ALGEBRA OF SUBOBJECTS

"Since new paradigms are born from old ones, they ordinarily incorporate much of the vocabulary and apparatus, both conceptual and manipulative, that the traditional paradigm had previously employed. But they seldom employ these borrowed elements in quite the traditional way."

Thomas Kuhn

7.1. Complement, intersection, union

At the beginning of Chapter 6 it was asserted that the structure of $(\mathcal{P}(D), \subseteq)$ as **BA** depends on the rules of classical logic, through the properties of the connectives "and", "or", and "not". This can be made quite explicit by the consideration of characteristic functions. We see from the following result just how set operations depend on truth-functions.

THEOREM 1. *If A and B are subsets of D, with characters $\chi_A : D \to 2$, $\chi_B : D \to 2$, then*

(i) $\chi_{-A} = \neg \circ \chi_A$

(ii) $\chi_{A \cap B} = \chi_A \cap \chi_B \qquad (= \cap \circ \langle \chi_A, \chi_B \rangle)$

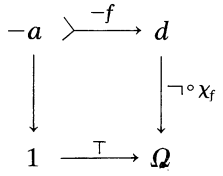
(iii) $\chi_{A \cup B} = \chi_A \cup \chi_B$.

PROOF. If $\chi_{-A}(x) = 1$, for $x \in D$, then $x \in -A$, so $x \notin A$, whence $\chi_A(x) = 0$, so $\neg\chi_A(x) = 1$. But if $\chi_{-A}(x) = 0$, then $x \notin -A$, so $x \in A$, whence $\chi_A(x) = 1$ and $\neg\chi_A(x) = 0$. Thus χ_{-A} and $\neg \circ \chi_A$ give the same output for the same input, and are identical. The proofs of (ii) and (iii) follow similar lines, using the definitions of $\cap$, $\curlywedge$, $\cup$, $\curlyvee$. □

Theorem 1 suggests a generalisation – the result in one context becomes the definition in another, as follows.

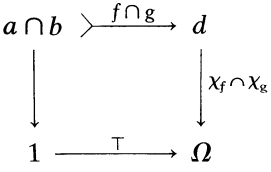
Let $\mathscr{E}$ be a topos, and d an $\mathscr{E}$-object. We define operations on the collection Sub(d) of subobjects of d in $\mathscr{E}$ thus:

(1) *Complements:* Given $f : a \rightarrowtail d$, the complement of f (relative to d) is the subobject $-f : -a \rightarrowtail d$ whose character is $\neg \circ \chi_f$. Thus $-f$ is defined to be the pullback



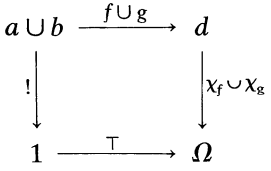
of $\top$ along $\neg \circ \chi_f$, yielding $\chi_{-f} = \neg \circ \chi_f$, by definition.

(2) *Intersections:* The intersection of $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ is the subobject $f \cap g : a \cap b \rightarrowtail d$ obtained by pulling $\top$ back along $\chi_f \cap \chi_g = \cap \circ \langle \chi_f, \chi_g \rangle$.



Hence $\chi_{f \cap g} = \chi_f \cap \chi_g$.

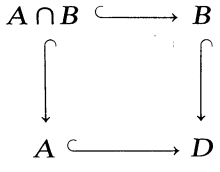
(3) *Unions:* $f \cup g : a \cup b \rightarrowtail d$ is the pullback of $\top$ along $\chi_f \cup \chi_g = \cup \circ \langle \chi_f, \chi_g \rangle$,



and so $\chi_{f \cup g} = \chi_f \cup \chi_g$. □

There is in fact a completely different approach available to the description of intersections and unions in **Set**.

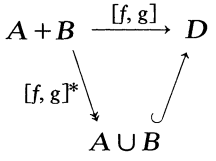
(a) *Intersection:* The diagram



is a pullback. Now in the poset $(\mathcal{P}(D), \subseteq)$, $A \cap B$ is the g.l.b. of A and B, hence their product, and indeed pullback. But we are saying something stronger than this, namely that the diagram is a pullback, not just in $\mathcal{P}(D)$, but in **Set** itself, as the reader may verify.

(b) *Unions:* In $\mathcal{P}(D)$, $A \cup B$ is the co-product of A and B. This description cannot be generalised as we do not yet know if $\mathrm{Sub}(d)$ has co-products, and moreover in **Set** itself the co-product $A+B$ is the *disjoint* union of A and B, so $A+B \neq A \cup B$ unless A and B are disjoint.

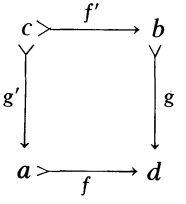
However, $A \cup B$ can be described as the union of the images of the inclusions $f: A \hookrightarrow D$ and $g: B \hookrightarrow D$, and in §6.6, in defining the disjunction arrow $\vee$, we gave a general construction for the union of two images. We form the co-product arrow $[f, g]: A+B \to D$, and then $A \cup B$ obtains as the image of $A+B$ under $[f, g]$, i.e.



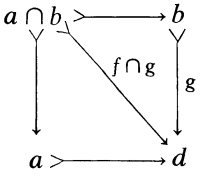
commutes as an epi-monic factorisation of $[f, g]$.

Although we have two descriptions of $\cap$ and $\cup$ in **Set** we are about to see that they present us with no choice in $\mathcal{E}$, i.e. that they lead to the same operations on $\mathrm{Sub}(d)$ (topoi really are the right generalisations of **Set**). The full proof is somewhat lengthy and intricate, and so we shall confine ourselves to outlining the basic strategy and leave the details to the reader who has developed a penchant for "arrow-chasing".

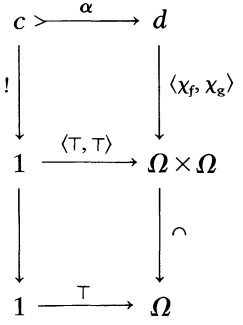
THEOREM 2. *In any topos $\mathcal{E}$, if $f: a \rightarrowtail d$ and $g: b \rightarrowtail d$ have pullback*



then $\alpha: c \rightarrowtail d$, where $\alpha = g \circ f' = f \circ g'$ has character $\chi_f \cap \chi_g$. Thus $\chi_\alpha = \chi_{f \cap g}$, so $\alpha \simeq f \cap g$ and there is a pullback of the form



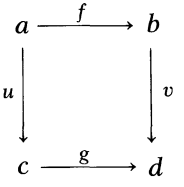
STRATEGY OF PROOF. The heart of the matter is to show that the top square of



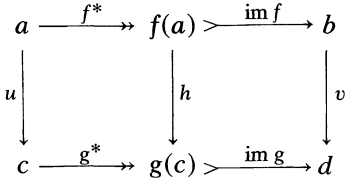
is a pullback. The bottom square is a pullback, by definition of $\cap$, so by the PBL the outer rectangle is a pullback, which by the Ω-axiom leads to the desired result that $\chi_\alpha = \cap \circ \langle \chi_f, \chi_g \rangle$. □

The analogous result for unions needs a preliminary

LEMMA. *In any* $\mathscr{E}$, *if*

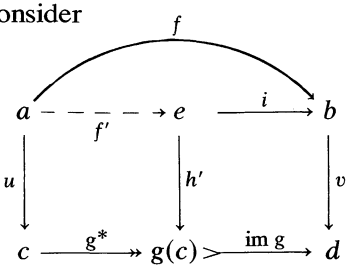


is a pullback, then there is an arrow $h : f(a) \to g(c)$ *that makes the right hand square of*

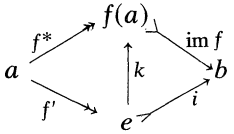


a pullback.

PROOF. Consider

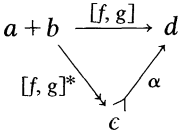


The right hand square obtains by pulling back im g along v, so i is monic. The existence of f', making the whole diagram commute follows from the universal property of the right hand square as a pullback, given that the "boundary" of the diagram is the pullback given in the hypothesis of the Lemma. The PBL then gives the left hand square as a pullback, and since the latter preserve epics (Fact 1, §5.3), $i \circ f'$ is an epi-monic factorisation of f. Hence there is a unique iso $k : e \to f(a)$ such that



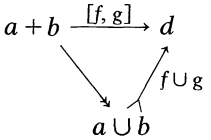
commutes. Then $h = h' \circ k^{-1}$ is the arrow required for the conclusion of the Lemma. □

THEOREM 3. *Given $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ in a topos $\mathscr{E}$, then the $\mathscr{E}$-arrow $\alpha : c \rightarrowtail d$ which is the image arrow of $[f, g] : a + b \to d$,*

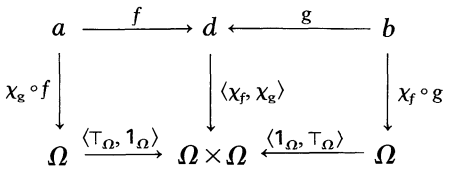


has character $\chi_f \cup \chi_g$.

Thus $\chi_\alpha = \chi_{f \cup g}$, so $\alpha \simeq f \cup g$ and there is an epi-monic factorisation



STRATEGY OF PROOF. The idea is to show that the two smaller squares of



are pullbacks. Since co-products preserve pullbacks (Fact 2, §5.3) we then get a pullback of the form

$$\begin{array}{ccc} a+b & \xrightarrow{[f,\,g]} & d \\ \downarrow & & \downarrow{\scriptstyle \langle \chi_f, \chi_g \rangle} \\ \Omega+\Omega & \xrightarrow{[\langle \top_\Omega, \mathbf{1}_\Omega \rangle \langle \mathbf{1}_\Omega, \top_\Omega \rangle]} & \Omega \times \Omega \end{array}$$

The Lemma then yields a pullback of the form

$$\begin{array}{ccc} c & \xrightarrow{\alpha} & d \\ \downarrow & & \downarrow{\scriptstyle \langle \chi_f, \chi_g \rangle} \\ e & \xrightarrow{i} & \Omega \times \Omega \end{array}$$

where i is the image arrow of $[\langle \top_\Omega, \mathbf{1}_\Omega \rangle, \langle \mathbf{1}_\Omega, \top_\Omega \rangle]$; But i is the arrow whose character is $\cup : \Omega \times \Omega \to \Omega$, i.e.

$$\begin{array}{ccc} e & \xrightarrow{i} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \cup} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is by definition a pullback. Putting these last two diagrams together and invoking the PBL shows that $\chi_\alpha = \cup \circ \langle \chi_f, \chi_g \rangle$. □

In view of Theorem 3 we can now describe the disjunction truth arrow $\cup$ as the character of

$$\Omega \cup \Omega \xrightarrow{\langle \top_\Omega, \mathbf{1}_\Omega \rangle \cup \langle \mathbf{1}_\Omega, \top_\Omega \rangle} \Omega \times \Omega$$

7.2. Sub(d) as a lattice

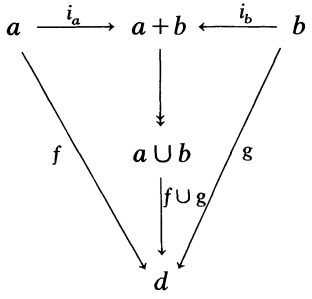
THEOREM 1. (Sub(d), $\subseteq$) *is a lattice in which*

(1) $f \cap g$ *is the g.l.b. (lattice meet) of* f *and* g;

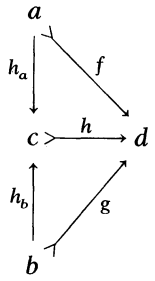
(2) $f \cup g$ *is the l.u.b.(join) of* f *and* g.

PROOF. (1) The characterisation of $f \cap g$ as a pullback of f and g makes it relatively easy to see why $f \cap g$ is the g.l.b. of f and g. The details are left to the reader.

(2) The characterisation of $f \cup g$ in Theorem 3 and the co-universal property of $[f, g]$ shows that



commutes and so each of f and g factors through $f \cup g$. Thus $f \subseteq f \cup g$, $g \subseteq f \cup g$, and $f \cup g$ is an upper bound of f and g. To show it is the least such, suppose $f \subseteq h$ and $g \subseteq h$. Then f and g each factor through h, so there are h_a, h_b making



commute. Then

$$[f, g] = [h \circ h_a, h \circ h_b]$$
$$= h \circ [h_a, h_b] \qquad \text{(dual of Exercise 3.8.3)}$$

and so $[f, g]$ is the composite of

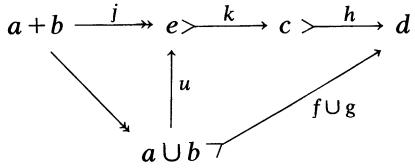
$$[h_a, h_b] : a + b \to c \quad \text{and} \quad h : c \rightarrowtail d.$$

Replacing $[h_a, h_b]$ by its epi-monic factorisation we get $[f, g]$ as the composite of

$$a + b \overset{j}{\twoheadrightarrow} e \overset{k}{\rightarrowtail} c \overset{h}{\to} d$$

for some j and k. But then j followed by $h \circ k$ is an epi-monic factorisation of $[f, g]$. By the uniqueness, up to isomorphism, of such things there

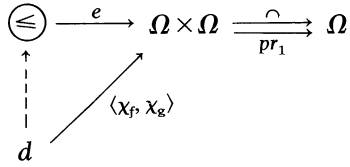
is an iso u such that



commutes. Then $k \circ u$ factors $f \cup g$ through h, yielding $f \cup g \subseteq h$ as required. □

COROLLARY. (1) $f \subseteq g$ *iff* $f \cap g \simeq f$ *iff* $f \cup g \simeq g$.

(2) $f \subseteq g$ *iff* $\langle \chi_f, \chi_g \rangle$ *factors* (*uniquely*) *through the equaliser*



of $\cap$ *and* pr_1.

PROOF. (1) In any lattice, $x \sqsubseteq y$ iff $x \sqcap y = x$ iff $x \sqcup y = y$.

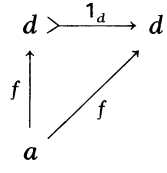
(2)
$$\begin{aligned} f \subseteq g \quad &\text{iff} \quad f \cap g \simeq f \\ &\text{iff} \quad \chi_{f \cap g} = \chi_f \\ &\text{iff} \quad \cap \circ \langle \chi_f, \chi_g \rangle = pr_1 \circ \langle \chi_f, \chi_g \rangle \end{aligned}$$

and the result follows by the universal property of equalisers. □

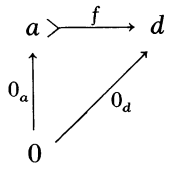
Part (2) of this Corollary is an analogue of the fact that in **Set** we have $A \subseteq B$ iff $\chi_A \leq \chi_B$ (the latter meaning $\chi_A(x) \leq \chi_B(x)$, all $x \in D$).

THEOREM 2. $(\mathrm{Sub}(d), \subseteq)$ *is a bounded lattice with unit* 1_d *and zero* 0_d.

PROOF. Given any $f : a \rightarrowtail d$, the commutativity of



and of



shows that $0_d \subseteq f$ and $f \subseteq \mathbf{1}_d$. □

EXERCISE 1. In Sub(d), $f \simeq \mathbf{1}_d$ iff f is iso, i.e. $f: a \cong d$. □

Sub(d) is in fact a distributive lattice, i.e. satisfies

$$f \cap (g \cup h) \simeq (f \cap g) \cup (f \cap h).$$

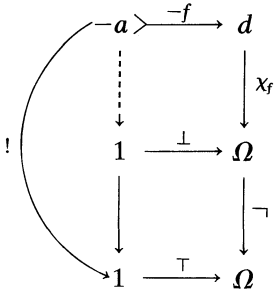
Again this is something that could be proved directly but in fact follows from some deeper results – this time a more detailed description of Sub(d) to be developed in the next chapter. We leave the matter till then (cf. §8.3).

What about complements? To date we have not used the definition $\chi_{-f} = \neg \circ \chi_f$. The first thing we shall prove in this connection is

THEOREM 3. *For $f: a \rightarrowtail d$, we have*

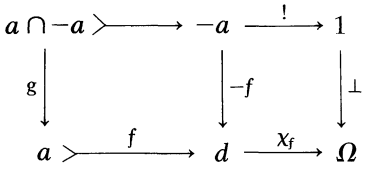
$$f \cap -f \simeq 0_d.$$

PROOF. The boundary of

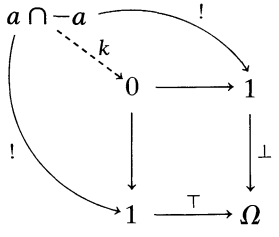


is the pullback defining $-f$, the bottom square is the pullback defining $\neg$, so the unique arrow $-a \to 1$ makes the whole diagram commute, and the top square a pullback.

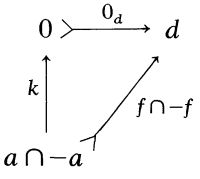
Then each square of



commutes (the left hand one is the pullback giving $f \cap -f$), so we get $\perp \circ ! = \chi_f \circ f \circ g$. But $\chi_f \circ f = true_a$ (Ω-axiom), so $\chi_f \circ f \circ g = true_a \circ g = true_{a \cap -a}$ (4.2.3). Hence the outer square of



commutes. But the inner square is a pullback, so the arrow $k : a \cap -a \to 0$ does exist. But then $a \cap -a \cong 0$ (§3.16), so $a \cap -a$ is an initial object and



must commute. Thus $f \cap -f \subseteq 0_d$, and since 0_d is the minimum element of Sub(d), the result follows. □

We seem to be well on the way to a proof that Sub(d) is a Boolean algebra, and hence complete the analogy with $\mathcal{P}(D)$ in **Set**. We know it to be a bounded distributive lattice, with $f \cap -f$ always the zero. It remains only to show that $f \cup -f$ is the unit. But we cannot do this! There are topoi in which it is false. To give an example we need

THEOREM 4. *In* Sub(Ω), (*for any topos*),

$$\perp \simeq -\top.$$

PROOF. $\chi_{\perp} = \neg$ (definition of $\neg$)

$= \neg \circ \mathbf{1}_{\Omega}$

$= \neg \circ \chi_{\top}$

$= \chi_{-\top}$. □

So in any topos, $\top \cup -\top = \top \cup \perp$. Now in our favourite example $\mathbf{M_2}$, $\mathbf{1}_{\Omega}$ in $\mathrm{Sub}(\Omega)$ can be identified with the set L_M, while $\top \cup \perp$, as the image of $[\top, \perp]$ (recall the description of the latter in Theorem 5.4.6), can be identified with the set $\{M_2, \emptyset\} \neq L_2$. Hence

$$\top \cup \perp \neq \mathbf{1}_{\Omega},$$

and so $-\top$ $(= \perp)$ is not the lattice complement of $\top$ in $\mathrm{Sub}(\Omega)$. But then, as the next result shows, $\mathrm{Sub}(\Omega)$ is not a Boolean algebra at all.

THEOREM 5. *In any topos, if* $\top : 1 \to \Omega$ *has a complement in* $\mathrm{Sub}(\Omega)$, *then this complement is the subobject* $\perp : 1 \to \Omega$.

PROOF. If $\top$ has a complement, f say, then $\top \cap f \simeq 0_{\Omega}$, so

$$\begin{array}{ccc} 0 & \xrightarrow{0_a} & a \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is a pullback. The Ω-axiom then gives $f = \chi_{0_a} = \perp \circ !_a$ (cf. Exercise 5.4.3). But $\perp \circ !_a$ obviously factors through $\perp$, so $f \subseteq \perp$. Lattice properties then give $\top \cup f \subseteq \top \cup \perp$, and since $\top \cup f \simeq \mathbf{1}_{\Omega}$, $\top \cup \perp \simeq \mathbf{1}_{\Omega}$. But by Theorems 3 and 4 above, $\top \cap \perp = 0_{\Omega}$, and so $\perp$ is a complement of $\top$. But in a distributive lattice, complements are unique, hence $f \simeq \perp$. □

7.3. Boolean topoi

A topos $\mathscr{E}$ will be called *Boolean* if for *every* $\mathscr{E}$-object d, $(\mathrm{Sub}(d), \subseteq)$ is a Boolean algebra.

THEOREM 1. *For any topos* $\mathscr{E}$, *the following statements are equivalent:*

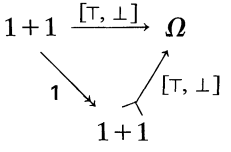
(1) $\mathscr{E}$ *is Boolean*

(2) $\mathrm{Sub}(\Omega)$ *is a* **BA**

(3) $\top : 1 \to \Omega$ *has a complement in* $\mathrm{Sub}(\Omega)$

(4) $\bot : 1 \to \Omega$ *is the complement of* $\top$ *in* $\mathrm{Sub}(\Omega)$
(5) $\top \cup \bot \simeq \mathbf{1}_\Omega$ *in* $\mathrm{Sub}(\Omega)$
(6) $\mathscr{E}$ *is classical, i.e.* $[\top, \bot] : 1+1 \to \Omega$ *is iso*
(7) $i_1 : 1 \to 1+1$ *is a subobject classifier.*

PROOF. (1) implies (2): definition of "Boolean"
(2) implies (3): definition of "**BA**"
(3) implies (4): Theorem 7.2.5
(4) implies (5): definition of "complement"
(5) implies (6): $[\top, \bot]$ is always monic, so

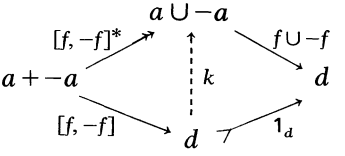


is an epi-monic factorisation of $[\top, \bot]$, i.e. in $\mathrm{Sub}(\Omega)$, $\top \cup \bot \simeq [\top, \bot]$. Then if $\top \cup \bot \simeq \mathbf{1}_\Omega$, we get $[\top, \bot] \simeq \mathbf{1}_\Omega$, making $[\top, \bot]$ iso by Exercise 7.2.1.

(6) implies (7): Exercise – the essential point being that anything isomorphic to a classifier will be one itself.

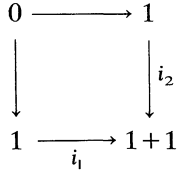
(7) implies (1): Given $f : a \rightarrowtail d$, we wish to show that $f \cup -f \simeq \mathbf{1}_d$, and so by the work of §7.2 $-f$ will be a complement for f, and $\mathrm{Sub}(d)$ will be a **BA**.

The basic strategy can be seen in the diagram



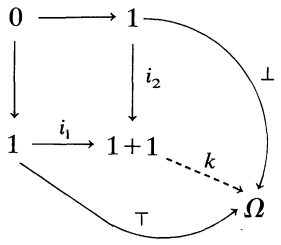
If we can show that $[f, -f]$ is epic, then the iso k as shown will exist to factor $\mathbf{1}_d$ through $f \cup -f$ to make $f \cup -f \simeq \mathbf{1}_d$. We need first the following:

LEMMA. *In any topos,*



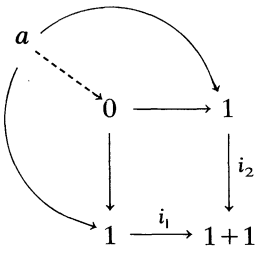
is a pullback, where i_1, i_2 *are the two injections for the co-product* $1+1$.

PROOF. The square commutes as 0 is initial. It is also a pushout by the co-universal property of the pair (i_1, i_2). But the outer square of

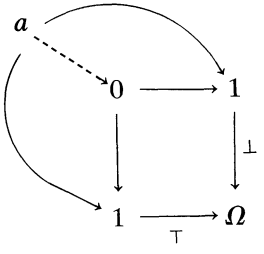


commutes, indeed is a pullback by the Ω-axiom, so the unique k exists as shown to make the diagram commute.

Then if the outer square of



commutes, k can be used to show the outer square of



commutes, giving the unique $a \to 0$ for the previous diagram as required. □

To finish our Theorem we shall denote by χ'_f, $\perp'$ etc. the arrows defined in the same way as χ_f, $\perp$, etc., but using $i_1 : 1 \to 1+1$ in place of $\top : 1 \to \Omega$. Now the Lemma tells us that $i_2 = \perp'$, so by the argument at the

beginning of Theorem 3 of §7.2,

$$\begin{array}{ccc} -a & \xrightarrow{-f} & d \\ \downarrow & & \downarrow \chi'_f \\ 1 & \xrightarrow{i_2} & 1+1 \end{array}$$

is a pullback. But so is

$$\begin{array}{ccc} a & \xrightarrow{f} & d \\ \downarrow & & \downarrow \chi'_f \\ 1 & \xrightarrow{i_1} & 1+1 \end{array}$$

and co-products preserve pullbacks, so

$$\begin{array}{ccc} a+-a & \xrightarrow{[f,-f]} & d \\ !+! \downarrow & & \downarrow \chi'_f \\ 1+1 & \xrightarrow{[i_1,i_2]} & 1+1 \end{array}$$

is a pullback. But $[i_1, i_2] = 1_{1+1}$ is epic, whence $[f, -f]$ is the pullback of an epic, i.e. an epic itself. □

7.4. Internal vs. External

THEOREM 1. *If $\mathscr{E}$ is Boolean, then $\mathscr{E} \vDash \alpha \vee \sim\alpha$, for any sentence α.*

PROOF. Let V be an $\mathscr{E}$-valuation. Form the pullback

$$\begin{array}{ccc} a & \xrightarrow{f} & 1 \\ \downarrow & & \downarrow V(\alpha) \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

of $\top$ along $V(\alpha)$, so that $\chi_f = V(\alpha)$.

Now if $\mathscr{E}$ is Boolean, Sub(1) is a **BA**, so $f \cup -f \simeq \mathbf{1}_1$, whence $\chi_{f\cup -f} = \chi_{\mathbf{1}_1} = \top$. But

$$\begin{aligned}\chi_{f\cup -f} &= \chi_f \cup \neg \circ \chi_f \\ &= V(\alpha) \cup \neg \circ V(\alpha) \\ &= V(\alpha \vee \sim\alpha).\end{aligned}$$

Hence $V(\alpha \vee \sim\alpha) = \top$. □

One might think that if our theory was working well then the converse of Theorem 1 should hold. However our example $\mathbf{M}_2$ is non-Boolean, since in it Sub(Ω) is not a **BA**, and yet $\mathbf{M}_2 \models \alpha \vee \sim\alpha$, as observed at the end of Chapter 6. The proof of Theorem 1 in fact only required that Sub(1) be a **BA**. That *this* is the relevant condition is shown by

THEOREM 2. *In any topos $\mathscr{E}$, the following are equivalent:*

(1) $\mathscr{E} \models \alpha$ *iff* $\vdash_{CL} \alpha$, *all sentences* α

(2) $\mathscr{E} \models \alpha \vee \sim\alpha$, *all* α

(3) Sub(1) *is a* **BA**.

PROOF. Clearly (1) implies (2). Assuming (2) we take a subobject $f : a \rightarrowtail 1$ in Sub(1) and observe that χ_f is a truth value $1 \to \Omega$. Taking an $\mathscr{E}$-valuation that has $V(\pi_0) = \chi_f$, we have $\chi_{f\cup -f} = \chi_f \cup \neg\chi_f = V(\pi_0) \cup \neg V(\pi_0) = V(\pi_0 \vee \sim\pi_0) = \top = \chi_{\mathbf{1}_1}$. Hence $f \cup -f \simeq \mathbf{1}_1$. This means that Sub(1) is a **BA**.

Finally assume (3), in order to derive (1). The "only if" part of (1) holds in any topos. The "if" part requires a proof that the CL-axioms are $\mathscr{E}$-valid and that detachment preserves $\mathscr{E}$-validity. We shall explain later why axioms I–XI are valid in any topos, and why *Detachment* is always validity preserving. For the present we note only that the proof of Theorem 1 shows that if Sub(1) is a **BA**, then axiom XII is $\mathscr{E}$-valid. □

COROLLARY. "Sub(1) *is a* **BA**" *does not imply that $\mathscr{E}$ is Boolean.*

The situation seems at first sight anomolous (at least it did to the author). In **Set** the logic is based on the **BA** 2, and in the general topos it seems to be intimately related to Sub(1). In **Set,** $\mathrm{Sub}(1) \cong \mathscr{P}(1) \cong 2$ – so far so good. But the work of the previous sections shows that the properties of the "generalised power-sets" Sub(d) are determined by Sub(Ω), whereas in **Set**, Sub(Ω) is a four-element set that has played no special role to date.

Some clarification of this situation is afforded by the observation that Sub(d) is a collection of subobjects of d and may well not be itself an actual $\mathscr{E}$-object. Thinking of $\mathscr{E}$ as a "general universe of mathematical discourse" then a person living in that universe, i.e. one who uses only the individuals that exist in that universe, does not "see" Sub(d) at all as a single entity. Sub(d) is *external* to $\mathscr{E}$. What the topos-dweller does see is the power object Ω^d, which is the "object of subsets" of the object d. Ω^d is an individual in the universe $\mathscr{E}$, and is the *internal* version of the notion of power set, while Sub(d) is the external version.

Now the Law of Excluded Middle does have an internal version. The validity of $\alpha \vee \sim\alpha$ in **Set** corresponds to the truth of the equation

$$x \cup \neg x = 1, \quad \text{for} \quad x \in 2.$$

The truth of this equation is equivalent to the commutativity of

$$\begin{array}{ccc} 2 & \xrightarrow{\langle \mathrm{id}_2, \neg \rangle} & 2 \times 2 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \cup} \\ 1 & \xrightarrow{\ true\ } & 2 \end{array}$$

(since $\langle \mathrm{id}_2, \neg \rangle(x) = \langle x, \neg x \rangle$).

Now this diagram has an analogue in any topos $\mathscr{E}$, and we have the interesting

THEOREM 3. Sub(Ω) *is a* **BA** *iff the diagram*

$$\begin{array}{ccc} \Omega & \xrightarrow{\langle 1_\Omega, \neg \rangle} & \Omega \times \Omega \\ \downarrow & & \downarrow{\scriptstyle \cup} \\ 1 & \xrightarrow{\ \top\ } & \Omega \end{array} \qquad \text{(EM)}$$

commutes.

PROOF. EM commutes when

$$\cup \circ \langle 1_\Omega, \neg \rangle = \top_\Omega$$

i.e.

$$1_\Omega \cup \neg = \top_\Omega$$

But we know that $1_\Omega = \chi_\top$, $\neg = \chi_\bot$, and $\top_\Omega = \chi_{1_\Omega}$, so

$$\begin{aligned} \text{Sub}(\Omega) \text{ is a } \mathbf{BA} \quad &\text{iff} \quad \top \cup \bot \simeq 1_\Omega \qquad (\S 7.3) \\ &\text{iff} \quad \chi_{\top \cup \bot} = \chi_{1_\Omega} \\ &\text{iff} \quad \chi_\top \cup \chi_\bot = \chi_{1_\Omega} \\ &\text{iff} \quad 1_\Omega \cup \neg = \top_\Omega. \end{aligned}$$

□

EXERCISE 1. Show explicitly why EM does not commute in $\mathbf{M}_2$. □

Now in our theory of topos semantics we use the collection $\mathscr{E}(1, \Omega)$ of truth-values. This again is an external thing – the internal version of the collection of arrows from 1 to Ω would be the *object of truth-values* $\Omega^1 \cong \Omega$. Also a valuation $V : \Phi \to \mathscr{E}(1, \Omega)$ is external, i.e. is not an actual $\mathscr{E}$-arrow.

Thus the semantical theory we have developed is an external one, and this is why there can be topoi like $\mathbf{M}_2$ that look classical "from the outside" and yet can have non-classical properties (curiously, $\mathbf{M}_2$ is internally bivalent while "from the outside" Ω has three elements). We now see that a topos also has an internal logic, in the form of commuting diagrams like EM (cf. Exercise 2 below). It is precisely when this internal logic is classical that the topos is Boolean.

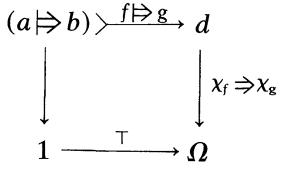
From the viewpoint that topoi offer a complete alternative to the category **Set** as a context for doing mathematics it is finally the internal structure that is important. Nonetheless the present external theory is very useful for elucidating the logical properties of topoi, and as we shall see, for describing the link between topoi and intuitionistic logic.

EXERCISE 2. Describe the validity of the CL-axioms I–XI in terms of commutativity of diagrams involving truth-arrows. (All of them commute in any topos – can you prove some of them?)

7.5. Implication and its implications

In the same way that we used the truth arrows $\wedge$, $\vee$, $\neg$ to define operations $\cap$, $\cup$, $-$ on $\text{Sub}(d)$ we can use implication $\Rightarrow$ to define the following operation: if $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ are subobjects of d, then $f \mathbin{\Rightarrow\!\!\!\!\!\!\rightarrowtail} g : (a \Rightarrow b) \rightarrowtail d$ is the subobject obtained by pulling $\top$ back along

$\chi_f \Rightarrow \chi_g = \Rightarrow \circ \langle \chi_f, \chi_g \rangle$. Thus



is a pullback, i.e. $\chi_{f \Rightarrow g} = \chi_f \Rightarrow \chi_g$.

In order to study the properties of this new operation we need some technical results.

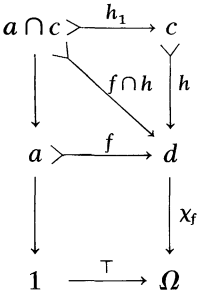
LEMMA 1. *If f, g, and h are subobjects of d (in any topos), then*

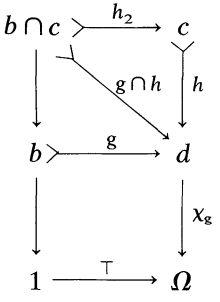
(1) $\quad f \cap h \simeq g \cap h \quad$ *iff* $\quad \chi_f \circ h = \chi_g \circ h,$

and hence

(2) $\quad \chi_f \cap \chi_h = \chi_g \cap \chi_h \quad$ *iff* $\quad \chi_f \circ h = \chi_g \circ h.$

PROOF. (1) Consider





In each diagram the bottom squares are pullbacks by the Ω-axiom, and the top squares are pullbacks by the characterisation of intersections. So by the PBL, $\chi_f \circ h = \chi_{h_1}$ and $\chi_g \circ h = \chi_{h_2}$. Thus $\chi_f \circ h = \chi_g \circ h$ iff $h_1 \simeq h_2$. But this last condition holds only if there is an iso k giving $h_1 \circ k = h_2$, and so $h \circ h_1 \circ k = h \circ h_2$, i.e. $(f \cap h) \circ k = g \cap h$, and so $f \cap h \simeq g \cap h$. The argument reverses to show $f \cap h \simeq g \cap h$ only if $h_1 \simeq h_2$. Part (2) is immediate from (1). □

COROLLARY

$$f \cap h \subseteq g \quad \textit{iff} \quad \chi_{f \cap g} \circ h = \chi_f \circ h.$$

PROOF

$$\begin{aligned} f \cap h \subseteq g \quad &\text{iff} \quad (f \cap h) \cap g \simeq f \cap h \\ &\text{iff} \quad (f \cap g) \cap h \simeq f \cap h \qquad \text{(lattice properties)} \\ &\text{iff} \quad \chi_{f \cap g} \circ h = \chi_f \circ h \qquad \text{(Lemma).} \end{aligned}$$ □

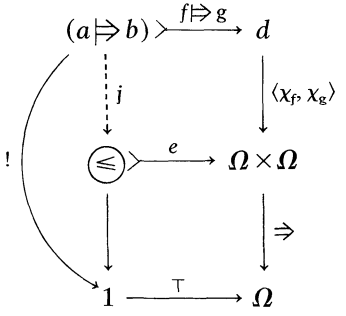
THEOREM 1. *In* Sub(d) *we have:*

(1) $h \subseteq f \Rightarrow g$ *iff* $f \cap h \subseteq g$

(2) $f \subseteq g$ *iff* $f \Rightarrow g \simeq 1_d$

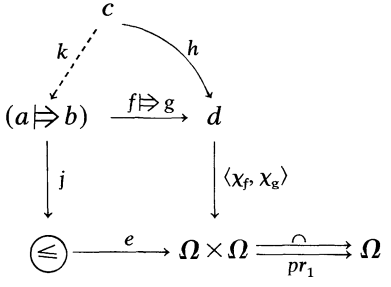
(3) $f \subseteq g$ *iff* $\chi_f \Rightarrow \chi_g = true_d$.

PROOF. (1) First consider



The boundary commutes by definition of $f \Rightarrow g$. The bottom square is a pullback, so the unique arrow j exists to make the whole thing commute. Then the PBL gives the top square as a pullback.

The basic strategy of the main proof is seen in the diagram

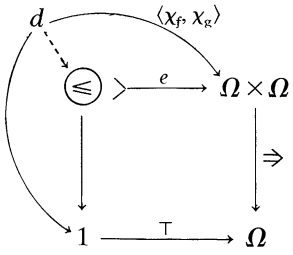


We have $h \subseteq f \Rightarrow g$ precisely when there is an arrow k as shown making the top triangle commute. Since the square is a pullback, such a k exists precisely when $\langle \chi_f, \chi_g \rangle \circ h$ factors through e. By the universal property of e as an equaliser, this happens precisely when $pr_1 \circ \langle \chi_f, \chi_g \rangle \circ h = \frown \circ \langle \chi_f, \chi_g \rangle \circ h$, i.e. $\chi_f \circ h = \chi_{f \cap g} \circ h$. But this last equality holds iff $f \cap h \subseteq g$, by the last Corollary.

(2) We use part (1). Suppose $f \subseteq g$. Then for any h in Sub(d), $f \cap h \subseteq f \subseteq g$, so by (1), $h \subseteq f \Rightarrow g$. This makes $f \Rightarrow g$ the unit 1_d of Sub(d). Conversely if $f \Rightarrow g = 1_d$, then $f \subseteq f \Rightarrow g$, so $f \cap f \subseteq g$, i.e. $f \subseteq g$.

(3) From (2), and the definition of $\Rightarrow$, since $\chi_{1_d} = true_d$. □

EXERCISE. Give a categorial proof of part (2), by using the Corollary to Theorem 1 of §7.2 and the diagram



□

COROLLARY TO THEOREM 1. *In* Sub(d):

(1) $1_d \Rightarrow 1_d = 0_d \Rightarrow 1_d = 0_d \Rightarrow 0_d = 1_d$.

(2) $1_d \Rightarrow 0_d = 0_d$.

PROOF. (1) By part (2) of the Theorem, as $1_d \subseteq 1_d$, $0_d \subseteq 1_d$, $0_d \subseteq 0_d$.

(2) Since $1_d \Rightarrow 0_d \subseteq 1_d \Rightarrow 0_d$, part (1) gives

$$1_d \cap (1_d \Rightarrow 0_d) \subseteq 0_d,$$

i.e.

$$1_d \Rightarrow 0_d \subseteq 0_d, \qquad (1_d \text{ is maximum})$$

and hence

$$1_d \Rightarrow 0_d = 0_d.$$ □

Now in $\mathcal{P}(D)$, $A \Rightarrow D$ is $-A \cup D$. (why?) The analogous situation does not obtain in all topoi. In $\mathbf{M_2}$, $\top \Rightarrow \top = 1_\Omega$ in Sub(Ω) (by Theorem 1(2)), while $-\top \cup \top = \bot \cup \top = \top \cup \bot$, and we saw in §7.2 that $\top \cup \bot \neq 1_\Omega$ in $\mathbf{M_2}$.

To determine the conditions under which $\Rightarrow$ can be defined from $\cup$ and $-$ we need

LEMMA 2. (1) *In any lattice, if m and n satisfy*

(i) $x \sqsubseteq m$ *iff* $a \sqcap x \sqsubseteq b$, *all* x

(ii) $x \sqsubseteq n$ *iff* $a \sqcap x \sqsubseteq b$, *all* x

then $m = n$.

(2) *In a Boolean algebra,*

$$x \sqsubseteq (a' \sqcup b) \quad \textit{iff} \quad a \sqcap x \sqsubseteq b,$$

and so the only m *that satisfies the condition of* (1)(i) *is* $m = a' \sqcup b$.

PROOF. (1) Exercise – use $m \sqsubseteq m$ etc.

(2) First, by properties of l.u.b.'s and g.l.b.'s, note that if $x \sqsubseteq z$, then $y \sqcap x \sqsubseteq y \sqcap z$ (any x, y, z). Next note that in a **BA**, $a \sqcap (a' \sqcup b) = (a \sqcap a') \sqcup (a \sqcap b) = 0 \sqcup (a \sqcap b) = a \sqcap b \sqsubseteq b$ so that if $x \sqsubseteq (a' \sqcup b)$ by the foregoing we have $a \sqcap x \sqsubseteq a \sqcap (a' \sqcup b) \sqsubseteq b$, i.e. $a \sqcap x \sqsubseteq b$. Conversely, if $a \sqcap x \sqsubseteq b$ then $x = 1 \sqcap x = (a' \sqcup a) \sqcap x = (a' \sqcap x) \sqcup (a \sqcap x) \sqsubseteq a' \sqcup b$. □

THEOREM 2. *In any topos* $\mathscr{E}$, *the following are equivalent:*

(1) $\mathscr{E}$ *is Boolean*

(2) *In each* Sub(d), $f \Rightarrow g \simeq -f \cup g$

(3) *In* Sub(Ω), $f \Rightarrow g \simeq -f \cup g$

(4) $\top \Rightarrow \top = \top \cup \bot$.

PROOF. (1) implies (2): Theorem 1(1) states that in the lattice Sub(d), $h \subseteq f \Rightarrow g$ iff $f \cap h \subseteq g$. But if Sub(d) is a **BA**, Lemma 2(2) tells us that $h \subseteq -f \cup g$ iff $f \cap h \subseteq g$. Lemma 2(1) then implies that $f \Rightarrow g = -f \cup h$.

(2) implies (3): obvious.

(3) implies (4): $-\top \cup \top = \top \cup \bot$ as noted prior to Lemma 2.

(4) implies (1): We always have $\top \Rightarrow \top \simeq \mathbf{1}_\Omega$. Use part (5) of the Theorem in §7.3. □

So we see that in a non-Boolean topos, $\Rightarrow$ does not behave like a Boolean implication operator. What its behaviour *is* like in general will be revealed in the next chapter. Before proceeding to that however, we pause for the purpose of

7.6. Filling two gaps

1. Theorem 1 of §6.7 gave some tables for the behaviour of the truth-values $\top$ and $\bot$ under the arrows $\wedge$, $\vee$, and $\Rightarrow$. We are now in a position to show why these tables are correct.

The key lies in the lattice structure of Sub(1), where the unit is $\mathbf{1}_1$ and the zero 0_1. Thus we have $\mathbf{1}_1 \cap \mathbf{1}_1 \simeq \mathbf{1}_1$, while $\mathbf{1}_1 \cap 0_1 \simeq 0_1 \cap \mathbf{1}_1 \simeq 0_1 \cap 0_1 \simeq$

0_1. But $\chi_{1_1} = \top$ and $\chi_{0_1} = \bot$, so we have

$$\top \cap \bot = \chi_{1_1} \cap \chi_{0_1} = \chi_{1_1 \cap 0_1} = \chi_{0_1} = \bot$$
$$\top \cap \top = \chi_{1_1} \cap \chi_{1_1} = \chi_{1_1 \cap 1_1} = \chi_{1_1} = \top,$$

and so on, yielding the table

$\cap$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

Now using the Corollary to Theorem 1 of §7.5 we find $\top \Rightarrow \bot = \chi_{1_1} \Rightarrow \chi_{0_1} = \chi_{1_1 \Rightarrow 0_1} = \chi_{0_1} = \bot$, $\bot \Rightarrow \top = \chi_{0_1 \Rightarrow 1_1} = \chi_{1_1} = \top$ etc. leading to

$\Rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

EXERCISE. Derive the table

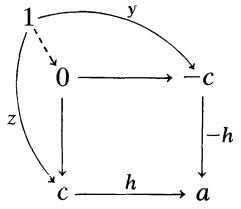
$\cup$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

□

2. Theorem 5 of §5.4 asserted without proof that a classical ($1+1 \cong \Omega$) topos in which every non-zero object is non-empty is in fact well-pointed. Now if $\mathscr{E}$ is classical, we now know it to be Boolean by §7.3. So let us take a pair of *distinct* parallel arrows $f,g : a \rightrightarrows b$ in $\mathscr{E}$ and look for an element $x : 1 \to a$ that distinguishes them, i.e. has $f \circ x \neq g \circ x$. We let $h : c \rightarrowtail a$ be the equaliser of f and g, and $-h : -c \rightarrowtail a$ the complement of h in Sub(a) (remember $\mathscr{E}$ is Boolean). Then $-c$ is non-zero (in **Set**, $-c \neq \emptyset$ as f and g differ at some point of a). For, if $-c \cong 0$, then $-h \simeq 0_a$, so $h \simeq h \cup 0_a \simeq h \cup -h \simeq 1_a$, whence h is iso and since $f \circ h = g \circ h$ we would get $f = g$.

Now if all non-zero $\mathscr{E}$-objects are non-empty there must then be an arrow $y : 1 \to -c$. Then let x be $-h \circ y : 1 \to a$. Then if $f \circ x = g \circ x$, as h

equalises f and g there would be some $z : 1 \to c$ such that $h \circ z = x$. Hence the boundary of



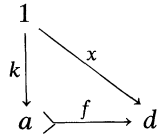
would commute, giving an arrow $1 \to 0$. But this would make $\mathscr{E}$ degenerate, contrary to the fact that $c \not\cong 0$. We conclude $f \circ x \neq g \circ x$.

7.7. Extensionality revisited

In Chapter 5 we considered well-pointedness as a categorial formulation of the extensionality principle for functions. For sets themselves, extensionality simply means that sets with the same elements are identical. It follows from this that identity of sets is characterised by the set inclusion relation: $A = B$ iff $A \subseteq B$ and $B \subseteq A$, since

$$A \subseteq B \quad \text{iff every member of } A \text{ is a member of } B.$$

This definition of the subset relation is readily lifted to the general category. If $f : a \rightarrowtail d$ is a subobject of d, and $x : 1 \to d$ an element of d, then as in §4.8 we say that x *is an element of* f, $x \in f$, iff x factors through f.



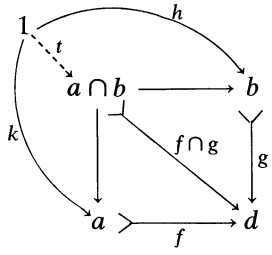
i.e. for some $k : 1 \to a$, $x = f \circ k$.

THEOREM 1. *In any topos* $\mathscr{E}$, *in* Sub(d) *we have*

$$x \in f \cap g \quad \textit{iff} \quad x \in f \quad \textit{and} \quad x \in g.$$

PROOF. If x factors through $f \cap g$, then since $f \cap g$ factors through both f and g, so too will x.

Conversely, suppose that $x \in f$ and $x \in g$, so that



$x = f \circ k$ and $x = g \circ h$ for some elements $k : 1 \to a$ and $h : 1 \to b$. But the inner square of the diagram is a pullback (§7.1) so the arrow t exists as shown making $f \cap g \circ t = f \circ k = x$. This t factors x through $f \cap g$, giving $x \in f \cap g$. □

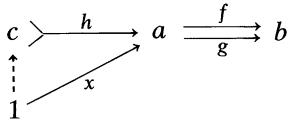
A topos in which subobjects are determined by their elements will be called *extensional*. That is, $\mathscr{E}$ is extensional iff for any $\mathscr{E}$-object d, the condition

$$f \subseteq g \quad \text{iff} \quad \text{for all} \quad x : 1 \to d,\ x \in f \quad \text{implies} \quad x \in g$$

holds in Sub(d).

THEOREM 2. *$\mathscr{E}$ is extensional iff well-pointed.*

PROOF. Let $f, g : a \rightrightarrows b$ be a pair of parallel $\mathscr{E}$-arrows, with $f \circ x = g \circ x$, all $x : 1 \to a$. Let $h : c \rightarrowtail a$ be the equaliser of f and g. Then if $x \in \mathbf{1}_a$,



(which holds for any $x : 1 \to a$), we get $x \in h$ by the universal property of h as equaliser. Extensionality of $\mathscr{E}$ then gives $\mathbf{1}_a \subseteq h$, and so $h \circ k = \mathbf{1}_a$, for some k. Since $f \circ h = g \circ h$, this yields $f = h$ upon composition with k.

Conversely, suppose that $\mathscr{E}$ is well-pointed. The "only if" part of the extensionality condition is straightforward and holds in any category. For the "if" part, suppose that every $x \in f$ has $x \in g$. In order to establish $f \subseteq g$, it suffices to show $f \cap g \simeq f$, i.e. $\chi_{f \cap g} = \chi_f$. Since in general $f \cap g \subseteq f$, Theorem 7.5.1 (3), gives

$$\Rightarrow \circ \langle \chi_{f \cap g}, \chi_f \rangle = true_d.$$

Then if $x : 1 \to d$ is any element of d,

$$\Rightarrow \circ \langle \chi_{f \cap g}, \chi_f \rangle \circ x = true_d \circ x$$

i.e.

$$\chi_{f \cap g} \circ x \Rightarrow \chi_f \circ x = true$$

(Exercise 3.8.3 and 4.2.3).

Now $\chi_{f \cap g} \circ x$ and $\chi_f \circ x$ are both truth-values $1 \to \Omega$, and $\mathscr{E}$ is bivalent (being well-pointed), so that each is *true* or *false*. But by Exercise 4.8.2, if $\chi_f \circ x = true$, then $x \in f$, so by our hypothesis $x \in g$, and hence by Theorem 1, $x \in f \cap g$, yielding $\chi_{f \cap g} \circ x = true$. In view of the last equation derived above, and the table for $\Rightarrow$ established in §7.6, $\chi_{f \cap g} \circ x$ and $\chi_f \circ x$ must be either both *true*, or both *false*.

What we have shown then is that the parallel arrows $\chi_{f \cap g}, \chi_f : d \to \Omega$ are not distinguished by any element $x : 1 \to d$ of their domain. Since $\mathscr{E}$ is well-pointed, this implies $\chi_{f \cap g} = \chi_g$ as required. □

Theorem 2 points up the advance of topos theory over Lawvere's earlier work [64] on a theory of the category of sets. That system included well-pointedness as an axiom, but the derivation of extensionality required an essential use of a version of the "axiom of choice" (cf. Chapter 12).

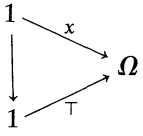
It is noteworthy that the analogues of Theorem 1 for the other set operations, viz

(a) $\quad x \in -f \quad$ iff not $\quad x \in f$

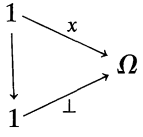
and

(b) $\quad x \in f \cup g \quad$ iff $\quad x \in f \quad$ or $\quad x \in g$

fail in some topoi. Take for instance any $\mathscr{E}$ that is Boolean but not bivalent – the simplest example would be the topos $\mathbf{Set}^2$ of pairs of sets. Then $\mathscr{E}$ has a truth value $x : 1 \to \Omega$ distinct from $\top$ and $\bot$. Then neither of



and



commute, so $x \notin \top$ and $x \notin -\top$ (since $\bot = -\top$ always). Moreover as $\mathscr{E}$ is Boolean, $\top \cup -\top \simeq 1_\Omega$, and so $x \in \top \cup -\top$. Hence both (a) and (b) fail.

THEOREM 3. *$\mathscr{E}$ is bivalent iff* (a) *holds in every* Sub(d).

PROOF. The argument just given to show that (a) fails at least in Sub(Ω) if $\mathscr{E}$ is not bivalent works in any $\mathscr{E}$. On the other hand if $\mathscr{E}$ is bivalent, then if $y : 1 \to \Omega$ is a truth-value with $y \neq \top$, then $y = \bot$ and so $\neg \circ y = \top$. Using this, we find, for $f : a \rightarrowtail d$ and $x : 1 \to d$,

$$\begin{aligned} x \in -f \quad &\text{iff} \quad \chi_{-f} \circ x = \top && \text{(Exercise 4.8.2)} \\ &\text{iff} \quad \neg \circ \chi_f \circ x = \top \\ &\text{iff} \quad \chi_f \circ x \neq \top \\ &\text{iff} \quad \text{not } x \in f. \end{aligned}$$

□

THEOREM 4. *$\mathscr{E}$ satisfies* (b) *for all $\mathscr{E}$-objects d iff $\mathscr{E}$ satisfies the condition* (c):

For any truth values $y : 1 \to \Omega$ and $z : 1 \to \Omega$, $y \cup z = true$ iff $y = true$ or $z = true$.

PROOF. If (b) holds in Sub(1), then let $f : a \rightarrowtail 1$ and $g : b \rightarrowtail 1$ be such that $\chi_f = y$, $\chi_g = z$. Then taking $x : 1 \to 1$, i.e. $x = 1_1$,

$$\begin{aligned} y \cup z = \top \quad &\text{iff} \quad (y \cup z) \circ x = \top \\ &\text{iff} \quad \chi_{f \cup g} \circ x = \top \\ &\text{iff} \quad x \in f \cup g \\ &\text{iff} \quad x \in f \quad \text{or} \quad x \in g \\ &\text{iff} \quad \chi_f \circ x = \top \quad \text{or} \quad \chi_g \circ x = \top \\ &\text{iff} \quad y = \top \quad \text{or} \quad z = \top. \end{aligned}$$

Conversely if (c) holds, then in any Sub(d) we find that

$$\begin{aligned} x \in f \cup g \quad &\text{iff} \quad \chi_{f \cup g} \circ x = \top \\ &\text{iff} \quad \cup \circ \langle \chi_f, \chi_g \rangle \circ x = \top \\ &\text{iff} \quad \cup \circ \langle \chi_f \circ x, \chi_g \circ x \rangle = \top \\ &\text{iff} \quad \chi_f \circ x = \top \quad \text{or} \quad \chi_g \circ x = \top \\ &\text{iff} \quad x \in f \quad \text{or} \quad x \in g. \end{aligned}$$

□

A topos satisfying (c), equivalently (b), will be called *disjunctive*. Obviously every bivalent topos is disjunctive. However, the converse is not true, and so (b) does not imply (a) in general. The category **Set**$^{\to}$ of set

functions has three truth values, and so violates (a). However, it does satisfy (c), since the disjunction arrow yields the table

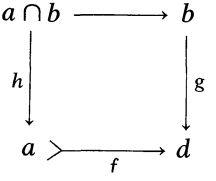
$\cup$	$\top$	x	$\bot$
$\top$	$\top$	$\top$	$\top$
x	$\top$	x	x
$\bot$	$\top$	x	$\bot$

where x is the third element of Ω. This will perhaps be easier to see from the alternative description of $\mathbf{Set}^{\rightarrow}$ to emerge from Chapters 9 and 10. Indeed, Exercise 4 of §10.6 will provide a method of constructing an infinity of disjunctive, non-bivalent, and non-Boolean topoi.

THEOREM 5. *If $\mathscr{E}$ is Boolean and non-degenerate, then $\mathscr{E}$ is disjunctive iff $\mathscr{E}$ is bivalent.*

PROOF. Since $f \cup -f \simeq 1_d$ in a Boolean topos, for any $x : 1 \to d$ we have $x \in f \cup -f$. Thus if $\mathscr{E}$ is disjunctive, from (b) we get $x \in f$ or $x \in -f$. However, we cannot have $x \in f$ and $x \in -f$, for then $x \in f \cap -f \simeq 0_d$, and so $1 \cong 0$. Thus exactly one of "$x \in -f$" and "$x \in f$" obtains, making $\mathscr{E}$ bivalent. □

EXERCISE. Suppose that $\mathscr{E}$ is well-pointed, and $x \in f$ implies $x \in g$. Use Theorem 5.5.1 to show that the pullback h



of g along f is iso, making $f \cap g \simeq f$. Hence give an alternative proof that any well-pointed topos is extensional.