But since $S \subseteq [p)$, $q \in S$ implies $[q)_p = [q)$.

(2) Since $[p) \in \mathbf{P}^+$, $S \in \mathbf{P}^+$ implies $S \cap [p) \in \mathbf{P}^+$, i.e. $S_p \in \mathbf{P}^+$. Since $S_p \subseteq [p)$, the result follows by part (1).

(3) Exercise.

(4) We have to show that

$$q \in S \quad \text{iff for some } p, \quad q \in S_p = S \cap [p).$$

Since in general, $S_p \subseteq S$, the implication from right to left is immediate. Conversely, if $q \in S$ then if S is hereditary we have $q \in [q) \subseteq S$, and so $q \in S \cap [q)$, i.e. the proof is completed by taking $p = q$. □

Now we know from §8.4 that the poset $([p)^+, \subseteq)$ of hereditary subsets of $[p)$ under the *subset* ordering is a Heyting algebra (in fact – for the interest of the reader familiar with such things – $[p)^+$ is a subdirectly irreducible **HA**). The lattice meet $\cap_p$ and join $\cup_p$ are simply the operations $\cap$ and $\cup$ of set intersection and union. The pseudo-complement

$\neg_p:[p)^+ \to [p)^+$ is defined for $S \subseteq [p)$ by

$$\neg_p S = \{q\colon q \in [p) \text{ and } [q)_p \subseteq -S\}$$

while the relative pseudo-complement $\Rightarrow_p\colon [p)^+ \times [p)^+ \to [p)^+$ has

$$S \Rightarrow_p T = \{q\colon q \in [p) \text{ and } S \cap [q)_p \subseteq T\}.$$

Now given any $S \subseteq P$, we may first relativise S to $[p)$, i.e. form S_p, and then apply $\neg_p$, or we may apply $\neg$ to S first, and then relativise. The two procedures prove to be commutative, for **P**-hereditary S, and more generally we have

THEOREM 2. *For any* $S, T \in \mathbf{P}^+$

(1) $(S_p) \cap_p (T_p) = (S \cap T)_p$;
(2) $(S_p) \cup_p (T_p) = (S \cup T)_p$;
(3) $\neg_p(S_p) = (\neg S)_p$;
(4) $(S_p) \Rightarrow_p (T_p) = (S \Rightarrow T)_p$.

PROOF. (1) Exercise.

(2)
$$\begin{aligned} S_p \cup_p T_p &= S_p \cup T_p \\ &= (S \cap [p)) \cup (T \cap [p)) \\ &= (S \cup T) \cap [p) \qquad \text{(distributive law)} \\ &= (S \cup T)_p. \end{aligned}$$

(3) Since $[q) = [q)_p$ for $p \sqsubseteq q$, we have

$$\begin{aligned} \neg_p(S_p) &= \{q\colon q \in [p) \text{ and } [q) \subseteq -S\} \\ &= [p) \cap \neg S \\ &= (\neg S)_p. \end{aligned}$$

(4) Exercise. □

The algebraically minded reader will note that Theorem 2 states that the assignment of S_p to S is an **HA** homomorphism from $\mathbf{P}^+$ to $[p)^+$, which is surjective by Theorem 1 (3).

10.3. The subobject classifier in $\mathbf{Set}^{\mathbf{P}}$

That $\mathbf{Set}^{\mathbf{P}}$ is a topos is a special case of the fact that $\mathbf{Set}^{\mathscr{C}}$ is a topos for any small category $\mathscr{C}$. The definition of the subobject classifier for $\mathbf{Set}^{\mathscr{C}}$ given in §9.3 proves in the case $\mathscr{C} = \mathbf{P}$ to be expressible in terms of the **HA**'s of the form $[p)^+$. According to §9.3, $\Omega\colon \mathbf{P} \to \mathbf{Set}$ has

$$\Omega(p) = \text{the set of } p\text{-sieves}.$$

Now a p-sieve is a subset S of

$$\mathbf{P}_p = \left\{f\text{: for some } q, p \xrightarrow{f} q \text{ in } \mathbf{P}\right\}$$

that is closed under left multiplication, i.e. has $g \circ f \in S$ whenever $f \in S$ and $g : q \to r$ is a **P**-arrow. But as **P** is a preorder category, there is at most one arrow from p to q, and this exists precisely when $p \sqsubseteq q$. So for a fixed p, we may identify the arrow $f : p \to q$ with its codomain q. Hence $\mathbf{P}_p$ becomes

$$\{q : p \sqsubseteq q\} = [p)!,$$

and the description of S as a p-sieve becomes

$$r \in S \quad \text{whenever} \quad q \in S \quad \text{and} \quad q \sqsubseteq r$$

i.e. S is $[p)$-hereditary!

Thus $\Omega(p) = [p)^+$, the collection of hereditary subsets of $[p)$.

In general for a functor $F : \mathbf{P} \to \mathbf{Set}$ we will write F_p for the image $F(p)$ of p in **Set**. Whenever $p \sqsubseteq q$, F yields a function from F_p to F_q, which will be denoted F_{pq}. We may thus view F as a collection $\{F_p : p \in P\}$ of sets indexed by P and provided with "transition maps" $F_{pq} : F_p \to F_q$ whenever $p \sqsubseteq q$. In particular F_{pp} is the identity function on F_p.

In the case of Ω, the modification as above of the definition of §9.3 shows that when $p \sqsubseteq q$, $\Omega_{pq} : \Omega_p \to \Omega_q$ takes $S \in [p)^+$ to $S \cap [q) \in [q)^+$, i.e.

$$\Omega_{pq}(S) = S_q.$$

The terminal object for $\mathbf{Set}^{\mathbf{P}}$ is the "constant" functor $1 : \mathbf{P} \to \mathbf{Set}$ having $1_p = \{0\}$, all $p \in P$, and $1_{pq} = \mathrm{id}_{\{0\}}$ for $p \sqsubseteq q$. The subobject classifier $true : 1 \to \Omega$ is the natural transformation whose "p-th" component $true_p : \{0\} \to \Omega_p$ is given by

$$true_p(0) = [p).$$

Thus *true* picks out the unit element from each **HA** $[p)^+$.

Now if $\tau : F \rightarrowtail G$ is a subobject of G in $\mathbf{Set}^{\mathbf{P}}$ then each component τ_p will be injective, and will whenever convenient be assumed to be the inclusion function $F_p \hookrightarrow G_p$. Again by modifying the §9.3 definition we find that the character $\chi_\tau : G \to \Omega$ has p-th component $(\chi_\tau)_p : G_p \to [p)^+$ given by

$$\text{for each} \quad x \in G_p, \qquad (\chi_\tau)_p(x) = \{q : p \sqsubseteq q \text{ and } G_{pq}(x) \in F_q\}$$

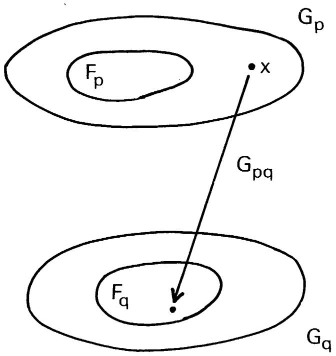


Fig. 10.1.

EXERCISE 1. Show that $(\chi_\tau)_p(x)$ is hereditary in $[p)$.

EXERCISE 2. Show that χ_τ *is* a natural transformation from G to Ω, i.e. that

$$\begin{array}{ccc} G_p & \xrightarrow{(\chi_\tau)_p} & \Omega_p \\ {\scriptstyle G_{pq}}\downarrow & & \downarrow{\scriptstyle \Omega_{pq}} \\ G_q & \xrightarrow{(\chi_\tau)_q} & \Omega_q \end{array}$$

commutes whenever $p \sqsubseteq q$. □

Notice that if $x \in F_p$, then for any $q \in [p)$, since

$$\begin{array}{ccc} F_p & \overset{\tau_p}{\hookrightarrow} & G_p \\ {\scriptstyle F_{pq}}\downarrow & & \downarrow{\scriptstyle G_{pq}} \\ F_q & \overset{\tau_q}{\hookrightarrow} & G_q \end{array}$$

commutes we must have $G_{pq}(x) = F_{pq}(x) \in F_q$, and so $q \in (\chi_\tau)_p(x)$. On the other hand if $x \notin F_p$, then $G_{pp}(x) = x \notin F_p$, and so $p \notin (\chi_\tau)_p(x)$, i.e. $(\chi_\tau)_p(x) \neq [p)$. Altogether then we have that

$$F_p = \{x: (\chi_\tau)_p(x) = [p)\} \cong \{\langle 0, x\rangle: (\chi_\tau)_p(x) = true_p(0)\}$$

and hence

$$\begin{array}{ccc}
F_p & \xhookrightarrow{\tau_p} & G_p \\
\downarrow & & \downarrow{\scriptstyle (\chi_\tau)_p} \\
\{0\} & \xhookrightarrow{true_p} & [p)^+
\end{array}$$

is a pullback in **Set**. Since this holds for all p,

$$\begin{array}{ccc}
F & \overset{\tau}{\rightarrowtail} & G \\
\downarrow & & \downarrow{\scriptstyle \chi_\tau} \\
1 & \xrightarrow{true} & \Omega
\end{array}$$

is a pullback in $\mathbf{Set}^{\mathbf{P}}$. The verification of the rest of the Ω-axiom is rather delicate. Suppose $\sigma : G \to \Omega$ makes

$$\begin{array}{ccc}
F & \xrightarrow{\tau} & G \\
\downarrow & & \downarrow{\scriptstyle \sigma} \\
1 & \xrightarrow{true} & \Omega
\end{array}$$

a pullback. Then for each q,

$$\begin{array}{ccc}
F_q & \hookrightarrow & G_q \\
\downarrow & & \downarrow{\scriptstyle \sigma_q} \\
\{0\} & \xrightarrow{true_q} & \Omega_q
\end{array}$$

will be a pullback, and so by the nature of pullbacks in **Set** we may assume

$$(*) \qquad F_q = \{x : \sigma_q(x) = [q)\}$$

Now let us take a particular p. Then whenever $p \sqsubseteq q$,

$$\begin{array}{ccc}
G_p & \xrightarrow{\sigma_p} & \Omega_p \\
{\scriptstyle G_{pq}}\downarrow & & \downarrow{\scriptstyle \Omega_{pq}} \\
G_q & \xrightarrow{\sigma_q} & \Omega_q
\end{array}$$

commutes, and hence

$$
\begin{aligned}
q\in(\chi_\tau)_p(x) \quad &\text{iff} \quad G_{pq}(x)\in F_q \\
&\text{iff} \quad \sigma_q(G_{pq}(x))=[q) && \text{(by }(*))\\
&\text{iff} \quad \Omega_{pq}(\sigma_p(x))=[q) && \text{(last diagram)}\\
&\text{iff} \quad \sigma_p(x)\cap[q)=[q) && \text{(definition }\Omega_{pq})\\
&\text{iff} \quad [q)\subseteq\sigma_p(x) \\
&\text{iff} \quad q\in\sigma_p(x) && \text{(Exercise 10.2.3)}
\end{aligned}
$$

Thus $(\chi_\tau)_p(x)=\sigma_p(x)$. Since this holds of all $p\in P$ and all $x\in G_p$, it follows that $\sigma=\chi_\tau$.

EXAMPLE 1. We saw in §9.3 that the topos $\mathbf{Set}^{\rightarrow}$ of set functions is essentially the same as $\mathbf{Set}^{\mathbf{2}}$ where $\mathbf{2}$ is the poset category $\{0, 1\}$ with $0\sqsubseteq 1$. In $\mathbf{2}$ we have

$$\Omega_0=\{\{0, 1\}, \{1\}, \emptyset\}$$
$$\Omega_1=\{\{1\}, \emptyset\}$$

and Ω_{01} maps $\{0, 1\}$ and $\{1\}$ to $\{1\}$, and $\emptyset$ to $\emptyset$. If we denote $\{0, 1\}$, $\{1\}$ and $\emptyset$ by 1, $\frac{1}{2}$, and 0 respectively in Ω_0, and $\{1\}$ and $\emptyset$ in Ω_1 by 1 and 0, Ω_{01} becomes the function t providing the $\mathbf{Set}^{\rightarrow}$-classifier defined in §4.4.

EXAMPLE 2. Let $\boldsymbol{\omega}=(\omega, \leqslant)$ be the poset of all finite ordinals $0, 1, 2, \ldots, m, \ldots,$ under their natural ordering. $\mathbf{Set}^{\boldsymbol{\omega}}$ is described by Maclane [75] as the category of "sets through time", an object being thought of as a string

$$F_0 \xrightarrow{F_{01}} F_1 \xrightarrow{F_{12}} F_2 \longrightarrow \ldots \longrightarrow F_m \xrightarrow{F_{mm+1}} F_{m+1} \longrightarrow \ldots$$

Now in $\boldsymbol{\omega}$, $[m)=\{m, m+1, m+2, \ldots\}$. Moreover if $S\subseteq\omega$ is non-empty, S has a first member m_S, so that if S is hereditary, $S=[m_S)$. Thus all non-empty hereditary sets are principal and can be identified with their first elements. Introducing a symbol ∞ to stand for the empty set we may then simplify Ω by identifying $\boldsymbol{\omega}^+$ with

$$\{0, 1, 2, \ldots, m, \ldots, \infty\}$$

and for $m\in\omega$, putting

$$\Omega_m=\{m, m+1, \ldots, \infty\}.$$

Whenever $m \leqslant n$, Ω_{mn} becomes

$$\Omega_{mn}(p) = \begin{cases} n & \text{if} \quad m \leqslant p \leqslant n \\ p & \text{if} \quad n \leqslant p \\ \infty & \text{if} \quad p = \infty, \end{cases}$$

while $true_m(0) = m$, for each $m \in \omega$.

Given $\tau : F \rightarrowtail G$, the character χ_τ has $(\chi_\tau)_m : F_m \to G_m$, given by

$$\begin{aligned} (\chi_\tau)_m(x) = {} & \text{the first } n \text{ after } m \text{ that has} \\ & G_{mn}(x) \in F_n, \text{ if such exists,} \end{aligned}$$

while

$$(\chi_\tau)_m(x) = \infty \text{ if } G_{mn}(x) \notin F_n \text{ whenever } m \leqslant n.$$

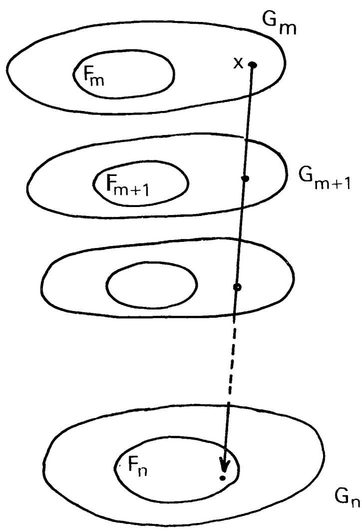


Fig. 10.2.

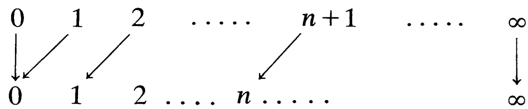
Thus $(\chi_\tau)_m(x)$ denotes the first time that x lands in the subobject F, the "time till truth" as Maclane puts it. Maclane's description of the subobject classifier for $\mathbf{Set}^{\omega}$ is even simpler than the one just given. The effect of the map Ω_{mm+1} can be displayed as

$$\begin{array}{ccccccc} \Omega_m & m & m+1 & m+2 & m+3 & \dots & \infty \\ & \downarrow & \swarrow & \swarrow & \swarrow & & \downarrow \\ \Omega_{m+1} & m+1 & m+2 & m+3 & \dots & & \infty \end{array}$$

The picture looks the same for each m, and indeed it is the structure of the map that is significant, not the labelling of the entries in the "order-isomorphic" sequences Ω_m and Ω_{m+1}. We may replace each Ω_m by the single set

$$\Omega = \{0, 1, 2, \ldots, \infty\}$$

and each Ω_{mm+1} by the single map $t: \Omega \to \Omega$, displayed as



Then the object of truth values becomes, as in Maclane, the constant functor $\Omega \xrightarrow{t} \Omega \xrightarrow{t} \Omega \xrightarrow{t} \ldots$ and the arrow *true* has the inclusion $\{0\} \hookrightarrow \Omega$ for each component.

So now we have seen three set-theoretically distinct objects in $\mathbf{Set}^{\omega}$ that serve as objects of truth-values, underlining again the point that the Ω-axiom characterises $\top: 1 \to \Omega$ uniquely up to isomorphism only.

10.4. The truth arrows

I. False

The initial object $0: \mathbf{P} \to \mathbf{Set}$ in $\mathbf{Set}^{\mathbf{P}}$ is the constant functor having $0_p = \emptyset$ and $0_{pq} = \mathrm{id}_{\emptyset}$ for $p \sqsubseteq q$. The unique transformation $0 \dot{\to} 1$ has components $\emptyset \hookrightarrow \{0\}$ (i.e. the same component for each p). The character of $!: 0 \to 1$ is $false: 1 \to \Omega$, with $false_p: \{0\} \to \Omega_p$ having

$$\begin{aligned} false_p(0) &= \{q: p \sqsubseteq q \text{ and } \mathbf{1}_{pq}(0) \in 0_q\} \\ &= \{q: p \sqsubseteq q \text{ and } 0 \in \emptyset\} \\ &= \emptyset. \end{aligned}$$

Thus false picks out the zero element from each **HA** $[p)^+$.

II. Negation

$\neg: \Omega \to \Omega$ is the character of *false*. Identifying $false_p$ with $\{\emptyset\} \subseteq \Omega_p$ we find then the p-th component $\neg_p: \Omega_p \to \Omega_p$ of $\neg$ has

$$\begin{aligned} \neg_p(S) &= \{q: p \sqsubseteq q \text{ and } \Omega_{pq}(S) \in \{\emptyset\}\} \\ &= \{q: p \sqsubseteq q \text{ and } S \cap [q) = \emptyset\} \\ &= [p) \cap \neg S \\ &= (\neg S)_p. \end{aligned}$$

We had already used the symbol $\neg_p$ in §10.2 to denote the pseudo-complementation operation in $[p)^+$. The equation just derived shows that *the latter operation is precisely the same as the p-th component of the negation truth arrow in* $\mathbf{Set}^{\mathbf{P}}$, and so the notation remains consistent.

III. Conjunction

The functor $\Omega \times \Omega$ has

$$(\Omega \times \Omega)_p = \langle \Omega_p, \Omega_p \rangle$$

and for $p \sqsubseteq q$, $(\Omega \times \Omega)_{pq}$ is the product map $\Omega_{pq} \times \Omega_{pq}$ (cf. §3.8).

The arrow $\langle \top, \top \rangle : 1 \to \Omega \times \Omega$ in $\mathbf{Set}^{\mathbf{P}}$ has components

$$\langle \top, \top \rangle_p : \{0\} \to \Omega_p \times \Omega_p$$

given by $\langle \top, \top \rangle_p(0) = \langle [p), [p) \rangle$.

Its character is the conjunction arrow

$$\Omega \times \Omega \xrightarrow{\cap} \Omega$$

with components $\cap_p : \Omega_p \times \Omega_p \to \Omega_p$ having

$$\begin{aligned}
\cap_p(\langle S, T \rangle) &= \{q : p \sqsubseteq q \text{ and } \langle \Omega_{pq}(S), \Omega_{pq}(T) \rangle = \langle [q), [q) \rangle\} \\
&= \{q : p \sqsubseteq q \text{ and } S \cap [q) = [q) = T \cap [q)\} \\
&= \{q : p \sqsubseteq q \text{ and } [q) \subseteq S \text{ and } [q) \subseteq T\} \\
&= \{q : p \sqsubseteq q \text{ and } q \in S \text{ and } q \in T\} \\
&= S \cap T \cap [p) \\
&= (S \cap T)_p \\
&= S \cap T. \qquad \text{(Theorem 10.2.1)}
\end{aligned}$$

IV. Implication

The equaliser $e : \circledleq \rightarrowtail \Omega \times \Omega$ of $\cap : \Omega \times \Omega \to \Omega$ and $pr_1 : \Omega \times \Omega \to \Omega$, has as domain the functor $\circledleq : \mathbf{P} \to \mathbf{Set}$, with

$$\begin{aligned}
\circledleq_p &= \{\langle S, T \rangle : \cap_p(\langle S, T \rangle) = S\} \\
&= \{\langle S, T \rangle : S \subseteq T\} \subseteq \Omega_p \times \Omega_p,
\end{aligned}$$

and $\circledleq_{pq}$, for $p \sqsubseteq q$, giving output $\langle S_q, T_q \rangle$ for input $\langle S, T \rangle$.

The components of e are the inclusions $e_p : \circledleq_p \hookrightarrow \Omega_p \times \Omega_p$.

The implication arrow $\Rightarrow : \Omega \times \Omega \to \Omega$, being the character of e, has component $\Rightarrow_p$ given by

$$\begin{aligned}\Rightarrow_p(\langle S, T\rangle) &= \{q : p \sqsubseteq q \text{ and } \langle \Omega_{pq}(S), \Omega_{pq}(T)\rangle \in \circledleq_p\}\\ &= \{q : p \sqsubseteq q \text{ and } S \cap [q) \subseteq T \cap [q)\}\\ &= \{q : p \sqsubseteq q \text{ and } S \cap [q) \subseteq T\}\\ &= (S \Rightarrow T) \cap [p)\\ &= (S \Rightarrow T)_p.\end{aligned}$$

Thus the p-th component of the implication arrow is the relative pseudo-complementation for the **HA** $[p)^+$.

V. Disjunction

EXERCISE 1. Show that the p-th component of the transformation

$$[\langle \top_\Omega, \mathbf{1}_\Omega\rangle, \langle \mathbf{1}_\Omega, \top_\Omega\rangle]$$

is "essentially" the set

$$\{\langle [p), S\rangle : S \in \Omega_p\} \cup \{\langle S, [p)\rangle : S \in \Omega_p\}$$

and hence that *the disjunction arrow* $\vee : \Omega \times \Omega \to \Omega$ *has components* $\vee_p(\langle S, T\rangle) = S \cup T$.

It is worth pausing here to reflect on what has been accomplished. We now know that the truth arrows in $\mathbf{Set}^{\mathbf{P}}$ are precisely those natural transformations whose components interpret the corresponding connectives on the Heyting algebras in **P**. But remember that the truth arrows were defined long before intuitionistic logic and **HA**'s were mentioned. They arose from a categorial description of the classical truth functions in **Set**. *Subsequently*, when interpreted in the particular topos $\mathbf{Set}^{\mathbf{P}}$, they yield the intuitionistic truth functions. Thus the theory of "topos logic" abstracts a structure common to classical and intuitionistic logic. What better example could there be of the advancement of understanding through the interplay of generalisation and specialisation (§2.4)?

10.5. Validity

In view of the results of the last section one would anticipate an intimate relationship between validity in $\mathbf{Set}^{\mathbf{P}}$ and algebraic semantics on the **HA**'s

$[p)^+$. In fact the main result of this section, indeed of this chapter, is the

VALIDITY THEOREM. *For any poset* $\mathbf{P}$, *and propositional sentence* $\alpha \in \Phi$,

$$\mathbf{Set}^{\mathbf{P}} \models \alpha \quad \textit{iff} \quad \mathbf{P} \models \alpha.$$

In the left-hand expression we mean topos-validity as defined in §6.7. The right-hand expression refers to Kripke-style validity as in §8.4.

There is some choice as to how we go about proving the Validity Theorem. We know from §8.4 that

$$\mathbf{P} \models \alpha \quad \textit{iff} \quad \mathbf{P}^+ \models \alpha,$$

and from §8.3 that

$$\mathbf{Set}^{\mathbf{P}} \models \alpha \quad \textit{iff} \quad \mathbf{Set}^{\mathbf{P}}(1, \Omega) \models \alpha \quad \textit{iff} \quad \mathrm{Sub}(1) \models \alpha,$$

so we could proceed to establish relationships between the **HA**'s $\mathbf{P}^+$, $\mathbf{Set}^{\mathbf{P}}(1, \Omega)$, and Sub(1). Ultimately these are all variations on the same underlying theme. We choose to approach the Validity Theorem directly in terms of the definitions of validity concerned.

Let $\mathcal{M} = (\mathbf{P}, V)$ be a model based on $\mathbf{P}$, where $V: \Phi_0 \to \mathbf{P}^+$ is a $\mathbf{P}$-valuation. We use V to define a $\mathbf{Set}^{\mathbf{P}}$-valuation $V': \Phi_0 \to \mathbf{Set}^{\mathbf{P}}(1, \Omega)$ à la §6.7. V' assigns to each sentence letter π a truth value $V'(\pi): 1 \to \Omega$ in $\mathbf{Set}^{\mathbf{P}}$. The p-th component $V'(\pi)_p : \{0\} \to \Omega_p$ is defined by

$$(*) \qquad \begin{aligned} V'(\pi)_p(0) &= V(\pi) \cap [p) \\ &= V(\pi)_p \end{aligned}$$

Thus $V'(\pi)_p$ picks out the set of points in $[p)$ at which π is true in $\mathcal{M}$.

Now if $p \sqsubseteq q$ then $V(\pi) \cap [p) \cap [q) = V(\pi) \cap [q)$ (Exercise 10.2.2) and so

$$\begin{array}{ccc} \{0\} & \xrightarrow{V'(\pi)_p} & [p)^+ \\ \big\downarrow & & \big\downarrow \Omega_{pq} \\ \{0\} & \xrightarrow{V'(\pi)_q} & [q)^+ \end{array}$$

commutes. Hence $V'(\pi)$ *is* a natural transformation.

By the rules of §8.4 the model $\mathcal{M}$ produces for each sentence $\alpha \in \Phi$ a subset $\mathcal{M}(\alpha) = \{q : \mathcal{M} \models_q \alpha\}$ of P, and hence, for each $p \in P$, a subset $\mathcal{M}(\alpha)_p = \mathcal{M}(\alpha) \cap [p)$ of $[p)$. On the other hand by the rules of §6.7, V'

provides each α with a $\mathbf{Set^P}$-arrow $V'(\alpha):1 \twoheadrightarrow \Omega$ and hence, for each $p \in P$, a function $V'(\alpha)_p:\{0\} \to \Omega_p$. We have

LEMMA 1. *For any α, the p-th component*

$$V'(\alpha)_p:\{0\} \to [p)^+$$

of $V'(\alpha)$ has $V'(\alpha)_p(0) = \mathcal{M}(\alpha)_p$.

PROOF. By induction on the formation of α. Since $\mathcal{M}(\pi) = V(\pi)$, for $\alpha = \pi$ the result is immediate from (*). If $\alpha = {\sim}\beta$, and the result holds for β, then

$$\begin{aligned} V'({\sim}\beta)_p &= (\neg \circ V'(\beta))_p \\ &= \neg_p \circ V'(\beta)_p \end{aligned}$$

and so

$$\begin{aligned} V'(\alpha)_p(0) &= \neg_p(V'(\beta)_p(0)) \\ &= \neg_p(\mathcal{M}(\beta)_p) && \text{(induction hypothesis)} \\ &= (\neg \mathcal{M}(\beta))_p && \text{(Part II of §10.4, and Theorem 10.2.2(3))} \\ &= \mathcal{M}({\sim}\beta)_p && \text{((4′), §8.4)} \\ &= \mathcal{M}(\alpha)_p, \end{aligned}$$

hence the result holds for α. □

EXERCISE 1. Complete the proof of Lemma 1 for the cases of the connectives $\vee$, $\wedge$, $\supset$, using the other parts of §10.4, the rest of Theorem 2 of §10.2, and clauses (2′), (3′), and (5′) from §8.4. □

COROLLARY 2. $\mathbf{Set^P} \models \alpha$ *only if* $\mathbf{P} \models \alpha$.

PROOF. Let $\mathcal{M} = (\mathbf{P}, V)$ be any $\mathbf{P}$-based model, and V' the $\mathbf{Set^P}$-valuation corresponding to V as in (*). Since $\mathbf{Set^P} \models \alpha$, $V'(\alpha) = true$, and so for each p, $V'(\alpha)_p(0) = true_p(0) = [p)$. Since $p \in [p)$, Lemma 1 gives $p \in \mathcal{M}(\alpha)_p \subseteq \mathcal{M}(\alpha)$. Thus $\mathcal{M}(\alpha) = P$. As this holds for any model on $\mathbf{P}$, α is valid on $\mathbf{P}$.

To prove the converse of Corollary 2, we begin with a $\mathbf{Set^P}$-valuation $V':\Phi_0 \to \mathbf{Set^P}(1, \Omega)$ and construct from it a $\mathbf{P}$-valuation $V:\Phi_0 \to \mathbf{P}^+$. The arrow $V'(\pi):1 \twoheadrightarrow \Omega$ picks out, for each $q \in P$, an hereditary subset $V'(\pi)_q(0)$ of $[q)$. We form the union of all of these sets to get $V(\pi)$. Thus

$$V(\pi) = \bigcup\{V'(\pi)_q(0) \colon q \in P\}$$

i.e.

(**) $\quad r \in V(\pi)$ iff for some q, $\quad r \in V'(\pi)_q(0)$.

Having now obtained a **P**-valuation V we could apply $(*)$ to get another $\mathbf{Set}^{\mathbf{P}}$-valuation V'', with $V''(\pi)_p(0) = V(\pi) \cap [p)$. However this just gives us back the original V', as we see from

LEMMA 3. *For any* $p \in P$,

$$V(\pi) \cap [p) = V'(\pi)_p(0),$$

where $V(\pi)$ *is defined by* (**).

PROOF. It is clear from (**) that $V'(\pi)_p(0) \subseteq V(\pi)$. But since $V'(\pi): 1 \rightarrow \Omega$, $V'(\pi)_p: \{0\} \rightarrow \Omega_p$, and so $V'(\pi)_p(0) \subseteq [p)$. Hence $V'(\pi)_p(0) \subseteq V(\pi) \cap [p)$. Conversely, suppose $r \in V(\pi) \cap [p)$. Then $p \sqsubseteq r$, and for some q, $r \in V'(\pi)_q(0)$. Since $V'(\pi)_q(0) \subseteq [q)$, it follows that $q \sqsubseteq r$, and hence

$$\begin{array}{ccc} \{0\} & \xrightarrow{V'(\pi)_q} & \Omega_q \\ \downarrow & & \downarrow \Omega_{qr} \\ \{0\} & \xrightarrow{V'(\pi)_r} & \Omega_r \end{array}$$

commutes, because $V'(\pi)$ is a natural transformation. Thus $V'(\pi)_q(0) \cap [r) = V'(\pi)_r(0)$.

Analogously, since $p \sqsubseteq r$,

$$V'(\pi)_p(0) \cap [r) = V'(\pi)_r(0).$$

Then, knowing that $r \in V'(\pi)_q(0)$ and $r \in [r)$, we may apply these last two equations to conclude that $r \in V'(\pi)_p(0)$. Hence $V(\pi) \cap [p) \subseteq V'(\pi)_p(0)$. □

Now if V is a **P**-valuation, and V' is defined by $(*)$, i.e. $V'(\pi)_p(0) = V(\pi)_p$, then by Theorem 1(4) of §10.2,

$$\begin{aligned} \bigcup\{V'(\pi)_p(0) \colon p \in P\} &= \bigcup\{V(\pi)_p \colon p \in P\} \\ &= V(\pi), \end{aligned}$$

so the application of (**) just gives us V back again. The upshot of this, and Lemma 3, is that the definitions $(*)$ and (**) are inverse to each other and establish a bijection between **P**-valuations and $\mathbf{Set}^{\mathbf{P}}$-valuation. Thus in Lemma 1 we may alternatively regard V as having been defined from V' by (**).

COROLLARY 4. $\mathbf{P} \models \alpha$ *only if* $\mathbf{Set}^{\mathbf{P}} \models \alpha$.

PROOF. Let V' be any $\mathbf{Set}^{\mathbf{P}}$-valuation, and $\mathcal{M} = (\mathbf{P}, V)$ the corresponding model defined by (**). Since $\mathbf{P} \models \alpha$, $\mathcal{M}(\alpha) = P$, and so for any p, $\mathcal{M}(\alpha)_p = \mathcal{M}(\alpha) \cap [p) = [p) = true_p(0)$. Thus by Lemma 1, $V'(\alpha)_p(0) = true_p(0)$. Hence $V'(\alpha) = true$. □

Corollaries 2 and 4 together give the Validity Theorem.

10.6. Applications

(1) The most important immediate consequence of the Validity Theorem is the characterisation of the class of topos-valid sentences. If $\mathbf{P}_{\mathrm{IL}}$ is the canonical frame for IL described in §8.4 then, for any $\alpha \in \Phi$

$$\vdash_{\mathrm{IL}} \alpha \quad \textit{iff} \quad \mathbf{P}_{\mathrm{IL}} \models \alpha,$$

and hence by the Validity Theorem

$$\vdash_{\mathrm{IL}} \alpha \quad \textit{iff} \quad \mathbf{Set}^{\mathbf{P}_{\mathrm{IL}}} \models \alpha.$$

From this we get the:

COMPLETENESS THEOREM FOR $\mathscr{E}$-VALIDITY. *If α is valid on every topos, then*

$$\vdash_{\mathrm{IL}} \alpha.$$

Together with the Soundness Theorem given in §8.3 this yields the result that the sentences valid on all topoi are precisely the IL-theorems.

(2) It was stated in §6.7 that the category $\mathbf{Set}^{\rightarrow}$ does not validate $\alpha \vee \sim\alpha$. To see this, recall that $\mathbf{Set}^{\rightarrow}$ is essentially the same as $\mathbf{Set}^{\mathbf{2}}$. But in the Example of §8.4 it was shown that $\mathbf{2} \not\models \alpha \vee \sim\alpha$. The Validity Theorem then gives $\mathbf{Set}^{\mathbf{2}} \not\models \alpha \vee \sim\alpha$.

(3) The logic LC, mentioned in §8.4, is generated by adjoining to the IL-axioms the classical tautology

$$(\alpha \supset \beta) \vee (\beta \supset \alpha)$$

LC is what is known as an *intermediate logic*, i.e. its theorems include all IL-theorems and are included in the CL-theorems.

Now it is known (cf. Dummett [59] or Segerberg [68]) that

$$\boldsymbol{\omega} \models \alpha \quad \textit{iff} \quad \vdash_{\text{LC}} \alpha,$$

and so we have

$$\vdash_{\text{LC}} \alpha \quad \textit{iff} \quad \mathbf{Set}^{\boldsymbol{\omega}} \models \alpha,$$

i.e. LC is the logic of the topos of "sets through time" described in §10.3. This is the appropriate context if time is considered to be made up of discrete moments. However the logic is not altered by the assumption that time is dense, or even continuous. If $\mathbb{Q}$ and $\mathbb{R}$ denote respectively the posets of rational, and of real, numbers under their natural (arithmetic) ordering, then from Section 5 of Segerberg we conclude that

$$\boldsymbol{\omega} \models \alpha \quad \textit{iff} \quad \mathbb{Q} \models \alpha \quad \textit{iff} \quad \mathbb{R} \models \alpha.$$

and so the topoi $\mathbf{Set}^{\boldsymbol{\omega}}$, $\mathbf{Set}^{\mathbb{Q}}$, and $\mathbf{Set}^{\mathbb{R}}$ all have the same logic.

In fact the most general conclusion we can make is that if $\mathbf{P}$ is *any* infinite linearly order poset (i.e. $p \sqsubseteq q$ or $q \sqsubseteq p$, for all $p, q \in P$), then

$$\mathbf{Set}^{\mathbf{P}} \models \alpha \quad \textit{iff} \quad \vdash_{\text{LC}} \alpha.$$

EXERCISE 1. Let $\{0, 1, 2, \ldots, \infty\}$ be the modified version of $\boldsymbol{\omega}^+$ described in §10.3. Define **HA** operations on this set by modifying the operations on $\boldsymbol{\omega}^+$. Relate these operations to the definition of the "LC-matrix" given in Dummett [59]. □

PROBLEM. Let $\mathscr{E}$ be any topos, and put

$$L_{\mathscr{E}} = \{\alpha : \mathscr{E} \models \alpha\}$$

then $L_{\mathscr{E}}$ is closed under *Detachment*, and is an intermediate logic. A canonical frame $\mathbf{P}_{L_{\mathscr{E}}}$ may be defined for $L_{\mathscr{E}}$ by replacing IL by $L_{\mathscr{E}}$ everywhere in the definition of $\mathbf{P}_{\text{IL}}$.

Is there a general categorial relationship between the topoi $\mathscr{E}$ and $\mathbf{Set}^{\mathbf{P}_{L_{\mathscr{E}}}}$?

□

Exercises (for Heyting-algebraists)

EXERCISE 2. Given a truth value $\tau: 1 \to \Omega$ in $\mathbf{Set}^{\mathbf{P}}$, define $S_\tau \in \mathbf{P}^+$ by

$$S_\tau = \bigcup\{\tau_p(0): p \in P\}.$$

Show that the assignment of S_τ to τ gives a Heyting algebra isomorphism

$$\mathbf{Set}^{\mathbf{P}}(1, \Omega) \cong \mathbf{P}^+.$$

EXERCISE 3. Let $\sigma: F \rightarrowtail 1$ be a subobject of 1 in $\mathbf{Set}^{\mathbf{P}}$. Then for each p, σ_p can be taken as the inclusion $F_p \hookrightarrow \{0\}$, and so we have either $F_p = \emptyset$, or $F_p = \{0\} = 1$. Define

$$S_\sigma = \{p: F_p = 1\}.$$

Show that S_σ is hereditary and that the assignment of S_σ to σ yields an **HA** isomorphism

$$\mathrm{Sub}(1) \cong \mathbf{P}^+.$$

What is the inverse of this isomorphism?

EXERCISE 4. Suppose that the poset $\mathbf{P}$ has a least (initial) element. Show then that if $S, T \in \mathbf{P}^+$, $S \cup T = P$ iff $S = P$ or $T = P$.

Derive from this that the topos $\mathbf{Set}^{\mathbf{P}}$ is disjunctive, in the sense of §7.7. □