Sixth International Conference on
Geometry, Integrability and Quantization
June 3–10, 2004, Varna, Bulgaria
Ivaïlo M. Mladenov and Allen C. Hirshfeld, Editors
SOFTEX, Sofia 2005, pp 11–61

ISOCHRONOUS SYSTEMS

FRANCESCO CALOGERO

Dipartimento di Fisica, Università degli Studi di Roma "La Sapienza"
Istituto Nazionale di Fisica Nucleare, Sezione di Roma
Piazzale Aldo Moro, 5, 00185 Roma, Italy

Abstract. Recently – via a simple trick, amounting essentially to a change of independent, and possibly as well of dependent, variables – the possibility has been noted to modify a quite general evolution system so that the modified system possess a lot of *completely periodic*, indeed *isochronous*, solutions. Generally these *isochronous* solutions emerge out of an *open* domain of initial data having full dimensionality in the space of initial data. And many of the isochronous systems obtained in this manner seem rather interesting. In this paper these developments are reviewed, mainly in the context of dynamical systems (systems of ODEs – in particular, systems interpretable as many-body problems), and some specific examples are discussed in detail, including an analysis of the transition (to motions with higher periods, or *aperiodic*, or perhaps *chaotic*) occurring when the initial data get outside of the region producing *isochronous* motions. The applicability of this approach in the context of nonlinear evolution PDEs is also outlined.

Introduction

This review paper covers the material presented at the International Conference on Geometry, Integrability and Quantization held in Varna (Bulgaria) in June 2004, via four lectures organized as follows: 1. Overview: "isochronous systems are not rare"; 2. The "goldfish": theory and simulations; 3. Novel technique to identify solvable dynamical systems and a solvable extension of the goldfish many-body problem. 4. Isochronous PDEs.

Lecture 1

1. Overview

An **isochronous system** is characterized by the property to possess an *open* domain having full dimensionality in phase space such that *all* the motions evolving from a set of initial data in it are *completely periodic* with the same fixed period. The natural measure of this open domain might, or it might not, be *infinite* when the measure of the entire phase space is itself *infinite*: for instance, if the entire phase space is the two-dimensional Euclidian plane, such a domain might be the exterior, or the interior, of a circle of finite radius.

It is justified to call such systems **superintegrable**, or perhaps **partially superintegrable** inasmuch as the property of *isochronicity* of all their motions holds only in a subregion of the entire phase space. This terminology is justified by the observation that, roughly speaking, all confined motions of a *superintegrable* system – in which all but one of the degrees of freedom are constrained by the existence of the *maximal* possible number of constants of motion – are *completely periodic*, although not necessarily all with a fixed period – entailing that *isochronicity* entails *superintegrability*, while the converse is not the case.

For instance a well-known isochronous system is the one-dimensional N-body problem characterized by the (normal) Hamiltonian

$$H(\underline{z}, \underline{p}) = \frac{1}{2} \sum_{n=1}^{N} (p_n^2 + \omega^2 z_n^2) + \frac{1}{4} \sum_{m,n=1; m \neq n}^{N} \frac{g^2}{(z_n - z_m)^2}$$

and correspondingly by the Newtonian equations of motion

$$\ddot{z}_n + \omega^2 z_n = \sum_{m=1, m \neq n}^{N} \frac{g^2}{(z_n - z_m)^3}.$$

Here and always below ω is a positive constant, $\omega > 0$, and the rest of the notation is, we trust, self-evident; in particular superimposed dots denote differentiations with respect to the *real* independent variable t (“time”). Indeed, in the *real* domain, *all* the solutions of these equations of motion are *isochronous*, completely periodic,

$$\underline{z}(t + T) = \underline{z}(t)$$

with the fixed period

$$T = \frac{2\pi}{\omega}. \tag{1}$$

This is not quite true in the *complex* domain, namely if we consider the dependent variables z_n to be *complex* rather than *real* (in which case we might as well allow the “coupling constant” g to be complex, while we always consider the constant ω to be real, indeed, without loss of generality, positive). Then *all* motions, which take of course place in the *complex* plane, are again *completely periodic*, but the period may be an integer multiple of T. Indeed also in this case the particle configuration does repeat itself with period T, but the individual particles might exchange their roles, entailing that the period of the motion become an integer multiple of T (this cannot happen in the *real* case, when the motion takes place on the real line and the ordering of the particles cannot change throughout the motion due to the singular character of the repulsive two-body interaction).

Hence the many-body problem characterized by this Hamiltonian is *isochronous*, both in the *real* and in the *complex* contexts; and the open domain of initial data for which it possesses the isochronicity property coincides in this case with the entire phase space, with the only exclusion of a lower-dimensional set of initial data yielding motions leading to particle collisions (this can of course only happen in the *complex* case). And it is of course well known that this N-body problem is *superintegrable* (see for instance [14]).

Another *isochronous* system (albeit only if it is considered in the *complex* case) is the more general N-body problem characterized by the (normal) Hamiltonian

$$H(\underline{z}, \underline{p}) = \frac{1}{2} \sum_{n=1}^{N} (p_n^2 + \omega^2 z_n^2) + \frac{1}{4} \sum_{m,n=1; m \neq n}^{N} \frac{g_{nm}^2}{(z_n - z_m)^2} \tag{2a}$$

and correspondingly by the Newtonian equations of motion

$$\ddot{z}_n + \omega^2 z_n = \sum_{m=1, m \neq n}^{N} \frac{g_{nm}^2}{(z_n - z_m)^3}. \tag{2b}$$

This N-body problem differs from the previous one because the coupling constants g_{nm} are now permitted to be *arbitrarily different* (except for the obvious symmetry restriction $g_{nm}^2 = g_{mn}^2$, see (2a)). Indeed it has recently been shown [14, 15, 32] that these Newtonian equations of motion yield a *completely periodic* motion with period (1), provided the initial data fall in an appropriate (open) domain, which however generally does not include only real data.

1.1. Isochronous Systems are not Rare

To convince the (possibly skeptical) reader of the validity of the assertion that constitutes the title of this section, we now show that, given a largely arbitrary dynamical system, it is possible to introduce a deformed version of it featuring a *real* constant ω, that has the following properties: for $\omega = 0$, it coincides with the original, undeformed system. For $\omega > 0$, it possesses an *open* region having full dimensionality in its phase space such that *all* solutions evolving from an initial datum in it are *completely periodic* with a period $\widetilde{T}$ which is a finite integer multiple of the basic period T, see (1), namely the deformed system is *isochronous* [18, 25, 27].

Let us indeed consider a quite general dynamical system, which we write as follows

$$\zeta' = F(\zeta; \tau). \tag{3}$$

Here $\zeta \equiv \zeta(\tau)$ is the dependent variable, which might be a scalar, a vector, a tensor, a matrix, you name it. The independent variable is τ, and the main limitation on the dynamical system (3) is that it be permissible to treat this variable as *complex*. This requires that the derivative with respect to this *complex* variable τ that appears in the left-hand side of the evolution equation (3) make sense, namely that this dynamical system be *analytic*, entailing that the dependent variable ζ be an *analytic* function of the complex variable τ (but this does not require $\zeta(\tau)$ to be a holomorphic nor a meromorphic function of τ and $\zeta(\tau)$ might feature all sorts of singularities, including branch points, in the complex τ-plane, indeed this will generally happen since we generally assume the evolution equation (3) to be *nonlinear*). The quantity F in the right-hand side of (3) – which has of course the same scalar, vector, matrix ... character as ζ – might depend (arbitrarily but analytically) on ζ as well as on τ. (The possibility that this dynamical system might also feature other, "spacelike", independent variables – for instance, be a system of PDEs rather than ODEs – is treated in the last lecture.)

In spite of the generality of this dynamical system, (3), there generally holds a result (like "Theorem of existence, uniqueness and analyticity") that characterizes the solution $\zeta(\tau)$ of its initial-value problem determined by the assignment

$$\zeta(0) = \zeta_0.$$

Here, for notational simplicity, we assign the initial datum ζ_0 at $\tau = 0$, and we assume of course that the right-hand side of (3) is not singular for $\tau = 0$ and $\zeta = \zeta_0$. The relevant result (see for instance Section 12.21 of [35]) guarantees, not only for the initial datum ζ_0, but for a (sufficiently small but open) set of initial data in its neighborhood, the existence of a circular disk in the complex τ-plane, centered at $\tau = 0$ (where the initial data are assigned) and having a *nonvanishing* radius ρ, such that the solutions $\zeta(\tau)$ corresponding to these initial data are *holomorphic* in it, namely for $|\tau| < \rho$ (and note that if $\zeta(\tau)$ is a multicomponent object, the property to be *holomorphic* is featured by each and everyone of its components).

Let us now introduce the following changes of dependent and independent variables [10]

$$z(t) = \exp(\mathrm{i}\lambda\omega t)\zeta(\tau) \tag{4a}$$

$$\tau \equiv \tau(t) = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}. \tag{4b}$$

This transformation is called "the trick". The essential part of it is the change of *independent* variable (4b): and let us re-emphasize that, here and hereafter, the new *independent* variable t is considered as the *real*, "**physical time**" variable. Note that (4b) entails

$$\tau(0) = 0, \qquad \dot{\tau}(0) = 1$$

and, most importantly, that $\tau(t)$ is a periodic function of t with period T, see (1). More specifically, as the time t increases from zero onwards, the complex variable τ travels counterclockwise round and round on the circle C the diameter of which, of length $\frac{2}{\omega}$, lies on the imaginary axis in the complex τ-plane, with one extreme at the origin, $\tau = 0$, and the other at the point $\tau = \frac{2\mathrm{i}}{\omega}$, making a full circle in the time interval T. As for the prefactor $\exp(\mathrm{i}\lambda\omega t)$ that multiplies $\zeta(\tau)$ in the right-hand side of (4a), its purpose is to allow, via an appropriate choice of the parameter λ, the deformed system, see below, to have a neater look; however this choice is hereafter restricted by the condition that λ be *real* and *rational*, say

$$\lambda = \frac{p}{q}$$

with p and q two coprime integers and $q > 0$. This restriction is essential to guarantee, via (4), that if $\zeta(\tau)$ is *holomorphic* in τ in the (closed) disk encircled by the circle C, then $z(t)$ is *completely periodic* (namely, each and everyone of its components is *periodic*) with the period

$$\tilde{T} = qT. \tag{5}$$

The deformed dynamical system is the one that obtains from (3) via the trick (4). It clearly reads as follows

$$\dot{z} = \mathrm{i}\lambda\omega z + \exp[\mathrm{i}(\lambda+1)\omega t]F\left(\exp(-\mathrm{i}\lambda\omega t)z; \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}\right). \tag{6}$$

And it is plain, on the basis of the arguments we just gave, that this system is *isochronous*, a sufficient condition for the *complete periodicity* with period $\tilde{T}$, see (5), of its solutions being provided by the inequality

$$\frac{2}{\omega} < \rho$$

which can clearly be satisfied by initial data situated inside an *open* domain of such data, at least provided ω is sufficiently large (actually, in all the examples reported below no restriction on the value of ω is required, namely such an open domain exists for any arbitrary value of $\omega > 0$).

The *isochronicity* of this deformed dynamical system, (6), is a rather obvious consequence of the way it has been manufactured. But, trivial as the emergence of this property might indeed be, the remarkable fact is that the class of *isochronous* dynamical systems that can be manufactured in this manner is not only vast, but it does include many instances which this author considers quite interesting (although the final assessment on the validity of such a value judgement must be left to the reader). Some of these examples are reported below, and for others the interested reader will be referred to the relevant literature. Note however that in this review paper as a rule we limit our presentation to exhibiting *isochronous* dynamical systems obtained via this approach, without providing any detail of their derivation nor any characterization of the (open) region of their phase space where they behave as such.

1.2. Examples

In this section we report tersely several examples of *isochronous* dynamical systems; in each case we also provide the reference where more information can be found. Except when explicitly otherwise mentioned, these dynamical systems are to be considered in the *complex* context.

The first example [18] we report is a Hamiltonian N-body problem which is a generalization of (2). It is characterized by the (normal) Hamiltonian

$$\begin{aligned} H(\underline{z}, \underline{p}) &= \frac{1}{2}\sum_{n=1}^{N}(p_n^2 + \omega^2 z_n^2) + \frac{1}{4}\sum_{m,n=1;m\neq n}^{N} \frac{g_{nm}^2}{(z_n - z_m)^2} \\ &+ \frac{1}{4}\sum_{m,n=1;m\neq n}^{N}\sum_{k=1}^{K} \frac{f_{nm}^{(k)}}{(1+k)(z_n - z_m)^{2(1+k)}} \end{aligned} \tag{7a}$$

and correspondingly by the Newtonian equations of motion

$$\ddot{z}_n + \omega^2 z_n = \sum_{m=1,m\neq n}^{N} \frac{g_{nm}^2}{(z_n - z_m)^3} + \sum_{m=1,m\neq n}^{N}\sum_{k=1}^{K} \frac{f_{nm}^{(k)}}{(z_n - z_m)^{3+2k}}. \tag{7b}$$

The next example [14,24,29] we report is a *real* N-body problem in the horizontal plane, characterized by the Newtonian equations of motions

$$\begin{aligned} &\ddot{\vec{r}}_n = \omega \hat{k} \wedge \dot{\vec{r}}_n \\ &+ 2\sum_{m=1,m\neq n}^{N} (\alpha_{nm} + \beta_{nm}\hat{k}\wedge) \frac{\left[\dot{\vec{r}}_n\left(\dot{\vec{r}}_m \cdot \vec{r}_{nm}\right) + \dot{\vec{r}}_m\left(\dot{\vec{r}}_n \cdot \vec{r}_{nm}\right) - \vec{r}_{nm}\left(\dot{\vec{r}}_n \cdot \dot{\vec{r}}_m\right)\right]}{r_{nm}^2}. \end{aligned} \tag{8}$$

Here $\vec{r}_n \equiv (x_n, y_n, 0)$ is a *real* two-vector in the horizontal plane, $\hat{k} \equiv (0, 0, 1)$ is the unit vector orthogonal to the horizontal plane, the symbol $\wedge$ denotes the (three-dimensional) vector product so that $\hat{k} \wedge \vec{r}_n = (-y_n, x_n, 0)$, and we use the short-hand notation $\vec{r}_{nm} = \vec{r}_n - \vec{r}_m$ entailing $r_{nm}^2 = r_n^2 + r_m^2 - 2\vec{r}_n \cdot \vec{r}_m$. Note that these equations are *translation-* and *rotation-invariant*, and they are *Hamiltonian*, although the corresponding Hamiltonian function is *not* of normal type (kinetic plus potential energy).

The $N(N-1)$ "coupling constants" α_{nm} and β_{nm} are of course *real*, but they are otherwise *arbitrary* except for the symmetry restrictions $\alpha_{nm} = \alpha_{mn}$, $\beta_{nm} = \beta_{mn}$ which are required in order that this system be *Hamiltonian*. If *all* these coupling constants *vanish*, this dynamical system has a clear physical interpretation: it describes the motion of N equal, electrically charged, point particles, moving in the horizontal plane under the effect of a magnetic field orthogonal to that plane (in the approximation in which the electrostatic interparticle interaction is neglected). In that case each particle moves on a circle, the center and radius of which depend on the initial data, while the time taken to go round it is, in all cases, T, see (1). If the $\frac{1}{2}N(N-1)$ coupling constants β_{nm} vanish, $\beta_{nm}=0$, and the $\frac{1}{2}N(N-1)$ coupling constants α_{nm} all equal unity, $\alpha_{nm}=1$, the system is a well-known *integrable* (indeed *solvable*) system (see for instance [14]). This is as well the case if the $\frac{1}{2}N(N-1)$ coupling constants β_{nm} vanish, $\beta_{nm}=0$, and the $\frac{1}{2}N(N-1)$ coupling constants α_{nm} equal minus one half, and only act among "nearest neighbors", $\alpha_{nm}=-\frac{1}{2}(\delta_{m,n+1}+\delta_{m,n-1})$ [19].

A more detailed discussion of this model, (8) – including its behavior for initial data *outside* of the region yielding *isochronous* motions – is made in Section 2.

Several interesting classes of *isochronous* dynamical systems are reported in [23]. We only mention here a remarkably general example, characterized by the Newtonian equations of motion

$$\ddot{\underline{z}}+\mathrm{i}\omega\dot{\underline{z}}=\sum_{k=1}^{K}\underline{f}^{(-k)}(\underline{z},\dot{\underline{z}}+\mathrm{i}\omega\underline{z})$$

where $\underline{z}\equiv(z_1,\ldots,z_N)$ is the N-vector whose *complex* components $z_n\equiv z_n(t)$ are the dependent variables, while the "forces" $\underline{f}^{(-k)}(\underline{z},\underline{\tilde{z}})$ are required to be *analytic* in all their arguments and to satisfy the scaling properties

$$\underline{f}^{(-k)}(\alpha\underline{z},\underline{\tilde{z}})=\alpha^{-k}\underline{f}^{(-k)}(\underline{z},\underline{\tilde{z}})$$

which however entail no restriction on the velocity-dependence of these forces, namely on the dependence of $\underline{f}^{(-k)}(\underline{z},\underline{\tilde{z}})$ on the (components of the) second, $\underline{\tilde{z}}$, of its two N-vector arguments.

The next example [16] we report is characterized by the Newtonian equations of motion

$$\ddot{\vec{r}}_n+\mathrm{i}\omega\dot{\vec{r}}_n+2\omega^2\vec{r}_n=\sum_{m=1,m\neq n}^{N}\frac{M_m\vec{r}_{mn}}{r_{mn}^3}$$

where we assume the N dependent variables $\vec{r}_n\equiv\vec{r}_n(t)$ to be three-vectors (although the property of *isochronicity* of this deformed system would hold no less if these were S-vectors, with S an arbitrary positive integer) and we use the shorthand notation $\vec{r}_{mn}\equiv\vec{r}_m-\vec{r}_n$. This system is (perhaps) remarkable inasmuch as it represents a (*complex*) deformation of the classical N-body gravitational problem, to which it clearly reduces for $\omega=0$.

The next example [27] we report is characterized by the following (first-order) equations of motion of oscillator type

$$\begin{aligned} \dot{x}_n - \mathrm{i} p_n \omega x_n &= f_n(\underline{x}, \underline{y}), \qquad n = 1, \dots, N \\ \dot{y}_m + \mathrm{i} q_m \omega y_m &= g_m(\underline{x}, \underline{y}), \qquad m = 1, \dots, M. \end{aligned} \tag{9}$$

Here the N-vector $\underline{x}$, respectively the M-vector $\underline{y}$, have as components the $N+M$ *complex* dependent variables $x_n \equiv x_n(t)$, $y_m \equiv y_m(t)$; the $N+M$ parameters p_n, q_m are all *nonnegative integers* (or they could be *nonnegative rational* numbers), and the $N+M$ *complex* functions f_n, g_m are restricted by the following conditions (which are sufficient to guarantee the *isochronicity* of this dynamical system):

1) $f_n(\underline{x}, \underline{y})$ and $g_m(\underline{x}, \underline{y})$ are *holomorphic* at $\underline{x} = 0$, $\underline{y} = 0$

2) $\lim_{\varepsilon \to 0} \varepsilon^{-1} \underline{f}(\varepsilon \underline{x}, \varepsilon \underline{y}) = \underline{0}$, $\lim_{\varepsilon \to 0} \varepsilon^{-1} \underline{g}(\varepsilon \underline{x}, \varepsilon \underline{y}) = \underline{0}$

3) $\underline{f}(\underline{x}, \underline{y})$ and $\underline{g}(\underline{x}, \underline{y})$ are *polynomial* in the y_m

4a) $\lim_{\varepsilon \to 0} \varepsilon^{-1-p_n} f_n(\varepsilon^{\underline{p}} \underline{x}, \varepsilon^{-\underline{q}} \underline{y}) =$ nondivergent, $n = 1, \dots, N$

4b) $\lim_{\varepsilon \to 0} \varepsilon^{-1+q_m} g_m(\varepsilon^{\underline{p}} \underline{x}, \varepsilon^{-\underline{q}} \underline{y}) =$ nondivergent, $m = 1, \dots, M$.

In the conditions (4a) and (4b) the notation $\varepsilon^{\underline{p}} \underline{x}$ indicates of course the N-vector of components $\varepsilon^{p_n} x_n$, and likewise $\varepsilon^{-\underline{q}} \underline{y}$ is the M-vector of components $\varepsilon^{-q_m} y_m$. Note that this dynamical system, (9), includes the *Hamiltonian* case characterized by the restrictions

$$N = M, \quad p_n = q_n, \quad f_n(\underline{x}, \underline{y}) = \frac{\partial V(\underline{x}, \underline{y})}{\partial y_n}, \quad g_n(\underline{x}, \underline{y}) = -\frac{\partial V(\underline{x}, \underline{y})}{\partial x_n}$$

which imply that the equations of motion (9) are just the Hamiltonian equations entailed by the Hamiltonian function

$$H(\underline{x}, \underline{y}) = \mathrm{i}\omega \sum_{n=1}^{N} p_n x_n y_n + V(\underline{x}, \underline{y})$$

isochronicity being now guaranteed by the following conditions on the function $V(\underline{x}, \underline{y})$:

1) $V(\underline{x}, \underline{y})$ is *holomorphic* at $\underline{x} = 0$, $\underline{y} = 0$

2) $\lim_{\varepsilon \to 0} \varepsilon^{-2} V(\varepsilon \underline{x}, \varepsilon \underline{y}) = \underline{0}$

3) $V(\underline{x}, \underline{y})$ is *polynomial* in the y_n

4) $\lim_{\varepsilon \to 0} \varepsilon^{-1} V(\varepsilon^{\underline{p}} \underline{x}, \varepsilon^{-\underline{p}} \underline{y}) =$ nondivergent.

Next, let us exhibit two classes of *isochronous* Hamiltonian systems [20]. The first is defined by the Hamiltonian

$$H(\underline{p},\underline{z}) = \sum_{n=1}^{N}\left[\cosh(p_n)\sqrt{c_n + z_n^2}\right] + U(\underline{z}) \tag{10a}$$

$$U(\underline{z}) = \sum_{k=1}^{K} U^{(-2k)}(\underline{z}) \tag{10b}$$

$$U^{(-2k)}(\alpha\underline{z}) = \alpha^{-2k}U^{(-2k)}(\underline{z}) \tag{10c}$$

where the N (possibly *complex*) constants c_n are *arbitrary* and the dependence of the functions $U^{(-2k)}(\underline{z})$ on the (N components of the) N-vector $\underline{z}$ is required to be *analytic* and to satisfy the scaling property (10c), but is otherwise as well *arbitrary*. The corresponding Newtonian equations of motion read of course as follows

$$\ddot{z}_n + \omega^2 z_n = -\sqrt{\dot{z}_n^2 + \omega^2(c_n + z_n^2)}\,\frac{\partial U(\underline{z})}{\partial z_n}.$$

The second of these models is defined by the Hamiltonian

$$H(\underline{p},\underline{z}) = \sum_{n=1}^{N}[\varphi(p_n)z_n] + W(\underline{z})$$

where the function $\varphi(p)$ is required to satisfy the ODE

$$\varphi(p) = [\varphi'(p) + \mathrm{i}(1+\mu)\omega]^{\frac{1+\mu}{2}}[\varphi'(p) - \mathrm{i}(1-\mu)\omega]^{\frac{1-\mu}{2}}$$

with μ a *real rational* number different from minus unity, $\mu \neq -1$, and satisfying either one (or both) of the following two inequalities

$$\mu > -3 \quad \text{or} \quad \mu < -(1+2K)$$

where K is an *arbitrary nonnegative* integer that, together with μ, characterizes the function $W(\underline{z})$ via the formula

$$W(\underline{z}) = W^{(0)}(\underline{z}) + \sum_{k=1}^{K} W^{(\mu - \frac{2k}{1+\mu})}(\underline{z})$$

with $W^{(a)}(\underline{z})$ a function the dependence of which on the (N components of the) N-vector $\underline{z}$ is *analytic* and only required to satisfy the scaling property

$$W^{(a)}(\alpha\underline{z}) = \alpha^a W^{(a)}(\underline{z}).$$

The corresponding Newtonian equations of motion read of course as follows

$$\ddot{z}_n + 2\mathrm{i}\mu\omega\dot{z}_n + (1-\mu^2)\omega^2 z_n$$
$$= [\dot{z}_n + \mathrm{i}(1+\mu)\omega z_n]^{\frac{1-\mu}{2}}[\dot{z}_n - \mathrm{i}(1-\mu)\omega z_n]^{\frac{1+\mu}{2}}\,\frac{\partial W(\underline{z})}{\partial z_n}.$$

The last two examples we report can be characterized as assemblies of *nonlinear harmonic oscillators* [31], inasmuch as these two dynamical systems (which are actually special cases of more general systems [31]) have the remarkable property that their *generic* solutions (namely, *all* their solutions, except for a lower-dimensional set of *singular* solutions in which one or more of the "moving particles" shoot off to infinity at a finite time) are *completely periodic* with the fixed period T, see (1). Their Newtonian equations of motion read

$$\ddot{\underline{z}}_{nm} - 3\mathrm{i}\omega\dot{\underline{z}}_{nm} - 2\omega^2\underline{z}_{nm} = c\sum_{\nu=1}^{N}\sum_{\mu=1}^{M}\underline{z}_{n\mu}(\underline{z}_{\nu\mu}\cdot\underline{z}_{\nu m})$$

$$\ddot{\underline{z}}_{nm} - 3\mathrm{i}\omega\dot{\underline{z}}_{nm} - 2\omega^2\underline{z}_{nm} = c\sum_{\nu=1}^{N}\sum_{\mu=1}^{M}\underline{z}_{\nu\mu}(\underline{z}_{\nu\mu}\cdot\underline{z}_{nm}).$$

These are two (different) systems of NM Newtonian equations of motion satisfied by the NM *complex* S-vectors $\underline{z}_{nm}$ (with S an *arbitrary* positive integer); hence here the index n runs from 1 to N, and the index m runs from 1 to M, with N and M two *arbitrary* positive integers, while c is of course an *arbitrary* complex constant (which might actually be rescaled away). The dot sandwiched between two S-vectors denotes the standard (Euclidian) scalar product, entailing the rotation-invariant character, in S-dimensional space, of these equations of motion. Since these systems only feature *linear* and *cubic* forces, these models are remarkably close to physics; and they become even more applicable if they are written in their *real* versions, that obtain in an obvious manner by setting

$$\underline{z}_{nm} = \underline{x}_{nm} + \mathrm{i}\underline{y}_{nm}, \qquad c = a + \mathrm{i}b.$$

In contrast to what we did for the previous examples, let us outline here the derivation of these results. Actually the two systems of Newtonian equations written above are merely special subcases, corresponding to appropriate parametrizations (see, for instance, [14]) of a square matrix M (of appropriate rank) in terms of S-vectors, of the following nonlinear *matrix* evolution equation

$$\ddot{M} - 3\mathrm{i}\omega\dot{M} - 2\omega^2 M = cM^3. \tag{11}$$

Hence the findings reported above are merely special cases of the more general result according to which the *generic* solution of this nonlinear *matrix* evolution equation – with $M \equiv M(t)$ a square matrix of *arbitrary* rank – is *periodic* with period T, see (1)

$$M(t+T) = M(t).$$

And this result is an immediate consequence, via the following *matrix* version of the trick,

$$M(t) = \exp(\mathrm{i}\omega t)\Psi(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega} \tag{12}$$

of a previous result due to Inozemtsev [36], according to which the *matrix* evolution equation

$$\Psi'' = c\Psi^3$$

which clearly corresponds to (11) via (12), is *integrable* and *all* its solutions $\Psi(\tau)$ are *meromorphic* functions of the independent variable τ.

Let us end this section by re-emphasizing – to underscore the significance of the examples exhibited in this section – that, as noted in the introductory part of this section, *isochronous* systems are, at least *partially*, *superintegrable*.

Lecture 2

2. The Goldfish

"A mathematician, using the dressing method to find a new integrable system, could be compared with a fisherman, plunging his net into the sea. He does not know what a fish he will pull out. He hopes to catch a goldfish, of course. But too often his catch is something that could not be used for any known purpose to him. He invents more and more sophisticated nets and equipments, and plunges all that deeper and deeper. As a result, he pulls on the shore after a hard work more and more strange creatures. He should not despair, nevertheless. The strange creatures may be interesting enough if you are not too pragmatic, and who knows how deep in the sea do goldfishes live?"

This sentence (copied from page 622 of [46]) provided the motivation to call "goldfish" [13] – due to its neat appearance, and the simplicity of its solution – the *integrable*, indeed *solvable*, system characterized by the Newtonian equations of motion

$$\zeta_n'' = 2 \sum_{m=1, m\neq n}^{N} \frac{\zeta_n' \zeta_m'}{\zeta_n - \zeta_m} \tag{13}$$

as well as some of its variants (see below). Here the N dependent variables $\zeta_n \equiv \zeta_n(\tau)$ evolve as functions of the independent variable τ, and we always consider them in the *complex*; while appended primes denote of course differentiations with respect to τ.

The solution of the initial-value problem for this model is given by the following simple prescription [8]: *the N values $\zeta_n(\tau)$ coincide with the N roots of the following algebraic equation in ζ*

$$\sum_{n=1}^{N} \frac{\zeta'(0)}{\zeta - \zeta_n(0)} = \frac{1}{\tau}. \tag{14}$$

Note that this equations gets transformed into a polynomial equation of degree N in ζ after multiplication by the factor $\prod_{n=1}^{N}[\zeta-\zeta_n(0)]$, hence it indeed has N solutions.

Via the (particularly simple) version of the trick appropriate to this model,

$$z_n(t)=\zeta_n(\tau),\qquad \tau=\frac{\exp(\mathrm{i}\omega t)-1}{\mathrm{i}\omega} \tag{15a}$$

entailing

$$z_n(0)=\zeta_n(0),\qquad \dot{z}_n(0)=\zeta_n'(0) \tag{15b}$$

the Newtonian equations of motion (13) become

$$\ddot{z}_n=\mathrm{i}\omega\dot{z}_n+2\sum_{m=1,m\neq n}^{N}\frac{\dot{z}_n\dot{z}_m}{z_n-z_m} \tag{16}$$

and the prescription providing the solution of the initial-value problem for these Newtonian equations of motion, (16), states that *the N values $z_n(t)$ coincide with the N roots of the following algebraic equation in z*

$$\sum_{n=1}^{N}\frac{\dot{z}(0)}{z-z_n(0)}=\frac{\mathrm{i}\omega}{\exp(\mathrm{i}\omega t)-1}. \tag{17}$$

As always in this paper we consider the constant ω to be *real* (and, for definiteness, *positive*), and the variable t to be as well *real* and to represent "time". The periodicity of (the right-hand side of) this algebraic equation in z, (17), entails that the set of its N zeros is *periodic* with period T, see (1), and hence the solution $\underline{z}(t)\equiv(z_1(t),\ldots,z_N(t))$ is as well *completely periodic*, but due to the possibility that, through the motion, some zeros exchange their roles, the periodicity of this solution may turn out to be a (finite, integer) multiple of T. In conclusion the system (16) is *isochronous*, indeed it also qualifies as an assembly of *nonlinear harmonic oscillators*, since *all* its solutions $\underline{z}(t)$ are *completely periodic* with the same period T or a finite integer multiple of it; and the phase space of its initial data $\underline{z}(0)$, $\underline{\dot{z}}(0)$ is divided into domains out of which the system evolves with the same periodicity, these domains being separated from each other by (lower-dimensional) boundaries yielding trajectories along which the equations of motions become singular due to the collision of two or more particles (note that, at a collision point, the solutions remain finite but the speed of the colliding particles diverge, and their individual identity gets lost).

In this section we focus on the more general version of this model, the solutions of which cannot generally be obtained by algebraic operations, yet can be analyzed in considerable detail and do exhibit a richness of behaviors that justifies attributing also to this model the name "goldfish" [29]. This more general model is characterized by the Newtonian equations of motion

$$\ddot{z}_n = \mathrm{i}\omega \dot{z}_n + 2 \sum_{m=1, m\neq n}^{N} \frac{a_{nm} \dot{z}_n \dot{z}_m}{z_n - z_m} \tag{18}$$

which differ from (16) due to the presence of the *arbitrary* "coupling constants" a_{nm} (and of course reduces back to (16) for $a_{nm} = 1$). But before discussing the behavior of the solutions of this model let us interject two remarks that are relevant to justify our calling it a "goldfish".

The first remark is, that these Newtonian equations of motion, (18), are Hamiltonian (provided $a_{nm} = a_{mn}$, as we hereafter assume). Indeed it is easily seen that from the Hamiltonian

$$H(\underline{p}, \underline{z}) = \sum_{n=1}^{N} \left[-\frac{\mathrm{i}\omega z_n}{c} + \exp(cp_n) \prod_{m=1, m\neq n}^{N} (z_n - z_m)^{-a_{nm}} \right] \tag{19a}$$

one can obtain the following Hamiltonian equations

$$\dot{z}_n = \frac{\partial H}{\partial p_n} = c \exp(cp_n) \prod_{m=1, m\neq n}^{N} (z_n - z_m)^{-a_{nm}} \tag{19b}$$

$$\dot{p}_n = -\frac{\partial H}{\partial z_n} = c^{-1} \left[\mathrm{i}\omega + \sum_{m=1, m\neq n}^{N} \frac{a_{nm}(\dot{z}_n - \dot{z}_m)}{z_n - z_m} \right]. \tag{19c}$$

To write in more convenient form the second set, (19c), of these equations we used the first set, (19b). And it is now clear that (logarithmic) t-differentiation of (19b) yields, using (19c), precisely (18). Note that this result obtains for any *arbitrary* (nonvanishing) choice of the constant c, that appears in the definition of the Hamiltonian and in the Hamiltonian equations of motion, see (19), but is not present in the Newtonian equations (18).

Secondly, we observe that these N *complex* equations of motion, (18), obeyed by the N *complex* dependent variables $z_n(t)$ moving in the *complex* plane, can be reformulated [11,12] as N *real* equations of motion satisfied by N *real* two-vectors $\vec{r}_n(t)$ moving in the horizontal plane, by setting

$$z_n = x_n + \mathrm{i}y_n, \quad \vec{r}_n = (x_n, y_n, 0), \quad \hat{k} = (0, 0, 1), \quad a_{nm} = \alpha_{nm} + \mathrm{i}\beta_{nm}. \tag{20}$$

These Newtonian equations of motion read as follows

$$\ddot{\vec{r}}_n = \omega \hat{k} \wedge \dot{\vec{r}}_n + 2 \sum_{m=1, m\neq n}^{N} r_{nm}^{-2} (\alpha_{nm} + \beta_{nm} \hat{k} \wedge) \cdot [\dot{\vec{r}}_n (\dot{\vec{r}}_m \cdot \vec{r}_{nm}) + \dot{\vec{r}}_m (\dot{\vec{r}}_n \vec{r}_{nm}) - \vec{r}_{nm} (\dot{\vec{r}}_n \cdot \dot{\vec{r}}_m)]. \tag{21}$$

Here we use the short-hand notation $\vec{r}_{nm} = \vec{r}_n - \vec{r}_m$, entailing $r_{nm}^2 = r_n^2 + r_m^2 - 2\vec{r}_n \cdot \vec{r}_m$.

The fact that these equations of motion are both *translation-* and *rotation-invariant* is remarkable, as well as the fact that, when *all* the coupling constants α_{nm}, β_{nm} vanish, these Newtonian equations of motion have a clear physical interpretation: they describe a "cyclotron", namely the motion of N equal, electrically charged, point-like particles moving in the horizontal plane in the presence of a constant magnetic field orthogonal to that plane, in the approximation in which their mutual (electrostatic) interactions are neglected. And these Newtonian equations of motion (of course: see for instance [10]) are no less Hamiltonian than the (*complex*) equations of motion (18). Indeed a *real* Hamiltonian that generates directly the *real* equations of motion (21) is provided by the *real part* of the Hamiltonian that obtains by inserting the assignment (20) in (19a), together with

$$p_n = p_{nx} - \mathrm{i} p_{ny}, \qquad \vec{p}_n = (p_{nx}, p_{ny}, 0).$$

Note the minus sign in the first set of these equations where p_{nx} respectively p_{ny} are the x-component respectively the y-component of the two-vector $\vec{p}_n$, that plays the role of canonically conjugate momentum to the canonical two-vector variable $\vec{r}_n$.

Hereafter we discuss the behavior of this model on the basis of the (neater if less "physical") *complex* equations of motion (18), describing N points z_n that evolve over time in the *complex* z-plane; but the reader should not forget that these evolutions can as well be interpreted as describing the (physical) motions of N equal (pointlike) particles in the (*real*) horizontal plane. And the main tool of our analysis is the trick (15), that relates our equations of motion (18) to the equations of motion

$$\zeta_n'' = 2 \sum_{m=1, m \neq n}^{N} \frac{a_{nm} \zeta_n' \zeta_m'}{\zeta_n - \zeta_m} \tag{22}$$

that, together with the initial data $\zeta_n(0)$, $\zeta_n'(0)$ (see (15b)) define the solutions $\zeta_n \equiv \zeta_n(\tau)$ in the *complex* τ-plane. The "physical" evolution of the points $z_n \equiv z_n(t)$ as functions of the *real* time variable t is then given by the evolution of the corresponding coordinates $\zeta_n(\tau)$, see (15a), as the *complex* variable τ travels round and round on the circle C in the *complex* τ-plane, the diameter of which, of length $\frac{2}{\omega}$, has one extreme at the origin $\tau = 0$ and the other on the positive imaginary axis at $\tau = \frac{2\mathrm{i}}{\omega}$. It is therefore clear that the behavior of $z_n(t)$ as a function of the *real*, "physical time" variable t depends on the analytic structure of $\zeta_n(\tau)$ as function of the *complex* variable τ, in particular of the singularities, if any, of this function $\zeta_n(\tau)$ that fall in the disk D encircled by the circle C in the *complex* τ-plane. We tersely review here the relevant analysis, following [29] where the interested reader will find a more detailed report, including the display of the trajectories of several numerical simulations of the motions of the points $z_n(t)$ obtained via a computer code created by M. Sommacal. A more rewarding experience is of course to watch the evolution over time of these numerical simulations – as it were, like a movie show: this will eventually become possible when, hopefully pretty soon, the Sommacal simulation code will be made available via the web for general use.

We note first of all that there exists in phase space an *open* region of initial data $z_n(0)$, $\dot{z}_n(0)$, characterized by *large* values of the moduli $|z_n(0) - z_m(0)|$ of the initial interparticle distances and by *small* values of the moduli of the initial particle velocities $|\dot{z}_n(0)|$ (see (22) and (15b)), that guarantees (all components $\zeta_n(\tau)$ of) the corresponding solution $\underline{\zeta}(\tau)$ of (22) to be *holomorphic* in (a disk of radius ρ centered at the origin $\tau = 0$ of the *complex* τ-plane that includes) the circle C, hence the corresponding solution $\underline{z}(t)$ to be *completely periodic* with period T, see (15a) and (1). This was firstly proven in [24], to which the interested reader is referred for more details, including an explicit evaluation of a lower bound to ρ in terms of the (moduli of) the coupling constants a_{nm} and of the minimum value of the moduli $|z_n(0) - z_m(0)|$ of the initial interparticle distances and the maximum value of the moduli of the initial particle velocities $|\dot{z}_n(0)|$. This result guarantees the *isochronous* character of this model, (18), for any arbitrarily given assignment of the coupling constants a_{nm}.

Next, let us restrict here, for simplicity, our consideration to models (18) in which *all* the coupling constants a_{nm} are *real* and *nonnegative*,

$$a_{nm} \geq 0 \tag{23}$$

(for other cases see [29]). Then the singularities of the generic solution $\underline{\zeta}(\tau)$ of (22) – which occur at values τ_b of τ where *two* coordinates $\zeta_n(\tau)$ coincide, say $\zeta_\mu(\tau_b) = \zeta_\nu(\tau_b) = b$ (see the right-hand side of (22)) – are *branch points* characterized by the exponent, say,

$$\gamma = \gamma_{\mu\nu} = \frac{1}{1 + a_{\mu\nu}} \tag{24}$$

so that in their neighborhood, namely for $\tau \approx \tau_b$,

$$\begin{aligned} \zeta_s(\tau) &= b \pm c(\tau - \tau_b)^\gamma + v(\tau - \tau_b) \\ &+ \sum_{k=1}^{\infty} \sum_{\substack{\ell,m=0 \\ \ell+m\geq 1}}^{\infty} \varphi^{(s)}_{k\ell m}(\tau - \tau_b)^{k+\ell\gamma+m(1-\gamma)}, \quad s = \mu, \nu \end{aligned} \tag{25a}$$

$$\begin{aligned} \zeta_n(\tau) &= b_n + v_n(\tau - \tau_b) \\ &+ \sum_{k=1}^{\infty} \sum_{\ell=\delta_{k1}}^{\infty} \sum_{m=0}^{\infty} \varphi^{(n)}_{k\ell m}(\tau - \tau_b)^{k+\ell\gamma+m(1-\gamma)}, \quad n \neq \mu, \nu. \end{aligned} \tag{25b}$$

The $\pm$ sign in front of c in the right-hand side of the first, (25a), of these formulas indicates that one sign must be chosen for $s = \mu$, the opposite for $s = \nu$. Note that here the $4 + 2(N - 2) = 2N$ constants τ_b, b, c, v, b_n, v_n are *a priori* arbitrary – except for the obvious restrictions $b_n \neq b$, $b_n \neq b_m$ – while the coefficients $\varphi^{(s)}_{k\ell m}$, $\varphi^{(n)}_{k\ell m}$ can be computed from these constants, recursively, by inserting this *ansatz*, (25), in the equations of motion (22). The fact that the number, $2N$, of *a priori* undetermined coupling constants equals the number of arbitrary initial data for this system of ODEs, (22), indicates that this kind of branch points, characterized by the exponents γ_{nm}, see (24), is the typical singularity featured by the generic solution $\underline{\zeta}(\tau)$ of (22). (Branch points with different exponents may appear, but only in *nongeneric* solutions $\underline{\zeta}(\tau)$ which, at some value τ_b of τ, feature the coincidence of more than two components, say $\zeta_\mu(\tau_b) = \zeta_\nu(\tau_b) = \zeta_\lambda(\tau_b)$; for a more detailed discussion of this question see [29]).

We conclude therefore that the *generic* solution $\underline{\zeta}(\tau)$ of (22) features a, generally infinite, number of branch points, that generally affect each of its components $\zeta_n(\tau)$, and which are characterized, for the class of models to which we are restricting attention here, see (23), by (*real*) exponents γ_{nm}, see (24), which are then clearly characterized by the inequalities

$$0 < \gamma_{nm} \leq 1.$$

What does this tell us about the *generic* solution $\underline{z}(t)$ of the equations of motions of primary interest to us, (18), in particular about its evolution as function of the *real* "time" variable t?

To the solution $\underline{\zeta}(\tau)$ is associated a Riemann surface the structure of which is determined by the character and distribution of the branch points of $\underline{\zeta}(\tau)$ in the complex τ-plane (each of which is generally featured by each component $\zeta_n(\tau)$ of $\underline{\zeta}(\tau)$, although generally not in the same way: see (25)), and we know from (15a) that the values take by $\underline{z}(t)$ as t evolves from $t = 0$ towards $t = \infty$ coincide with the values taken by $\underline{\zeta}(\tau)$ as the independent variable τ travels, on that Riemann surface associated with $\underline{\zeta}(\tau)$, counterclockwise round and round on the circle C defined above (the diameter of which lies on the imaginary axis in the complex τ-plane, with one end at $\tau = 0$ and the other at $\tau = \frac{2\mathrm{i}}{\omega}$), employing a period T, see (1), to make each full round. Hence the behavior of the solution $\underline{z}(t)$ of (18) depends on the structure of the Riemann surface associated with the corresponding solution $\underline{\zeta}(\tau)$ of (22), and specifically on the number of different sheets of that surface that are visited as one travels on it before returning, if ever, to the main sheet from which the travel started at $t = \tau = 0$.

If no other sheet is visited besides the main one, the corresponding solution $\underline{z}(t)$ is of course periodic with period T, see (1) and (15a),

$$\underline{z}(t + T) = \underline{z}(t). \tag{26}$$

This happens provided no branch point is featured by $\underline{\zeta}(\tau)$ on its main sheet *inside* the circle C, and as already indicated in the preceding section, it has been proven [24] (even in the more general case with arbitrary coupling constants a_{nm}) that there is an *open* region having full dimensionality in the phase space of initial data, see (15b), that yields such an outcome, implying the *isochronicity* of the model characterized by the Newtonian equations of motion (18). This region R (of initial data that yield solutions $\underline{\zeta}(\tau)$ of (22) which are *holomorphic* in the closed disk encircled by C, hence solutions $\underline{z}(t)$ of (18) that are completely periodic in t, see (26)) has a boundary – a lower-dimensional domain in the phase space of initial data – out of which emerge motions leading, at a time t_b smaller than T, to a "particle collision", say $z_\nu(t_b) = z_\mu(t_b)$.

The character of the solution $\underline{z}(t)$ yielded by initial data that are *outside* of the region R depends on the structure of the Riemann surface associated with the corresponding solution $\underline{\zeta}(\tau)$. This is mainly determined by the values of the branch point exponents γ_{nm}, which are themselves determined by the values of the coupling constants a_{nm}, see (25) and (24). Let us focus on the (more interesting) case in which these constants a_{nm} are *rational* numbers, entailing that the coefficients γ_{nm} determining the character of the branch points are as well *rational*, see (24), so that each of the cuts associated with them opens the way, in the Riemann surface, to a *finite* number of sheets. There are then two possibilities, each generally characterized by *open* regions of initial data having full dimensionality in phase space, the boundaries of which always are (lower-dimensional) domains out of which emerge motions leading, in a time t_b smaller than T, to a "particle collision".

One possibility is that the number B of sheets visited before returning to the main sheet be *finite*, $B < \infty$ and the corresponding solutions $\underline{z}(t)$ are then *completely periodic* with period $\tilde{T} = (B+1)T$, $\underline{z}(t+\tilde{T}) = \underline{z}(t)$.

Another possibility is that the number of new sheets visited be *unlimited*, namely that the structure of the Riemann surface be such that, by traveling round and round on it along the circle C one *never* returns back to the main sheet. This can happen, even if the exponents γ_{nm} are *all rational* so that via the cuts associated to each of them access is gained to only a *finite* number of new sheets, because of the possibility that an *infinity* of branch points be located inside the circle C on the *infinite* sheets associated to these branch points, via a never ending mechanism of branch points nesting. Whenever this happens the corresponding solution $\underline{z}(t)$ is *aperiodic*, and it is moreover likely that it then be *chaotic*, in the sense of displaying a *sensitive dependence* on its initial data. Indeed this will happen whenever some ones out of this infinity of branch points fall arbitrarily close to the contour C, because then a *minute* change in the initial data, to which there will correspond a *minute* change in the pattern of these branch points of $\underline{\zeta}(\tau)$ in the complex τ-plane, will cause some relevant branch point to cross over from outside the circle

C to inside it, or viceversa, and this will eventually effect quite significantly the time evolution of $\underline{z}(t)$, by causing a change in the sequence of sheets that get visited by traveling along the circle C on the Riemann surface associated to the corresponding $\underline{\zeta}(\tau)$.

This phenomenology has a clear "physical interpretation", which can be qualitatively understood as follows. The many-body problem characterized by the Newtonian equations of motion (18) generally yields confined motions, the trajectory of each particle tending to wind round and round – it would indeed reduce to a circle were it not for the interaction with the other particles. A possibility, as we know, is that this N-body motion be *completely periodic*, with the same period T that characterizes the circular motion of each particle when the two-body interparticle interaction is altogether missing ($a_{nm} = 0$). Another possibility, in the case discussed above with *rational* coupling constants, is that there exist other motions which are as well *completely periodic*, but with periods which are *integer multiples* of T. A third possibility, which cannot *a priori* be excluded, is that there also exist motions which are *aperiodic* but in some way overall *ordered*, perhaps featuring trajectories that eventually wind up around *limit cycles*. And still another possibility is that the motions described by the solution $\underline{z}(t)$ be *aperiodic* and *disordered*. In this case the physical mechanism causing a sensitive dependence on the initial data can be understood as follows. Such disordered motions necessarily feature *near misses*, in which, typically, two particles pass *quite close* to each other (while the probability that an actual collision occur among *point particles* moving *in a plane* is of course *a priori* nil). Such a *near miss* in the motion described by $\underline{z}(t)$ corresponds – see the discussion above – to a branch point of the corresponding solution $\underline{\zeta}(\tau)$ occurring *quite close* to the circle C in the complex τ-plane (which is the one-dimensional region of the two-dimensional complex τ-plane in which the values of $\underline{\zeta}(\tau)$ correspond to the values $\underline{z}(t)$ describing the motion of physical particles moving as functions of the time t); and in the generic case of a *two-body* near miss, there is a correspondence between the fact that such a branch point occur *just inside*, or *just outside*, the circle C, and the way the particles pass, on one or the other side, by each other. Likewise, the tiny change in the initial data that causes, in the context of the solutions $\underline{\zeta}(\tau)$ – see the discussion above – a branch point of $\underline{\zeta}(\tau)$ to pass from inside to outside the circle C, or viceversa, corresponds, in the context of the "physical" solutions $\underline{z}(t)$, to a change occurring in the corresponding *near miss*, from the case in which the two particles involved in it slide by each other on one side to the case in which they instead slide by each other on the other side – entailing a significant change in the subsequent motion (indeed, the closer a near miss, the more it affects the motion, due to the singularity of the two-body interaction at zero separation, see (18)).

The phenomenology outlined here does indeed occur in this goldfish model, as demonstrated and discussed in [29]. It also occurs – rather similarly if more simply, since in this case only square-root branch points occur, irrespective of the values of the coupling constants – in the model 2, which has been treated in detail, along the same lines discussed above, in [32], where the interested reader will also find a representative set of numerically simulated trajectories. And it is rather clear that this phenomenology provides a paradigm of rather general applicability for the transition from *isochronicity* to *deterministic chaos*, indeed perhaps for the generic *onset* of *deterministic chaos*, as will be discussed in [30].

Lecture 3

3. A Novel Technique to Identify Solvable Dynamical Systems, and a Solvable Extension of the Goldfish Many-Body Problem

In this section we review some recent developments [21, 22, 28] (occasionally we follow the last of these papers, [22], *verbatim*).

3.1. The Technique

The point of departure of our technique is a matrix evolution equation "of Newtonian type",

$$\ddot{U} = F(U, \dot{U}). \tag{27}$$

Here (and below) the $N \times N$ matrix $U \equiv U(t)$ is the dependent variable, the independent variable t ("time") is *real* and superimposed dots indicate derivatives with respect to it; and we assume the matrix function $F(U, \dot{U})$ to depend on no other matrix besides its two arguments (the order of which is of course important, since these two matrices, U and $\dot{U}$, need not commute), so that there hold the identity

$$RF(U, \dot{U})R^{-1} = F(RUR^{-1}, R\dot{U}R^{-1})$$

where R is *any* (invertible) $N \times N$ matrix.

We then introduce the parameterization of the $N \times N$ matrix $U(t)$ in terms of its N eigenvalues $z_n(t)$ and of its diagonalizing $N \times N$ matrix $R(t)$

$$U = RZR^{-1} \tag{28a}$$

$$Z = \text{diag}[z_n]. \tag{28b}$$

But before proceeding to obtain the evolution equations implied by (27) for the diagonal matrix $Z(t)$ and for the diagonalizing matrix $R(t)$, or rather (see below) for the matrix $M(t)$ defined in terms of $R(t)$ by the formula

$$M = R^{-1}\dot{R} \tag{29}$$

let us note that the formulas (28) define the matrix R only up to multiplication from the right by an *arbitrary* diagonal matrix, say

$$D = \mathrm{diag}[d_n] \tag{30}$$

since replacing in (28a) R with

$$\tilde{R} = RD \tag{31}$$

is clearly of no consequence. The corresponding "gauge transformation" of the matrix M,

$$\widetilde{M} = \tilde{R}^{-1}\dot{\tilde{R}} = D^{-1}MD + D^{-1}\dot{D} \tag{32a}$$

namely

$$\widetilde{M}_{nn} = M_{nn} + \frac{\dot{d}_n}{d_n} \tag{32b}$$

$$\widetilde{M}_{nm} = d_n^{-1} M_{nm} d_m, \quad n \neq m \tag{32c}$$

entails that in our parameterization of the $N \times N$ matrix $U(t)$ (via (28) with (29)) the N^2 matrix elements of this matrix get replaced by the N elements $z_n(t)$ of the diagonal matrix $Z(t)$ (namely by the N eigenvalues of the matrix $U(t)$: see (28)) and by the $N(N-1)$ off-diagonal elements $M_{nm}(t)$ (with $n \neq m$) of the $N \times N$ matrix $M(t)$, while the N diagonal elements $M_{nn}(t)$ can be arbitrarily adjusted by choosing appropriately the elements $d_n(t)$ of the diagonal matrix $D(t)$, see (32b) (of course, up to a corresponding adjustment of the corresponding off-diagonal elements, see (32c)).

Differentiation of (28a) with respect to the independent variable t yields, using (29),

$$\dot{U} = R\{\dot{Z} + [M, Z]\}R^{-1} \tag{33a}$$

$$\ddot{U} = R\{\ddot{Z} + [\dot{M}, Z] + 2[M, \dot{Z}] + [M, [M, Z]]\}R^{-1}. \tag{33b}$$

Here and throughout we use of course the standard notation $[X, Y] \equiv XY - YX$ for the commutator of two matrices.

We now insert these formulas, (28) and (33), in the matrix evolution equation (27) and we thereby obtain the $N \times N$ matrix evolution equation

$$\ddot{Z} + [\dot{M}, Z] + 2[M, \dot{Z}] + [M, [M, Z]] = F(Z, \dot{Z} + [M, Z]). \tag{34}$$

Up to now, the treatment has been general, namely applicable to any matrix evolution equation (27). Hereafter, following [22], we focus on a specific *integrable* (indeed *solvable*) matrix equation; analogous treatments of other *solvable* equations are given in [21] and [28]; and this is the appropriate place to emphasize that this approach is not new, indeed it has been employed often, in analogous or quite different contexts, see for instance [2–6, 33, 37, 39–45].

3.2. Extended Goldfish

The *solvable* matrix evolution equation which we now take as starting point to apply the technique described above reads as follows

$$\ddot{U} = a(\dot{U}U + U\dot{U}). \tag{35}$$

Here a indicates a *scalar* constant, which might of course be eliminated by a trivial rescaling of the dependent or independent variables, although we prefer not to do so in order to keep track of the contributions coming from the (nonlinear) right-hand side of this matrix evolution equation, (35), and also to maintain open the option to set a to zero, going thereby back to the standard goldfish many-body problem (see preceding section and related results [21], [28], [13]).

It is easily seen that the *general* solution of this matrix evolution equation (35), reads

$$U(t) = a^{-1}[\cos(At) - BA^{-1}\sin(At)]^{-1}[A\sin(At) + B\cos(At)] \tag{36a}$$

where A and B are two arbitrary *constant* $N \times N$ matrices. In terms of the initial-value problem for the matrix evolution equation (35) clearly (36a) entails

$$U(0) = a^{-1}B, \qquad \dot{U}(0) = a^{-1}(A^2 + B^2) \tag{36b}$$

and these two matrix equations can be inverted to yield

$$A^2 = -a^2[U(0)]^2 + a\dot{U}(0), \qquad B = aU(0). \tag{36c}$$

Note that the explicit expression (36a) entails that the matrix $U(t)$ is actually a function of the matrix A^2 rather than A.

We now insert the formulas (28) and (33) in the matrix evolution equation (35) and we thereby obtain the $N \times N$ matrix evolution equation

$$\ddot{Z} + [\dot{M}, Z] + 2[M, \dot{Z}] + [M, [M, Z]] = a\{(\dot{Z} + [M, Z])Z + Z(\dot{Z} + [M, Z])\}.$$

We now separate the diagonal and off-diagonal terms of this matrix evolution equation, making moreover the notational assignment

$$M_{nn} \equiv F_n$$

to help us keeping in mind that, consistently with the observation made above, we retain the freedom to assign arbitrarily these N quantities F_n. We thus obtain the following Newtonian equations of motion, interpretable as those of an N-body problem (which is, of course, *solvable*, thanks to its relation to the *solvable* matrix evolution equation (35))

$$\ddot{z}_n = 2a\dot{z}_n z_n - 2\sum_{m=1, m\neq n}^{N} (z_n - z_m)M_{nm}M_{mn} \tag{37a}$$

$$(z_n - z_m)\dot{M}_{nm} + 2(\dot{z}_n - \dot{z}_m)M_{nm} = a(z_n^2 - z_m^2)M_{nm}$$
$$+ \sum_{\ell=1;\ell\neq n,m}^{N} (z_n + z_m - 2z_\ell)M_{n\ell}M_{\ell m} - (z_n - z_m)M_{nm}(F_n - F_m), \; n \neq m. \tag{37b}$$

Here the N coordinates $z_n \equiv z_n(t)$ denote the positions of the N moving particles, the $N(N-1)$ "auxiliary variables" $M_{nm} \equiv M_{nm}(t)$ evolve according to the system of $N(N-1)$ first-order ODEs (37b), and the N **source terms** F_n can be assigned as arbitrary functions of time, or even of the other dependent variables z_m and $M_{m\ell}$, without spoiling the solvable character of the model. Here and throughout indices like n, m, ℓ range from 1 to N, unless otherwise indicated.

An interesting redefinition of the auxiliary variables obtains by setting

$$M_{nm} = (z_n - z_m)^{-1}(\dot{z}_n\dot{z}_m)^{1/2}u_{nm} \tag{38}$$

(this assignment is suggested by the form of the Lax matrix introduced in [7]). The equations of motion (37) of the N-body model take thereby the form

$$\ddot{z}_n = 2a\dot{z}_n z_n + 2\sum_{m=1,m\neq n}^{N} \frac{\dot{z}_n\dot{z}_m}{z_n - z_m}u_{nm}u_{mn}, \tag{39a}$$

$$\dot{u}_{nm} + \frac{\dot{z}_n - \dot{z}_m}{z_n - z_m}u_{nm}(1 - u_{nm}u_{mn}) = -\sum_{\ell=1;\ell\neq n,m}^{N} \dot{z}_\ell\left[\frac{u_{n\ell}(u_{\ell m} + u_{nm}u_{\ell n})}{z_n - z_\ell}\right.$$
$$\left. + \frac{u_{\ell m}(u_{n\ell} + u_{nm}u_{m\ell})}{z_m - z_\ell}\right] - u_{nm}(F_n - F_m), \quad n \neq m. \tag{39b}$$

Here the auxiliary variables are of course the $N(N-1)$ quantities $u_{nm}(t)$; and note that, while the (Newtonian) equations of motion (39a) that characterize the evolution of the "particle coordinates" $z_n(t)$ follow straightforwardly from (37a) via (38), in order to obtain from (37b) the equations (39b) that characterize the evolution of the auxiliary variables $u_{nm}(t)$ one must use, in addition to (38), the equations of motion (39a). It is now clear that, for $F_n = 0$ (or, equivalently, for $F_n = F$), the $N(N-1)$ evolution equations (39b) admit the (special) solution

$$u_{nm} = -1, \qquad n \neq m \tag{40}$$

entailing that the Newtonian equations of motion (39a) become then

$$\ddot{z}_n = 2a\dot{z}_n z_n + 2\sum_{m=1,m\neq n}^{N} \frac{\dot{z}_n\dot{z}_m}{z_n - z_m}. \tag{41}$$

For $a = 0$ these equations of motion coincide with those of the standard "goldfish" model (see the preceding section). For $a \neq 0$ they characterize a novel *solvable* extension of the goldfish N-body problem.

Before discussing the solvability of this model, (41), let us note its Hamiltonian character. Indeed the Hamiltonian

$$H=\sum_{n=1}^{N}\left\{-\frac{a}{s}z_n^2+\exp(sp_n)\prod_{m=1,m\neq n}^{N}(z_n-z_m)^{-1}\right\} \tag{42a}$$

where s is an arbitrary (nonvanishing) constant, yields the Hamiltonian equations

$$\dot{z}_n=\frac{\partial H}{\partial p_n}=s\exp(sp_n)\prod_{m=1,m\neq n}^{N}(z_n-z_m)^{-1} \tag{42b}$$

$$\dot{p}_n=-\frac{\partial H}{\partial z_n}=\frac{1}{s}\left\{2az_n+\sum_{m=1,m\neq n}^{N}\frac{\dot{z}_n+\dot{z}_m}{z_n-z_m}\right\}. \tag{42c}$$

Note that to write more neatly the second set of these Hamiltonian equations (42c), we used the first, (42b). It is then obvious that t-differentiation of (the logarithm of) the first set of Hamiltonian equations, (42b), yields, via the second set, (42c), just the Newtonian equations of motion (41), demonstrating thereby their Hamiltonian character.

The solution of the initial-value problem for this N-body model, (41), is given by the following result (entailed by (28) with (36)): the coordinates $z_n(t)$ are the N eigenvalues of the $N\times N$ matrix (36a) with

$$(A^2)_{nm}=-\delta_{nm}a^2z_n^2(0)+a[\dot{z}_n(0)\dot{z}_m(0)]^{1/2},\qquad B_{nm}=\delta_{nm}az_n(0). \tag{43}$$

Here δ_{nm} is the standard Kronecker symbol, $\delta_{nm}=1$ if $n=m$, $\delta_{nm}=0$ if $n\neq m$. These formulas indicate that the $N\times N$ matrix B is diagonal, while the $N\times N$ matrix A^2 is the sum of a diagonal matrix and a dyadic matrix.

Via the standard trick (outlined at the end of this section) the following *isochronous* variant of the model (41) is obtained

$$\begin{aligned}&\ddot{z}_n-3\mathrm{i}\omega\dot{z}_n-2\omega^2z_n\\&=2a(\dot{z}_n-\mathrm{i}\omega z_n)z_n+2\sum_{m=1,m\neq n}^{N}\frac{\dot{z}_n\dot{z}_m-\mathrm{i}\omega(\dot{z}_nz_m+\dot{z}_mz_n)-\omega^2z_nz_m}{z_n-z_m}.\end{aligned} \tag{44}$$

Here and throughout ω is a *real* (for definiteness, *positive*) constant, and because of the way this *modified* system is obtained from (41) *all* its nonsingular solutions are *completely periodic* with period T, see (1), or with an integer multiple of this basic period. Indeed the solutions $z_n(t)$ of these equations of motion, (44), are the N eigenvalues of the *modified* matrix $\widetilde{U}(t)$ defined as follows

$$\widetilde{U}(t)=\exp(\mathrm{i}\omega t)U(\tau),\qquad \tau=\frac{\exp(\mathrm{i}\omega t)-1}{\mathrm{i}\omega} \tag{45a}$$

where the matrix $U(\tau)$ is defined by (36a) (of course with t replaced by τ), but now with

$$\tilde{U}(0) = a^{-1}B, \qquad \dot{\tilde{U}}(0) = a^{-1}(A^2 + B^2) + \mathrm{i}\omega a^{-1}B \tag{45b}$$

entailing

$$A^2 = -a^2[\tilde{U}(0)]^2 + a\dot{\tilde{U}}(0) - \mathrm{i}a\omega\tilde{U}(0), \qquad B = a\tilde{U}(0). \tag{45c}$$

In terms of the initial-value data for the model (44) these expressions read

$$\begin{aligned}(A^2)_{nm} &= -\delta_{nm}a^2 z_n^2(0) + a\{[\dot{z}_n(0) - \mathrm{i}\omega z_n(0)][\dot{z}_m(0) - \mathrm{i}\omega z_m(0)]\}^{1/2} \\ B_{nm} &= \delta_{nm} a z_n(0).\end{aligned} \tag{45d}$$

Note that the assertion made above about the complete periodicity of *all* the non-singular solutions of the system (44) is implied by the assertion made now about the solution of this system, since it is clear that the matrix $\tilde{U}$ is periodic in the time variable t with period T, see (1), (45a) and (36a). This incidentally allows to consider this generalized goldfish model, (44), as describing an assembly of "non-linear harmonic oscillators" [31]. The behavior of this system in the neighborhood of its equilibrium configuration is discussed in Section 3.4.

The solutions $z_n(t)$ of the generalized goldfish model (44) are necessarily *complex* (for *positive* ω, $\omega > 0$), but these equations of motion can be reformulated [11,12] as *real* and *covariant* equations describing the motion in the (horizontal) plane of N particles the positions of which there are identified by the *real* two-vectors $\vec{r}_n$ related to the *complex* numbers z_n via the standard relations (see (20))

$$z_n = x_n + \mathrm{i}y_n, \qquad a = a_x - \mathrm{i}a_y \tag{46a}$$

$$\vec{r}_n = (x_n, y_n, 0), \qquad \vec{a} = (a_x, a_y, 0), \qquad \hat{k} = (0, 0, 1) \tag{46b}$$

which entail that the equations of motion (44) read then as follows

$$\begin{aligned}&\ddot{\vec{r}}_n - 3\omega\hat{k}\wedge\dot{\vec{r}}_n - 2\omega^2\vec{r}_n = 2[\dot{\vec{r}}_n(\vec{a}\cdot\vec{r}_n) + \vec{r}_n(\vec{a}\cdot\dot{\vec{r}}_n) - \vec{a}(\dot{\vec{r}}_n\cdot\vec{r}_n)] \\ &\quad - 2\omega\hat{k}\wedge[2\vec{r}_n(\vec{a}\cdot\vec{r}_n) - \vec{a}r_n^2] + 2\sum_{m=1, m\neq n}^{N} r_{nm}^{-2}\{\dot{\vec{r}}_n(\dot{\vec{r}}_m\cdot\vec{r}_{nm}) + \dot{\vec{r}}_m(\dot{\vec{r}}_n\cdot\vec{r}_{nm}) \\ &\quad - \vec{r}_{nm}(\dot{\vec{r}}_n\cdot\dot{\vec{r}}_m) - \omega\hat{k}\wedge[\dot{\vec{r}}_n(\vec{r}_m\cdot\vec{r}_{nm}) + \dot{\vec{r}}_m(\vec{r}_n\cdot\vec{r}_{nm}) \\ &\quad - \vec{r}_n([\dot{\vec{r}}_n + \dot{\vec{r}}_m]\cdot\vec{r}_m) + \vec{r}_m([\dot{\vec{r}}_n + \dot{\vec{r}}_m]\cdot\vec{r}_m)] + \omega^2[\vec{r}_n r_m^2 - \vec{r}_m r_n^2]\}\end{aligned} \tag{46c}$$

where we use again the short-hand notation $\vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m$ entailing $r_{nm}^2 = r_n^2 + r_m^2 - 2\vec{r}_n\cdot\vec{r}_m$. This equation is *covariant* (thanks to the definition (46) of a; note the minus sign there!), but it is *not rotation-invariant* because the constant two-vector $\vec{a}$ identifies a preferred direction in the plane.

Before closing this section let us, for completeness, outline how the *isochronous* model (44) is obtained from (41), which to this end we rewrite here in the following (merely notationally) modified guise

$$\zeta_n'' = 2a\zeta_n'\zeta_n + 2 \sum_{m=1, m\neq n}^{N} \frac{\zeta_n'\zeta_m'}{\zeta_n - \zeta_m} \tag{47}$$

where $\zeta_n \equiv \zeta_n(\tau)$ and appended primes indicate of course differentiations with respect to the (*complex*: see immediately below) independent variable τ. We then set

$$z_n(t) = \exp(\mathrm{i}\omega t)\zeta_n(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\omega t)}{\mathrm{i}\omega}.$$

And it is then easily verified that (22) implies (44).

3.3. An Alternative Approach

In this section we describe an alternative approach (see for instance Section 2.3 of [14]). Let $\psi(z,t)$ be a monic polynomial of degree N in the (complex) variable z, and let us denote with $c_m(t)$ its coefficients and with $z_n(t)$ its zeros (which will be eventually identified with the coordinates of the particles evolving according to the Newtonian equations of motion (41))

$$\psi(z,t) = \prod_{n=1}^{N} [z - z_n(t)] = z^N + \sum_{m=1}^{N} c_m(t) z^{N-m}. \tag{48}$$

Note that this formula implies the relation

$$c_1(t) = -\sum_{n=1}^{N} z_n(t). \tag{49}$$

We now recall the relations (that obtain by logarithmic differentiation of the representation of $\psi(z,t)$ via its zeros, see (48), or see the equations (2.3.2–8,11) of [14])

$$\psi_t(z,t) = -\psi(z,t) \sum_{n=1}^{N} [z - z_n(t)]^{-1} \dot{z}_n(t) \tag{50}$$

that clearly implies (via (49))

$$z\psi_t(z,t) - \dot{c}_1(t)\psi(z,t) = \psi(z,t) \sum_{n=1}^{N} [z - z_n(t)]^{-1} (-z_n(t)\dot{z}_n(t)) \tag{51a}$$

and

$$\psi_{tt}(z,t) = \psi(z,t) \sum_{n=1}^{N} [z - z_n(t)]^{-1} \left(-\ddot{z}_n(t) + 2 \sum_{m=1, m\neq n}^{N} \frac{\dot{z}_n(t)\dot{z}_m(t)}{z_n(t) - z_m(t)} \right). \tag{51b}$$

Here and below subscripted variables denote of course partial differentiations.

It is clear from these formulas, (51), that the equations of motion (41) imply that the polynomial (48) satisfies the PDE

$$\psi_{tt}(z,t) - 2a[z\psi_t(z,t) - \dot{c}_1(t)\psi(z,t)] = 0 \tag{52}$$

and clearly this PDE, via the (second) relation (48), entails the system of ODEs

$$\ddot{c}_m - 2a\dot{c}_{m+1} + 2a\dot{c}_1 c_m = 0 \tag{53a}$$

supplemented with the "boundary conditions"

$$c_0 = 1, \qquad c_{N+1} = 0. \tag{53b}$$

Note that (53a) is trivially satisfied for $m = 0$ (see (53b)), and that it can be integrated once for $m = 1$, yielding

$$c_2 = \frac{1}{2}c_1^2 + \frac{1}{2a}(\dot{c}_1 + C)$$

where C is an integration constant. Insertion of this expression of c_2 in (53a) with $m = 2$ yields

$$\dot{c}_3 = \left(\frac{1}{2a}\right)^2 \dddot{c}_1 + \frac{\ddot{c}_1 c_1 + 2\dot{c}_1^2 + C\dot{c}_1}{2a} + \frac{1}{2}\dot{c}_1 c_1^2 \tag{54}$$

the right-hand side of which is however not an exact differential. Alternatively one could start from $m = N$ (which yields, see (53), the Schrödinger-like linear equation

$$\ddot{c}_N + 2a\dot{c}_1 c_N = 0$$

with $c_N(t)$ playing the role of eigenfunction and $\dot{c}_1(t)$ playing the role of "potential") and work all the way down by solving sequentially (but only formally!) the series of second-order, nonhomogeneous, linear ODEs for $c_m(t)$ with $m = N-1, N-2, \ldots$, arriving in the end, for $m = 1$, to a highly nonlinear (integro-differential) equation for $c_1(t)$.

Clearly the fact that the system (53) is *solvable* is far from trivial. It is obviously implied by the results described above, since the coefficients $c_m(t)$ can be explicitly written in terms of the zeros $z_n(t)$ (see for instance Section 2.3.1 of [14]); in particular (49) and (28) clearly entail the relations

$$c_1(t) = -\operatorname{trace}[U(t)], \qquad c_N(t) = (-1)^N \det[U(t)] \tag{55}$$

with the $N \times N$ matrix $U(t)$ evolving according to (36a).

The *isochronous* variant of this system, (53), can be obtained by first rewriting it in the following guise

$$\gamma_m'' - 2a\gamma_{m+1}' + 2a\gamma_1'\gamma_m = 0, \qquad \gamma_0 = 1, \qquad \gamma_{N+1} = 0 \tag{56}$$

with $\gamma_m \equiv \gamma_m(\tau)$, and by then setting

$$c_m(t) = \exp(\mathrm{i}m\omega t)\gamma_m(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}. \tag{57}$$

It reads

$$\begin{aligned}\ddot{c}_m - \mathrm{i}(2m+1)\omega\dot{c}_m - m(m+1)\omega^2 c_m - 2a[\dot{c}_{m+1} - \mathrm{i}(m+1)\omega c_{m+1}] \\ + 2a[\dot{c}_1 - \mathrm{i}\omega c_1]c_m = 0, \quad c_0 = 1, \quad c_{N+1} = 0.\end{aligned} \tag{58}$$

Again, the *isochronous* character of the *general* solution of this system, implied by our treatment, is a nontrivial finding (up to the observation that all true mathematical results are indeed trivial!).

In the next section we consider the behavior of the *isochronous* models, (44) and (58), in the neighborhood of their equilibrium configurations.

3.4. Equilibrium Configurations of the Isochronous Models, Small Oscillations and Diophantine Relations

In this section we discuss the equilibrium configuration of the two (clearly related) *isochronous* models characterized by the equations of motion (44) and (58), as well as the behavior of these systems in the neighborhood of their equilibrium configurations. In this manner we also obtain some results for "remarkable matrices", following an approach already employed in the past in analogous contexts, see for instance [1], [9], [14] and the references cited there.

The equilibrium configuration

$$z_n(t) = \overline{z}_n, \qquad \dot{z}_n(t) = 0 \tag{59a}$$

of the system (44) is clearly characterized by the following system of N algebraic equations which determine (up to permutations) the N unknowns $\overline{z}_n$

$$1 = \mathrm{i}\left(\frac{a}{\omega}\right)\overline{z}_n + \sum_{m=1, m\neq n}^{N} \frac{\overline{z}_m}{\overline{z}_n - \overline{z}_m}. \tag{59b}$$

This suggests setting

$$\overline{z}_n = \left(\frac{\mathrm{i}\omega}{2a}\right) x_n. \tag{60}$$

This notation is convenient to make contact with other results. The reader should of course note that the numbers x_n introduced here have nothing to do with the real parts of the quantities z_n, see (46). Indeed these N numbers x_n need not be real and they satisfy the N algebraic equations

$$x_n = -2 + 2\sum_{m=1, m\neq n}^{N} \frac{x_m}{x_n - x_m} \tag{61a}$$

or equivalently

$$x_n = -2N + 2 \sum_{m=1, m \neq n}^{N} \frac{x_n}{x_n - x_m} \tag{61b}$$

which can therefore be identified as the N zeros of the generalized Laguerre polynomial $L_N^{(-2N-1)}(x)$

$$L_N^{(-2N-1)}(x_n) = 0. \tag{61c}$$

Let us provide, for completeness, a proof of this result (not new, see [1, 9] and the literature quoted there). The (conveniently normalized) generalized Laguerre polynomial

$$(-1)^N N! L_N^{(-2N-1)}(x) \equiv \chi(x) = \sum_{m=0}^{N} \frac{(N+m)!}{(N-m)!m!} x^{N-m} \tag{62}$$

is characterized by the ODE

$$x\chi''(x) - (x + 2N)\chi'(x) + N\chi(x) = 0 \tag{63}$$

while its representation via its zeros,

$$\chi(x) = \prod_{n=1}^{N} (x - x_n) \tag{64a}$$

entails clearly (by logarithmic differentiation; or see the equations (7), (12) and (13) of Section 2.3.2 of [14])

$$\chi'(x) = \chi(x) \sum_{n=1}^{N} (x - x_n)^{-1} \tag{64b}$$

$$x\chi'(x) - N\chi(x) = \chi(x) \sum_{n=1}^{N} (x - x_n)^{-1} x_n \tag{64c}$$

$$x\chi''(x) = \chi(x) \sum_{n=1}^{N} (x - x_n)^{-1} 2 \sum_{m=1, m \neq n}^{N} (x_n - x_m)^{-1} x_n \tag{64d}$$

where the appended primes denote of course differentiations. It is then clear that the insertion of these three formulas in (63) entails (61b).

Let us now consider the behavior of our *isochronous* system (44) in the neighborhood of its equilibrium configuration. To this end we set

$$z_n(t) = \overline{z}_n + \varepsilon w_n(t) \tag{65a}$$

and we then insert this assignment in the equations of motion (44) treating ε as a small parameter. We thus get the *linearized* equations of motion

$$\ddot{w}_n - 3\mathrm{i}\omega\dot{w}_n - 2\omega^2 w_n = 2a\overline{z}_n(\dot{w}_n - 2\mathrm{i}\omega w_n)$$
$$+ 2\sum_{m=1,m\neq n}^{N}\left[\frac{-\mathrm{i}\omega(\overline{z}_m\dot{w}_n + \overline{z}_n\dot{w}_m)}{\overline{z}_n - \overline{z}_m} + \frac{\omega^2(\overline{z}_m^2 w_n - \overline{z}_n^2 w_m)}{(\overline{z}_n - \overline{z}_m)^2}\right] \tag{66a}$$

namely

$$\underline{\ddot{w}} + \mathrm{i}\omega\underline{\Gamma}\,\underline{\dot{w}} + \omega^2\underline{\Lambda}\,\underline{w} = 0. \tag{66b}$$

Here and throughout this section, to underline the vector and matrix character of our formulas, N-vectors are denoted by *lower* case *underlined* letters, hence $\underline{w} \equiv \underline{w}(t)$ denotes the N-vector of components $w_n \equiv w_n(t)$, and likewise $N \times N$ matrices are denoted by *upper* case *underlined* letters. In particular the two (constant) matrices $\underline{\Gamma}$ and $\underline{\Lambda}$ are defined (componentwise) as follows, via (60), in terms of the N zeros x_n of the generalized Laguerre polynomial $L_N^{(-2N-1)}(x)$

$$\Gamma_{nm} = -\delta_{nm} + (1 - \delta_{nm})\frac{2x_n}{x_n - x_m} \tag{67a}$$

$$\Lambda_{nm} = -\delta_{nm}2\left[1 + x_n + \sum_{\ell=1,\ell\neq n}^{N}\frac{x_\ell^2}{(x_n - x_\ell)^2}\right] + (1 - \delta_{nm})\frac{2x_n^2}{(x_n - x_m)^2}. \tag{67b}$$

Note that to simplify the expression of the diagonal part of the matrix $\underline{\Gamma}$ we used (61a).

The *general* solution of the linear evolution equations (66) is provided by the formula

$$\underline{w}(t) = \sum_{k=1}^{2N} a_k \exp(\mathrm{i}\lambda_k\omega t)\underline{v}^{(k)} \tag{68}$$

where the $2N$ constants a_k are arbitrary (to be determined, in the context of the initial-value problem, from the $2N$ initial data $w_n(0)$ and $\dot{w}_n(0)$), while the $2N$ numbers λ_k, respectively the corresponding N-vectors $\underline{v}^{(k)}$, are the eigenvalues, respectively the eigenvectors, of the following (generalized) N-vector eigenvalue equation (see (66b))

$$-\lambda_k^2\underline{v}^{(k)} - \lambda_k\underline{\Gamma}\,\underline{v}^{(k)} + \underline{\Lambda}\,\underline{v}^{(k)} = 0, \qquad k = 1, \ldots, 2N.$$

Hence the numbers λ_k are the $2N$ roots of the following equation (polynomial of degree $2N$) in λ

$$\det[\lambda^2\underline{1} + \lambda\underline{\Gamma} - \underline{\Lambda}] = 0. \tag{69}$$

Here and throughout $\underline{1}$ denotes of course the $N \times N$ unit matrix, $(\underline{1})_{nm} = \delta_{nm}$.

But we already know, from our previous treatment, that the solutions of the *isochronous* model (44) are *completely periodic* with period T, see (1). Actually solutions with a (larger) period which is an integer multiple of T can also emerge, due to the exchange of the identity of the eigenvalues of the matrix $U(t)$, see (45), through the motion; and some exceptional singular solutions also exist, in which two or more particles collide at some finite time; but neither one of these two phenomena can occur for the *small* oscillations around the equilibrium configuration considered here. The same periodicity property must therefore characterize the behavior (68) in the neighborhood of the equilibrium configuration of this system. We thus arrive at the following *diophantine* finding: *the $2N$ numbers λ_k are all integers*. In fact, motivated by this finding and by some numerical checks, we make the following

Conjecture. *Let the two $N \times N$ matrices $\underline{\Gamma}$ and $\underline{\Lambda}$ be defined by* (67)*, in terms of the N zeros x_n of the generalized Laguerre polynomial $L_N^{(-2N-1)}(x)$, namely the polynomial of degree N characterized by the ODE* (63)*. Then*

$$\det[\lambda^2 \underline{1} + \lambda \underline{\Gamma} - \underline{\Lambda}] = \prod_{k=1}^{N} [(\lambda - 2k)(\lambda + 2k - 1)]. \tag{70}$$

A related *diophantine* conjecture – more explicit hence more suitable for numerical checks – is now obtained in the context of the investigation of the behavior of the *isochronous* system (58) in the neighborhood of its equilibrium configuration

$$c_m(t) = \bar{c}_m = \left(\frac{\mathrm{i}\omega}{2a}\right)^m \frac{(N+m)!}{(N-m)!m!}. \tag{71}$$

The fact that this formula provides a time-independent solution of (58) can be easily verified, as well as its consistency with (62) and (60).

To study the behavior of the system (58) in the neighborhood of this equilibrium configuration, (71), we now set

$$c_m(t) = \bar{c}_m + \varepsilon \left(\frac{\mathrm{i}\omega}{2a}\right)^m \eta_m(t) \tag{72}$$

and by treating ε as a small parameter we obtain the following *linearized* system for the time evolution of the quantities $\eta_m(t)$

$$\begin{aligned} &\ddot{\eta}_m - \mathrm{i}(2m+1)\omega\dot{\eta}_m + [N(N+1) - m(m+1)]\omega^2\eta_m - \mathrm{i}\omega\dot{\eta}_{m+1} \\ &\quad - (m+1)\omega^2\eta_{m+1} + \frac{(N+m)!}{(N-m)!m!}[\mathrm{i}\omega\dot{\eta}_1 + \omega^2\eta_1] = 0, \quad \eta_0 = 0, \; \eta_{N+1} = 0 \end{aligned} \tag{73a}$$

namely

$$\underline{\ddot{\eta}} + \mathrm{i}\omega\underline{C}\underline{\dot{\eta}} + \omega^2\underline{L}\underline{\eta} = 0 \tag{73b}$$

where the two matrices $\underline{C}$ and $\underline{L}$ are defined (componentwise) as follows

$$C_{nm} = -\delta_{nm}(2n+1) - \delta_{n+1,m} + \delta_{1m}\frac{(N+n)!}{(N-n)!n!} \tag{74a}$$

$$L_{nm} = \delta_{nm}[N(N+1) - n(n+1)] - \delta_{n+1,m}(n+1) + \delta_{1m}\frac{(N+n)!}{(N-n)!n!}. \tag{74b}$$

The *general* solution of this system of linear equations reads

$$\underline{\eta}(t) = \sum_{k=1}^{2N} b_k \exp(\mathrm{i}\lambda_k \omega t)\underline{w}^{(k)} \tag{75}$$

where the $2N$ constants b_k are arbitrary (to be determined, in the context of the initial-value problem, from the $2N$ initial data $\eta_n(0)$ and $\dot{\eta}_n(0)$), while the $2N$ numbers λ_k, respectively the corresponding N-vectors $\underline{w}^{(k)}$, are the eigenvalues, respectively the eigenvectors, of the following (generalized) matrix eigenvalue equation (see (66b))

$$-\lambda_k^2 \underline{w}^{(k)} - \lambda_k \underline{C}\,\underline{w}^{(k)} + \underline{L}\,\underline{w}^{(k)} = 0, \qquad k = 1, \ldots, 2, N. \tag{76}$$

Hence the numbers λ_k are the $2N$ roots of the following equation (polynomial of degree $2N$) in λ

$$\det[\lambda^2 \underline{1} + \lambda \underline{C} - \underline{L}] = 0. \tag{77}$$

But obviously these numbers coincide with those defined above, see (69). We may therefore assert with certainty that they are *all* integers, and the **Conjecture** made above can now be reformulated to read

$$\det[\lambda^2 \underline{1} + \lambda \underline{C} - \underline{L}] = \prod_{k=1}^{N} [(\lambda - 2k)(\lambda + 2k - 1)]. \tag{78}$$

Other versions of this conjecture obtain via some additional manipulations – see [22]. We display here just one specific example (of course true) of formulas obtained in this manner

$$\begin{vmatrix} \lambda + 29 & -14 & 0 & 0 & 0 \\ 30(\lambda - 1) & (\lambda - 8)(\lambda + 3) & -8(\lambda - 3) & 0 & 0 \\ 0 & -(\lambda - 8)(\lambda + 3) & (\lambda - 6)(\lambda + 7) & -\frac{9}{2}(\lambda - 4) & 0 \\ 0 & 0 & -(\lambda - 9)(\lambda + 2) & (\lambda - 8)(\lambda + \frac{7}{2}) & -2(\lambda - 5) \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$
$$= (\lambda - 4)(\lambda - 6)(\lambda - 8)(\lambda + 3)(\lambda + 5)(\lambda + 7)(\lambda + 9). \tag{79}$$

3.5. Additional Remarks

Recently, via an approach analogous to that described in Section 3.3, a model remarkably similar to that considered in this section has been investigated by David Gomez-Ullate, Andy Hone and Matteo Sommacal [34]. The equations of motions of their model, in its "many-body problem" formulation, read

$$\ddot{z}_n = 2(1 - N)z_n^3 + 2 \sum_{m=1, m \neq n}^{N} \frac{\dot{z}_n \dot{z}_m + z_n^4}{z_n - z_m} \tag{80}$$

and those of the model related to it via the approach of the preceding section read

$$\ddot{c}_m + (m+1)(m+2)c_{m+2} - 2c_2 c_m = 0 \tag{81a}$$

with the "boundary conditions"

$$c_0 = 1, \qquad c_{N+1} = 0. \tag{81b}$$

Via the simple change of dependent variables

$$z_n = b\tilde{z}_n, \qquad b = \left(\frac{a}{1-N}\right)^{1/2} \tag{82}$$

the equations of motion (80) can be recast in the form

$$\ddot{\tilde{z}}_n = 2a\tilde{z}_n^3 + 2 \sum_{m=1, m\neq n}^{N} \frac{\dot{\tilde{z}}_n \dot{\tilde{z}}_m + b^2 \tilde{z}_n^4}{\tilde{z}_n - \tilde{z}_m} \tag{83}$$

demonstrating that they are indeed rather similar to (41). Likewise, via the simple change of dependent variables

$$c_{2m} = \frac{(-a)^m \tilde{c}_m}{(2m)!} \tag{84}$$

the evolution equations (81a) (with even m; note that they are decoupled from those for odd m) become

$$\ddot{\tilde{c}}_m - 2a\tilde{c}_{m+1} + 2a\tilde{c}_1 \tilde{c}_m = 0 \tag{85}$$

which are remarkably similar to the evolution equations (53a). However, in contrast to the systems treated here, (41) and (53), which are *solvable*, as explained above, for *all* values of the positive integer N, the systems (80) and (81), or equivalently (83) and (85), have been shown to be *solvable* only for $N < 4$; moreover, for $N = 3$ their solution generally involves transcendental (more precisely: elliptic) functions of the time variable, as well as algebraic functions of such elliptic functions (typically roots of N-degree polynomials the coefficients of which evolve in time as elliptic functions), in contrast to the solution of the models treated in this paper, which clearly only involve, for all values of N, algebraic functions of elementary (more precisely: exponential, or equivalently trigonometric) functions of the time variable (again, typically, via roots of N-degree polynomials the coefficients of which evolve exponentially, or equivalently trigonometrically, in the time variable, see (36a)).

Lecture 4

4. Isochronous PDEs

In this section we outline a version of "the trick" suitable to deform PDEs so that the deformed versions are *isochronous*, and we then apply it to quite a few "well-known" PDEs, many of them *integrable* or *solvable*. In this context by *isochronous* PDEs we mean evolution partial differential equations that possess *lots* of *completely periodic* solutions, generally, in the context of the initial-value problem, obtainable from *open* domains of initial data having full dimensionality in the phase space of such data. The main thrust of our presentation is to emphasize the wide applicability of this approach, as well as the neat look of the *isochronous* PDEs yielded by it, providing thereby some support for the slogan "isochronous PDEs are not rare". Hence the major part of this section consists merely of a listing of *unmodified* and modified PDEs, lifted from the recent paper [38] which we occasionally reproduce below *verbatim* and to which we refer for additional information on these PDEs, including the appropriate references that justify attributing to them – as we shall do on a case by case basis – the property to be *integrable* or *solvable*.

4.1. Notation and Preliminaries: The Trick

The independent variables of the *unmodified* evolution PDE are denoted as $\underline{\xi} \equiv (\xi_1, \ldots, \xi_N)$ and τ; the dependent variable of the *unmodified* evolution PDE is denoted as $w(\underline{\xi}; \tau) \equiv w(\xi_1, \ldots, \xi_n; \tau)$, and if a second dependent variable also enters, it is denoted as $\tilde{w}(\underline{\xi}; \tau) \equiv \tilde{w}(\xi_1, \ldots, \xi_n; \tau)$; upper case letters, $W(\underline{\xi}; \tau) \equiv W(\xi_1, \ldots, \xi_n; \tau)$, $\widetilde{W}(\underline{\xi}; \tau) \equiv \widetilde{W}(\xi_1, \ldots, \xi_n; \tau)$, are used for matrices. The independent variables of the *modified* evolution PDE are denoted as $\underline{x} \equiv (x_1, \ldots, x_n)$ and t; the dependent variable of the *modified* evolution PDE is denoted as $u(\underline{x}; t) \equiv u(x_1, \ldots, x_n; t)$, and if a second dependent variable also enters, it is denoted as $\tilde{u}(\underline{x}; t) \equiv \tilde{u}(x_1, \ldots, x_n; t)$; and again upper case letters, $U(\underline{x}; t) \equiv U(x_1, \ldots, x_n; t)$, $\widetilde{U}(\underline{x}; t) \equiv \widetilde{U}(x_1, \ldots, x_n; t)$, are used for matrices. The relation among the (independent and dependent) variables of the *unmodified* evolution PDE and the *modified* evolution PDE are given by the following formulas ("the trick")

$$\tau = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega} \tag{86}$$

$$\xi_n = \xi_n(t) = x_n \exp(\mathrm{i}\mu_n \omega t), \qquad n = 1, \ldots, N \tag{87}$$

$$u(\underline{\xi}; t) = \exp(\mathrm{i}\lambda\omega t) w(\underline{\xi}; \tau) \tag{88a}$$

$$\tilde{u}(\underline{\xi}; t) = \exp(\mathrm{i}\tilde{\lambda}\omega t) \tilde{w}(\underline{\xi}; \tau) \tag{88b}$$

with analogous formulas, see (88), in the matrix case. Here and hereafter constants such as μ_n, λ, $\widetilde{\lambda}$, α, β (Greek letters) denote *rational* numbers (not necessarily positive), which whenever necessary shall be properly assigned, while Latin letters such as a, b, c denote *complex* (or, as the case may be, *real*) constants (sometimes we keep such constants even when they could be eliminated by trivial rescaling transformations; and of course by such transformations additional such constants might instead be introduced). When $N = 1$ we drop the index n, namely we write ξ instead of ξ_1, x instead of x_1, μ instead of μ_1, and for $N = 2$ we also, to simplify the notation, write η instead of ξ_2, y instead of x_2, ν instead of μ_2.

Note that this transformation, (86)–(88), entails that, at the initial time, $\tau = t = 0$, the change of variables disappears altogether

$$\underline{\xi}(0) = \underline{x}, \qquad u(\underline{x}; 0) = w(\underline{\xi}; 0), \qquad \widetilde{u}(\underline{\xi}; 0) = \widetilde{w}(\underline{\xi}; 0). \tag{89}$$

Hereafter subscripted variables denote partial differentiations, $w_\tau \equiv \frac{\partial w(\underline{\xi};\tau)}{\partial \tau}$, $u_{x_n} \equiv \frac{\partial u(\underline{x};t)}{\partial x_n}$ and so on.

Let us emphasize – obvious as this may be – that, since the transition from an *unmodified* PDE satisfied by $w(\underline{\xi}; \tau)$ to the corresponding *modified* PDE satisfied by $u(\underline{x}; t)$ is performed via the explicit change of variables (86)–(88) ("the trick"), properties such as *integrability* or *solvability*, if possessed by the *unmodified* PDE satisfied by $w(\underline{\xi}; \tau)$, carry over to the corresponding (*modified*) PDE satisfied by $u(\underline{x}; t)$ – which generally has in addition the property of *isochronicity*, as defined above. Let us moreover note that the property of *isochronicity* of the *modified* evolution PDE – which does *not* require that the original, *unmodified* PDE from which it has emerged be itself *integrable* – implies that in some open set of its phase space the *modified* equation is generally *integrable*, indeed, in some sense, *superintegrable* (for a discussion of this question in the ODEs context see Section 1).

Let us end this section by pointing out that, in most cases, the *modified* PDEs are *complex*; they can of course be rewritten in *real* form by introducing the *real* and *imaginary* parts (or, instead, the amplitudes and phases) of all the quantities that enter in these PDEs, and by then considering the two, generally coupled, *real* PDEs that obtain from each *complex* PDE by considering separately its *real* and *imaginary* parts; below, in a few cases, we also exhibit the system of *real* evolution PDEs obtained in this manner.

4.2. Examples

In this section we display, with minimal commentary, a list of nonlinear evolution PDEs, each of them firstly in its *unmodified* avatar, then in its *modified* version. It is remarkable that so many "well-known" *autonomous* evolution PDEs possess *modified* versions which are as well autonomous, at least as regards their time dependence; in several cases this appears to be due to some minor miracle, inasmuch as the number of relevant parameters λ, μ, ν, $\widetilde{\lambda}$, ..., is smaller than the equations they are required to satisfy in order to guarantee the *autonomous* character of the *modified* equations, yet nontrivial parameters satisfying these conditions do exist. All the *modified* evolution PDEs displayed below possess many *isochronous* solutions – but only in rare cases we do exhibit below examples of these solutions.

A discussion of each of these nonlinear evolution PDEs would indeed require much more space. Let us also emphasize that the list reported below includes only some kind of "representative" instances of this phenomenology. Obviously many more examples could be added.

The list is arranged in a user-friendly manner, being ordered according to the following taxonomic rules: of primary importance is the number of *independent* variables; next, the number of *dependent* variables; next, the *order* of the differential equation, with primary attention to the "time" variable; finally, the type of *nonlinearity* (except when an equation is presented as a special case of a more general equation, see for instance (97) and (124)).

The following *unmodified* $(1+1)$-dimensional "generalized shock-type" PDE is *integrable*, indeed *solvable*

$$w_\tau = a w^\alpha w_\xi. \tag{90a}$$

By setting $\mu = 1 - \alpha\lambda$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\lambda\omega u + \mathrm{i}(\alpha\lambda - 1)\omega x u_x = a u^\alpha u_x. \tag{90b}$$

The *general* solution of the initial-value problem for this PDE, (90b), is given, in implicit form, by the following formula

$$u(x;t) = \mathrm{e}^{\mathrm{i}\lambda\omega t} u_0\left(\exp[\mathrm{i}(1-\alpha\lambda)\omega t]\left\{x + a\frac{1-\mathrm{e}^{-\mathrm{i}\omega t}}{\mathrm{i}\omega}[u(x;t)]^\alpha\right\}\right) \tag{90c}$$

where of course $u_0(x) = u(x;0)$.

The following *unmodified* $(1+1)$-dimensional "generalized Burgers-Hopf" PDE reads

$$w_\tau = a w w_\xi + b(w^\alpha w_\xi)_\xi. \tag{91a}$$

By setting $\lambda = \frac{1}{2-\alpha}$ and $\mu = \frac{1-\alpha}{2-\alpha}$ one gets the corresponding *modified* evolution PDE

$$u_t + \mathrm{i}\frac{1}{\alpha-2}\omega u + \mathrm{i}\frac{1-\alpha}{\alpha-2}\omega x u_x = a u u_x + b(u^\alpha u_x)_x. \tag{91b}$$

Note that, for $\alpha = 1$, this PDE becomes *autonomous* also with respect to the 'space' variable x, while for $\alpha = 0$ the PDE (91a) becomes the standard (*solvable*) Burgers-Hopf equation.

The following *unmodified* $(1+1)$-dimensional dispersive KdV-like PDE is *integrable*, indeed *solvable*

$$w_\tau = w_{\xi\xi\xi} + 3[w_{\xi\xi}w^2 + 3(w_\xi)^2] + 3w_\xi w^4. \tag{92a}$$

By setting $\lambda = \frac{1}{6}$, $\mu = \frac{1}{3}$ and (for notational simplicity) $\Omega = \frac{\omega}{6}$ one gets the corresponding Ω-modified evolution PDE

$$u_t - \mathrm{i}\Omega u - 2i\Omega x u_x = u_{xxx} + 3(u_{xx}u^2 + 3u_x^2) + 3u_x u^4. \tag{92b}$$

The symmetry properties, and some explicit solutions, of the following *unmodified* $(1+1)$-dimensional "generalized KdV equation" have been investigated recently

$$w_\tau = a(w^\alpha)_{\xi\xi\xi} + b(w^\beta)_\xi. \tag{93a}$$

By setting $\lambda = \frac{2}{3\beta-\alpha-2}$, $\mu = \frac{\beta-\alpha}{3\beta-\alpha-2}$ one gets the *modified* evolution PDE

$$u_t - \frac{2i\omega}{3\beta-\alpha-2}u - \frac{\mathrm{i}(\beta-\alpha)\omega}{3\beta-\alpha-2}xu_x = a(u^\alpha)_{xxx} + b(u^\beta)_x. \tag{93b}$$

We assume here of course that $3\beta - \alpha - 2 \neq 0$. Particularly interesting is the case with $\alpha = \beta$, when this *modified* PDE becomes *autonomous* also in the space variable x. In the even more special case with $\alpha = \beta = 2$, and by setting $u = u_1 + \mathrm{i}u_2$, $a = c_1 + \mathrm{i}c_2$, $b = c_3 + \mathrm{i}c_4$, we re-write this evolution PDE as a system of two coupled PDEs

$$\begin{aligned} u_{1_t} + \omega u_2 &= [c_1(u_1^2 - u_2^2) - 2c_2u_1u_2]_x + [c_3(u_1^2 - u_2^2) - 2c_4u_1u_2]_{xxx} \\ u_{2_t} - \omega u_1 &= [c_2(u_1^2 - u_2^2) + 2c_1u_1u_2]_x + [c_4(u_1^2 - u_2^2) + 2c_3u_1u_2]_{xxx}. \end{aligned} \tag{93c}$$

Here of course we assume that the two dependent variables $u_1 \equiv u_1(x,t), u_2 \equiv u_2(x,t)$ are *real*, and that as well *real* are the 4 arbitrary constants c_1, c_2, c_3, c_4.

The following *unmodified* $(1+1)$-dimensional "Schwarzian KdV" equation is *integrable*

$$w_\tau = w_{\xi\xi\xi} + a\frac{w_{\xi\xi}^2}{w_\xi}. \tag{94a}$$

By setting $\mu = 1/3$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\lambda\omega u - \mathrm{i}\frac{\omega}{3}xu_x = u_{xxx} + a\frac{u_{xx}^2}{w_x}. \tag{94b}$$

The following *unmodified* $(1+1)$-dimensional **Cavalcante-Tenenblat equation** is *integrable*

$$w_\tau = a\left(\frac{1}{\sqrt{w_\xi}}\right)_{\xi\xi} + b(w_\xi)^{3/2}. \tag{95a}$$

By setting $\mu = 1/3$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\lambda\omega u - \mathrm{i}\frac{\omega}{3}xu_x = a\left(\frac{1}{\sqrt{u_x}}\right)_{xx} + b(u_x)^{3/2}. \tag{95b}$$

The KdV class of *unmodified* $(1+1)$-dimensional *integrable* evolution PDEs reads

$$w_\tau = \Lambda^m w_\xi, \qquad m = 1, 2, \ldots \tag{96a}$$

where Λ is the integrodifferential operator (depending on the dependent variable $w(\xi;\tau)$) that acts on a generic (twice-differentiable, and integrable at infinity) function $\phi(\xi)$ as follows

$$\Lambda\phi(\xi) = \phi_{\xi\xi}(\xi) - 4w(\xi;\tau)\phi(\xi) + 2w_\xi(\xi;\tau)\int_\xi^\infty \phi(\xi')\,\mathrm{d}\xi'. \tag{96b}$$

By setting $\lambda = \frac{2}{2m+1}$, $\mu = \frac{1}{2m+1}$ and (for notational simplicity) $\Omega_m = \frac{\omega}{2m+1}$ one gets the corresponding class of Ω_m-modified evolution PDEs

$$u_t - \mathrm{i}\Omega_m(2u + xu_x) = \mathrm{L}^m u_x \tag{96c}$$

where L is the integrodifferential operator (depending on the dependent variable $u(x;t)$) analogous to Λ, namely the operator that acts on a generic (twice-differentiable, and integrable at infinity) function $f(x)$ as follows

$$\mathrm{L}f(x) = f_{xx}(x) - 4u(x;t)f(x) + 2u_x(x;t)\int_x^\infty f(x')\,\mathrm{d}x', \qquad m = 1, 2, \ldots \tag{96d}$$

For $m = 1$ the PDE (96a) becomes the well-known KdV equation

$$w_\tau + w_{\xi\xi\xi} = 6ww_\xi \tag{97a}$$

and the corresponding *modified* equation reads

$$u_t + u_{xxx} - \mathrm{i}\frac{\omega}{3}(2u + u_x) = 6uu_x. \tag{97b}$$

The *unmodified* (1+1)-dimensional "Monge-Ampère" *integrable* PDE reads

$$w_{\tau\tau}w_{\xi\xi} - w_{\xi\tau}^2 = 0. \tag{98a}$$

The corresponding *modified* PDE is in this case t-autonomous for any choice of λ and μ

$$\begin{aligned} u_{tt}u_{xx} - (u_{tx})^2 + \mathrm{i}\omega[-(2\lambda+1)u_t u_{xx} + 2(\lambda+\mu)u_{tx}u_x] \\ + \omega^2[-\lambda(\lambda+1)uu_{xx} + \mu(\mu-1)xu_x u_{xx} + (\lambda+\mu)^2(u_x)^2] = 0. \end{aligned} \tag{98b}$$

The *general* solution of this PDE (98b) is given in two steps: first, for any arbitrary function $F(r)$, find the function $r(x;t)$ from the *nondifferential* equation

$$r(x;t) = x\mathrm{e}^{\mathrm{i}(\mu-1)\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}F[r(x;t)] \tag{98c}$$

then, for any arbitrary function $G(r)$ and constant a, evaluate the solution

$$u(x;t) = \mathrm{e}^{\mathrm{i}\lambda\omega t} \int_a^t \mathrm{d}t' \mathrm{e}^{\mathrm{i}\omega t'} G[r(x;t')]. \tag{98d}$$

A class of explicit solutions of this PDE (98b) is

$$u(x;t) = \mathrm{e}^{\mathrm{i}(\lambda+\mu)\omega t} x f \left[\frac{\sin\left(\frac{\omega t}{2}\right)}{x \mathrm{e}^{\mathrm{i}(\mu-1/2)\omega t}} \right] \tag{98e}$$

where $f(z)$ is an arbitrary function. Three particularly neat cases of the *modified* evolution PDE (69) are worth explicit display: for $\lambda = \mu = 0$,

$$u_{tt}u_{xx} - u_{tx}^2 - \mathrm{i}\omega u_t u_{xx} = 0 \tag{98f}$$

for $\lambda = -1$, $\mu = 1$

$$u_{tt}u_{xx} - u_{tx}^2 + \mathrm{i}\omega u_t u_{xx} = 0 \tag{98g}$$

for $\lambda = -1/2$, $\mu = 1/2$

$$u_{tt}u_{xx} - u_{tx}^2 + \left(\frac{\omega}{2}\right)^2 (u - x u_x) u_{xx} = 0. \tag{98h}$$

The following *unmodified* $(1+1)$-dimensional *solvable* PDE reads

$$w_{\tau\tau} w_\xi - w_{\tau\xi} w_\tau = 0. \tag{99a}$$

The corresponding *modified* PDE is in this case t-autonomous for any choice of λ and μ

$$\begin{aligned} u_{tt}u_x - u_{tx}u_t + \mathrm{i}\omega[(-\lambda+\mu+1)u_t u_x + \lambda u u_{tx} - \mu x u_{tx} u_x + \mu x u_t u_{xx}] \\ + \omega^2[\lambda(\mu-1)u u_x + (-\lambda+\mu-2)\mu x (u_x)^2 + \lambda\mu x u u_{xx}] = 0. \end{aligned} \tag{99b}$$

The *general* solution of this PDE reads

$$u(x;t) = \mathrm{e}^{\mathrm{i}\lambda\omega t} f[g(x\mathrm{e}^{\mathrm{i}\mu\omega t}) + \mathrm{e}^{\mathrm{i}\omega t}] \tag{99c}$$

where $f(z)$, $g(z)$ are two arbitrary analytic functions.

The following *unmodified* **$(1+1)$-dimensional Boussinesq equation** is *integrable*

$$w_{\tau\tau} = (w_{\xi\xi\xi} + w w_\xi)_\xi. \tag{100a}$$

By setting $\lambda = 1$ and $\mu = 1/2$ one gets the corresponding *modified* evolution PDE

$$u_{tt} - \mathrm{i}\omega u - \frac{\mathrm{i}\omega}{2} x u_x = (u_{xxx} - u u_x)_x. \tag{100b}$$

The following *unmodified* $(1+1)$-dimensional “nonlinear wave equation” reads

$$w_{\tau\tau} = (w^\alpha w_\xi)_\xi. \tag{101a}$$

By setting $\mu = 1 - \frac{\alpha\lambda}{2}$ one gets the corresponding *modified* evolution PDE

$$u_{tt} - \mathrm{i}(2\lambda + 1)\omega u_t - \mathrm{i}\omega(2 - \alpha\lambda)xu_{tx} - \lambda(\lambda + 1)\omega^2 u$$
$$- (2 - \alpha\lambda)\left(\lambda + 1 - \frac{\alpha\lambda}{4}\right)\omega^2 x u_x - \left(1 - \frac{\alpha\lambda}{2}\right)^2 x^2 u_{xx} = (u^\alpha u_x)_x. \tag{101b}$$

Two special cases of this nonlinear PDE warrant explicit display

$$u_{tt} + \left(\frac{\omega}{2}\right)^2 u = \left(\frac{u_x}{u^4}\right)_x \tag{101c}$$

corresponding to $\alpha = -4$, $\lambda = -1/2$, and

$$u_{tt} - 5\omega u_t - 6\omega^2 u = (uu_x)_x \tag{101d}$$

corresponding to $\alpha = 1$, $\lambda = 2$.

Another class (out of many possible ones) of *unmodified* **$(1 + 1)$-dimensional nonlinear wave equations** reads

$$w_{\tau\tau} = \sum_k \frac{a_k}{w^{3+\alpha_k}} \left(\frac{\partial^{p_k} w}{\partial \xi^{p_k}}\right)^{\alpha_k} \tag{102a}$$

where the numbers p_k are *nonnegative integers* (or possibly just *integers*). By setting $\lambda = -1/2$, $\mu = 0$ one gets the corresponding *modified* evolution PDE

$$u_{tt} + \left(\frac{\omega}{2}\right)^2 u = \sum_k \frac{a_k}{u^{3+\alpha_k}} \left(\frac{\partial^{p_k} u}{\partial x^{p_k}}\right)^{\alpha_k}. \tag{102b}$$

The following *unmodified* $(1 + 1)$-dimensional system of two coupled PDEs is *integrable*

$$w_\tau = w_{\xi\xi} + \tilde{w}^2$$
$$\tilde{w}_\tau = w_{\xi\xi}. \tag{103a}$$

By setting $\lambda = \tilde{\lambda} = 1$, $\mu = 1/2$ one gets the corresponding *modified* system

$$u_t - \mathrm{i}\omega u - \frac{\mathrm{i}\omega}{2} x u_x = u_{xx} + \tilde{u}^2$$
$$\tilde{u}_u - \mathrm{i}\omega\tilde{u} - \frac{\mathrm{i}\omega}{2} x \tilde{u}_x = u_{\xi\xi}. \tag{103b}$$

The following *unmodified* $(1 + 1)$-dimensional system of two coupled PDEs is *integrable*

$$w_\tau = a(w\tilde{w})_\xi$$
$$\tilde{w}_\tau = (bw + c\tilde{w}^2)_\xi. \tag{104a}$$

By setting $\lambda = 2(1-\mu)$, $\tilde{\lambda} = 1-\mu$ one gets the corresponding *modified* system

$$\begin{aligned} u_t - 2\mathrm{i}(1-\mu)\omega u - \mathrm{i}\mu\omega x u_x &= a(u\tilde{u})_x \\ \tilde{u}_t - \mathrm{i}(1-\mu)\omega\tilde{u} - \mathrm{i}\mu\omega x\tilde{u}_x &= (bu + c\tilde{u}^2)_x. \end{aligned} \tag{104b}$$

The following *unmodified* $(1+1)$-dimensional "Zakharov-Shabat" system of two coupled PDEs is *integrable*

$$\begin{aligned} w_\tau + w_{\xi\xi} &= w^2\tilde{w} \\ \tilde{w}_\tau - \tilde{w}_{\xi\xi} &= -\tilde{w}^2 w. \end{aligned} \tag{105a}$$

By setting $\tilde{\lambda} = 1-\lambda$, $\mu = 1/2$ one gets the corresponding *modified* system

$$\begin{aligned} u_t - \mathrm{i}\lambda\omega u - \mathrm{i}\frac{\omega}{2}x u_x + u_{xx} &= u^2\tilde{u} \\ \tilde{u}_t - \mathrm{i}(1-\lambda)\omega\tilde{u} - \mathrm{i}\frac{\omega}{2}x\tilde{u}_x - \tilde{u}_{xx} &= -u\tilde{u}^2. \end{aligned} \tag{105b}$$

The following *unmodified* **$(1+1)$-dimensional Wadati-Konno-Ichikawa system** of two coupled PDEs is *integrable*

$$\begin{aligned} w_\tau &= a\left(\frac{w}{\sqrt{1+w\tilde{w}}}\right)_{\xi\xi} \\ \tilde{w}_\tau &= b\left(\frac{\tilde{w}}{\sqrt{1+w\tilde{w}}}\right)_{\xi\xi}. \end{aligned} \tag{106a}$$

By setting $\tilde{\lambda} = -\lambda$, $\mu = 1/2$ one gets the corresponding *modified* system

$$\begin{aligned} u_t - \mathrm{i}\lambda\omega u - \mathrm{i}\frac{\omega}{2}x u_x &= a\left(\frac{u}{\sqrt{1+u\tilde{u}}}\right)_{xx} \\ \tilde{u}_t + \mathrm{i}\lambda\omega\tilde{u} - \mathrm{i}\frac{\omega}{2}x\tilde{u}_x &= b\left(\frac{\tilde{u}}{\sqrt{1+u\tilde{u}}}\right)_{xx}. \end{aligned} \tag{106b}$$

The following *unmodified* **$(1+1)$-dimensional Landau-Lifshitz system** of two coupled PDEs is *integrable*

$$\begin{aligned} w_\tau &= -\sin(w)\tilde{w}_{\xi\xi} - 2\cos(w)w_\xi\tilde{w}_\xi + (a-b)\sin(w)\cos(\tilde{w})\sin(\tilde{w}) \\ \tilde{w}_\tau &= \frac{w_{\xi\xi}}{\sin(w)} - \cos(w)(\tilde{w}_\xi)^2 + \cos(w)(a\cos^2(\tilde{w}) + b\sin^2(\tilde{w}) + c). \end{aligned} \tag{107a}$$

By setting $\lambda = \tilde{\lambda} = 0$, $\mu = 1/2$ one gets the corresponding *modified* system

$$\begin{aligned} u_t - \frac{\mathrm{i}\omega}{2}x u_x &= -\sin(u)\tilde{u}_{xx} - 2\cos(u)u_x\tilde{u}_x + (a-b)\sin(u)\cos(\tilde{u})\sin(\tilde{u}) \\ \tilde{u}_t - \frac{\mathrm{i}\omega}{2}x\tilde{u}_x &= \frac{u_{xx}}{[\sin(u)]} - \cos(u)\tilde{u}_x^2 + \cos(u)(a\cos^2(\tilde{u}) + b\sin^2(\tilde{u}) + c). \end{aligned} \tag{107b}$$

Note the simplification if $a = b$, and, moreover, if $c = -a$.

The following *unmodified* $(2+1)$-dimensional PDE is *integrable*

$$w_\tau = a w_\eta + b w w_\xi. \tag{108a}$$

By setting $\lambda = 0$, $\mu = \nu = 1$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\omega(x u_x + y u_y) = a u_y + b u u_x. \tag{108b}$$

The following *unmodified* $(2+1)$-dimensional PDE is *integrable*

$$w_\tau = a w_\eta + b (w_\xi)^2. \tag{109a}$$

By setting $\lambda = 0$, $\mu = 1/2$, $\nu = 1$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\omega\left(\frac{x}{2}u_x + y u_y\right) = a u_y + b(u_x)^2. \tag{109b}$$

The following *unmodified* $(2+1)$-dimensional PDE reads

$$w_\tau = a(w_\xi w_\eta - w w_{\xi\eta})^\alpha. \tag{110a}$$

By setting $\lambda = \frac{1}{2\alpha-1}$, $\mu = \nu = 0$ one gets the corresponding *modified* evolution PDE

$$u_t - \frac{\mathrm{i}\omega}{2\alpha-1}\omega u = a(u_x u_y - u u_{xy})^\alpha. \tag{110b}$$

A (rather trivial) separable solution of this PDE reads

$$u(x, y; t) = \exp\left(\frac{\mathrm{i}\omega t}{2\alpha-1}\right) f(x) g(y) \tag{110c}$$

where $f(x)$, $g(y)$ are two arbitrary functions.
The following *unmodified* $(2+1)$-dimensional PDE reads

$$w_{\tau\tau} = a(w_\xi w_\eta - w w_{\xi\eta})^\alpha. \tag{111a}$$

By setting $\lambda = \frac{2}{2\alpha-1}$, $\mu = \nu = 0$ one gets the corresponding *modified* evolution PDE

$$u_{tt} - \frac{2\alpha+3}{2\alpha-1}\mathrm{i}\omega u_t - \frac{2(2\alpha+1)}{(2\alpha-1)^2}\omega^2 u = a(u_x u_y - u u_{xy})^\alpha. \tag{111b}$$

A (rather trivial) separable solution of this PDE reads

$$u(x, y; t) = \left[b \exp\left(\frac{2\mathrm{i}\omega t}{2\alpha-1}\right) + c \exp\left(\frac{2\alpha+1}{2\alpha-1}\mathrm{i}\omega t\right)\right] f(x) g(y) \tag{111c}$$

where a, b are two arbitrary constants and $f(x)$, $g(y)$ are two arbitrary functions.
The following *unmodified* $(2+1)$-dimensional system of two coupled PDEs is *integrable*

$$\begin{aligned} w_\tau + w_{\xi\xi\xi} &= (w\tilde{w})_\xi \\ \tilde{w}_\eta &= w_\xi. \end{aligned} \tag{112a}$$

By setting $\lambda = \nu + 1/3$, $\tilde{\lambda} = 2/3$, $\mu = 1/3$ and, for notational convenience, $\Omega = \omega/3$, one gets the corresponding Ω-modified system

$$\begin{aligned} u_t + u_{xxx} - \mathrm{i}(3\nu + 1)\Omega u - \mathrm{i}\Omega x u_x - 3\mathrm{i}\nu\Omega y u_y &= (u\tilde{u})_x \\ \tilde{u}_y &= u_x . \end{aligned} \tag{112b}$$

The following *unmodified* **(2 + 1)-dimensional long-wave equation system** of two coupled PDEs is *integrable*

$$\begin{aligned} w_{\tau\eta} + \tilde{w}_{\xi\xi} &= \frac{1}{2}(w^2)_{\xi\eta} \\ \tilde{w}_\tau + w_{\xi\xi} &= (w\tilde{w} + w_{\xi\eta})_\xi . \end{aligned} \tag{113a}$$

By setting $\lambda = 1/2$, $\tilde{\lambda} = 0$, $\mu = 1/2$, $\nu = -1/2$ and, for notational convenience, $\Omega = \omega/2$, one gets the corresponding *modified* system

$$\begin{aligned} u_{ty} - \mathrm{i}\Omega u_y - \mathrm{i}\Omega(x u_{xy} - y u_{yy}) + \tilde{u}_{xx} &= \frac{1}{2}(u^2)_{xy} \\ \tilde{u}_t - \mathrm{i}\Omega(x u_x - y u_y) + u_{xx} &= (u\tilde{u} + u + u_{xy})_x . \end{aligned} \tag{113b}$$

The following *unmodified* $(2 + 1)$-dimensional system of two coupled PDEs is *integrable*

$$\begin{aligned} w_\tau + w_{\xi\xi\eta} &= (w^2)_\eta + w_\xi \tilde{w} \\ \tilde{w}_\xi &= w_\eta . \end{aligned} \tag{114a}$$

By setting $\tilde{\lambda} = 1 - \lambda/2$, $\mu = \lambda/2$, $\nu = 1 - \lambda$ one gets the corresponding *modified* system

$$\begin{aligned} u_t - \mathrm{i}\lambda\omega u - \frac{\mathrm{i}\lambda\omega}{2} x u_x - \mathrm{i}(1 - \lambda)\omega y u_y + u_{xxy} &= (u^2)_y + u_x \tilde{u} \\ \tilde{u}_x &= u_y . \end{aligned} \tag{114b}$$

A nontrivial family of solutions of this system, (114b), reads as follows

$$\begin{aligned} u(x, y; t) &= \frac{-4a^2 \mathrm{e}^{\mathrm{i}\lambda\omega t}}{\left\{\cosh\left[a x \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}\omega t} - f(t) - g\left(y \mathrm{e}^{\mathrm{i}(1-\lambda)\omega t} - b\mathrm{e}^{\mathrm{i}\omega t}\right)\right]\right\}^2} \\ \tilde{u}(x, y; t) &= \frac{\mathrm{e}^{\mathrm{i}\frac{\lambda}{2}\omega t}}{a}\left\{ - f'(t) + \mathrm{e}^{\mathrm{i}\omega t} g'\left(y \mathrm{e}^{\mathrm{i}(1-\lambda)\omega t} - b\mathrm{e}^{\mathrm{i}\omega t}\right)\left[\mathrm{i}\omega b - 4a^2\right.\right. \\ &\quad \left.\left. + \left(\frac{2a}{\cosh\left[a x \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}\omega t} - f(t) - g\left(y \mathrm{e}^{\mathrm{i}(1-\lambda)\omega t} - b\mathrm{e}^{\mathrm{i}\omega t}\right)\right]}\right)^2\right]\right\} \end{aligned} \tag{114c}$$

where a, b are two arbitrary constants and $f(t)$, $g(z)$ are arbitrary functions (and of course $f'(t)$, $g'(z)$ denote their derivatives).

The following *unmodified* **(2+1)-dimensional matrix KP system** of two coupled matrix PDEs is *integrable*

$$\begin{aligned} W_\tau + W_{\xi\xi\xi} - 3\widetilde{W}_\eta &= 3(W^2)_\xi + 3\mathrm{i}[W, \widetilde{W}] \\ \widetilde{W}_\xi &= W_\eta . \end{aligned} \tag{115a}$$

Here $W \equiv W(\xi, \eta; \tau)$ and $\widetilde{W} \equiv \widetilde{W} \equiv \widetilde{W}(\xi, \eta; \tau)$ are matrices (of course, of the same rank), and the notation $[W, \widetilde{W}]$ denotes their commutator. By setting $\lambda = 2/3$, $\widetilde{\lambda} = 1$, $\mu = 1/3$, $\nu = 1$ one gets the corresponding *modified* system

$$\begin{aligned} U_t + U_{xxx} - 3\widetilde{U}_y - \frac{2}{3}\mathrm{i}\omega U - \frac{1}{3}\mathrm{i}\omega x U_x - \mathrm{i}\omega y U_y &= 3(U^2)_x + 3\mathrm{i}[U, \widetilde{U}], \\ \widetilde{U}_x &= U_y . \end{aligned} \tag{115b}$$

The following class of *unmodified* $(N+1)$-dimensional PDEs,

$$\frac{\partial^{m+1} w}{\partial\tau\partial\xi^m} = f(w) \tag{116a}$$

where m is a *positive integer* (or possibly just an *integer*) and $f(w)$ is an arbitrary analytic function, gets transformed, by setting $\lambda = 0$, $\mu = -1/m$ into the corresponding *modified* evolution PDE

$$\frac{\partial^{m+1} u}{\partial t\partial x^m} + \mathrm{i}\omega \frac{\partial^m u}{\partial x^m} + \mathrm{i}\frac{\omega}{m} x \frac{\partial^{m+1} u}{\partial x^{m+1}} = f(u). \tag{116b}$$

For instance for $m = 1$ and $f(w) = \exp(aw)$, (116a) becomes the "Liouville" equation

$$w_{\tau\xi} = \exp(aw) \tag{117a}$$

and the corresponding *modified* evolution PDE (116b) reads

$$u_{tx} + \mathrm{i}\omega u_x + \mathrm{i}\omega x u_{xx} = \exp(au). \tag{117b}$$

The general solution of this *modified* **Liouville equation** reads

$$\begin{aligned} u(x;t) &= g(x\mathrm{e}^{-\mathrm{i}\omega t}) + f(t) \\ &\quad - \frac{2}{a}\ln\left\{b\mathrm{e}^{-\mathrm{i}\omega t}\int_{x_0}^{x} \mathrm{d}y \exp[ag(y\mathrm{e}^{-\mathrm{i}\omega t})] - \frac{a}{2b}\int_0^t \mathrm{d}s \exp[\mathrm{i}\omega s - af(s)]\right\} \end{aligned}$$

where b is an arbitrary (nonvanishing) *complex* constant, x_0 is an arbitrary *real* constant, $g(x)$ is an arbitrary analytic function and $f(t)$ is as well an arbitrary function of the *real* 'time' variable t, but it must of course be periodic with period T, see (1), for the *isochronicity* property to hold.

The following *unmodified* $(N+1)$-dimensional PDE reads

$$\frac{\partial^{m+2} w}{\partial\xi^m\partial\tau^2} = f(w) \tag{118a}$$

where m is an *integer* and $f(w)$ is an analytic function. By setting $\lambda = 0$, $\mu = -2/m$ one gets the corresponding *modified* evolution PDE

$$\begin{aligned}&\frac{\partial^{m+2}u}{\partial t^2\partial x^m} + 3\mathrm{i}\omega\frac{\partial^{m+1}u}{\partial t\partial x^m} + 4\mathrm{i}\omega x\frac{\partial^{m+2}u}{\partial t\partial x^{m+1}} - 2\omega^2\frac{\partial^m u}{\partial x^m}\\ &\quad - \frac{2}{m}\omega\left(3+\frac{2}{m}\right)x\frac{\partial^{m+1}u}{\partial x^{m+1}} - \left(\frac{2}{m}\right)^2\omega^2x^2\frac{\partial^{m+2}u}{\partial x^{m+2}} = f(u).\end{aligned} \tag{118b}$$

The following *unmodified* $(N+1)$-dimensional PDE reads

$$w_{\tau\tau} = w_\tau\sum_{n=1}^N\left[a_n\left(w_{\xi_n}\right)^{\alpha_n}\right] + \sum_{n=1}^N\left[b_n\left(w_{\xi_n}\right)^{2(\beta\alpha_n-1)}w_{\xi_n\xi_n}\right]. \tag{119a}$$

By setting $\lambda = \beta - 1$, $\mu_n = 1 - \beta + \frac{1}{\alpha_n}$ one gets the corresponding *modified* evolution PDE

$$\begin{aligned}&u_{tt} - \mathrm{i}\omega\left[(2\beta-1)u_t + \sum_{n=1}^N\left(1-\beta+\frac{1}{\alpha_n}\right)x_nu_{tx_n}\right]\\ &\quad - \omega^2\left[\beta(\beta-1)u + \beta\sum_{n=1}^N\left(1-\beta+\frac{1}{\alpha_n}\right)x_nu_{x_n}\right]\\ &= \left\{u_t - \mathrm{i}\omega\left[(\beta-1)u + \sum_{n=1}^N\left(1-\beta+\frac{1}{\alpha_n}\right)x_nu_{x_n}\right]\right\}\sum_{n=1}^N\left[a_n\left(u_{x_n}\right)^{\alpha_n}\right]\\ &\quad + \sum_{n=1}^N\left[b_n\left(u_{x_n}\right)^{2(\beta\alpha_n-1)}u_{x_nx_n}\right].\end{aligned} \tag{119b}$$

The following *unmodified* $(N+1)$-dimensional "nonlinear diffusion" PDE reads

$$w_\tau = w^\alpha\sum_{n=1}^N w_{\xi_n\xi_n}. \tag{120a}$$

By setting $\mu_n = \frac{1}{2}(1-\alpha\lambda)$ one gets the corresponding *modified* evolution PDE

$$u_t - \mathrm{i}\lambda\omega u - \mathrm{i}\frac{1-\alpha\lambda}{2}\omega\sum_{n=1}^N x_nu_{x_n} = u^\alpha\sum_{n=1}^N u_{x_nx_n}. \tag{120b}$$

The following *unmodified* $(N+1)$-dimensional "nonlinear heat equation with a source" reads

$$w_\tau = \sum_{n=1}^N a_n\left(w^{\alpha_n}w_{\xi_n}\right)_{\xi_n} + bw^\beta. \tag{121a}$$

By setting $\lambda = \frac{1}{\beta-1}$, $\mu_n = \frac{\beta-\alpha_n}{2(\beta-1)}$ one gets the corresponding *modified* evolution PDE

$$u_t - \frac{\mathrm{i}\omega}{\beta-1}u - \frac{\mathrm{i}(\beta-\alpha_n)\omega}{2(\beta-1)}\sum_{n=1}^{N} x_n u_{x_n} = \sum_{n=1}^{N} a_n \left(u^{\alpha_n} u_{x_n}\right)_{x_n} + bu^{\beta}. \quad (121\text{b})$$

The following *unmodified* $(N+1)$-dimensional "Bateman" PDE is *solvable*

$$\det\begin{pmatrix} 0 & w_\tau & w_{\xi_1} & \cdots & w_{\xi_n} \\ w_\tau & w_{\tau\tau} & w_{\xi_1\tau} & \cdots & w_{\tau\xi_n} \\ w_{\xi_1} & w_{\tau\xi_1} & w_{\xi_1\xi_1} & \cdots & w_{\xi_n\xi_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{\xi_n} & w_{\tau\xi_n} & w_{\xi_1\xi_n} & \cdots & w_{\xi_n\xi_n} \end{pmatrix} = 0. \quad (122\text{a})$$

The corresponding *modified* PDE is in this case t-autonomous for any choice of λ and μ_n. For $\lambda = \mu_n = 0$ it reads

$$\det\begin{pmatrix} 0 & u_t & u_{x_1} & \cdots & u_{x_n} \\ u_t & u_{tt} & u_{x_1 t} & \cdots & u_{x_n t} \\ u_{x_1} & u_{x_1 t} & u_{x_1 x_1} & \cdots & u_{x_n x_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x_n} & u_{t x_n} & u_{x_1 x_n} & \cdots & u_{x_n x_n} \end{pmatrix} - \mathrm{i}\omega u_t \det\begin{pmatrix} 0 & u_{x_1} & \cdots & u_{x_n} \\ u_{x_1} & u_{x_1 x_1} & \cdots & u_{x_n x_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_n} & u_{x_1 x_n} & \cdots & u_{x_n x_n} \end{pmatrix} = 0.$$
(122b)

The general solution $u \equiv u(\underline{x}; t)$ of this PDE is given by the implicit formula

$$\left(\mathrm{e}^{\mathrm{i}\omega t} - 1\right) f_0(u) + \sum_{k=1}^{n} x_k f_k(u) = c \quad (122\text{c})$$

where the $N+1$ functions $f_k(z)$, $k = 0, 1, \ldots, N$ are arbitrary.
For $N = 1$, the *unmodified* **$(1+1)$-dimensional Bateman equation** reads

$$w_{\tau\tau}\left(w_\xi\right)^2 + w_{\xi\xi} w_\tau^2 - 2 w_\xi w_\tau w_{\xi\tau} = 0 \quad (123\text{a})$$

and the *modified* version of this equation reads

$$u_{tt}u_x^2 + u_{xx}u_t^2 - 2u_{xt}u_x u_t + \mathrm{i}\omega[2\lambda u(u_x u_{xt} - u_t u_{xx}) + (2\mu - 1)(u_x)^2 u_t]$$
(123b)
$$+ \omega^2[\lambda^2 u(u_x)^2 - \lambda^2 u^2 u_{xx} + \lambda(2\mu - 1)u(u_x)^2 + \mu(\mu - 1)x(u_x)^3] = 0.$$

The *general* solution of this equation reads (in implicit form)

$$(\mathrm{e}^{\mathrm{i}\omega t} - 1) f\left(\mathrm{e}^{-\mathrm{i}\lambda\omega t} u(x; t)\right) + x\mathrm{e}^{\mathrm{i}\mu\omega t} g\left(\mathrm{e}^{-\mathrm{i}\lambda\omega t} u(x; t)\right) = c \quad (123\text{c})$$

with $f(z)$ and $g(z)$ two arbitrary functions which can be easily determined in terms of the initial data, say $u_0(x) = u(x; 0)$ and $u_1(x) = u_t(x; 0)$.

By setting $\lambda = 0$, $\mu = 1/2$ and (for notational convenience) $\Omega = \omega/2$ the *modified* $(1+1)$-dimensional Bateman equation (123b) takes the simple (*real*) form

$$u_{tt}u_x^2 + u_{xx}u_t^2 - 2u_{xt}u_xu_t = \Omega^2 x u_x^3. \tag{124a}$$

Note that, if $u(x;t)$ is a solution of this PDE, $v(x;t) = f[u(ax;t-b)]$ is also a solution, with $f(z)$ an arbitrary function and a, b two arbitrary constants.

The initial-value problem for this equation is solved by the implicit formula

$$u(x;t) = u_0\left(\frac{\tan(\Omega t)}{\Omega}\frac{u_1(u_0^{(\mathrm{inv})}[u(x;t)])}{u_0'(u_0^{(\mathrm{inv})}[u(x;t)])} + \frac{x}{\cos[\Omega t]}\right) \tag{124b}$$

where of course $u_0^{(\mathrm{inv})}(z)$ respectively $u_0'(z)$ are the inverse respectively the derivative of the function $u_0(z)$ (namely, $u_0^{(\mathrm{inv})}[u_0(x)] = x$, $u_0'(x) = \mathrm{d}u_0(x)/\mathrm{d}x$). And two explicit solutions of this equation, (124a), read as follows

$$u(x;t) = f\left\{\frac{c_1x + c_2\cos[\Omega(t-c_3)]}{c_4x + c_5\cos[\Omega(t-c_6)]}\right\} \tag{124c}$$

$$u(x;t) = f\left\{\frac{c_1\cos[\Omega(t-c_2)]}{\cos[2\Omega(t-c_3)] + \cos[2\Omega(c_2-c_3)]}x + c_4\tan[\Omega(t+c_2-2c_3)]\right\}. \tag{124d}$$

Here $f(z)$ denotes an arbitrary (twice differentiable) function, and the constants c_k are arbitrary. Clearly these solutions are *real* if the function $f(z)$ and the constants c_k are themselves *real*; and the conditions that the function $f(z)$ and the constants c_k must satisfy in order to guarantee the *isochronicity* of these solutions are rather obvious.

The following *unmodified* $(N+1)$-dimensional PDE reads

$$w_{\tau\tau} = w_\tau^2 f(w, w_{\xi_1}, \ldots, w_{\xi_N}, w_{\xi_1\xi_1}, \ldots, w_{\xi_N\xi_N}, \ldots) \tag{125a}$$

where $f(w, w_{x_1}, \ldots, w_{\xi_N}, w_{\xi_1\xi_1}, \ldots, w_{\xi_N\xi_N}, \ldots)$ is an arbitrary analytic function. By setting $\lambda = \mu_n = 0$ one gets the corresponding *modified* evolution PDE

$$u_{tt} - \mathrm{i}\omega u = u_t^2 f(u, u_{x_1}, \ldots, u_{x_N}, u_{x_1x_1}, \ldots, u_{x_Nx_N}, \ldots). \tag{125b}$$

4.3. Periodicity in the Space Variable

Throughout this section we focussed on *modified* PDEs that feature many *isochronous* solutions. By a variant of the trick (86)–(88) it is in some cases possible to generate equations that feature many solutions which are *completely periodic* not only in the time variable t, but as well in the space variable x. We only exhibit here a single example, obtained from the *unmodified* PDE (90a) via the following change of variables

$$\tau = \frac{\mathrm{e}^{\mathrm{i}\omega t} - 1}{\mathrm{i}\omega}, \tag{126a}$$

$$\xi = \frac{\mathrm{e}^{\mathrm{i}kx} - 1}{ik}, \tag{126b}$$

$$u(x;t) = \mathrm{e}^{\mathrm{i}\lambda\omega t}\mathrm{e}^{\mathrm{i}\rho kx}w(\xi;\tau) \tag{126c}$$

where, to apply this transformation to the *unmodified* PDE (90a), we set $\lambda = 1/\alpha$ and $\rho = -1/\alpha$, while ω and k are two *real* (indeed, without loss of generality, *positive*) constants, that determine the basic period in the (*real*) time variable t, see (1), as well as the basic period, $L = 2\pi/k$, in the (also *real*) space variable x. The modified PDE corresponding to (90a) reads then

$$u_t - \frac{\mathrm{i}\omega}{\alpha}u = au^{\alpha}(u_x + \frac{\mathrm{i}k}{\alpha}u) \tag{127a}$$

and its *general* solution, in implicit form, reads

$$\begin{aligned} u(x;t) = \mathrm{e}^{\frac{\mathrm{i}\omega t}{\alpha}} &\left\{1 + \frac{ak}{\omega}(1 - \mathrm{e}^{-\mathrm{i}\omega t})[u(x;t)]^{\alpha}\right\}^{1/\alpha} \\ \times\, u_0 &\left(x - \frac{\mathrm{i}}{k}\log\left\{1 + \frac{ak}{\omega}(1 - \mathrm{e}^{-\mathrm{i}\omega t})[u(x;t)]^{\alpha}\right\}\right) \end{aligned} \tag{127b}$$

where of course the initial datum, $u_0(x) = u(x;0)$ should be itself periodic with period L (or some appropriate integer multiple, or fraction, of L), in order for this solution to be periodic for all time with period L (or some appropriate integer multiple, or fraction, of L) – in addition of course to being periodic in t with period T (see (1) – or with some integer multiple of T, depending on the analyticity properties of $u_0(z)$ as a function of the complex variable z). An explicit special case of this implicit equation (corresponding to $u_0(x) = \mathrm{e}^{\mathrm{i}qx}$) reads

$$u(x;t) = b\mathrm{e}^{\frac{\mathrm{i}\omega t}{\alpha}}\mathrm{e}^{\mathrm{i}qx}\left\{1 + \frac{ak}{\omega}(1 - \mathrm{e}^{-\mathrm{i}\omega t})[u(x;t)]^{\alpha}\right\}^{\frac{1}{\alpha}+\frac{q}{k}} \tag{127c}$$

yielding, for $q = -k/\alpha$, the trivial solution of (127a)

$$u(x;t) = b\exp\left[\mathrm{i}\frac{(\omega t - kx)}{\alpha}\right]. \tag{127d}$$

For $q = k/\alpha$ one obtains instead from (127c) the explicit solution of (127a)

$$u(x;t) = \mathrm{e}^{\frac{\mathrm{i}(\omega t - kx)}{\alpha}} \left\{ \frac{\omega}{2a^2bk^2(\mathrm{e}^{\mathrm{i}\omega t}-1)^2} \times \left[\omega - 2ab\mathrm{e}^{\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}\omega t}-1) - \sqrt{\omega^2 - 4abk\omega\mathrm{e}^{\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}\omega t}-1)} \right] \right\}^{\frac{1}{\alpha}}.$$

Of course many other explicit solutions could be exhibited in specific cases, for instance in the special case with $q = -k$ and $\alpha = 2$

$$u(x;t) = \pm\sqrt{\frac{\omega \pm \sqrt{4ab^2k\omega\mathrm{e}^{-2\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}\omega t}-1)+\omega^2}}{2ak(1-\mathrm{e}^{\mathrm{i}\omega t})}}. \tag{127e}$$

Finally, we also rewrite below the evolution PDE (127a) with $\alpha = 1$ in *real* form, setting $u = u_1 + \mathrm{i}u_2$, $a = c_1 + \mathrm{i}c_2$ where the two dependent variables $u_1 \equiv u_1(x;t)$ and $u_2 \equiv u_2(x;t)$, as well as the two constants c_1, c_2, are of course now *real*

$$\begin{aligned} u_{1_t} + \omega u_2 &= c_1[u_1u_{1_x} + u_2u_{2_x} - 2ku_1u_2] - c_2[u_1u_{2_x} + u_2u_{1_x} + k(u_1^2 - u_2^2)] \\ &\qquad (128) \\ u_{2_t} - \omega u_1 &= c_2[u_1u_{1_x} + u_2u_{2_x} - 2ku_1u_2] + c_1[u_1u_{2_x} + u_2u_{1_x} + k(u_1^2 - u_2^2)]. \end{aligned}$$

References

[1] Ahmed S., Bruschi M., Calogero F., Olshanetsky M. and Perelomov A., *Properties of the Zeros of the Classical Polynomials and of Bessel Functions*, Nuovo Cimento B **49** (1979) 173–199.

[2] Arnlind J. and Hoppe J., *Eigenvalue-Dynamics off Calogero-Moser*, preprint IHES/P/03/41, July 2003.

[3] Babelon O. and Talon M., *The Symplectic Structure of the Spin Calogero Model*, Phys. Lett. A **236** (1997) 462–468.

[4] Barucchi G. and Regge T., *Conformal Properties of a Class of Exactly Solvable N-Body Problems in Space Dimension One*, J. Math. Phys. **18** (1977) 1149–1153.

[5] Billey E., Avan J. and Babelon O., *The r-Matrix Structure of the Euler-Calogero-Moser Model*, Phys. Lett. A **186** (1994) 114–118.

[6] Billey E., Avan J. and Babelon O., *Exact Yangian Symmetry in the Classical Euler-Calogero-Moser Model*, Phys. Lett. A **188** (1994) 263–271.

[7] Bruschi M. and Calogero F., *The Lax Representation for an Integrable Class of Relativistic Dynamical Systems*, Commun. Math. Phys. **109** (1987) 481–492.

[8] Calogero F., *Motion of Poles and Zeros of Special Solutions of Nonlinear and Linear Partial Differential Equations, and Related "Solvable" Many Body Problems*, Nuovo Cimento B **43** (1978) 177–241.

[9] Calogero F., *Matrices, Differential Operators and Polynomials*, J. Math. Phys. **22** (1981) 919–932.

[10] Calogero F., *A Class of Integrable Hamiltonian Systems whose Solutions are (perhaps) All Completely Periodic*, J. Math. Phys. **38** (1997) 5711–5719.

[11] Calogero F., *Integrable and Solvable Many-Body Problems in the Plane via Complexification*, J. Math. Phys. **39** (1998) 5268–5291.

[12] Calogero F., *Tricks of the Trade: Relating and Deriving Solvable and Integrable Dynamical Systems*, In: Calogero-Moser-Sutherland Models, J. F. van Diejen and L. Vinet (Eds), Proceedings of the Workshop on Calogero-Moser-Sutherland Models held in Montreal, 10–15 March 1997, CRM Series in Mathematical Physics, Springer, Berlin, 2000, 93–116.

[13] Calogero F., *The "Neatest" Many-Body Problem Amenable to Exact Treatments (a "Goldfish"?)*, Physica D **152–153** (2001) 78–84.

[14] Calogero F., *Classical Many-Body Problems Amenable to Exact Treatments*, Lecture Notes in Physics Monograph **m 66**, Springer, Heidelberg, 2001.

[15] Calogero F., *Periodic Solutions of a System of Complex ODEs*, Phys. Lett. A **293** (2002) 146–150.

[16] Calogero F., *A Complex Deformation of the Classical Gravitational Many-Body Problem that Features a Lot of Completely Periodic Motions*, J. Phys. A: Math. Gen. **35** (2002) 3619–3627.

[17] Calogero F., *Differential Equations Featuring Many Periodic Solutions*, In: Geometry and Integrability, L. Mason and Y. Nutku Y. (Eds), London Mathematical Society Lecture Notes, vol. **295**, Cambridge University Press, 2003, 9–21.

[18] Calogero F., *Partially Superintegrable (Indeed Isochronous) Systems are not Rare*, In: New Trends in Integrability and Partial Solvability, A. Shabat, A. Gonzalez-Lopez, M. Manas, L. Martinez Alonso and M. A. Rodriguez (Eds), NATO Science Series, II. Mathematics, Physics and Chemistry, vol. **132**, Proceedings of the NATO Advanced Research Workshop held in Cadiz, Spain, 2–16 June 2002, Kluwer, 2004, 49–77.

[19] Calogero F., *Solution of the Goldfish N-Body Problem in the Plane with (only) Nearest-Neighbor Coupling Constants all Equal to Minus One Half*, J. Nonlinear Math. Phys. **11** (2004) 1–11.

[20] Calogero F., *Two New Classes of Isochronous Hamiltonian Systems*, J. Nonlinear Math. Phys. **11** (2004) 208–222.

[21] Calogero F., *A Technique to Identify Solvable Dynamical Systems, and a Solvable Generalization of the Goldfish Many-Body Problem*, J. Math. Phys. **45** (2004) 2266–2279.

[22] Calogero F., *A Technique to Identify Solvable Dynamical Systems, and Another Solvable Extension of the Goldfish Many-Body Problem*, J. Math. Phys. **45** (2004) 4661–4678.

[23] Calogero F., *Isochronous Dynamical Systems*, Applicable Analysis (in press).

[24] Calogero F. and Françoise J.-P., *Periodic Solutions of a Many-Rotator Problem in the Plane*, Inverse Problems **17** (2001) 1–8.

[25] Calogero F. and Françoise J.-P., *Periodic Motions Galore: How to Modify Nonlinear Evolution Equations so that They Feature a Lot of Periodic Solutions*, J. Nonlinear Math. Phys. **9** (2002) 99–125.

[26] Calogero F. and Françoise J.-P., *Nonlinear Evolution ODEs Featuring Many Periodic Solutions*, Theor. Mat. Fis. **137** (2003) 1663–1675.

[27] Calogero F. and Françoise J.-P., *Isochronous Motions Galore: Nonlinearly Coupled Oscillators with Lots of Isochronous Solutions*. In: Superintegrability in Classical and Quantum Mecanics, Proceedings of the Workshop on Superintegrability in Classical and Quantum Systems, Centre de Recherches Mathématiques (CRM), Université de Montréal, September 16–21, 2002, CRM Proceedings and Lecture Notes, vol. **37**, American Mathematical Society, 2004, pp 15–27.

[28] Calogero F. and Françoise J.-P., *New Solvable Many-Body Problems in the Plane*, Annales Henri Poincaré (submitted to).

[29] Calogero F., Françoise J.-P. and Sommacal M., *Periodic Solutions of a Many-Rotator Problem in the Plane. II. Analysis of Various Motions*, J. Nonlinear Math. Phys. **10** (2003) 157–214.

[30] Calogero F., Gomez-Ullate D., Santini P. and Sommacal M., *The Transition from Ordered to Disordered Motions, Explained as Travel on Riemann Surfaces*, (in preparation).

[31] Calogero F. and Inozemtsev V. I., *Nonlinear Harmonic Oscillators*, J. Phys. A: Math. Gen. **35** (2002) 10365–10375.

[32] Calogero F. and Sommacal M., *Periodic Solutions of a System of Complex ODEs. II. Higher Periods*, J. Nonlinear Math. Phys. **9** (2002) 483–516.

[33] Gibbons J. and Hermsen T., *A Generalization of the Calogero-Moser System*, Physica **11D** (1984) 337–348.

[34] Gomez-Ullate D., Hone A. and Sommacal M, *New Many-Body Problems in the Plane with Periodic Solutions*, New J. Phys. **6** (24) (2004) 1–23.

[35] Ince E., *Ordinary Differential Equations*, Dover, New York, 1956.

[36] Inozemtsev V., *Matrix Analogues of Elliptic Functions*, Funct. Anal. Appl. **23** (1990) 323-325 [Russian original: Funct. Anal. Pril. **23** (1989) 81–82].

[37] Krichever I., Babelon O., Billey E. and Talon M., *Spin Generalization of the Calogero-Moser System and the Matrix KP Equation*, Amer. Math. Soc. Transl. **170** (1995) 83–119.

[38] Mariani M. and Calogero F., *Isochronous PDEs*, Yadernaya Fisika (in press).

[39] Nakamura K. and Lakshmanan M., *Complete Integrability in a Quantum Description of Chaotic Systems*, Phys. Rev. Lett. **57** (1986) 1661–1664.

[40] Nekrasov N., *Holomorphic Bundle and Many-Body Systems*, hep-tj/9503157.

[41] Pechukas P., *Distribution of Energy Eigenvalues in the Irregular Spectrum*, Phys. Rev. Lett. **51** (1983) 943–946.

[42] Wojciechowski S., *An Integrable Marriage of the Euler Equations with the Calogero-Moser System*, Phys. Lett. **111A** (1984) 101–103.

[43] Wojciechowski S., *Generalized Integrable Many-Body Systems in One Dimension*, Phys. Scripta **34** (1986) 304–308.

[44] Yukawa T., *New Approach to the Statistical Properties of Energy Levels*, Phys. Rev. Lett. **54** (1985) 1883–1886.

[45] Yukawa T., *Lax Form of the Quantum Mechanical Eigenvalue Problem*, Phys. Lett. A **116** (1986) 227–230.

[46] Zakharov V., *On the Dressing Method*, In: Sabatier P. (editor), *Inverse Methods in Action*, Springer, Heidelberg, 1990, 602–623.