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INTERGABLE SYSTEMS IN MÖBIUS GEOMETRY

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Abstract. This is an overview of classical and recent results on the geometry of isothermic surfaces and conformally flat hypersurfaces in conformal geometry and their relation with curved flats, a particularly simple type of integrable system.

1. Introduction

This text is an account of a series of five lectures given by the author at the 8th International Conference on Geometry, Integrability and Quantization in Varna, Bulgaria. Many of the discussed results are not new and a very wide range of results is covered, in the form of an overview. They are, or in some cases will be, published elsewhere (some of the material has already been known to the classical geometers) in greater detail and references are included for the interested reader, to facilitate further work. In fact, most of the discussed material is elaborated in the author's book [13], where also more pointers to the relevant literature can be found. Thus this text should be understood as an introduction and advertisement for the discussed problems and methods and, as such, is kept in a rather informal and colloquial style.

One of the main ideas, besides providing appropriate background material, was to show how curved flats, a particularly simple type of integrable system, appears in conformal (Möbius) geometry. Here the term "integrable system" is understood in the sense that the set of partial differential equations describing the geometry can be formulated as a zero curvature condition on a loop of connections by introducing a (spectral) parameter. As a consequence, known techniques from integrable systems theory, such as the finite gap integration scheme, can be applied [9]. However, the author's interest is rather in the associated geometry and more exactly in the geometric interpretation of the obtained spectral family and the relation with other (Bäcklund or Darboux type) transformations of the described geometric objects.

Besides background material on Möbius geometry covered in Section 1 and Section 3, two main topics are discussed: isothermic surfaces and conformally flat hypersurfaces, both of which allow a description in terms of curved flats. The theory of isothermic surfaces is, after a 15 year period of intensive recent research (and rediscovery of a wealth of classical results) rather well developed and their transformation theory, in many aspects intimately related to the theory of curved flats in the symmetric space of point pairs in the conformal three-sphere, is well understood. Some core aspects of the theory are discussed in Section 2 and Section 4. The theory of conformally flat hypersurfaces, on the other hand, is less well developed and more work will be necessary to understand the implications of their relations with curved flats better. It is the author's hope that the aspects of the theory discussed in Section 5 will raise interest and promote further work in the field.

2. The Classical Model

In this section we shall discuss the projective model of Möbius geometry. This model provides a slick approach to many calculations and arguments in Möbius geometry and can be considered as the heart of Möbius geometry as the other models or formalisms in Möbius geometry are derived from it.

After introducing the basic ideas of how to view Möbius geometry as a subgeometry of projective geometry we shall discuss sphere pencils and sphere complexes, which will help to formulate some arguments in the discussion of the group of Möbius transformations. Then we shall see how the space form geometries appear as subgeometries of Möbius geometry. In the last section we will discuss the notions of sphere congruence and envelope, providing the tools to define the central sphere congruence (or, conformal Gauss map) of a hypersurface in the conformal n-sphere. This notion plays, for example, a central role in the theory of Willmore surfaces (which are classically called "conformally minimal surfaces").

We shall sketch the basic facts and some of the proofs but for more details the reader is referred to either [1] or to [13, Chapter 1].

2.1. Motivation

We wish to study the *conformal* geometry of submanifolds $M^m \subset S^n$. Thus we think of S^n *not* as the round sphere $S^n \subset \mathbb{R}^{n+1}$ equipped with its induced metric but rather equipped with just a method of "angle measurement" given by the round metric on S^n, that is, with the conformal class of the standard round metric. Or, otherwise said, we are interested in those geometric properties of a submanifold that are invariant under conformal changes of the ambient metric, that is, under multiplication of the standard round metric of S^n by any positive function.

One possible approach is to study the transformation behaviour of the usual geometric invariants of a submanifold in a Riemannian manifold – such as its covariant derivative, second fundamental form and its normal connection. This allows to determine conformal invariants for the submanifold such as, for example, the Weyl curvature tensor of its induced covariant derivative, its traceless second fundamental form or its normal curvature – the geometry of which can then be studied. This is a common approach when coming from a Riemannian geometry background or when studying the conformal geometry of submanifolds in arbitrary conformal manifolds.

More specifically, if $M^m \subset (N^n, g)$ is a submanifold in a Riemannian manifold with induced metric I, induced covariant derivative ∇, (normal bundle valued) second fundamental form II, shape operator S and normal connection $\nabla^\perp$ then these geometric invariants of the submanifold transform as follows when the ambient metric is replaced by a conformally equivalent metric $\tilde{g} := \mathrm{e}^{2u} g$, $u \in C^\infty(N^n)$, and

$$\begin{aligned}
\widetilde{\nabla}_v w &= \nabla_v w + \mathrm{d}u(v)\, w + \mathrm{d}u(w)\, v - I(v,w)\, \nabla^{(M,I)} u \\
\widetilde{\nabla}^\perp_v n &= \nabla^\perp_v n + \mathrm{d}u(v)\, n \\
\widetilde{II}(v,w) &= II(v,w) - I(v,w)\, (\nabla^{(N^n,g)} u)^\perp \\
\widetilde{S}_n v &= A_n v - \mathrm{d}u(n)\, v.
\end{aligned}$$

From these a variety of conformal invariants for submanifolds can be defined but – unlike, for example, in the case of surfaces in Euclidean ambient space – until now there seems to be no common agreement over which to use as a complete set of conformal invariants that determine (and describe!) a submanifold in conformal geometry, cf. [13, Section P.6]. The formulation of Bonnet-type theorems for submanifolds in conformal or, more specifically, Möbius geometry is a topic of active research (see, for example, [3, 17]) and an ultimate formulation, without the usual non-degeneracy restrictions, has recently been developed [4].

For submanifolds in Möbius geometry, that is, submanifolds in the n-sphere S^n equipped with its standard conformal structure, there is another more direct approach considering the ambient Möbius geometry as a subgeometry of projective geometry, following the ideas of F. Klein in his Erlanger Program [15]. As a motivation of this approach, which we will discuss in more detail below, consider

$$S^n \subset \mathbb{R}^{n+1} \cong \{1\} \times \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}_1$$

where we equip $\mathbb{R}^{n+2}_1$ with the Minkowski scalar product (the index one refers to the signature of the metric on $\mathbb{R}^{n+2}$) given by

$$|y|^2 = -y_0^2 + y_1^2 + \cdots + y_{n+1}^2.$$

Thus the round n-sphere becomes

$$S^n = \{y \in \mathbb{R}^{n+2}_1;\ |y|^2 = 0,\ y_0 = 1\}.$$

Now observe that, for any function $u \in C^\infty(S^n)$, we have

$$|\mathrm{d}(\mathrm{e}^u y)|^2 = |\mathrm{e}^u (y\,\mathrm{d}u + \mathrm{d}y)|^2 = \mathrm{e}^{2u}(|y|^2\,\mathrm{d}u^2 + 2\mathrm{d}u\langle y, \mathrm{d}y\rangle + |\mathrm{d}y|^2) = \mathrm{e}^{2u}|\mathrm{d}y|^2$$

since y lives in the light cone so that $y \perp y, \mathrm{d}y$. Thus a conformal change of metric can be modelled by a rescaling and any section of the null line bundle $L^{n+1} \to S^n$ over the n-sphere S^n, where

$$L^{n+1} = \{y \in \mathbb{R}^{n+2}_1;\ |y| = 0\}$$

represents a choice of metric in the conformal class of the round metric of S^n. Or, otherwise said, we can consider the **projective light cone**

$$L^{n+1}/\mathbb{R} \cong S^n \subset \mathbb{RP}^{n+1}$$

as the *conformal n-sphere*, with any choice of homogeneous coordinates in the light cone providing a choice of Riemannian metric in the standard conformal class of metrics.

In the remainder of this sections we shall investigate this approach further.

2.2. The Projective Model

As discussed above, we consider the *projective light cone* $\mathbb{RP}^{n+1} \supset L^{n+1}/\mathbb{R} \cong S^n$ as the *conformal n-sphere* S^n, equipped with the conformal structure of the standard round metric. Any choice of homogeneous coordinates (section of the null line bundle $L^{n+1} \to S^n$) provides a choice of metric in the conformal class.

Besides the points of S^n, which can be considered as the first type of "elements" in Möbius geometry, the hyperspheres of S^n form a second type of "elements" – just as points and hyperplanes (or, lines) are the "elements" of projective geometry. Note that the notion of a "hypersphere", that is, of a totally umbilic hypersurface in S^n, is a Möbius geometric notion and using, for example, the above transformation formula for the second fundamental form, it becomes apparent that the notion of an umbilic is conformally invariant since a conformal change of the ambient metric does not affect the traceless part of the second fundamental form (the vanishing of which characterizes umbilics).

Thinking of $S^n \subset \mathbb{R}^{n+1} \subset \mathbb{RP}^{n+1}$, a hypersphere in S^n can be considered as the intersection of a hyperplane in $\mathbb{RP}^{n+1}$ with S^n as an absolute quadric. Identifying a hyperplane that intersects S^n transversally with its pole (with respect to S^n as absolute quadric) in the "outer space" $\mathbb{RP}^{n+1}_O$ of S^n – the part of $\mathbb{RP}^{n+1}$ that consists of points of (real) tangent lines to S^n – we obtain an identification of the hyperspheres in S^n with the points of $\mathbb{RP}^{n+1}$ "outside" S^n [13, Section 1.1].

On the level of homogeneous coordinates, polarity is expressed by orthogonality in $\mathbb{R}_1^{n+2}$ as the absolute quadric S^n is the projective light cone of the Minkowski product on $\mathbb{R}_1^{n+2}$. Thus a hypersphere in S^n is represented by the spacelike line in $\mathbb{R}_1^{n+2}$ that is orthogonal to the hyperplane of the hypersphere.

For example, suppose that a hypersphere is given by its center $m \in S^n \subset \mathbb{R}^{n+1}$ and its radius $r \in (0, \pi)$. Then the (spacelike) orthogonal complement of the hyperplane of this sphere is spanned by the vector $(1, \frac{m}{\cos r}) \in \{1\} \times \mathbb{R}^{n+1}$. Normalizing we obtain a representative

$$S \simeq \tfrac{1}{\sin r}(\cos r, m) \in S_1^{n+1} := \{y \in \mathbb{R}_1^{n+2};\, |y|^2 = 1\}$$

of the hypersphere in the Lorentz sphere S_1^{n+1}. Namely, $x \in S^n \subset \mathbb{R}^{n+1}$ is a point of this hypersphere iff

$$x \cdot m = \cos r \Leftrightarrow \langle (1, x), (1, \tfrac{m}{\cos r}) \rangle = 0.$$

Note how the first obtained formula fails for great hyperspheres in S^n, where the radius becomes $\frac{\pi}{2}$ and the point of the outside of $S^n \subset \mathbb{RP}^{n+1}$ lies in the hyperplane at infinity of $\mathbb{RP}^{n+1}$.

Using the same line of argument one derives a formula for a hypersphere given in terms of its center $m \in \mathbb{R}^n$ and radius $r \in (0, \infty)$ in $\mathbb{R}^n \subset S^n$, where $\mathbb{R}^n \cup \{\infty\} \leftrightarrow S^n$ via stereographic projection

$$\mathbb{R}^n \cup \{\infty\} \ni x \leftrightarrow \mathbb{R}\left(\tfrac{1+|x|^2}{2}, x, \tfrac{1-|x|^2}{2}\right) \in S^n \subset \mathbb{RP}^{n+1}.$$

Here,

$$|x - m|^2 = r^2 \Leftrightarrow \left\langle \left(\tfrac{1+|x|^2}{2}, x, \tfrac{1-|x|^2}{2}\right), \left(\tfrac{1+(|m|^2-r^2)}{2}, m, \tfrac{1-(|m|^2-r^2)}{2}\right) \right\rangle = 0$$

so that $\frac{1}{r}\left(\frac{1+(|m|^2-r^2)}{2}, m, \frac{1-(|m|^2-r^2)}{2}\right) \in S_1^{n+1}$ is a normalized representative of the hypersphere in $\mathbb{R}^n$.

In the same way as the incidence of a hypersphere and a point is given by polarity (orthogonality of their homogeneous coordinates), the intersection angle of two hyperspheres is given by the Minkowski scalar product. Consider a point $x \in S_1 \cap S_2 \subset \mathbb{R}^n$ in the intersection of two hyperspheres $S_i \subset \mathbb{R}^n$ given in terms of their centers m_i and their radii r_i as above and then

$$\langle S_1, S_2 \rangle = \tfrac{r_1^2 + r_2^2 - |m_1 - m_2|^2}{2 r_1 r_2} = \tfrac{(x-m_1)\cdot(x-m_2)}{r_1 r_2} = \cos\alpha$$

where α is the intersection angle of the two hyperspheres. In particular, the two hyperspheres intersect orthogonally iff the two corresponding spacelike lines in Minkowski space are orthogonal.

At this point, we have a basic description of the elements of Möbius geometry and their relations in terms of the classical (projective) model of Möbius geometry. However, in order to complete the description of the model, we will need to describe the Möbius group, that is, the group of conformal or (equivalently) hypersphere preserving transformations of S^n in that model.

2.3. Sphere Pencils and Complexes

Lines and hyperplanes are basic objects in projective geometry. Having described the points and hyperspheres in the conformal n-sphere S^n as points in projective $(n+1)$-space $\mathbb{RP}^{n+1}$, it seems natural to investigate the hypersphere configurations given by lines and hyperplanes in this projective space [13, Section 1.2]. We discuss these two hypersphere configurations in turn.

Definition. *A configuration of hyperspheres in S^n whose representatives form a line in the projective space $\mathbb{RP}^{n+1}$ is called a* ***hypersphere pencil****.*

Sphere pencils come in three flavours, depending on whether the corresponding line does not intersect S^n ("elliptic sphere pencil"), touches S^n in one point ("parabolic sphere pencil"), or intersects S^n transversally ("hyperbolic sphere pencil").

An elliptic sphere pencil can be thought of as being given by two orthogonally intersecting hyperspheres: we may choose an orthonormal basis for the (spacelike) two-plane corresponding to the line in projective space $\mathbb{RP}^{n+1}$. Clearly, these two hyperspheres intersect in a codimension two sphere in S^n. Now, the homogeneous coordinate vector of any point in this codimension two sphere is orthogonal to both hyperspheres (incidence of a point and a hypersphere) and, hence, is orthogonal to any hypersphere of the pencil. Thus all the hyperspheres of the pencil intersect in this codimension two sphere, which can be identified with the pencil. So, elliptic sphere pencils can be identified with codimension two spheres in S^n. A suitable stereographic projection into $\mathbb{R}^n$ gives a one-parameter family of hyperplanes intersecting in an $(n-2)$-plane.

A parabolic sphere pencil contains exactly one "point sphere", given by the intersection point of the line in projective space with $S^n \subset \mathbb{RP}^{n+1}$. As the corresponding two-plane in the space of homogeneous coordinates is tangent to the light cone, the corresponding light line in the two-plane is orthogonal to all vectors in the two-plane (the induced metric on the two-plane is degenerate) and, consequently, this point is contained in all spheres of the pencil. Choosing homogeneous coordinates of this point and of one of the hyperspheres of the pencil as a basis for this degenerate two-plane we realize that a hypersphere that intersects this base hypersphere in the base point orthogonally will intersect all the hyperspheres of the pencil orthogonally. As a consequence, all the hyperspheres of the pencil touch (have first order contact) in the point sphere of the pencil. So, parabolic sphere pencils can be identified with *contact elements* in S^n, that is, with a one-parameter family of hyperspheres that touch in the point sphere of the pencil. A suitable stereographic projection into $\mathbb{R}^n$ gives a one-parameter family of parallel hyperplanes (which touch in the point at infinity).

A hyperbolic sphere pencil contains two "point spheres", given by the intersection of the line in projective space with S^n. Choosing homogeneous coordinates of these two points as a basis for the (Minkowski) two-plane in $\mathbb{R}^{n+2}_1$ corresponding to the line in projective space we learn that any hypersphere that contains the two points (whose representative is orthogonal to the basis vectors of the plane) intersects every hypersphere of the pencil orthogonally. Thus a suitable stereographic projection yields a one-parameter family of concentric hyperspheres in $\mathbb{R}^n$ with their common center and the point at infinity being the two point spheres of the pencil. Now we turn to the second topic of this section.

Definition. *A configuration of hyperspheres in S^n whose representatives form a hyperplane in projective space $\mathbb{RP}^{n+1}$ is called a* ***hypersphere complex****.*

In terms of homogeneous coordinates, a hypersphere complex is given by a non-zero vector in $\mathcal{K} \in \mathbb{R}^{n+2}_1 \setminus \{0\}$

$$\{S \in S^{n+1}_1;\ \langle S, \mathcal{K}\rangle = 0\}.$$

Again, sphere complexes come in three different flavours, depending on whether the hyperplane of spheres intersects S^n ("elliptic sphere complex", $|\mathcal{K}|^2 > 0$), the hyperplane touches S^n ("parabolic sphere complex", $|\mathcal{K}|^2 = 0$), or the hyperplane does not meet S^n ("hyperbolic sphere pencil", $|\mathcal{K}|^2 < 0$).

In the case of an elliptic sphere complex, we can interpret $\mathcal{K}$ as a hypersphere. Thus the hyperspheres of the elliptic sphere complex consists of all hyperspheres intersecting a given hypersphere orthogonally. Interpreting this hypersphere $\mathcal{K} \simeq \partial H^n$ as the boundary at infinity of a hyperbolic space, the hyperspheres of the complex become the hyperplanes of this hyperbolic space (and the complementary hyperbolic space obtained by inverting H^n in $\mathcal{K} \simeq \partial H^n$).

In the case of a parabolic sphere complex $\mathcal{K}$ becomes isotropic and can therefore be interpreted as a point in S^n. Thinking of this point as the point at infinity of $\mathbb{R}^n \subset S^n$ we characterize the hyperspheres of the complex as the hyperspheres containing the point at infinity of $\mathbb{R}^n$, that is, as the hyperplanes of $\mathbb{R}^n$.

In the case of a hyperbolic sphere pencil, $|\mathcal{K}|^2 < 0$, we can go back to our original construction and think of $\mathcal{K} = (1, 0) \in \mathbb{R}^{n+2}_1$. Now, the hyperplanes of hyperspheres orthogonal to $\mathcal{K}$ all contain $\mathcal{K}$, the "center" of $S^n \subset \mathbb{R}^{n+1}$. Thus, in this case, we see that our hyperspheres become great spheres in S^n, that is, hyperplanes in the round S^n.

Thus, in all three cases, we can interpret the hyperspheres of a sphere complex as hyperplanes in a suitably chosen space form geometry. We shall see later how this relates to the "metric subgeometries" of Möbius geometry – the space form geometries are all subgeometries of Möbius geometry, as one might expect as any motion in a space form is, in particular, also a conformal transformation and as all the space form geometries are (locally) conformally equivalent (via suitable stereographic projections).

2.4. Möbius Transformations

Clearly, since umbilics are invariant under conformal changes of the ambient metric, any conformal transformation of S^n maps hyperspheres to hyperspheres. In fact, every conformal transformation of S^n does induce a transformation on the space S_1^{n+1} of hyperspheres in S^n.

Definition. *A hypersphere preserving diffeomorphism of S^n is called a* ***Möbius transformation*** *of S^n.*

Thus, every conformal transformation of S^n is a Möbius transformation.

In order to see the converse, one shows that Möbius transformations come from Lorentz transformations of $\mathbb{R}_1^{n+2}$, which immediately implies that they are conformal.

Also, it is clear that any Lorentz transformation of $\mathbb{R}_1^{n+2}$ descends to a projective transformation of $\mathbb{RP}^{n+1}$ preserving S^n as the absolute quadric. These, in turn, give rise to Möbius transformations as they preserve hyperplanes and polarity.

To see that any Möbius transformation gives rise to a Lorentz transformation of $\mathbb{R}_1^{n+2}$, one first convinces oneself that Möbius transformations preserve sphere pencils of each type. Elliptic and parabolic sphere pencils can be characterized entirely in terms of incidence and contact and are hence preserved by Möbius transformations, hyperbolic sphere pencils are seen to be preserved as soon as one knows that orthogonal intersection is preserved, which can be proved using an interpretation of a parabolic sphere pencil as the tangent line of a curve intersecting the spheres of the pencil orthogonally. Next, one shows that Möbius transformations preserve sphere complexes. With these two facts at hand one can argue that any Möbius transformation of S^n extends (in a unique way) to a projective transformation of the ambient $\mathbb{RP}^{n+1}$ that preserves $S^n \subset \mathbb{RP}^{n+1}$ as absolute quadric (see [13, Section 1.3]). Now, the fundamental theorem of projective geometry takes care of the rest of the proof: every Möbius transformation extends to a projective transformation of $\mathbb{RP}^{n+1}$ preserving S^n that, in turn, can then be lifted to a Lorentz transformation of $\mathbb{R}_1^{n+2}$.

In fact, the Lorentz group $O_1(n+2)$ is a (trivial) double cover of the group of Möbius transformations of S^n, $\text{Möb}(n) \cong O_1(n+2)/\pm 1$.

In particular, inversions in hyperspheres extend to polar reflections in projective space which, in turn, lift to ordinary (Lorentz) reflections in $\mathbb{R}^{n+2}_1$

$$L^{n+1} \ni v \mapsto v - 2\langle v, S\rangle S \in L^{n+1}$$

where $S \in S^{n+1}_1$ is a unit spacelike vector representing the hypersphere of inversion.

2.5. Quadrics of Constant Curvature

Above we have discussed that the hyperspheres of a sphere complex, given by a non-zero vector $\mathcal{K} \in \mathbb{R}^{n+2}_1$, can be thought of as the hyperplanes in a space form geometry. We shall now put that interpretation of a linear complex into context (see [13, Section 1.4]). First note that, given a fixed vector $\mathcal{K} \in \mathbb{R}^{n+2}_1 \setminus \{0\}$, the hyperplane section

$$\mathcal{Q}^n_\kappa := \{y \in L^{n+1};\, \langle y, \mathcal{K}\rangle = -1\}, \qquad \kappa := -|\mathcal{K}|^2$$

of the light cone L^{n+1}, has constant sectional curvature κ. If $\kappa \neq 0$ then $\mathcal{Q}^n_\kappa$ is, up to a Lorentz transformation, a standard model of $S^n\left(\frac{1}{\sqrt{\kappa}}\right) \subset \mathbb{R}^{n+1}$ (if $\kappa > 0$) or of (two copies of) $H^n\left(\frac{1}{\sqrt{-\kappa}}\right) \subset \mathbb{R}^{n+1}_1$ (if $\kappa < 0$) and if $\kappa = 0$ then observe that

$$\mathbb{R}^n \ni x \mapsto \left(\tfrac{1+|x|^2}{2}, x, \tfrac{1-|x|^2}{2}\right) \in \mathcal{Q}^n_0 \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+2}_1$$

is an isometry.

Again, we can argue that the hyperspheres of the linear complex defined by $\mathcal{K}$ provide the hyperplanes of the space form geometry of $\mathcal{Q}^n_\kappa$. In the cases $\kappa \neq 0$ the orthogonal hyperplane $S^\perp \subset \mathbb{R}^{n+2}_1$ of a hypersphere $S \perp \mathcal{K}$ in the linear complex contains the point $\frac{1}{\kappa}\mathcal{K}$ which takes the role of the origin when thinking of $\mathcal{Q}^n_\kappa$ as a standard model of a space form in the affine space $\{y;\, \langle \mathcal{K}, y\rangle = -1\}$, and for the case $\kappa = 0$ we just stick with our previous argument that the hyperplanes in $\mathcal{Q}^n_\kappa$ are just the hyperspheres that contain the point at infinity (given by $\mathcal{K}$).

However, more can be said: the (mean) curvature H of a hypersphere $S \in S^{n+1}_1$ in $\mathcal{Q}^n_\kappa$ is given by

$$H = -\langle S, \mathcal{K}\rangle.$$

In this setup, the classical stereographic projection $S^n \to \mathbb{R}^n$ just becomes a central projection from one hyperplane section of the light cone onto another

$$\mathcal{Q}_0 \ni \left(\tfrac{1+|x|^2}{2}, x, \tfrac{1-|x|^2}{2}\right) \longleftrightarrow \left(1, \tfrac{2x}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2}\right) \in \mathcal{Q}_1 \setminus \{(1,0,-1)\}$$

where the respective light cones are $\mathcal{Q}_0 = \{y \in L^{n+1};\, \langle y, (1,0,-1)\rangle = -1\}$ and $\mathcal{Q}_1 = \{y \in L^{n+1};\, \langle y, (1,0,0)\rangle = -1\}$.

More generally, one can consider the central projection from any hyperplane section of the light cone to another as a stereographic projection (the points on the light cone generators that are parallel to either of the two hyperplanes need to be removed for these projections). Clearly these maps are conformal and hence can qualify as "stereographic projections."

To obtain the group of motions of one of these space form geometries observe that it must be a subgroup of the Möbius group and, therefore, must come from a Lorentz transformation of $\mathbb{R}^{n+2}_1$ (in the case $\kappa < 0$ we only consider motions that extend smoothly through the infinity boundary, separating the two hyperbolic spaces). Obviously, all Lorentz transformations that fix $\mathcal{K}$ provide isometries of $\mathcal{Q}^n_\kappa$ and hence motions of the space form geometry. Again, an argument using the standard models of the space forms proves the converse.

2.6. Sphere Congruences and Their Envelopes

One of the most central notion in Möbius differential geometry is that of a (hyper)sphere congruence and its envelopes. For example, in the geometry of surfaces in three-dimensional Möbius geometry the "conformal Gauss map," which takes the role of the ordinary Gauss map of surfaces in Euclidean space, is a certain sphere congruence enveloped by the surface.

Definition. *A* ***sphere congruence*** *is a smooth map $S : M^m \to S^{n+1}_1$ into the space of hyperspheres in S^n. We say that an immersion $f : M^m \to S^n$* ***envelops*** *a sphere congruence $S : M^m \to S^{n+1}_1$ if, for all $p \in M^m$, $f(p) \in S(p)$ and $\mathrm{d}_pf(T_pM) \subset T_{f(p)}S(p)$, that is, the submanifold touches the spheres in corresponding points.*

As we already discussed above, the incidence of $f(p)$ and $S(p)$ can, analytically, be expressed as

$$0 = \langle f(p), S(p)\rangle.$$

The touching condition means that the points of $f(M^m)$ that are "infinitely close" to $f(p)$ also lie on that sphere, that is,

$$0 \equiv \langle f(p) + \mathrm{d}_pf, S(p)\rangle = \langle \mathrm{d}_pf, S(p)\rangle \quad \Leftrightarrow \quad 0 \equiv \langle f(p), \mathrm{d}_pS\rangle.$$

Thus, an enveloped sphere congruence can be interpreted as a unit normal vector field (Gauss map) of f, or conversely, an envelope of a sphere congruence S can be viewed as an isotropic normal field of S. Both of these viewpoints are useful at times.

Also note that, if f envelops a sphere congruence S, then it envelops any sphere congruence $S + a\,f$, where $a : M^m \to \mathbb{R}$ is an arbitrary function. In the hypersurface case, $m = n - 1$, any two enveloped sphere congruences are related in this way: at each point $p \in M^{n-1}$ the sphere must lie in the parabolic sphere pencil given by the point $f(p)$ and the sphere $S(p)$ of the original sphere congruence enveloped by f.

As an example consider an immersion $g : M^{n-1} \to S^n \subset \mathbb{R}^{n+1}$ with unit normal field $n : M^{n-1} \to S^n$. Then $T := (0, n)$ is the *tangent plane congruence* of $f = (1, g)$ in the round sphere S^n which is, clearly, enveloped by f. Note that $S = k\,f + T$ yields a *curvature sphere* of f at $p \in M^{n-1}$ when

$$\partial_v S(p) = \partial_v k(p)\, f(p) + (0, \partial_v n + k\partial_v g)(p) \parallel f(p)$$

for some $v \in T_pM^{n-1}$. This shows that the curvature directions v and the curvature spheres of f at a point $p \in M^{n-1}$ are Möbius invariant.

Definition. *Let $g : M^{n-1} \to S^n \subset \mathbb{R}^{n+1}$ be an immersion with unit normal field $n : M^{n-1} \to S^n$. Then the sphere congruence*

$$Z := H\,f + T = (H, H\,g + n)$$

where H is the mean curvature of f, is called the ***central sphere congruence*** *of f.*

The central sphere congruence is connected to the hypersurface in a conformally invariant way. This is most easily seen by considering the second fundamental form of f with respect to S as a unit normal field

$$-\langle \mathrm{d}Z, \mathrm{d}f\rangle = -(\mathrm{d}n + H\mathrm{d}g) \cdot \mathrm{d}g$$

is the unique choice of sphere congruence for which the second fundamental form becomes trace free – remember from the introduction that the trace free second fundamental form is conformally invariant.

In the case $n = 3$, the central sphere congruence is also called the *conformal Gauss map* of f. The conformal Gauss map is, away from umbilics, the unique sphere congruence that induces a conformally equivalent metric on M^2. This provides another argument for the conformal invariance of the central sphere congruence in the surface case.

3. Curved Flats

In this section we shall consider a classical problem discussed in Blaschke's [1] book: when do the two envelopes of a sphere congruence in the conformal three-sphere induce conformally equivalent metrics? Neglecting the case of dual Willmore surfaces, which also appear as a solution of this problem, we shall see how we arrive at a very simple type integrable system, at "curved flats," cf. [9]. In this context we will also discover a close relationship with Christoffel's problem, which is a similar problem as Blaschke's problem but in Euclidean ambient geometry.

A more in-depth discussion of the material in this section can be found in [1] or in [13, Chapter 3].

3.1. Blaschke's Problem

We consider the following situation: let $S : M^2 \to S^4_1$ be a congruence of two-spheres in the conformal S^3 so that its induced metric $I = |dS|^2$ is positive definite. Then, since $dS \perp S$, the normal bundle of S has signature $(1,1)$ and, consequently, S has two isotropic normal fields $f, \hat{f} : M^2 \to L^4$. We think of $f, \hat{f} : M^2 \to S^3 \cong L^4/\mathbb{R}$ as the two envelopes of S.

Now we consider the following problem: when do the two envelopes f and $\hat{f}$ of the sphere congruence S induce conformally equivalent metrics $|df|^2$ and $|d\hat{f}|^2$ on M^2, that is, when is the map $f(p) \mapsto \hat{f}(p)$ assigning to each point of f the corresponding point on the other envelope $\hat{f}$ of S conformal?

Thinking of f as an (isotropic) normal field to S we can consider the shape operator $A := -dS^{-1} \circ df^T$ of S with respect to f, where $(.)^T$ denotes the projection onto the tangent plane of S. Note that, since $f \perp S, dS, f$, we learn that $df \perp S, f$ so that

$$df = df^T \bmod f \quad \Rightarrow \quad |df|^2 = |df^T|^2 = I(A, A) = I(A^2.,.).$$

Similarly $|d\hat{f}|^2 = I(\hat{A}^2.,.)$ with the shape operator of S with respect to $\hat{f}$ as a normal field. Hence, conformality of the induced metrics of f and $\hat{f}$ amounts to

$$\begin{aligned}\hat{\lambda}^2|df|^2 = \lambda^2|d\hat{f}|^2 &\Leftrightarrow \hat{\lambda}^2 A^2 = \lambda^2 \hat{A}^2\\ &\Leftrightarrow \hat{\lambda}^2(\operatorname{tr} A\, A - \det A \operatorname{id}) = \lambda^2(\operatorname{tr} \hat{A}\, \hat{A} - \det \hat{A} \operatorname{id})\end{aligned}$$

where we use Cayley–Hamilton's formula for the last equivalence.

Thus, we have two cases:

1) A, $\hat{A}$ and id are linearly independent – then we must have $\operatorname{tr} A = \operatorname{tr} \hat{A} = 0$, or
2) A, $\hat{A}$ and id are linearly dependent – then we have $[A, \hat{A}] = 0$, that is, A and $\hat{A}$ simultaneously diagonalize so that we obtain two subcases (a_i and $\hat{a}_i$ denote the eigenvalues of A and $\hat{A}$, respectively)
   - 2a) $\hat{\lambda} a_1 = \lambda \hat{a}_1$ and $\hat{\lambda} a_2 = \lambda \hat{a}_2$ with $\lambda > 0$, or
   - 2b) $\hat{\lambda} a_1 = \lambda \hat{a}_1$ and $\hat{\lambda} a_2 = -\lambda \hat{a}_2$ with $\lambda, \hat{\lambda} > 0$.

The case 1) leads to dual pairs of Willmore (conformally minimal) surfaces, which are the envelopes of a minimal sphere congruence in S^4_1, that is, a sphere congruence that is a minimal surface in the Lorentz sphere.

Here we shall concern ourselves with the case 2), which will lead us to Darboux pairs of isothermic surfaces and to curved flats in the (symmetric) space of point pairs in the conformal three-sphere. (For the simplicity of the discussion we shall neglect mixing cases.)

3.2. Ribaucour Sphere Congruences

To facilitate computations in this section we normalize f and $\hat{f}$ so that $\langle f, \hat{f} \rangle \equiv 1$ (here we use again that the normal bundle of S has signature $(1, 1)$ so that the two envelopes f and $\hat{f}$ of S are distinct at all points).

With this normalization we write the normal connection of S as

$$\nabla^{\perp} f = \nu f, \qquad \nabla^{\perp} \hat{f} = -\nu \hat{f}$$

with the one-form $\nu = \langle \mathrm{d}f, \hat{f} \rangle$. Now, since $\mathrm{d}f = \mathrm{d}f^T + \nu f$ and $\mathrm{d}\hat{f} = \mathrm{d}\hat{f}^T - \nu \hat{f}$, the normal curvature of S computes from a Ricci type equation

$$\mathrm{d}\nu = \langle \mathrm{d}\hat{f} \wedge \mathrm{d}f \rangle = \langle \mathrm{d}\hat{f}^T \wedge \mathrm{d}f^T \rangle - \underbrace{\nu \wedge \nu}_{=0} = I([A, \hat{A}]., .).$$

Thus the normal curvature of S vanishes if and only the shape operators with respect to the two isotropic normal fields commute.

Note that, as a consequence, the sphere congruences S in the above cases 2a) and 2b) have flat normal bundle.

In order to interpret this condition geometrically, we now switch viewpoint and think of S as a unit normal field of f rather than of f as an isotropic normal field of S. For this purpose we shall assume that f is an immersion, that is, its induced metric $|\mathrm{d}f|^2 = I(A^2., .)$ is positive definite. As a consequence, A is regular ($\det A \neq 0$) and, since

$$-\langle \mathrm{d}S, \mathrm{d}f \rangle = I(A., .) = I(A^2 \circ A^{-1}., .)$$

A^{-1} is the shape operator of f with respect to S as a normal field. Hence the eigendirections of A are the curvature directions of f as an immersion into S^3.

A similar statement holds for $\hat{f}$: the eigendirections of $\hat{A}$ are the curvature directions of $\hat{f} : M^2 \to S^3$ as soon as $\hat{f}$ is an immersion. From this we conclude that when $f, \hat{f} : M^2 \to S^3$ are immersions then their curvature directions correspond if and only if the normal bundle of S, spanned by f and $\hat{f}$, is flat.

In this way, we make contact with the classical notion of a "Ribaucour sphere congruence." Classically, a sphere congruence S is called Ribaucour if the curvature lines on its two envelopes do correspond (under the map $f(p) \mapsto \hat{f}(p)$ that maps a point of contact with a sphere $S(p)$ on one envelope to the point of contact with the same sphere on the other envelope). Our equivalent formulation is slightly more general (cf. [13, Section 3.1]).

Definition. *A sphere congruence* $S : M^2 \to S^4_1$ *is called a* ***Ribaucour sphere congruence*** *if the rank two vector bundle* $\mathrm{span}\{f, \hat{f}\}$ *spanned by its two envelopes is flat.*

Thus: the sphere congruences in our cases 2a) and 2b) above are Ribaucour sphere congruences.

3.3. Symmetry Breaking

We shall now consider case 2a). As the sphere congruence S is Ribaucour, as discussed above, we may normalize $f, \hat{f} : M^2 \to L^4$ so that $\langle f, \hat{f}\rangle \equiv 1$ (as before) and, additionally, $\nu \equiv 0$. That is, we choose f and $\hat{f}$ to be *parallel* sections of the flat normal bundle of S.

Now we set $\kappa := 2\frac{\hat{\lambda}}{\lambda}$ (we have $\lambda > 0$) so that, with $\mathrm{d}f^T = \mathrm{d}f$ and $\mathrm{d}\hat{f}^T = \mathrm{d}\hat{f}$

$$\hat{A} = \tfrac{\kappa}{2}A \quad \Leftrightarrow \quad 0 = \tfrac{\kappa}{2}\mathrm{d}f^T - \mathrm{d}\hat{f}^T = \tfrac{\kappa}{2}\mathrm{d}f - \mathrm{d}\hat{f}.$$

Employing the Codazzi equations $\mathrm{d}^2 f = \mathrm{d}^2\hat{f} = 0$ we then get

$$0 = \mathrm{d}\left(\tfrac{\kappa}{2}\mathrm{d}f - \mathrm{d}\hat{f}\right) = \tfrac{1}{2}\mathrm{d}\kappa \wedge \mathrm{d}f$$

so that $\mathrm{d}\kappa = 0$ since f is an immersion. That is, κ is a constant and we learn that

$$\mathcal{K} := \tfrac{\kappa}{2}f - \hat{f}$$

is a constant vector with $\langle f, \mathcal{K}\rangle \equiv -1$. The immersion $f : M^2 \to \mathcal{Q}^3_\kappa$ takes values in the quadric

$$\mathcal{Q}^3_\kappa = \{y \in L^4\,;\, \langle y, \mathcal{K}\rangle = -1\}$$

of constant curvature $\kappa = -|\mathcal{K}|^2$.

Moreover, $S(p)\perp\mathcal{K}$ for all $p \in M^2$ so that the spheres of the congruence are the tangent planes of f in its ambient constant curvature geometry. Also note that (unless $\kappa = 0$ in which case $\hat{f} \equiv -\mathcal{K}$)

$$\hat{f} = \tfrac{\kappa}{2}f - \mathcal{K} = \tfrac{\kappa}{2}\left(f - 2\tfrac{\langle f,\mathcal{K}\rangle}{|\mathcal{K}|^2}\mathcal{K}\right)$$

is Möbius equivalent to f – in fact, $\hat{f}$ is obtained as a polar reflection of f in the vector $\mathcal{K}$ defining the ambient quadric of constant curvature: hence $\hat{f}$ is the antipodal to f in the case $\kappa > 0$ and it is obtained by reflection at the infinity boundary of hyperbolic space in case $\kappa < 0$.

3.4. Darboux Pairs

Now we consider case 2b). As before, we normalize f and $\hat{f}$ to be parallel sections of the normal bundle of S with $\langle f, \hat{f}\rangle \equiv 1$ as S has flat normal bundle.

Using the Codazzi equations for f and $\hat{f}$ combined with the equations from 2b) it can then be shown that f and $\hat{f}$ have common conformal curvature line coordinates [13, §3.2.1]. Thus both envelopes, f and $\hat{f}$, of S are *isothermic surfaces*.

Definition. *A surface is called **isothermic** if it has (locally, away from umbilics) conformal curvature line coordinates.*

And, the two surfaces f and $\hat{f}$ are what is classically called *Darboux transforms* of each other.

Definition. *In the situation of case* 2b), *where the isothermic surfaces f and $\hat{f}$ are conformal Ribaucour transforms of each other, we say that f and $\hat{f}$ form a **Darboux pair** of isothermic surfaces.*

We shall see later how to construct a Darboux partner $\hat{f}$ for a given isothermic surface, given an isothermic f there are ∞^{1+3} Darboux transforms $\hat{f}$ of f.
For now, we are interested in a different aspect of Darboux pairs of isothermic surfaces: writing the second fundamental form of $S : M^2 \to \mathbb{R}^5_1$ in terms of the extended normal frame S, f and $\hat{f}$

$$II(v,w) = -I(v,w)\,S + I(Av,w)\,\hat{f} + I(\hat{A}v,w)\,f$$

we obtain

$$0 = K - \langle II(e_1,e_1), II(e_2,e_2)\rangle = K - \{1 + a_1\hat{a}_2 + \hat{a}_1 a_2\} = K - 1$$

as the Gauss equation of S, where (e_1, e_2) denotes an orthonormal principal frame. Thus, the extended tangent bundle $\mathrm{span}\{S, \mathrm{d}S\} = \{f, \hat{f}\}^\perp$ of S is flat (note that the other components of the curvature tensor vanish trivially, $R(v,w)S = \partial_v\partial_w S - \partial_w\partial_v S - \partial_{[v,w]}S = 0$). On the other hand, we already know that the normal bundle $\mathrm{span}\{f, \hat{f}\}$ of S is flat since S is Ribaucour. Consequently, the extended Gauss map

$$\gamma : M^2 \to \tfrac{O_1(5)}{O(3)\times O_1(2)}, \qquad p \mapsto \gamma(p) := \mathrm{span}\{f(p), \hat{f}(p)\}$$

of S into the symmetric *space of point pairs* in S^3 is a *curved flat* [9].

Definition. *Let $\gamma : M^m \to G/K$ a map into a symmetric space. Consider the symmetric decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$$

*of the Lie algebra of G, where $\mathfrak{k}$ is the isotropy algebra, and write the corresponding decomposition of the connection form $F^{-1}\mathrm{d}F = \Phi = \Phi_\mathfrak{k} + \Phi_\mathfrak{p}$ of a lift (frame) $F : M^m \to G$ of γ. Then γ is called a **curved flat** if*

$$[\Phi_\mathfrak{p} \wedge \Phi_\mathfrak{p}] \equiv 0.$$

In the case at hand, of Darboux pairs of isothermic surfaces, it is straightforward to see that this definition is equivalent to the flatness of the two complementary sub-bundles $\mathrm{span}\{f, \hat{f}\}$ and $\{f, \hat{f}\}^\perp$ of the trivial $\mathbb{R}^5_1$-bundle over the surface as discussed above.

Also note that, in general, the notion of a curved flat does not depend on a choice of lift F for the map γ. If $\widetilde{F} = FH$, $H : M^2 \to K$, denotes another lift for the same map into the symmetric space then

$$\widetilde{\Phi} = \mathrm{Ad}(H^{-1})\Phi + \underbrace{H^{-1}\mathrm{d}H}_{\in\,\mathfrak{k}} \quad \text{and} \quad \widetilde{\Phi}_{\mathfrak{p}} = \mathrm{Ad}(H^{-1})\Phi_{\mathfrak{p}}$$

because $\mathfrak{k}$ and $\mathfrak{p}$ are $\mathrm{Ad}(H^{-1})$-stable, so that

$$[\widetilde{\Phi}_{\mathfrak{p}} \wedge \widetilde{\Phi}_{\mathfrak{p}}] = \mathrm{Ad}(H^{-1})[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}]$$

showing that the curved flat condition is gauge invariant.

Now contemplate the integrability condition for the connection form of a curved flat

$$0 = \mathrm{d}\Phi + \tfrac{1}{2}[\Phi \wedge \Phi] = \underbrace{\mathrm{d}\Phi_{\mathfrak{k}} + \tfrac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] + \tfrac{1}{2}[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}]}_{\in\mathfrak{k}} + \underbrace{\mathrm{d}\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}]}_{\in\mathfrak{p}}$$

together with the curved flat condition $[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] = 0$. Because of the indicated splitting of the Maurer–Cartan equation into $\mathfrak{k}$- and $\mathfrak{p}$-parts we can introduce a (spectral) parameter μ to encode integrability and the curved flat condition as the integrability of a loop of connection forms

$$\mu \mapsto \Phi_\mu := \Phi_{\mathfrak{k}} + \mu\,\Phi_{\mathfrak{p}}.$$

Namely,

$$0 = \mathrm{d}\Phi_\mu + \tfrac{1}{2}[\Phi_\mu \wedge \Phi_\mu] = \mathrm{d}\Phi_{\mathfrak{k}} + \tfrac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] + \mu\{\mathrm{d}\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}]\} + \tfrac{\mu^2}{2}\,[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}]$$

splits into three equations for the coefficients of the μ-powers, giving that Φ_μ is integrable for all μ if and only if $\Phi_1 = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} = \Phi$ is the connection form of a lift of a curved flat. In this way, a curved flat gives rise to a one-parameter family of curved flats in the symmetric space G/K.

In the case at hand, this curved flat family gives rise to a one-parameter family of Darboux pairs $\{f_\mu, \hat{f}_\mu\}$, $\mu \in \mathbb{R}$. We shall see later that the f_μ can actually be defined without reference to a Darboux transform $\hat{f}$ of f and that the family $\mu \mapsto f_\mu$ is actually given by the classical T-transformation (or "Calapso transformation"). Equivalently, it arises from the conformal deformation studied by Cartan [7].

3.5. Christoffel Pairs

The curved flat family becomes singular for $\mu = 0$. Here $\Phi_0 = \Phi_{\mathfrak{k}}$ takes values in the isotropy algebra and the corresponding curved flat becomes constant, that is, the two isothermic surfaces degenerate to two (distinct) points in S^3. However, by appropriately rescaling one or the other surface (using a μ-dependent gauge transformation) or by using Sym's formula the two points can be "blown up" to discover two surfaces

$$f_{\mu=0}, \qquad \hat{f}_{\mu=0} : M^2 \to \mathbb{R}^3 \cong T_{f_0}S^3 \cong T_{\hat{f}_0}S^3.$$

These two limiting surfaces will also be isothermic and will be a solution to a problem in Euclidean space, *Christoffel's problem*, very similar to the Blaschke's problem in conformal geometry [8] or [13, Section 3.3]: "When do two surfaces $f, f^* : M^2 \to \mathbb{R}^3$ have parallel tangent planes and induce conformally equivalent metrics?"

The solutions to this problem are, in a very analogous way to the solutions of Blaschke's problem (again, as for Blaschke's problem, we neglect any mixing of the cases)

1) the curvature directions on the surfaces are not parallel – then we find pairs of minimal surfaces whose differentials are "holomorphically related"
2) the curvature directions on the surfaces are parallel – in that case we get two subcases:
   - 2a) pairs of similar surfaces (equivalent via a homothety and a translation)
   - 2b) *Christoffel pairs* of isothermic surfaces.

As in the Darboux pair case the properties of the solutions in 2b) can serve as a definition for the notion of a Chrsitoffel pair.

Definition. *In the case* 2b), *where the isothermic surfaces f and f^* have parallel principal directions and induce conformally equivalent metrics, we say that they form a* ***Christoffel pair*** *of isothermic surfaces.*

As discussed above, Christoffel pairs of isothermic surfaces appear as a limiting case in the associated family of Darboux pairs of isothermic surfaces, $f = f_{\mu=0}$ and $f^* = \hat{f}_{\mu=0}$. The converse is also true – any Christoffel pair arises in this way. In fact, a Christoffel pair gives rise to a family of Darboux pairs so that it arises as the limit of the family as $\mu \to 0$ [13, §3.3.9].

As with the Darboux and Calapso transformations we shall see later how to construct a Christoffel transform from a given isothermic surface in Euclidean space (in contrast to the other two transformations, the Euclidean ambient geometry plays a role for the Christoffel transformation, as the definition of a Christoffel pair suggests).

4. A Quaternionic Formalism

In this section we learn about another approach to Möbius geometry. Just as in the two-dimensional case, Möbius transformations of the (conformal) three- or four-sphere can be described by linear fractional transformations, using quaternions.

This approach provides a rather compact formalism, especially well suited to treat surface theory in the conformal three-sphere. A similar approach, using Vahlen matrices, can be used to study submanifolds in n-dimensional Möbius geometry – however, this latter approach lacks the possibility of a subversive but useful mixing of the complex structure on a conformal surface and the ambient algebraic structure. More details about the material discussed in this section can be found in [16] or in [13, Chapter 4].

4.1. The Idea

It is a well known fact that (orientation preserving) Möbius transformations of the two-sphere can be written as fractional linear transformations of the (compactified) complex plane $\mathbb{C} \cup \{\infty\} \cong S^2$. Or, otherwise said, they are the projective transformations of the complex projective line $\mathbb{CP}^1 \cong S^2$

$$\mathrm{PGL}(\mathbb{C}^2) \times \mathbb{CP}^1 \ni (A, \mathbb{C}v) \mapsto \mathbb{C}(Av) \in \mathbb{CP}^1.$$

Our goal here will be to see that the same ideas can be used to describe the Möbius transformations of the four-sphere or the three-sphere using quaternions. Identifying $\mathbb{H} \cong \mathbb{R}^4$ we find that $\mathbb{HP}^1 \cong \mathbb{R}^4 \cup \{\infty\} \cong S^4$ as in the complex case. However, here we have the first encounter of the problems arising from the quaternions not being commutative and we have to decide whether scalar multiplication on the space $\mathbb{H}^2$ of homogeneous coordinates is from the right or from the left. In order to keep the formalism as "normal looking" as possible, we will consider $\mathbb{H}^2$ as a *right* vector space so that

$$\mathbb{HP}^1 = \{v\mathbb{H};\, v \in \mathbb{H}^2\}$$

and $\mathrm{GL}(\mathbb{H}^2)$ acts by left multiplication of quaternionic 2×2-matrices on the coordinate vectors,

$$\mathrm{PGL}(\mathbb{H}^2) \times \mathbb{HP}^1 \ni (A, v\mathbb{H}) \mapsto (Av)\mathbb{H} \in \mathbb{HP}^1.$$

The kernel of this action is the real line as real multiples of A get absorbed into the quaternionic line but general quaternionic multiples will result in a different linear fractional transformation because of the non-commutativity of the quaternions. Thus "PGL" refers to the group projectivized over the reals.

With this idea in mind we wish to:

1. show that this gives indeed a model for four- and three-dimensional Möbius geometry, that is, $\mathrm{PGL}(\mathbb{H}^2)$ acts on $\mathbb{HP}^1 \cong S^4$ (equipped with the conformal structure inherited from the identification $\mathbb{H} \cong \mathbb{R}^4$) by Möbius transformations, and
2. describe (hyper-)spheres and other Möbius geometric objects or configurations in a convenient way in this model.

The main idea to achieve these two goals is to link the quaternionic setup to the classical projective model that we discussed above. In order to do so there are two different ansatzes:

- identify spheres with Möbius involutions, i.e., take a group theoretic approach – here the difficulty is that inversions in hyperspheres are orientation reversing and hence will not be described by fractional linear transformations; we will, however, later see that this is a very valuable idea in order to describe three-dimensional Möbius geometry (cf. [13, Section 4.8])
- use the six-dimensional real vector space of quaternionic Hermitian forms on $\mathbb{H}^2$ to model the underlying Minkowski space $\mathbb{R}^6_1$ of the classical projective model – this will be our first approach (cf. [13, Section 4.3]).

4.2. Quaternionic Hermitian Forms

We look at the space $\mathcal{H}(\mathbb{H}^2)$ of quaternionic Hermitian on $\mathbb{H}^2$, that is, of maps $S : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}$ satisfying

i) $S(v, w) = \overline{S(w, v)}$ for any $v, w \in \mathbb{H}^2$

ii) $S(v, w\lambda + \tilde{w}) = S(v, w)\lambda + S(v, \tilde{w})$ for $\lambda \in \mathbb{H}$ and $v, w, \tilde{w} \in \mathbb{H}^2$.

Writing such a Hermitian form in terms of a quaternionic 2×2-matrix

$$\mathbb{H}^2 \times \mathbb{H}^2 \ni (v, w) \mapsto \bar{v}^t S w \in \mathbb{H}$$

we learn that the quaternionic matrix S has to satisfy $\bar{S}^t = S$ so that it is instantly clear that the space of quaternionic Hermitian forms $\mathcal{H}(\mathbb{H}^2)$ is a real 6-dimensional vector space,

$$\mathcal{H}(\mathbb{H}^2) = \left\{ S = \begin{pmatrix} s & h \\ \bar{h} & t \end{pmatrix} ;\, h \in \mathbb{H},\ s, t \in \mathbb{R} \right\}.$$

Now we equip $\mathcal{H}(\mathbb{H}^2)$ with $-\det$ as a quadratic form: we let

$$|.|^2 : \mathcal{H}(\mathbb{H}^2) \to \mathbb{R}, \quad S = \begin{pmatrix} s & h \\ \bar{h} & t \end{pmatrix} \mapsto |S|^2 := -\det S = |h|^2 - st.$$

Polarization then yields a Minkowski scalar product on $\mathcal{H}(\mathbb{H}^2) \cong \mathbb{R}^6_1$.

Note that, in general, there is no sensible notion of determinant for quaternionic 2×2-matrices because of the quaternions being non-commutative. However, using the map

$$\mathrm{End}(\mathbb{H}^2) \ni A \mapsto \bar{A}^t A \in \mathcal{H}(\mathbb{H}^2)$$

one can introduce an order four map, the **Study determinant**

$$[.] : \mathrm{End}(\mathbb{H}^2) \to \mathbb{R}, \quad A \mapsto [A] := \det(\bar{A}^t A)$$

which shares some useful properties with the determinant, caused by the fact that the Study determinant of a quaternionic 2×2-matrix is the usual determinant of the complex 4×4-matrix corresponding to the quaternionic 2×2-matrix under the identification

$$\mathbb{C} \oplus \mathbb{C}\mathrm{j} = \mathbb{H} \ni a + b\mathrm{j} \leftrightarrow \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2).$$

In particular,

- $A \in \mathrm{End}(\mathbb{H}^2)$ is invertible if and only if $[A] \neq 0$, i.e., $\mathrm{GL}(\mathbb{H}^2) = \{A \in \mathrm{End}(\mathbb{H}^2);\ [A] \neq 0\}$
- $[AB] = [A]\,[B]$ for $A, B \in \mathrm{End}(\mathbb{H}^2)$, and
- $|\bar{A}^t S A|^2 = [A]\,|S|^2$ for $A \in \mathrm{End}(\mathbb{H}^2)$ and $S \in \mathcal{H}(\mathbb{H}^2)$.

The last property shows that the 15-dimensional group

$$\mathrm{SL}(\mathbb{H}^2) := \{A \in \mathrm{End}(\mathbb{H}^2);\ [A] = 1\}$$

acts by isometries on $\mathbb{R}^6_1 \cong \mathcal{H}(\mathbb{H}^2)$ via

$$\mathrm{SL}(\mathbb{H}^2) \times \mathcal{H}(\mathbb{H}^2) \ni (A, S) \mapsto A \cdot S := \overline{A}^{-1t} S A^{-1} \in \mathcal{H}(\mathbb{H}^2).$$

Thus we obtain a group homomorphism $r : \mathrm{SL}(\mathbb{H}^2) \to O_1(6)$. In fact, $\mathrm{SL}(\mathbb{H}^2)$ is the universal cover of the identity component of the Lorentz group $O_1(6)$ as is seen by showing that its differential at the identity,

$$\varrho : \mathfrak{sl}(\mathbb{H}^2) \to \mathfrak{o}_1(6), \quad \varrho(X)S = -SX - \bar{X}^t S$$

is a Lie algebra isomorphism.

At this point we have a full description of four-dimensional Möbius geometry in terms of quaternionic 2×2-matrices available to us by rewriting the projective model discussed above in terms of quaternions. However, we still need to make contact with the identification of the conformal four-sphere with the quaternionic projective line $\mathbb{HP}^1$.

Thus, to show that $\mathbb{HP}^1 \cong L^5/\mathbb{R} \subset P\mathcal{H}(\mathbb{H}^2)$ consider the map

$$\mathbb{HP}^1 \ni v\mathbb{H} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mathbb{H} \mapsto \mathbb{R}S_v := \mathbb{R} \begin{pmatrix} |v_2|^2 & -v_1\bar{v}_2 \\ -v_2\bar{v}_1 & |v_1|^2 \end{pmatrix} \in P\mathcal{H}(\mathbb{H}^2).$$

Note that this map is well defined since quaternionic multiples of v result in real multiples of S_v. Moreover, $|S_v|^2 = 0$ so that it takes values in the projective light cone. In order to see that every light line in $\mathcal{H}(\mathbb{H}^2)$ is the image of a point in $\mathbb{HP}^1$ note that v spans the light cone of S_v

$$\bar{v}^t S_v v = 0 \quad \text{and} \quad \bar{w}^t S_v w = 0 \Rightarrow w \parallel v.$$

To verify the second statement write $S_v = \left(\begin{smallmatrix} s & h \\ \bar{h} & t \end{smallmatrix}\right)$. If $s = 0$ then also $h = 0$ since $0 = |S_v|^2 = |h|^2 - st$ and, without loss of generality, $t = 1$ so that $v = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$; if

$s \neq 0$, without loss of generality, $s = 1$ and $S_v = \begin{pmatrix} 1 & h \\ \bar{h} & |h|^2 \end{pmatrix}$ so that $v = \begin{pmatrix} -h \\ 1 \end{pmatrix}$, then

$$(\bar{x}, 1)\begin{pmatrix} 1 & h \\ \bar{h} & |h|^2 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = |x + h|^2 = 0 \Leftrightarrow x = -h.$$

Consequently, the inverse of the above map assigns to a light line in $\mathcal{H}(\mathbb{H}^2)$, spanned by a non-zero vector $S \in L^5 \subset \mathcal{H}(\mathbb{H}^2)$, the (unique) light line of S in $\mathbb{H}^2$. Note also that, as a consequence, the action

$$\mathrm{GL}(\mathbb{H}^2) \times \mathbb{HP}^1 \ni (A, v\mathbb{H}) \mapsto (Av)\mathbb{H} \in \mathbb{HP}^1$$

of $\mathrm{PGL}(\mathbb{H}^2)$ on $\mathbb{HP}^1$ is compatible with the action

$$\mathrm{SL}(\mathbb{H}^2) \times \mathcal{H}(\mathbb{H}^2) \ni (A, S) \mapsto \overline{A^{-1}}^t S A^{-1} \in \mathcal{H}(\mathbb{H}^2)$$

of $\mathrm{SL}(\mathbb{H}^2)$ on $\mathbb{R}^6_1$ because the actions preserve the light cone relation between quaternionic Hermitian forms and points in $\mathbb{HP}^1$ since

$$(A \cdot S)(Av, Av) = \overline{Av}^t(\overline{A^{-1}}^t S A^{-1})Av = \bar{v}^t S v = S(v, v).$$

Finally we would like to establish the incidence relation for a point in $\mathbb{HP}^1$ and a (hyper-)sphere $S \in S^5_1 \subset \mathcal{H}(\mathbb{H}^2)$. Again we write $S = \begin{pmatrix} s & h \\ \bar{h} & t \end{pmatrix}$ and consider two cases:

- if $s = 0$, then $S = \begin{pmatrix} 0 & -n \\ -\bar{n} & 2d \end{pmatrix}$ with suitable $n \in S^3 \subset \mathbb{H}$ and $d \in \mathbb{R}$. Also, for $v = \begin{pmatrix} x \\ 1 \end{pmatrix}$

$$\langle S, S_v \rangle = -\tfrac{1}{2} S(v, v) = x \cdot n - d$$

  so that S describes a plane with normal n and distance d from the origin in $\mathbb{H}^4 \cong \mathbb{R}^4$
- if $s \neq 0$, then $S = \frac{1}{r}\begin{pmatrix} 1 & -m \\ -\bar{m} & |m|^2 - r^2 \end{pmatrix}$ with some $m \in \mathbb{H}$ and $r \in \mathbb{R}$, and now for $v = \begin{pmatrix} x \\ 1 \end{pmatrix}$

$$\langle S, S_v \rangle = -\tfrac{1}{2} S(v, v) = -\tfrac{1}{2r}\{|x - m|^2 - r^2\}$$

  so that the light cone of S becomes a sphere with center $m \in \mathbb{R}^4$ and radius r.

In any case, $\langle S, S_v \rangle = 0$ iff $S(v, v) = 0$ so that incidence translates into isotropy – a point $v\mathbb{H} \in \mathbb{HP}^1$ is on the hypersphere described by $S \in S^5_1 \subset \mathcal{H}(\mathbb{H}^2)$ if and only if $S(v, v) = 0$, that is, if and only if it is a light line with respect to S as an inner product on $\mathbb{H}^2$.

At this point it becomes entirely straightforward to reformulate the descriptions for geometric configurations discussed in the classical context into the quaternionic setup. For example, the notion of an envelope as we have to remember that an immersion $f : M^m \to S^4$ envelopes a sphere congruence $S : M^m \to S^5_1$ if and only if

$$f \perp S, \mathrm{d}S \quad \Leftrightarrow \quad S(f,f) = 0, \quad \mathrm{d}S(f,f) = 0$$

where f on the left side is interpreted as a lift of the immersion into $S^4 \cong L^5/\mathbb{R}$ into the light cone $L^5 \subset \mathbb{R}^6_1$ whereas, on the right side, it is interpreted as a lift from $S^4 \cong \mathbb{HP}^1$ into the space $\mathbb{H}^2$ of homogeneous coordinates of $\mathbb{HP}^1$.

4.3. Möbius Involutions

From the projective model of Möbius geometry we know how to describe two-spheres in S^4 in terms of elliptic (hyper-)sphere pencils. Choosing two base points for the line in S^5_1, we may describe it as the orthogonal intersection of two hyper-spheres. Thinking of a hypersphere as the fixed point set of the inversion in this hypersphere, we may identify a two-sphere with the Möbius involution obtained by two consecutive inversions in orthogonal hyperspheres intersecting in the two-sphere. Note that the resulting Möbius involution does neither depend on the order of the inversions (the inversions commute since the hyperspheres intersect orthogonally) nor on the choice of orthogonal hyperspheres. If $S_1, S_2 \in S^5_1$ denote two orthogonal hyperspheres and

$$\widetilde{S}_1 = \cos\alpha\, S_1 + \sin\alpha\, S_2, \quad \widetilde{S}_2 = -\sin\alpha\, S_1 + \cos\alpha\, S_2$$

then the composition of the two inversions in $\widetilde{S}_1$ and $\widetilde{S}_2$ is given by

$$v \mapsto v - 2\langle v, \widetilde{S}_1\rangle \widetilde{S}_1 - 2\langle v, \widetilde{S}_2\rangle \widetilde{S}_2 = v - 2\langle v, S_1\rangle S_1 - 2\langle v, S_2\rangle S_2.$$

As a composition of two inversions the resulting Möbius transformation is orientation preserving and can therefore be written as a fractional linear transformation $J \in \mathrm{SL}(\mathbb{H}^2)$ of $\mathbb{HP}^1$. And, as J is an involution we have $J^2 = \pm 1$. It turns out [13, §4.8.1] that the space of two-spheres in S^4 can be identified with the space

$$\mathcal{S}(\mathbb{H}^2) := \{J \in \mathrm{End}(\mathbb{H}^2)\,;\, J^2 = -1\}$$

of almost complex structures on $\mathbb{H}^2$ so that incidence is given as a fixed point relation: a point $v\mathbb{H} \in \mathbb{HP}^1$ is on the two-sphere given by $J \in \mathcal{S}(\mathbb{H}^2)$ if and only if $(Jv)\mathbb{H} = v\mathbb{H}$. Otherwise said, we must have $Jv = v\lambda$ with some $\lambda \in \mathbb{H}$ when $v\mathbb{H}$ is on the two-sphere J. Then, since $J^2 = -1$, we learn that $\lambda^2 = -1$ so that $\lambda \in S^2 \subset \mathrm{Im}\,\mathbb{H}$. This last observation is the core of a proof that $\mathcal{S}(\mathbb{H}^2)$ is indeed the space of two-spheres in $S^4 \cong \mathbb{HP}^1$.

To make contact with the hyperspheres of the elliptic pencil describing the two-sphere J observe the following: if $S \in \mathcal{H}(\mathbb{H}^2)$ and J is symmetric with respect to S then

$$0 = S(Jv, Jv) - S(v, J^2 v) = (1 + |\lambda|^2)\, S(v,v)$$

as soon as $Jv = v\lambda$ so that the two-sphere J is contained in the hypersphere S. In this case, a second hypersphere containing the two-sphere J and intersecting S orthogonally can be defined by

$$\widetilde{S} := SJ, \quad \text{i.e.,} \quad \widetilde{S}(v,w) := S(v, Jw).$$

The converse of the above statement is also true: a two-sphere $J \in \mathcal{S}(\mathbb{H}^2)$ is contained in a three-sphere $S \in \mathcal{H}(\mathbb{H}^2)$ if and only if J is symmetric with respect to S [13, §4.8.6].

We are especially interested in the two-spheres in a fixed $S^3 \subset S^4$ in order to study the geometry of surfaces in codimension one. For this it is convenient to take

$$S^3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{so that} \quad 0 = (\bar{x}, 1) S^3 \begin{pmatrix} x \\ 1 \end{pmatrix} \Leftrightarrow x \in \operatorname{Im}\mathbb{H}$$

that is, $S^3 \cong \operatorname{Im}\mathbb{H} \cup \{\infty\}$. Now, using the above description of hyperspheres and hyperplanes and apply our formula $\widetilde{S} = SJ$ with $S = S^3$, we obtain

$$J = \begin{pmatrix} n & 2d \\ 0 & -n \end{pmatrix} \quad \text{or} \quad J = \tfrac{1}{r}\begin{pmatrix} m & |m|^2 - r^2 \\ 1 & -m \end{pmatrix}$$

with $n \in S^2 \subset \operatorname{Im}\mathbb{H}$ or $m \in \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ and $r \in \mathbb{R}$ for hyperplanes and hyperspheres in $\mathbb{R}^3$: note that the above J's are symmetric with respect to S^3, i.e., $S^3 J = \bar{J}^t S^3$. With these J the fixed point set is, as expected, the plane orthogonal to n with distance d from the origin in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$

$$\begin{pmatrix} n & 2d \\ 0 & -n \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}\lambda \quad \Leftrightarrow \quad \lambda = -n, \quad nx + xn + 2d = 0$$

and the sphere with center m and radius r in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$

$$\tfrac{1}{r}\begin{pmatrix} m & |m|^2 - r^2 \\ 1 & -m \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}\lambda \quad \Leftrightarrow \quad \lambda = \tfrac{1}{r}(x - m), \quad |x - m|^2 = r^2.$$

It does not come as a surprise that the above matrix representations for two-spheres in S^3 are very similar to those given in terms of Vahlen matrices in $\mathbb{R}^n$, cf. [13, Section 7.1].

5. Isothermic Surfaces

In this section we come back to isothermic surfaces, which we have already discussed in Section 2 above. The class of isothermic surfaces is a very rich class of surfaces, containing so diverse surfaces like surfaces of revolution, cones and cylinders, quadrics, surfaces of constant mean curvature in any space form. A particularly interesting feature of isothermic surfaces is their very rich transformation theory (which gives rise to, for example, Bianchi's Bäcklund type transformation for surfaces of constant mean curvature), which we briefly touched upon earlier in

Section 2. The quaternionic formalism is rather well suited to discuss the transformation theory of isothermic surfaces in an efficient way. This will be the subject of the present section. More details about the material discussed in this section as well as many references to the original literature can be found in [13, Chapter 5].

5.1. The Christoffel Transformation

Recall that a surface $f : M^2 \to \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ is called isothermic if it has (locally) conformal curvature line parameters (x, y). Using quaternions we can formulate this as

i) conformality: $|f_x|^2 = |f_y|^2$ and $f_x \perp f_y \Leftrightarrow f_x f_y^{-1} + f_y f_x^{-1} = 0$ (here $(.)^{-1}$ denotes quaternionic inverse)
ii) conjugate net: $f_{xy} = a f_x + b f_y$ with suitable real functions a, b.

Now, given an isothermic surface f in terms of conformal curvature line parameters (x, y) we consider the $\mathbb{R}^3$-valued one-form

$$\mathrm{d}f^* := f_x^{-1}\mathrm{d}x - f_y^{-1}\mathrm{d}y.$$

This is a well defined differential since

$$\mathrm{d}(f_x^{-1}\mathrm{d}x - f_y^{-1}\mathrm{d}y) = (a f_y^{-1} + b f_x^{-1})(f_x f_y^{-1} + f_y f_x^{-1})\,\mathrm{d}x \wedge \mathrm{d}y = 0.$$

Moreover, the surface f^* such defined is isothermic (with the same choice of conformal curvature line parameters)

i) conformality: $f_x^{-1} f_y + f_y^{-1} f_x = \overline{f_x f_y^{-1} + f_y f_x^{-1}} = 0$, and
ii) conjugate net: $(f_x^{-1})_y = -f_x^{-1} f_{xy} f_x^{-1} = -a f_x^{-1} + b f_y^{-1}$.

Note that the surface f^* so defined and the original surface f have parallel curvature directions and induce conformally equivalent metrics. In other words, they form a Christoffel pair in the sense of Section 2 and we obtain the promised recipe for cooking up a Christoffel partner to any given isothermic surface, by integrating the above differential $\mathrm{d}f^*$.

Observe that

$$\mathrm{d}f \wedge \mathrm{d}f^* = -(f_x f_y^{-1} + f_y f_x^{-1})\,\mathrm{d}x \wedge \mathrm{d}y = 0.$$

As a consequence we find that

$$\mathrm{d}f^* = \varrho\,(\mathrm{d}n + H\,\mathrm{d}f)$$

where n is a unit normal field of f, H denotes the mean curvature of f and ϱ is an integrating factor for the one-form $\mathrm{d}n + H\mathrm{d}f$. Clearly this integrating factor ϱ can be chosen to be constant when the mean curvature $H \equiv \text{const}$ is constant so that

$$f^* = \varrho(n + H\,f) = f + \tfrac{1}{H}\,n$$

up to scaling and translation. Thus f^* becomes the parallel constant mean curvature surface of f (or, the Gauss map n of f if f is minimal, $H \equiv 0$).

5.2. The Goursat Transformation

The Christoffel transform f^* of an isothermic surface depends on the Euclidean ambient geometry of the surface whereas the notion of an isothermic surface is a conformal notion. Using this interplay between the two geometries we can define another transformation:

Definition. *Suppose $f : M^2 \to \mathbb{R}^3$ is isothermic and $A \in$ Möb(3) is a Möbius transformation. Then the isothermic surface $(Af^*)^*$ is called a **Goursat transform** of f.*

This generalizes the classical Goursat transformation for minimal surfaces in which the action of a complex orthogonal transformation on the holomorphic null curve in the Weierstrass representation of a minimal surface is equivalent to a Möbius transformation of its Gauss map. Note that, in order to reconstruct a minimal surface from its (totally umbilic) Gauss map, we have to specify further information such as a "curvature line net" for the Gauss map or a holomorphic quadratic ("Hopf") differential.

If the Möbius transformation A used in the Goursat transformation is a Euclidean motion then the resulting Goursat transform of an isothermic surface will just be similar to the original surface. If, however, we make use of an essential Möbius transformation then the resulting Goursat transform will in general be a new surface. So, when we take $A : x \mapsto (x - m)^{-1}$ we find that

$$\mathrm{d}(Af) = -(f - m)^{-1}\mathrm{d}f(f - m)^{-1} \quad \Rightarrow \quad \mathrm{d}(Af)^* = -(f - m)\mathrm{d}f^*(f - m)$$

because we must have $\mathrm{d}(Af) \wedge \mathrm{d}(Af)^* = 0$. Hence, for the differential of a Goursat transform of f, we find

$$\mathrm{d}(Af^*)^* = -(f^* - m)\,\mathrm{d}f\,(f^* - m)$$

(note how we could absorb m into a constant of the integration of f^*).

Another point of view is the following. If we define

$$\tau := \begin{pmatrix} f\,\mathrm{d}f^* & -f\,\mathrm{d}f^* f \\ \mathrm{d}f^* & -\mathrm{d}f^* f \end{pmatrix}$$

then we can extract the differential of the Christoffel transform of Af from τ by taking a suitable "off diagonal" element (in terms of a changed basis)

$$\mathrm{d}(Af)^* = (1, -m)\,\tau \begin{pmatrix} m \\ 1 \end{pmatrix}.$$

Since $d\tau = 0$ we can integrate τ into the algebra $\mathfrak{sl}(\mathbb{H}^2)$ to obtain a "master Christoffel transform" F^*, $dF^* = \tau$, of f, i.e., a matrix valued surface from which we can extract the Christoffel transforms of all possible Möbius transforms of f. Note that these surfaces are all Goursat transforms of each other.

This "retraction form" τ will play a central role in the transformation theory of isothermic surfaces.

An example of particular interest is the (local) classical Weierstrass representation for minimal surfaces – which can be interpreted as a special case of a Goursat transformation, where we consider a pair of holomorphic functions as a degenerate Christoffel pair [13, §5.3.21]. Take

$$f = -\mathrm{j}g \quad \text{and} \quad df^* = \tfrac{1}{2}dh\,\mathrm{j}$$

where g and h are two holomorphic functions (note that $df \wedge df^* = 0$) so that we obtain

$$\tau = \frac{1}{2}\begin{pmatrix} -\mathrm{j}g\,dh\mathrm{j} & \mathrm{j}g\,dhg \\ dh\mathrm{j} & -dhg \end{pmatrix}$$

and

$$(1, -\mathrm{i})\tau\begin{pmatrix}\mathrm{i}\\1\end{pmatrix} = \mathrm{Re}\,(g dh)\,\mathrm{i} + \mathrm{Re}\,\frac{(g^2-1)dh}{2}\,\mathrm{j} + \mathrm{Re}\,\frac{\mathrm{i}(g^2+1)dh}{2}\,\mathrm{k}.$$

5.3. The Darboux Transformation

Let us consider the following linear system ("*Darboux's linear system*") for a map $\hat{f}: M^2 \to \mathbb{R}^3$

$$(d + \lambda\tau)\begin{pmatrix}\hat{f}\\1\end{pmatrix} \parallel \begin{pmatrix}\hat{f}\\1\end{pmatrix}$$

where τ is the retraction form of an isothermic surface and $\lambda \in \mathbb{R}$ a real parameter. This linear system is the linearization of a Riccati type partial differential equation

$$d\hat{f} = \lambda(\hat{f} - f)\,df^*(\hat{f} - f)$$

so that the two equations are equivalent. Rewriting the Riccati equation as

$$d\hat{f} = \lambda(\hat{f} - f)^2 \cdot (\hat{f} - f)\,df^*(\hat{f} - f)^{-1}$$

we learn that f and $\hat{f}$ envelope a sphere congruence (which, in fact, is a Ribaucour sphere congruence) and that the induced metrics are conformally equivalent. Thus we obtain what we called a Darboux pair of isothermic surfaces in Section 2. We obtain the promised recipe for cooking up Darboux transforms from a given isothermic surface, by solving the above Riccati type partial differential equation or, equivalently, Darboux's linear system. Note that the Riccati type partial differential equation is completely integrable in the sense that there are no integrability conditions

$$d\{(\hat{f}-f)\,df^*(\hat{f}-f)\} = d\hat{f}\wedge df^*(\hat{f}-f) - (\hat{f}-f)\,df^*\wedge d\hat{f} = 0$$

when using the equation for $d\hat{f}$ again. Thus, apart from the real (spectral) parameter λ we have the choice of an initial condition for $\hat{f}$ when integrating the Riccati equation. This accounts for the well know four-parameter family of Darboux transforms of a given isothermic surface in S^3 (when considering surfaces in S^4 we obtain a five-parameter family of Darboux transforms for a given isothermic surface).

An interesting example for the Darboux transformation is given by pairs of parallel constant mean curvature surfaces. If we take f to have constant mean curvature $H \neq 0$ (and, therefore, to be isothermic) in $\mathbb{R}^3$ then its Christoffel transform can be taken to be $f^* = f + \frac{1}{H}n$. But then $\hat{f} := f^*$ is also a solution to the Riccati type equation above

$$d\hat{f} = df + \tfrac{1}{H}dn = H^2\,(\tfrac{1}{H}n)\,(df + \tfrac{1}{H}dn)\,(\tfrac{1}{H}n) = H^2\,(\hat{f}-f)\,df^*(\hat{f}-f).$$

It is a characteristic feature of constant mean curvature surfaces in Euclidean space to have a simultaneous Christoffel and Darboux transforms [11].

Finally, we should mention that there is another way to formulate the equations defining the Darboux transformation. Firstly, note that $\lambda \mapsto d + \lambda\tau$ is a loop of flat connections since

$$d\tau = 0 \quad \text{and} \quad [\tau \wedge \tau] = 0.$$

Hence there is a map $T^\lambda : M^2 \to \mathrm{SL}(\mathbb{H}^2)$ satisfying

$$dT^\lambda = T^\lambda\,\lambda\tau.$$

Secondly, note that $v : M^2 \to \mathbb{H}^2$ is $(d + \lambda\tau)$-parallel if and only if $T^\lambda v \equiv \mathrm{const}$. Thus, having determined T^λ for some λ, it is then a purely algebraic matter to determine any Darboux transform for that λ and its homogeneous coordinates are given by $T^{-\lambda}c$, where $c \in \mathbb{H}^2$ is a suitable vector.

5.4. The Calapso Transformation

As the above loop of maps $T^\lambda : M^2 \to \text{Möb}(3)$ is a central object in the theory of isothermic surfaces, we name it after one of its fathers (the other father being Bianchi, who called the transformation the "T-transformation" and for this reason we use "T^λ")

Definition. *The transformations $T^\lambda : M^2 \to \text{Möb}(3)$, where $dT^\lambda = T^\lambda\lambda\tau$, will be called the **Calapso transformations** of the isothermic surface f.*

As already discussed, the T^λ give rise to the Darboux transforms of an isothermic surface f via the integrated form

$$T^\lambda \begin{pmatrix} \hat{f} \\ 1 \end{pmatrix} \equiv \mathrm{const}$$

of Darboux's linear system. A second interesting application is that the above master Christoffel transform F^* of f can be obtained from the family of Calapso transformations $\lambda \to T^\lambda$ via Sym's formula

$$F^* = \left(\tfrac{\partial}{\partial\lambda}\big|_{\lambda=0} T^\lambda\right)(T^0)^{-1}$$

when we base the loop $\lambda \to T^\lambda$ at the identity (note that $\mathrm{d}T^0 = 0$ so that T^0 is a constant Möbius transformation which we can choose to be the identity). This is readily verified by taking derivatives to find

$$\mathrm{d}\left\{\left(\tfrac{\partial}{\partial\lambda}\big|_{\lambda=0} T^\lambda\right)(T^0)^{-1}\right\} = \mathrm{Ad}(T^0)\tau.$$

Note that the Calapso transformations can be considered as being attached to an isothermic surface in the conformal three-sphere as they do – even though the formula for τ contains reference to the Euclidean ambient geometry of f in the form of $\mathrm{d}f^*$ – not depend on how an isothermic surface is placed in Euclidean space in an essential way. If A is an essential Möbius transformation, say $A = \begin{pmatrix} 0 & 1 \\ 1 & -m \end{pmatrix}$, so that

$$\widetilde{f} = (f-m)^{-1} \quad \text{and} \quad \mathrm{d}\widetilde{f}^* = -(f-m)\,\mathrm{d}f^*(f-m)$$

as discussed above then

$$\widetilde{\tau} = \mathrm{Ad}(A)\tau \quad \text{and} \quad \widetilde{T}^\lambda = \mathrm{Ad}(A)T^\lambda$$

(T^λ is only determined by its differential equation up to post composition by a Möbius transformation, anyway – which we can fix by basing the loop of T^λ's at the identity, as above).

Thus, the surfaces $f^\lambda := T^\lambda f$ are defined in a conformally invariant way. These surfaces turn out to be isothermic and there are various way to prove this – the simplest way may be to see that the f^λ have retraction forms

$$\tau^\lambda = \mathrm{Ad}(T^\lambda)\tau$$

after convincing ourselves that the existence of a closed retraction form is a characterization for isothermic surfaces [13, §5.3.19].

Definition. *The surfaces* $f^\lambda = T^\lambda f$ *are called the* ***Calapso transforms*** *of an isothermic surface* f.

Our final mission is to relate these Calapso transforms to the curved flat family of Darboux pairs. For this consider the gauge transformation

$$F^\lambda := T^\lambda F^0 \quad \text{where} \quad F^0 = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

is a Euclidean frame of the isothermic surface $f : M^2 \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$. We learn that F^λ solves the system

$$dF^\lambda = F^\lambda \begin{pmatrix} 0 & \mu\, df \\ \frac{\lambda}{\mu} df^* & 0 \end{pmatrix}$$

which is, for $\lambda = \mu^2$, a quaternionic version of the curved flat system of the Christoffel pair $f = f_{\mu=0}$ and $f^* = \hat{f}_{\mu=0}$, as discussed in Section 2. Thus, $F^{\mu^2} \cong (f_\mu, \hat{f}_\mu)$ provide the curved flat associated family of Darboux pairs – but we also see, as promised in Section 2, that the $f_\mu = f^{\mu^2} = T^{\mu^2} f$ can be constructed from f without reference to the curved flat system, that is, without reference to suitable Darboux partners.

For constant mean curvature surfaces in space forms the Calapso transformation yields the Lawson correspondence [13, §5.5.29]. If κ denotes the constant curvature of the ambient space form then $H^2 + \kappa$ remains unchanged when the Calapso transforms f^λ are suitably placed in an ambient space form of curvature

$$\kappa_\lambda = \kappa + 4H\lambda - 4\lambda^2.$$

In particular, the Calapso transforms of a minimal surface in Euclidean space are "horospherical surfaces" in hyperbolic space, that is, up to scaling of the ambient metric they are constant mean curvature 1 surfaces in hyperbolic space and the Calapso transformations provide Bryant's (local) Weierstrass type representation as a special case [12].

5.5. Permutability and Discrete Isothermic Nets

There is a multitude of relations between the various transformations of isothermic surfaces, formalized in "permutability theorems" that are often named after Bianchi who seems to have been the first geometer to have recognized the importance of these permutability theorems. We already encountered one such permutability theorem in the curved flat picture of the Calapso transformation: the Calapso transformation takes Christoffel pairs to Darboux pairs and vice versa, $T^\lambda D^\lambda = CT^\lambda$ and $T^\lambda C = D^{-\lambda} T^\lambda$ [13, §5.6.10] (in these formulas, the transformation symbols C, D^λ, T^λ, etc., are to be understood symbolically, that is, as abstract transformations, in contrast to the very tangible way in which T^λ referred to a map into the Möbius group).

In fact, the entire permutability scheme $D^\lambda C = CD^\lambda$ [13, §5.6.3] for the Christoffel and Darboux transformations is taken, by the T^λ transformation, to an equivalent permutability scheme (just the parameter λ of the Darboux transformation changes sign) [13, §5.6.16].

This may indicate another permutability scheme for the Calapso transformation: $(T^\lambda)^{-1} = T^{-\lambda}$. In fact, more is true – it can be shown that the transformations T^λ satisfy a one-parameter group property, $T^{\lambda+\mu} = T^\lambda T^\mu$ [13, §5.5.9].

The most famous permutability theorem, however, is probably the Bianchi's permutability theorem for the Darboux transformation (there are many theorems of this type for Bäcklund type or Ribaucour type transformations): if $\hat{f}_1$ and $\hat{f}_2$ are two Darboux transforms of an isothermic surface, with parameters λ_1 and λ_2, respectively, then there is exactly one isothermic surface which is, at the same time, a λ_1-Darboux transform of $\hat{f}_2$ and a λ_2-Darboux transform of $\hat{f}_1$. Symbolically

$$D^{\lambda_1} D^{\lambda_2} = D^{\lambda_2} D^{\lambda_1}.$$

Moreover, this surface can be constructed from the first three in a purely algebraic way – the corresponding points of the four surfaces are concircular and have constant cross ratio $\frac{\lambda_1}{\lambda_2}$ [13, §5.6.6].

This permutability theorem can be taken as the starting point of a theory of discrete isothermic surfaces. By iterating the construction of the permutability theorem, one obtains a $\mathbb{Z}^2$-lattice of isothermic surfaces. The parameters of the Darboux transforms itself are attached to the edges of the lattice (and are equal on opposite edges of the faces of the lattice) and the cross ratios are attached to the faces. Just following the trace of a single point on the initial isothermic surface under the series of Darboux transformations, one arrives at the definition of a "discrete isothermic net" as a $\mathbb{Z}^2$-lattice in space so that the cross ratios of the faces factorize into two functions of one (discrete) variable on the edges of the lattice.

Interestingly, the isothermic nets so defined have a rather similar transformation theory as their smooth mates have. For example, one can define all the transformations discussed in this section and discovers that they satisfy the very same permutability theorems [13, Section 5.7]. More details along this line of thought can be found in [2].

6. Conformally Flat Hypersurfaces

In Section 2 we have seen how (Darboux pairs of) isothermic surfaces arise from curved flats in the space of point pairs in S^3. In this section we shall discuss another incarnation of curved flats in Möbius geometry: curved flats in the symmetric space of circles in S^4 – these are related to (one-parameter families of) conformally flat hypersurfaces.

For more details the reader is referred to [6] or [13, Chapter 2] and for most recent results see also [14].

6.1. Known Results and Open Problems

The theory we shall discuss is of local nature and we therefore understand the notion of conformal flatness in a local sense.

Definition. *A hypersurface $f : M^n \to S^{n+1}$ will be called* ***conformally flat*** *if, around each point $p \in M^n$, there is a (locally defined) function u so that $\mathrm{e}^{-2u}\langle \mathrm{d}f, \mathrm{d}f\rangle$ is flat.*

Otherwise said: the induced conformal class on the hypersurface contains locally flat representatives.

Equivalently, the hypersurface is conformally flat if there are conformal coordinates around each point, that is, the induced metric

$$\langle \mathrm{d}f, \mathrm{d}f\rangle = I = \mathrm{e}^{2u} \sum_{i=1}^{n} \mathrm{d}y_i^2$$

for a suitable coordinate system $y : M^n \supset U \to \mathbb{R}^n$ and some function $u \in C^\infty(U)$ defined on the coordinate neighbourhood.

Obviously, conformal flatness is a conformal notion so that it is best considered in a Möbius geometric setup.

Complete classifications of conformally flat hypersurfaces are known in dimensions $n = 2$ and $n \geq 4$, i.e.,

- $n = 2$. Every surface $f : M^2 \to S^3$ is conformally flat (this is Gauss' theorem on the existence of conformal coordinates) [13, Preliminaries P.4.6].
- $n \geq 4$. A hypersurface $f : M^n \to S^{n+1}$ is conformally flat if and only if it is quasi umbilic, that is, it has a principal curvature of multiplicity at least $n - 1$; or, equivalently, if and only if is is a branched channel hypersurface, that is, it is the envelope of a (branched) one-parameter family of hyperspheres [13, §1.8.17].

Conformally flat hypersurfaces of dimension $n \geq 4$ include, for example, hypersurfaces of revolution, cones, cylinders and tubes around curves. Note that all of these are foliated by $(n - 1)$-spheres. For more results in the higher dimensional case, see also [5].

Perhaps surprisingly, the case $n = 3$ turns out to be much more difficult (here the condition to be conformally flat becomes a third order condition on the metric, in contrast to the higher dimensional case, where the condition is algebraic for the curvature tensor). A classification of three-dimensional conformally flat hypersurfaces is still open. However, there are some known examples:

- Branched channel hypersurfaces are conformally flat, as in the higher dimensional case. As a consequence, it is sufficient to consider *generic* three-dimensional hypersurfaces when one seeks a classification, that is, hypersurfaces with three distinct principal curvatures.
- Surfaces of constant Gauss curvature in three-dimensional space forms can be used to construct examples of generic conformally flat hypersurfaces. Taking a surface of constant Gauss curvature in $S^3 \subset \mathbb{R}^4$, in $\mathbb{R}^3 \subset \mathbb{R}^4$, or in $H^3 \subset \mathbb{R}^4$ (where H^3 is considered as a Poincaré half space) the cone, cylinder or hypersurface of revolution in $\mathbb{R}^4$ constructed from the surface will be conformally flat and, generically, generic.
- However, not all three-dimensional hypersurfaces are conformally flat. Here, the tubes around the Veronese surface in S^4 provide rather symmetric examples as they are *not* conformally flat.

6.2. Curved Flats

In order to discover the curved flats arising from conformally flat hypersurfaces we go back to the classical model of Möbius differential geometry that we discussed in Section 1. Thus let $f : M^3 \to S^4$ be conformally flat and consider

$$S^4 = \{y \in \mathbb{R}^6_1;\, |y|^2 = 0, y_0 = 1\} \subset L^5 \subset \mathbb{R}^6_1$$

as a hyperplane section of the light cone. As f is conformally flat, there is a (locally defined) function u so that

$$\tilde{f} := \mathrm{e}^{-u} f : M^3 \supset U \to L^5$$

is a flat lift of the conformally flat hypersurface, that is, the induced metric

$$\tilde{I} = \langle \mathrm{d}\tilde{f}, \mathrm{d}\tilde{f}\rangle = \mathrm{e}^{-2u}\langle \mathrm{d}f, \mathrm{d}f\rangle$$

of $\tilde{f}$ is flat or, otherwise said, the vector bundle $p \mapsto \mathrm{d}_p\tilde{f}(T_pM)$ is a flat subbundle of the trivial $\mathbb{R}^6_1$-bundle over $U \subset M^3$. In this situation, one then proves easily that the normal bundle $p \mapsto (\mathrm{d}_p\tilde{f}(T_pM))^\perp$ of $\tilde{f}$ as an immersion into $\mathbb{R}^6_1$ is also flat [13, §2.1.4]. As a consequence, we find that the Gauss map

$$\gamma : M^3 \to \frac{O_1(6)}{O(3) \times O_1(3)}, \qquad p \mapsto \gamma(p) = \mathrm{d}_p\tilde{f}(T_pM)$$

is a curved flat in the Grassmannian of spacelike three-planes in $\mathbb{R}^6_1$ [13, §2.2.7]. Geometrically, this Grassmannian can be interpreted as the space of circles in S^4, much like the Grassmannian of spacelike two-planes (elliptic sphere pencils) is the space of two-spheres in S^4. With this geometry in mind we learn two things:

1. the flatness of the vector bundle $\gamma^{\perp}$ says that γ defines a *cyclic system*, that is, the three-parameter family of circles given by the map γ has a one-parameter family of orthogonal hypersurfaces in S^4 [13, §2.5.3]
2. the flatness of the vector bundle γ says that all the orthogonal hypersurfaces of this cyclic system are conformally flat [13, §2.2.13].

Thus, each such γ, obtained from a flat lift of our conformally flat hypersurface f in S^4, gives rise to a one-parameter family of conformally flat hypersurfaces that contains the original hypersurface f. Note that a flat lift of a conformally flat hypersurface is not unique and, therefore, this one-parameter family of conformally flat hypersurfaces attached to a given one is not unique.

Conversely, any curved flat in the space of circles yields, by the above arguments, a one-parameter family of conformally flat hypersurfaces that can be extracted from γ as parallel isotropic sections of the vector subbundle $\gamma^{\perp}$ of the trivial $\mathbb{R}^6_1$-bundle over M^3. Note that there will be no distinguished conformally flat hypersurface to be constructed from the curved flat γ.

On the other hand, curved flats come in one-parameter families. Hence, associated in a non-unique way to a given conformally flat hypersurface, we obtain a one-parameter family of one-parameter families of conformally flat hypersurfaces.

6.3. Guichard Nets

Curved flats come, in general, with special coordinate systems, associated to the roots of the symmetric space. In the case of conformally flat hypersurfaces, these are obtained by integrating the *conformal fundamental forms* of the hypersurface.

Definition. *Let $f : M^3 \to S^4$ be a hypersurface with principal orthonormal coframe $(\omega_1, \omega_2, \omega_3)$ and principal curvatures k_1, k_2 and k_3, that is, its first and second fundamental forms are given by*

$$I = \sum_{i=1}^{3} \omega_i^3 \quad \textit{and} \quad II = \sum_{i=1}^{3} k_i \omega_i^3.$$

Then the ***conformal fundamental forms*** *of f are:*

$$\begin{aligned}\gamma_1 &:= \sqrt{k_3 - k_1}\sqrt{k_1 - k_2}\,\omega_1\\ \gamma_2 &:= \sqrt{k_1 - k_2}\sqrt{k_2 - k_3}\,\omega_2\\ \gamma_3 &:= \sqrt{k_2 - k_3}\sqrt{k_3 - k_1}\,\omega_3.\end{aligned}$$

Note that only one of these forms is real, whereas the other two take values in $i\mathbb{R}$.

The conformal fundamental forms of a hypersurface in S^4 are invariant under conformal changes of the ambient metric, as one readily verifies using the transformation formulae listed at the beginning of this text (if one does not have a better argument like, for example, their relation to a conformally invariant metric attached to the hypersurface [13, §2.3.4]).

Conformal flatness of a hypersurface can now be characterized using these conformal fundamental forms, i.e., f is conformally flat if and only if these forms are closed, $\mathrm{d}\gamma_i = 0$ [13, §2.3.3].

As a consequence, a conformally flat hypersurface comes with a special coordinate system

$$x : M^3 \supset \mathbb{R}^3_2, \quad \mathrm{d}x_i = \gamma_i.$$

Writing $\omega_i = l_i \mathrm{d}x_i$ with the "Lamé functions" l_i (two of which are imaginary), we obtain

$$I = \sum_{i=1}^{3} l_i^2 \mathrm{d}x_i^2, \quad II = \sum_{i=1}^{3} k_i l_i^2 \mathrm{d}x_i^2, \quad \text{where} \quad \sum_{i=1}^{3} l_i^2 = 0$$

that is, the metric satisfies a trace zero condition, similar to the condition obtained on the conformal curvature line coordinates of an isothermic surface when writing those in complex form. This is the reason why Guichard considered conformally flat hypersurfaces as three-dimensional analogues of isothermic surfaces [10] and our reason for the following attribution:

Definition. *A* ***Guichard net*** *on a Riemannian manifold* (M^3, I) *is a (local) coordinate system* $x : M^3 \supset U \to \mathbb{R}^3_2$ *so that*

$$I = \sum_{i=1}^{3} l_i^2 \mathrm{d}x_i^2 \quad \textit{with} \quad \sum_{i=1}^{3} l_i^2 = 0.$$

Note that the notion of a Guichard net is a conformal notion and that the Guichard net condition is not affected by conformal changes of the metric I on M^3. We will often think of a Guichard net as a (parametrized) triply orthogonal system in M^3.

With this definition we can reformulate the above result as follows: every conformally flat hypersurface carries a principal Guichard net. On the other hand, as our hypersurface is conformally flat, there are (locally) conformal coordinates $y : M^3 \supset U \to \mathbb{R}^3$, which we can use to map the Guichard net on the hypersurface into $\mathbb{R}^3$

$$x \circ y^{-1} : \mathbb{R}^3 \supset y(U) \to \mathbb{R}^3_2.$$

Thus, a conformally flat hypersurface gives rise to a Guichard net in $\mathbb{R}^3$ which, because of Liouville's theorem and the conformal invariance of the Guichard net in M^3, is unique up to Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$.

It is lengthy (and anything but straightforward) to prove the converse: any Guichard net in $\mathbb{R}^3$ gives rise to a conformally flat hypersurface, which is unique up to Möbius transformation [13, §2.3.12].

In this way, we obtain a one-to-one correspondence between (Möbius equivalence classes of) conformally flat hypersurfaces and (Möbius equivalence classes of)

Guichard nets in $\mathbb{R}^3$. As a consequence, the classification of generic conformally flat hypersurfaces in S^4 is equivalent to a classification of Guichard nets in $\mathbb{R}^3$.

As an example, we consider the aforementioned generic conformally flat hypersurfaces constructed as cones, cylinders or hypersurfaces of revolution in $\mathbb{R}^4$ over surfaces of constant Gauss curvature in S^3, $\mathbb{R}^3$ or H^3. Respectively, the corresponding Guichard nets contain one family of totally umbilic surfaces which form part of a sphere pencil – so that the orthogonal trajectories are circles and the Guichard net is a very special case of a cyclic system (with totally umbilic orthogonal surfaces) in $\mathbb{R}^3$ [13, §2.4.12].

6.4. Conformally Flat Hypersurfaces with Cyclic Guichard Net

Motivated by the previous example we may attempt to classify the cyclic Guichard nets in $\mathbb{R}^3$, that is, those Guichard nets that form a cyclic system. As both, the notion of a Guichard net as well as the notion of a cyclic system are conformal notions we may as well classify the cyclic Guichard nets in the conformal three-sphere S^3 using Möbius geometric technology.

In doing this we encounter another symmetry breaking phenomenon: any cyclic Guichard net in the conformal three-sphere naturally "lives" in a quadric of constant curvature, where the circles of the system become straight lines and their orthogonal surfaces become linear Weingarten surfaces in parallel hyperspheres. That is, any cyclic Guichard net in the conformal three-sphere is the normal line congruence of a linear Weingarten surface in a space form subgeometry of Möbius geometry [13, §2.6.2].

The converse is true as well. Given a linear Weingarten surface, which does not have constant mean or principal curvatures, in a space form, its normal line congruence can be parametrized so that the resulting triply orthogonal system is a Guichard net [13, §2.6.5].

Stereographic projection into Euclidean space $\mathbb{R}^3$ then provides a cyclic Guichard net in $\mathbb{R}^3$.

The obvious question before the house is now: how do the corresponding conformally flat hypersurfaces look like?

The corresponding classification of conformally flat hypersurfaces with cyclic Guichard net is a very recent result [14]. Any conformally flat hypersurface with cyclic principal Guichard net naturally "lives" in a four-dimensional space form, where the orthogonal surfaces of the cyclic system are extrinsically (!) linear Weingarten surfaces in a family of parallel hyperspheres. Conversely, starting from a suitable linear Weingarten surface in a hyperplane of a space form, one can construct a unique conformally flat hypersurfaces with cyclic Guichard net.

Here the previously mentioned symmetry breaking phenomenon incarnates in a different flavour. In contrast to the one discussed in Section 2.3 and the one mentioned above, the fixed object becomes now not a vector specifying a space form but the pencil of parallel hyperspheres given by a fixed two-plane in Minkowski space.

The construction of the conformally flat hypersurface from a linear Weingarten surface is not unlike the construction of the generic conformally flat cones, cylinders and hypersurfaces of revolution mentioned earlier, and instead of pushing the initial surface out into the fourth dimension orthogonally, in this more general construction the surface is moved out of its hyperplane by a sort of screw motion in its normal plane, parametrized by elliptic functions.

Describing a conformally flat hypersurface in terms of a real function of three parameters, the hypersurface with cyclic Guichard net are the most general conformally flat hypersurfaces with a separation of variables, that is, where the describing real function splits into two functions of two and one variables, respectively. It remains open to investigate the generic case further.

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