CHAPTER 6

LOGIC CLASSICALLY CONCEIVED

"It is not easy, and perhaps not even useful, to explain briefly what logic is."

E. J. Lemmon

6.1. Motivating topos logic

In any systematic development of set theory one of the first topics to be examined is the so-called *algebra of classes*. This is concerned with ways of defining new sets, and when relativised to the subsets of a given set D focuses on the operations of

Intersection: $A \cap B = \{x \colon x \in A \text{ and } x \in B\}$
Union: $A \cup B = \{x \colon x \in A \text{ or } x \in B\}$
Complement: $-A = \{x \colon x \in D \text{ and not } x \in A\}$

The power set $\mathcal{P}(D)$ together with the operations $\cap$, $\cup$, $-$ exhibit the structure of what is known as a *Boolean algebra*. These algebras, to be defined shortly, are intimately connected with the classical account of logical truth.

Now the operations $\cap$, $\cup$, $-$ can be characterised by universal properties, and hence defined in any topos, yielding an "algebra of subobjects". It turns out that in some cases, this algebra does not satisfy the laws of Boolean algebra, indicating that the "logic" of the topos is not the same as classical logic. The proper perspective, it would seem, is that the algebra of subobjects is non-Boolean *because* the topos logic is non-classical, rather than the other way round. In defining $\cap$, $\cup$, $-$ we used the words "and", "or", and "not", and so the properties of the set operations are determined by the meaning, the logical behaviour, of these words. It is the rules of classical logic that dictate that $\mathcal{P}(D)$ should be a Boolean algebra.

The classical rules of logic are representable in **Set** by operations on the set $2 = \{0, 1\}$, and can then be developed in any topos $\mathscr{E}$ by using Ω in place of 2. This gives the "logic" of $\mathscr{E}$, which proves to characterise the behaviour of subobjects in $\mathscr{E}$. It is precisely when this logic fails to reflect all the principles of classical logic (i.e. the logic of **Set**) that the algebra of subobjects in $\mathscr{E}$ fails to be Boolean.

In this chapter we will briefly (in spite of Lemmon's caveat) outline the basics of classical logic and show how it generalises to the topos setting. Later chapters will deal with non-classical logic, and its philosophical motivation, leading eventually to a full account of what the logic of the general topos looks like.

6.2. Propositions and truth-values

A *proposition*, or *statement*, or *sentence*, is simply an expression that is either true or false. Thus

"2+2=4"

and

"2 plus 2 equals 5"

are to count as propositions, while

"Is 2+2 equal to 4?"

and

"Add 2 and 2!"

are not.

Thus each sentence has one of two *truth-values*. It is either *true*, which we indicate by assigning it the number 1, or *false*, indicated by the assignment of 0. The set of truth-values is $2=\{0, 1\}$ (hence the terminology used earlier for arrows $1 \to \Omega$).

We may construct compound sentences from given ones by the use of the *logical connectives* "and", "or", and "not", i.e. given sentences α and β we form the new sentences

"α and β" symbolised "$\alpha \wedge \beta$"

"α or β" symbolised "$\alpha \vee \beta$"

"not α" symbolised "$\sim\alpha$".

These are said to be obtained by *conjunction*, *disjunction*, and *negation*, respectively.

The truth-value of a compound sentence can be computed from the truth-values of its components, using some simple rules that we now describe.

Negation

The sentence $\sim\alpha$ is to be true (assigned 1) when α is false (assigned 0), and false (0) when α is true (1).

We present this rule in the form of a table

α	$\sim\alpha$
1	0
0	1

called the *truth-table* for negation. Alternatively we can regard it as determining a function $\neg$ from 2 to 2 that outputs 0 (resp. 1) for input 1 (resp. 0). This $\neg: 2 \to 2$, defined by $\neg 1 = 0$, $\neg 0 = 1$, is called the *negation truth-function.*

Conjunction

In order for $\alpha \wedge \beta$ to be true, both of α and β must be true. Otherwise $\alpha \wedge \beta$ is false.

Now, given two sentences α, and β, there are four ways their possible truth-values can be combined, as in the four rows of the truth-table

α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

for conjunction. The corresponding truth-value for $\alpha \wedge \beta$ in each row is determined according to the above rule.

The table provides a function $\sqcap$ from pairs of truth-values to truth-values, i.e. $\sqcap: 2 \times 2 \to 2$, defined by $1 \sqcap 1 = 1$, $1 \sqcap 0 = 0 \sqcap 1 = 0 \sqcap 0 = 0$. This is called the *conjunction truth-function*, which can also be presented in a tabular display as

$\sqcap$	1	0
1	1	0
0	0	0

Disjunction

$\alpha \vee \beta$ is true provided at least one of α and β are true, and is false only if both of α and β are false.

From this rule we obtain the disjunction truth-table

α	β	$\alpha \vee \beta$
1	1	1
1	0	1
0	1	1
0	0	0

and the corresponding *disjunction truth-function* $\cup : 2 \times 2 \to 2$, which has $1 \cup 1 = 1 \cup 0 = 0 \cup 1 = 1$, $0 \cup 0 = 0$, i.e.

$\cup$	1	0
1	1	1
0	1	0

Implication

The implication connective allows us to form the sentence "α implies β" symbolised "$\alpha \supset \beta$".

(synonyms: "if α then β", "α only if β".)

The classical interpretation of the connective "implies" is that $\alpha \supset \beta$ cannot be a true implication if it allows us to infer something false from something true. So we make $\alpha \supset \beta$ false if α is true while β is false. In all other cases $\alpha \supset \beta$ counts as true. The truth-table is

α	β	$\alpha \supset \beta$
1	1	1
1	0	0
0	1	1
0	0	1

The *implication truth function* $\Rightarrow : 2 \times 2 \to 2$ has $1 \Rightarrow 0 = 0$, $1 \Rightarrow 1 = 0 \Rightarrow 1 = 0 \Rightarrow 0 = 1$, or

$\Rightarrow$	1	0
1	1	0
0	1	1

Tautologies

By successive applications of the rules just given we can construct a truth-table for any compound sentence. For example

α	$\sim\alpha$	$\alpha\wedge\sim\alpha$	$\alpha\vee\sim\alpha$	$\alpha\wedge\alpha$	$\alpha\supset(\alpha\wedge\alpha)$
1	0	0	1	1	1
0	1	0	1	0	1

α	β	$\alpha\supset\beta$	$\beta\supset(\alpha\supset\beta)$	$\alpha\vee\beta$	$\alpha\supset(\alpha\vee\beta)$
1	1	1	1	1	1
1	0	0	1	1	1
0	1	1	1	1	1
0	0	0	1	0	1

A *tautology* is by definition a sentence whose truth-table contains only 1's. Thus $\alpha\vee\sim\alpha$, $\alpha\supset(\alpha\wedge\alpha)$, $\beta\supset(\alpha\supset\beta)$, $\alpha\supset(\alpha\vee\beta)$, are all tautologies. Such sentences are true no matter what truth-values their component parts have. The truth of $\alpha\vee\sim\alpha$ comes not from the truth or falsity of α, but from the logical "shape" of the sentence, the way its logical connectives are arranged. A tautology then expresses a logical law, a statement that is true for purely logical reasons, and not because of any facts about the world that happen to be the case.

6.3. The propositional calculus

In order to further our study of logic we need to give a somewhat more precise rendering of our description of propositions and truth-values. This is done by the device of a *formal language*. Such a language is presented as an alphabet (list of basic symbols) together with a set of *formation rules* that allow us to make *formulae* or *sentences* out of the alphabet symbols. The language we shall use, called PL, has the following ingredients:

Alphabet for PL

(i) an infinite list $\pi_0, \pi_1, \pi_2, \ldots$ of symbols, to be called *propositional variables*, or *sentence letters*;

(ii) the symbols $\sim$, $\wedge$, $\vee$, $\supset$;

(iii) the bracket symbols), (.

Formation Rules for PL-sentences

(1) Each sentence letter π_i is a sentence;
(2) If α is a sentence, so is $\sim\alpha$;
(3) If α and β are sentences, then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \supset \beta)$.

Notice that we are using the letters α and β as general names for sentences. Thus α might stand for a letter, like π_{24}, or something more complex, like $(\sim(\pi_2 \wedge \pi_{11}) \vee (\pi_0 \supset \pi_0))$. The collection of sentence letters is denoted Φ_0, while Φ denotes the set of all sentences, i.e.

$$\Phi_0 = \{\pi_0, \pi_1, \pi_2, \ldots\}$$

$$\Phi = \{\alpha : \alpha \text{ is a PL-sentence}\}.$$

To develop a theory of meaning, or *semantics*, for PL we use the truth-functions defined in §6.2. By a *value assignment* we shall understand any function V from Φ_0 to $\{0, 1\}$. Such a $V : \Phi_0 \to 2$ assigns a truth-value $V(\pi_i)$ to each sentence letter, and so provides a "meaning" or "interpretation" to the members of Φ_0. This interpretation can then be systematically extended to all sentences, so that V extends to a function from Φ to 2. This is done by "induction over the formation rules", through successive application of the rules

(a) $V(\sim\alpha) = \neg V(\alpha)$
(b) $V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta)$
(c) $V(\alpha \vee \beta) = V(\alpha) \cup V(\beta)$
(d) $V(\alpha \supset \beta) = V(\alpha) \Rightarrow V(\beta)$

EXAMPLE. If $V(\pi_0) = V(\pi_1) = 1$, and $V(\pi_2) = 0$, then

$$V(\sim\pi_1) = \neg V(\pi_1) = \neg 1 = 0$$

$$V(\sim\pi_1 \wedge \pi_2) = V(\sim\pi_1) \cap V(\pi_2) = 0 \cap 0 = 0$$

$$V(\pi_0 \supset (\sim\pi_1 \wedge \pi_2)) = V(\pi_0) \Rightarrow V(\sim\pi_1 \wedge \pi_2) = 1 \Rightarrow 0 = 0$$

etc. □

In this way any $V : \Phi_0 \to 2$ is "lifted" in a unique way to become a function $V : \Phi \to 2$.

A sentence $\alpha \in \Phi$ is then defined to be a *tautology*, or *classically valid*, if it receives the value "true" from every assignment whatsoever. Thus α is a tautology, denoted $\models \alpha$, iff for each value-assignment V, $V(\alpha) = 1$.

Axiomatics

The semantics for PL allows us to single out a special class of sentences – the tautologies. There is another way of characterising this class, namely by the use of an axiom system. Axiomatics are concerned with methods of generating new sentences from given ones, through the application of *rules of inference*. These rules, allowing us to "infer", or "derive", certain sentences, embodying principles of deduction and techniques of reasoning.

The basic ingredients of an axiom system then are

(i) a collection of sentences, called *axioms* of the system;

(ii) a collection of *rules of inference* which prescribe operations to be performed on sentences, to derive new ones.

Sentences derivable from the axioms are called *theorems*. To specify these a little more precisely we introduce the notion of a *proof sequence* as a finite sequence of sentences, each of which is either

(i) an axiom, or

(ii) derivable from earlier members of the sequence by one of the system's inferential rules.

A theorem can then be defined as a sentence which is the last member of some proof sequence. The set of theorems of an axiom system is said to be *axiomatised* by that system.

There are several known systems that axiomatise the classically valid sentences, i.e. whose theorems are precisely the tautologies of PL. The one we shall deal with will be called CL (for Classical Logic).

The axioms for CL comprise all sentences that are instances of one of the following twelve forms (α, β, and γ denote arbitrary sentences).

I	$\alpha \supset (\alpha \wedge \alpha)$
II	$(\alpha \wedge \beta) \supset (\beta \wedge \alpha)$
III	$(\alpha \supset \beta) \supset ((\alpha \wedge \gamma) \supset (\beta \wedge \gamma))$
IV	$((\alpha \supset \beta) \wedge (\beta \supset \gamma)) \supset (\alpha \supset \gamma)$
V	$\beta \supset (\alpha \supset \beta)$
VI	$(\alpha \wedge (\alpha \supset \beta)) \supset \beta$
VII	$\alpha \supset (\alpha \vee \beta)$
VIII	$(\alpha \vee \beta) \supset (\beta \vee \alpha)$
IX	$((\alpha \supset \gamma) \wedge (\beta \supset \gamma)) \supset ((\alpha \vee \beta) \supset \gamma)$
X	$\sim\alpha \supset (\alpha \supset \beta)$
XI	$((\alpha \supset \beta) \wedge (\alpha \supset \sim\beta)) \supset \sim\alpha$
XII	$\alpha \vee \sim\alpha$

The system CL has a single rule of inference;

RULE OF DETACHMENT. *From α and $\alpha \supset \beta$, the sentence β may be derived.*

This rule is known also by its medieval name, *modus ponens*, more correctly modus ponendo ponens. It operates on a pair of theorems, an implication and its *antecedent*, to "detach" the *consequent* as a new theorem.

By writing "$\vdash_{CL} \alpha$" to indicate that α is a CL-theorem the rule of detachment can be expressed as

$$\textit{if} \vdash_{CL} \alpha \quad \textit{and} \quad \vdash_{CL} (\alpha \supset \beta), \quad \textit{then} \quad \vdash_{CL} \beta.$$

The demonstration that the CL-theorems are precisely the tautologies falls into two parts:

SOUNDNESS THEOREM. *If* $\vdash_{CL} \alpha$, *then* α *is classically valid.*

COMPLETENESS THEOREM. *If* α *is classically valid, then* $\vdash_{CL} \alpha$.

In general a "soundness" theorem for an axiom system is a result to the effect that only sentences of a certain kind are derivable as theorems, while a "completeness" theorem states that all sentences of a certain kind are derivable. Together they give an exact characterisation of a particular type of sentence in terms of derivability. Thus the results just quoted state that theoremhood in CL characterises classical validity.

To prove the Soundness theorem is easy, in the sense that a computer could do it. First one shows that all of the axioms are tautologies (the truth-tables in §6.2 show that the axioms of the forms I, V, VII, and XII are tautologies). Then one shows that *detachment* "preserves" validity, i.e. if α and $\alpha \supset \beta$ are tautologies, then β is also a tautology. This implies that a proof sequence can consist only of valid sentences, hence every theorem of CL is valid.

The Completeness theorem requires more than a mechanical procedure for its verification. The first result of this kind for classical logic was established in 1921 by Emil Post, who proved that all tautologies were derivable in the system used by Russell and Whitehead in *Principia Mathematica*. Since then a number of methods have been developed for proving completeness of various axiomatisations of classical logic. A survey of these may be found in a paper by Surma [73].

6.4 Boolean algebra

The set 2, together with the truth-functions $\neg$, $\cap$, $\cup$ forms a Boolean algebra, a structure that we have mentioned several times and now at last are going to define. The definition proceeds in several stages.

Recall from Chapter 3 that a lattice is a poset $\mathbf{P}=(P, \sqsubseteq)$ in which any two elements $x, y \in P$ have

(i) a greatest lower bound (g.l.b.), $x \sqcap y$; and

(ii) a least upper bound (l.u.b.), $x \sqcup y$.

$x \sqcap y$ is also known as the lattice *meet* of x and y, while $x \sqcup y$ is the *join* of x and y. As observed in §§3.8, 3.9, when $\mathbf{P}$ is considered as a category, meets are products and joins are co-products.

Recall from §§3.5, 3.6 that a *zero* or *minimum* for a lattice is an element 0 having $0 \sqsubseteq x$, all $x \in P$, while a *unit* or *maximum* is an element 1 having $x \sqsubseteq 1$, all $x \in P$. A lattice is said to be *bounded* if it has a unit and a zero. Categorially, 0 is initial and 1 is terminal. Now a lattice always has pullbacks and pushouts (§3.13, Example 5 and its dual), so a bounded lattice is precisely (§3.15) a *finitely bicomplete skeletal pre-order category*.

EXAMPLE 1. $(\mathscr{P}(D), \subseteq)$ is a bounded lattice. The unit is D, the zero $\emptyset$, the meet of A and B is their intersection $A \cap B$, and the join is their union $A \cup B$.

EXAMPLE 2. The set $2=\{0, 1\}$ has the natural ordering $0 \leqslant 1$ which makes it into the ordinal pre-order $\mathbf{2}$ (Example 2, Chapter 2)

0 is the zero, and 1 the unit in this poset. $x \cap y$ is both the lattice meet and the result of applying the conjunction truth function to $\langle x, y\rangle \in 2 \times 2$. Likewise $x \cup y$ is both the join of x and y and their disjunction.

EXAMPLE 3. If I is a topological space with Θ its collection of open sets, then $(\Theta, \subseteq)$ is a poset exactly as in Example 1 – joins and meets are unions and intersections, the zero is $\emptyset$, and the unit is I.

EXAMPLE 4. $(L_M, \subseteq)$ is a bounded lattice, where L_M is the set of left ideals of monoid M. Joins and meets are as in Examples 1 and 3. □

A lattice is said to be *distributive* if it satisfies the following laws (each of which implies the other in any lattice):

(a) $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$

(b) $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$ all x, y, z.

EXAMPLE 5. All four examples above are distributive. □

To complete our description of a Boolean algebra we need one further notion – a lattice version of complementation.

In a bounded lattice, y is said to be a *complement* of x if

$$x \sqcup y = 1$$

and

$$x \sqcap y = 0.$$

A bounded lattice is *complemented* if each of its elements has a complement in the lattice.

EXAMPLE 6. $(\mathcal{P}(D), \subseteq)$ is complemented. The lattice complement of A is its set complement $-A$.

EXAMPLE 7. $(2, \leqslant)$ is complemented. The complement of x is its negation $\neg x$ (cf. truth-tables for $\alpha \vee \sim\alpha$, $\alpha \wedge \sim\alpha$).

EXAMPLE 8. In $(\Theta, \subseteq)$ the only candidate for the complement of $U \in \Theta$ is its set complement. But $-U \notin \Theta$ unless U is closed. Thus $(\Theta, \subseteq)$ will only be complemented in the event that every open set is also closed.

EXAMPLE 9. If $\mathbf{M}$ is the monoid $\mathbf{M}_2 = (2, \cdot, 1)$ then in $(L_M, \subseteq)$, $\{0\}$ has no lattice complement, as $\{1\} \notin L_M$. □

EXERCISE 1. In a *distributive* lattice each element has *at most* one complement, i.e. if $x \sqcap y = x \sqcap z = 0$ and $x \sqcup y = x \sqcup z = 1$, then $y = z$. □

A *Boolean algebra* (**BA**) is, by definition, a *complemented distributive lattice.*

EXAMPLE. $(\mathcal{P}(D), \subseteq)$ and $\mathbf{2} = (2, \leqslant)$. □

If $\mathbf{B} = (B, \sqsubseteq)$ is a **BA** then each $x \in B$ has, by the above exercise, exactly one complement. We denote it in general by x'.

EXERCISE 2. In any **BA** we have: (1) $(x')' = x$; (2) $x \sqcap y = 0$ iff $y \sqsubseteq x'$; (3) $x \sqsubseteq y$ iff $y' \sqsubseteq x'$; (4) $(x \sqcap y)' = x' \sqcup y'$; (5) $(x \sqcup y)' = x' \sqcap y'$. □

Boolean algebras are named after George Boole (1815–1864) who first described the laws they satisfy in his work, *The Mathematical Analysis of Logic* (1847).

6.5. Algebraic semantics

Each **BA** $\mathbf{B} = (B, \sqsubseteq)$ has operations $\sqcap$ (meet), $\sqcup$ (join), and $'$ (complement) corresponding to the conjunction, disjunction, and negation truth functions on 2. It also has an operation corresponding to implication. The sentence $\alpha \supset \beta$ has exactly the same truth-table as the sentence $\sim\alpha \vee \beta$, and hence on the classical account the two sentences have the same meaning. So for $x, y \in B$ we *define*

$$x \Rightarrow y = x' \sqcup y.$$

EXERCISE 1. Verify that $\alpha \supset \beta$ and $\sim\alpha \vee \beta$ have the same truth-table, and hence that the definition just reproduces the implication truth-function on 2. □

The operations on $\mathbf{B}$ can be used to generalise the semantics of 6.3.

A $\mathbf{B}$-valuation is a function $V: \Phi_0 \to B$. This is extended to a function $V: \Phi \to B$ by the rules

(a) $V(\sim\alpha) = V(\alpha)'$

(b) $V(\alpha \wedge \beta) = V(\alpha) \sqcap V(\beta)$

(c) $V(\alpha \vee \beta) = V(\alpha) \sqcup V(\beta)$

(d) $V(\alpha \supset \beta) = V(\alpha)' \sqcup V(\beta) = V(\alpha) \Rightarrow V(\beta)$.

Then a sentence α is $\mathbf{B}$-*valid*, $\mathbf{B} \models \alpha$, iff for every $\mathbf{B}$-valuation V, $V(\alpha) = 1$ (where 1 is the unit of $\mathbf{B}$). Notice that a $\mathbf{2}$-valuation is what we earlier called a value-assignment, and that $\mathbf{2} \models \alpha$ iff α is a tautology.

SOUNDNESS THEOREM FOR $\mathbf{B}$-VALIDITY: *If* $\vdash_{CL} \alpha$ *then* $\mathbf{B} \models \alpha$.

The proof of this is as for $\mathbf{2}$-validity. One shows that all the CL-axioms are $\mathbf{B}$-valid, and that *Detachment* preserves this property.

Now the zero and unit of $\mathbf{B}$ provide an "isomorphic copy" of $\mathbf{2}$ within $\mathbf{B}$. ($\mathbf{2}$ is a subobject of $\mathbf{B}$ in the category of **BA**'s). In this way any $\mathbf{2}$-valuation can be construed as a $\mathbf{B}$-valuation, hence $\mathbf{B} \models \alpha$ only if $\mathbf{2} \models \alpha$.

A sentence will be called **BA**-*valid* if it is valid in every **BA** (and hence in particular is $\mathbf{2}$-valid).

All of these notions of validity are connected by the observation that the following four statements are equivalent to each other:

$\vdash_{\mathrm{CL}} \alpha$
α is a tautology
α is **B**-*valid, for some particular* **B**
α is **BA**-*valid.*

EXERCISE 2. (*The Lindenbaum Algebra*). Define a relation $\sim_c$ on Φ by

$$\alpha \sim_c \beta \quad \text{iff} \quad \vdash_{\mathrm{CL}} \alpha \supset \beta \quad \text{and} \quad \vdash_{\mathrm{CL}} \beta \supset \alpha$$

Show that $\sim_c$ is an equivalence relation on Φ and that a partial ordering is well defined on the quotient set $\Phi/\sim_c$ by

$$[\alpha] \sqsubseteq [\beta] \quad \text{iff} \quad \vdash_{\mathrm{CL}} \alpha \supset \beta$$

The poset $\mathbf{B}_c = (\Phi/\sim_c, \sqsubseteq)$ is called the Lindenbaum Algebra of CL. Show that it is a **BA**, in which

$$[\alpha] \sqcap [\beta] = [\alpha \wedge \beta]$$

$$[\alpha] \sqcup [\beta] = [\alpha \vee \beta]$$

$$[\alpha] = [\sim\alpha]$$

$$[\alpha] = 1 \quad \text{iff} \quad \vdash_{\mathrm{CL}} \alpha.$$

Define a $\mathbf{B}_c$-valuation V_c by $V_c(\pi_i) = [\pi_i]$, and prove that $V_c(\alpha) = [\alpha]$, all sentences α. Hence show

$$\vdash_{\mathrm{CL}} \alpha \quad \text{iff} \quad \mathbf{B}_c \models \alpha. \qquad \square$$

The algebra $\mathbf{B}_c$ can be used to develop a proof that all tautologies are CL-theorems. The details of this can be found in Rasiowa and Sikorski [63], or Bell and Slomson [69].

6.6. Truth-functions as arrows

Each of the classical truth-functions has codomain 2, and so is the characteristic function of some subset of its domain. This observation will lead us to an arrows-only definition of the truth-functions that makes sense in any topos, through the Ω-axiom.

Negation

$\neg : 2 \to 2$ is the characteristic function of the set

$$\{x : \neg x = 1\} = \{0\} \subseteq 2.$$

But the inclusion function $\{0\} \hookrightarrow 2$ is the function we called *false* in §5.4. Hence in **Set** we have the pullback

$$\begin{array}{ccc} 1 & \xrightarrow{\textit{false}} & 2 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \neg} \\ 1 & \xrightarrow[\textit{true}]{} & 2 \end{array}$$

(recall that *false* is the characteristic function of $\emptyset \subseteq 1$).

Conjunction

The only input to $\cap : 2\times 2 \to 2$ that gives output 1 is $\langle 1, 1\rangle$. Hence $\cap = \chi_A$ where

$$A = \{\langle 1, 1\rangle\}$$

Now A being a one-element set can be identified with an arrow $1 \to 2\times 2$. We see that this arrow is the product map $\langle \textit{true}, \textit{true}\rangle$, which takes 0 to $\langle \textit{true}\ (0), \textit{true}\ (0)\rangle$, and hence

$$\begin{array}{ccc} 1 & \xrightarrow{\langle \textit{true}, \textit{true}\rangle} & 2\times 2 \\ \downarrow & & \downarrow{\scriptstyle \cap} \\ 1 & \xrightarrow[\textit{true}]{} & 2 \end{array}$$

is a pullback.

Implication

$\Rightarrow : 2\times 2 \to 2$ is the characteristic function of

$$\circledS = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\},$$

and so

$$\begin{array}{ccc} \circledS & \hookrightarrow & 2\times 2 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \Rightarrow} \\ 1 & \xrightarrow[\textit{true}]{} & 2 \end{array}$$

is a pullback. Now $\circledS$ is so named because, as a relation on 2, it is none other than the natural partial ordering on the ordinal **2**, i.e.

$$\circledS = \{\langle x, y\rangle : x \leqslant y \text{ in } \mathbf{2}\}$$

But in any lattice we in fact have

$$x \sqsubseteq y \quad \text{iff} \quad x \sqcap y = x$$

(why?) so

$$\textcircled{\leqslant} = \{\langle x, y\rangle : x \cap y = x\}$$

and so according to §3.10, $\textcircled{\leqslant} \hookrightarrow 2\times 2$ is the equaliser of

$$2\times 2 \underset{pr_1}{\overset{\cap}{\rightrightarrows}} 2$$

where pr_1 is the projection $pr_1(\langle x, y\rangle) = x$.

Disjunction

$\cup : 2\times 2 \to 2$ is χ_D, where

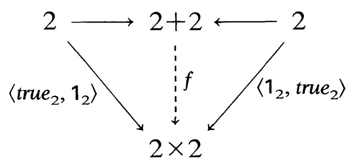
$$D = \{\langle 1, 1\rangle \langle 1, 0\rangle \langle 0, 1\rangle\}.$$

The description of D by arrows is a little more complex than in the other cases.

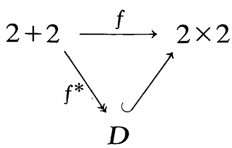
Notice first that $D = A \cup B$, where

$$A = \{\langle 1, 1\rangle, \langle 1, 0\rangle\}, \quad \text{and} \quad B = \{\langle 1, 1\rangle, \langle 0, 1\rangle\}.$$

Now $A \subseteq 2\times 2$ can be identified with the monic product map $\langle true_2, 1_2\rangle : 2 \to 2\times 2$ which takes 1 to $\langle 1, 1\rangle$ and 0 to $\langle 1, 0\rangle$. Similarly B is identifiable with $\langle 1_2, true_2\rangle$. We then form the co-product



i.e. $f = [\langle true_2, \mathbf{1}_2\rangle, \langle \mathbf{1}_2, true_2\rangle]$ and find that $\operatorname{Im} f = D$. Thus we have an epi-monic factorisation



This specifies D uniquely up to isomorphism by properties that can all be expressed in the language of categories, and so we can now define the

Truth-arrows in a topos

If $\mathscr{E}$ is a topos with classifier $\top : 1 \to \Omega$;

(1) $\neg : \Omega \to \Omega$ is the unique $\mathscr{E}$-arrow such that

$$\begin{array}{ccc} 1 & \xrightarrow{\perp} & \Omega \\ \downarrow & & \downarrow \neg \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is a pullback in $\mathscr{E}$. Thus $\neg = \chi_{\perp}$, where $\perp$ itself is the character of $! : 0 \to 1$.

(2) $\cap : \Omega \times \Omega \to \Omega$ is the character in $\mathscr{E}$ of the product arrow $\langle \top, \top \rangle : 1 \to \Omega \times \Omega$.

(3) $\cup : \Omega \times \Omega \to \Omega$ is defined to be the character of the image of the $\mathscr{E}$-arrow

$$[\langle \top_\Omega, 1_\Omega \rangle, \langle 1_\Omega, \top_\Omega \rangle] : \Omega + \Omega \to \Omega \times \Omega$$

(4) $\Rightarrow : \Omega \times \Omega \to \Omega$ is the character of

$$e : \textcircled{\leqslant} \rightarrowtail \Omega \times \Omega,$$

where the latter is the equaliser of

$$\Omega \times \Omega \overset{\cap}{\underset{pr_1}{\rightrightarrows}} \Omega,$$

$\cap$ being the conjunction truth arrow, and pr_1 the first projection arrow of the product $\Omega \times \Omega$.

Example 1. In **Set**, and **Finset** the truth arrows are the classical truth functions.

Example 2. In **Bn**(I), where $\Omega = (2 \times I, p_I)$, the stalk Ω_i over i is $2 \times \{i\}$, a "copy" of 2. The truth arrows in **Bn**(I) are essentially bundles of truth-functions, i.e. they consist of "copies" of the corresponding truth-functions acting on each stalk. Thus $\neg : \Omega \to \Omega$ is the function from $2 \times I$ to $2 \times I$ that takes $\langle 1, i \rangle$ to $\langle 0, i \rangle$ and $\langle 0, i \rangle$ to $\langle 1, i \rangle$. $\cap : \Omega \times \Omega \to \Omega$ takes a pair consisting of $\langle x, i \rangle$ and $\langle y, i \rangle$ to $\langle x \cap y, i \rangle$ (recall that $\Omega \times \Omega$ in **Bn**(I) consists only of those pairs that belong to the same stalk in Ω). The reader can readily define the other truth arrows in **Bn**(I).

Thus, whereas in **Set** Ω is the two-element **BA**, in **Bn**(I) Ω is a bundle of two-element **BA**'s, indexed by I.

EXAMPLE 3. In **M-Set**, where $\Omega = (L_M, \omega)$, the negation truth-arrow $\neg: L_M \to L_M$ is defined by

$$\neg(B) = \{m: m \in M \quad \text{and } \omega_m(B) = \emptyset\}$$
$$= \{m: \text{for all } n, n * m \notin B\}.$$

The conjunction arrow is given by set intersection, i.e. it is that function from $L_M \times L_M$ to L_M that takes $\langle B, C\rangle$ to $B \cap C$.

The disjunction arrow is given by set union.

Implication $\Rightarrow: L_M \times L_M \to L_M$ has the description

$$B \Rightarrow C = \{m: \omega_m(B) \subseteq \omega_m(C)\},$$

and $\textcircled{\leqslant}$ is the set inclusion relation on L_M.

EXAMPLE 4. In the particular case of our canonical (counter) example $\mathbf{M_2}$, the above definitions show the truth arrows to be given by the tables

	$\neg$
2	$\emptyset$
{0}	$\emptyset$
$\emptyset$	2

$\cap$	2	{0}	$\emptyset$
2	2	{0}	$\emptyset$
{0}	{0}	{0}	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$\cup$	2	{0}	$\emptyset$
2	2	2	2
{0}	2	{0}	{0}
$\emptyset$	2	{0}	$\emptyset$

$\Rightarrow$	2	{0}	$\emptyset$
2	2	{0}	$\emptyset$
{0}	2	2	$\emptyset$
$\emptyset$	2	2	2

EXAMPLE 5. The description of truth-arrows in **Top**(I), which in itself gives further indication of the unification achieved by the present theory, will be delayed till Chapter 8. □

EXERCISE 1. Describe the truth-arrows in $\mathbf{Set}^2$.

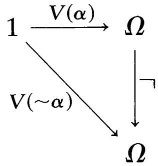
EXERCISE 2. Describe Ω and the truth-arrows in $\mathbf{Z_2}$-**Set**, where $\mathbf{Z_2} = \langle 2, +, 0\rangle$ is the monoid of the numbers 0 and 1 under addition. □

6.7. $\mathscr{E}$-semantics

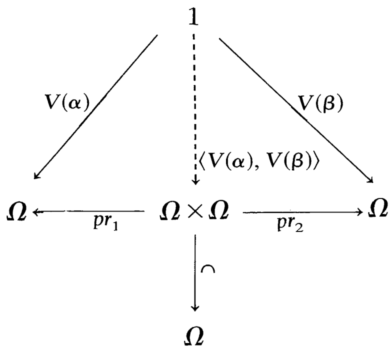
We are now able to do propositional logic in any topos $\mathscr{E}$. Recall that a truth value in $\mathscr{E}$ is an arrow $1 \to \Omega$ and that $\mathscr{E}(1, \Omega)$ denotes the collection of such $\mathscr{E}$-arrows.

An $\mathscr{E}$*-valuation* is a function $V:\Phi_0 \to \mathscr{E}(1, \Omega)$ assigning to each sentence letter π_i a truth value $V(\pi_i):1 \to \Omega$. This function is extended to the whole of Φ by the rules

(a) $V(\sim\alpha) = \neg \circ V(\alpha)$



(b) $V(\alpha \wedge \beta) = \cap \circ \langle V(\alpha), V(\beta)\rangle$



(c) $V(\alpha \vee \beta) = \cup \circ \langle V(\alpha), V(\beta)\rangle$

(d) $V(\alpha \supset \beta) = \Rightarrow \circ \langle V(\alpha), V(\beta)\rangle$.

In this way we extend V so that every sentence is assigned an $\mathscr{E}$-arrow $V(\alpha):1 \to \Omega$.

We shall say that α is $\mathscr{E}$*-valid*, denoted $\mathscr{E} \models \alpha$, iff for every $\mathscr{E}$-valuation V, $V(\alpha) = \top : 1 \to \Omega$.

EXERCISE 1. **Set**$\models\alpha$ iff **Finset**$\models\alpha$ iff **Finord**$\models\alpha$ iff α is a tautology iff $\vdash_{\overline{CL}} \alpha$.

EXERCISE 2. **Bn**$(I) \models \alpha$ iff $(\mathscr{P}(I), \subseteq) \models \alpha$, i.e. topos-validity in **Bn**(I) is equivalent to Boolean-algebra-validity in $(\mathscr{P}(I), \subseteq)$. Hence

Bn$(I)\models\alpha$ iff α is a tautology. □

In the topoi of these exercises, the system CL axiomatises the valid sentences. The natural question is – does this always happen? We are about to see that CL is complete for $\mathscr{E}$-validity, i.e. that any $\mathscr{E}$-valid sentence (whatever $\mathscr{E}$ is) is a CL-theorem. The question then reduces to – "is CL sound for $\mathscr{E}$-validity?" The short answer is – no! A slightly more revealing answer is that axioms I–XI of CL are $\mathscr{E}$-valid, but there are topoi in which the "law of excluded middle", $\alpha \vee \sim\alpha$, is not valid. An example is **Set**$^{\rightarrow}$, the category of set functions, for reasons that will emerge in Chapter 10, where the full story on topos validity will be told, at least for propositional logic.

To show that $\mathscr{E}$-valid sentences are tautologies we need the following result, which shows that the arrows $\top$ and $\bot$ behave under the application of the truth-arrows in $\mathscr{E}$ exactly as they do in **Set**. But first some terminology. If $\langle f, g\rangle : 1 \to \Omega \times \Omega$ is a "pair" of truth-values we write

$$f \cap g \quad \text{for} \quad \cap \circ \langle f, g\rangle : 1 \to \Omega$$

$$f \cup g \quad \text{for} \quad \cup \circ \langle f, g\rangle$$

$$f \Rightarrow g \quad \text{for} \quad \Rightarrow \circ \langle f, g\rangle \quad \text{etc.}$$

THEOREM 1. *In any* $\mathscr{E}$, $\top$ *and* $\bot$ *exhibit the behaviour displayed in the tables*

x	$\neg \circ x$
$\top$	$\bot$
$\bot$	$\top$

$\cap$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

(*i.e.* $\top \cap \top = \top, \top \cap \bot = \bot$ **etc.**)

$\cup$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\Rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

PROOF. That $\neg \circ \bot = \top$ follows by commutativity of the pullback that *defines* $\neg$ (cf. §6.6). To see why $\neg \circ \top = \bot$, consider

$$\begin{array}{ccc} 0 & \xrightarrow{\ !\ } & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \top} \\ 1 & \xrightarrow{\ \bot\ } & \Omega \\ \downarrow & & \downarrow{\scriptstyle \neg} \\ 1 & \xrightarrow{\ \top\ } & \Omega \end{array}$$

The bottom square is the pullback defining $\neg$. The top square is the pullback (inverted) defining $\perp$ as the character of $!:0 \to 1$. Hence by the PBL, the outer rectangle is a pullback showing $\neg \circ \top$ to be the character of $!:0 \to 1$.

It would be possible to derive the other tables from the relevant definitions, but in Chapter 7 some much deeper facts will be established which yield these tables as a rather easy corollary. So we will leave the details till then (cf. §7.6). □

Now suppose that $V : \Phi_0 \to 2$ is a classical value-assignment. We use V to define an $\mathscr{E}$-valuation $V' : \Phi_0 \to \mathscr{E}(1, \Omega)$ by putting

$$V'(\pi_i) = \begin{cases} \top & \text{if} \quad V(\pi_i) = 1 \\ \perp & \text{if} \quad V(\pi_i) = 0. \end{cases}$$

LEMMA. *For any sentence* $\alpha \in \Phi$,

(a) *either* $V'(\alpha) = \top$ *or* $V'(\alpha) = \perp$

(b) $V'(\alpha) = \top$ *iff* $V(\alpha) = 1$.

PROOF. The statement of the Lemma is true when $\alpha = \pi_i$ by definition. The proof itself is by induction over the formation rules for sentences. One proves the statement is true when $\alpha = {\sim}\beta$ on the inductive assumption that it is true for β, is true when $\alpha = \beta \wedge \gamma$ assuming it is true for β and for γ etc. In view of the exact correspondence of the tables of Theorem 1 to the classical truth-tables it should be clear why the Lemma works, and the details are left as an exercise. □

THEOREM 2. *For any topos* $\mathscr{E}$,

$$\textit{if} \quad \mathscr{E} \models \alpha \quad \textit{then} \quad \vdash_{\overline{\mathrm{CL}}} \alpha$$

PROOF. Let V be any classical valuation and V' its associated $\mathscr{E}$-valuation, as above. Since $\mathscr{E} \models \alpha$, $V'(\alpha) = \top$ and so by the Lemma, $V(\alpha) = 1$. Hence α is assigned 1 by every classical valuation, so is a tautology, whence $\vdash_{\overline{\mathrm{CL}}} \alpha$. □

THEOREM 3. *If* $\mathscr{E}$ *is bivalent, then*

$$\mathscr{E} \models \alpha \quad \textit{iff} \quad \vdash_{\overline{\mathrm{CL}}} \alpha$$

PROOF. Theorem 2 gives the "only if" part. Conversely, suppose $\vdash_{\overline{\mathrm{CL}}} \alpha$, i.e. α is a tautology. If V' is *any* $\mathscr{E}$-valuation, define a classical valuation by $V(\pi_i) = 1$ or 0 according as $V'(\pi_i) = \top$ or $\perp$. Since $\mathscr{E}$ is bivalent, $\top$ and

$\bot$ are its only truth-values, so this definition is legitimate. But then V' and V are related as in the Lemma, so as $V(\alpha) = 1$, we get $V'(\alpha) = \top$. □

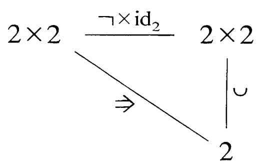
This last result suggests perhaps that bivalent topoi look more like **Set** than ones with more than two truth-values. However, our example $\mathbf{M_2}$ is bivalent and yet differs from **Set** in other ways, e.g. is non-classical in having $1+1$ not isomorphic to Ω. On the other hand the topos $\mathbf{Set}^2$ is not bivalent, but is classical, and does have its valid sentences axiomatised by CL. We could then conclude that bivalence does not of itself lead to a categorial axiomatisation of classical set theory. Or should we perhaps conclude that our definition of topos validity is not the right generalisation of the notion of logical truth in **Set**? Read on.

Appendix

Sentences α and β are *logically equivalent* when they have the same truth-table, i.e. when $V(\alpha) = V(\beta)$ for every classical valuation V. As was mentioned above, $\alpha \supset \beta$ is logically equivalent to $\sim\alpha \vee \beta$, and because of this some presentations of CL introduce $\supset$, not as a basic symbol of the alphabet, but as a *definitional abbreviation* for a combination involving $\sim$ and $\vee$. Since $\alpha \wedge \beta$ is logically equivalent to $\sim(\sim\alpha \vee \sim\beta)$, $\wedge$ may also be introduced in this way. Alternatively we can start with $\sim$ and $\wedge$ and define $\vee$ and $\supset$, and there are still other approaches.

The definability of $\supset$ from $\sim$ and $\vee$ is reflected by the fact that in 2, $x \Rightarrow y = \neg x \cup y$. In arrow-language this means that

$$\Rightarrow\ = \cup \circ (\neg \times id_2)$$



Now there are topoi in which the generalised truth-arrows do not satisfy this equation. So the question must be faced as to why the approach of this chapter is appropriate and why we do not simply define $\Rightarrow$ in $\mathscr{E}$ via $\neg$ and $\cup$ as above.

The point is that the connectives $\sim, \wedge, \vee, \supset$ were introduced separately, as they are all conceptually quite different, and each has its own intrinsic meaning. The construction of the truth-table was motivated independently in each case. That they prove to be inter-definable is *after the fact.* It is simply a feature of classical logic, a *consequence* of the classical account of truth and validity. Accordingly we defined the connectives independently, described them independently through the Ω-axiom, and lifted this description to the general topos. In so doing we find (in some cases) that the interdefinability is left behind. Later (Chapter 8) we shall see a different theory of propositional semantics in which the connectives are not inter-definable but in which they have exactly the same categorial description that they do in **Set**.