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THE HYPERBOLIC TRIANGLE DEFECT

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Abstract. The hyperbolic trigonometry, fully analogous to the common Euclidean trigonometry, is presented and employed to calculate the hyperbolic triangle defect in the Poincaré ball model of n-dimensional hyperbolic geometry. It is shown that hyperbolic trigonometry allows the hyperbolic triangle defect to be expressed in terms of the triangle hyperbolic side lengths by a remarkably elegant identity.

1. Introduction

Gyrovector spaces form the setting for hyperbolic geometry and its hyperbolic trigonometry just as vector spaces form the setting for Euclidean hyperbolic geometry and its Euclidean trigonometry. Gyrovector spaces, in turn, are gyrocommutative gyrogroups that admit scalar multiplication just as vector spaces are commutative groups that admit scalar multiplication.

Accordingly, to set the stage for the introduction of hyperbolic geometry and its applications, the definitions of abstract gyrogroups and gyrovector spaces are presented in Section 2. A concrete example of a family of n-dimensional gyrovector spaces (n finite or infinite), called the Möbius gyrovector spaces, is presented in Section 3. It is then demonstrated in Section 4 that Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry. Having the stage set, the hyperbolic geometry of the Poincaré ball model of hyperbolic geometry is presented in Section 5 including, in particular,

1) the Hyperbolic Law of Cosines and its resulting Hyperbolic Pythagorean Theorem that shares visual analogies with its Euclidean counterpart, Fig. 2, and
2) the Hyperbolic Law of Sines.

Hyperbolic geometry is applied in Section 6 and Section 7 to uncover the angular defect of the hyperbolic right angled triangle and, more generally, of the hyperbolic triangle in terms of its hyperbolic side lengths.

2. Gyrogroup And Gyrovector Spaces

Definition 1 (Gyrogroups). A groupoid is a non-empty set with a binary operation. The groupoid $(G, \oplus)$ is a **gyrogroup** if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

G1) $0 \oplus a = a$ Left Identity

for all $a \in G$. There is an element $0 \in G$ satisfying axiom G1) such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, satisfying

G2) $\ominus a \oplus a = 0$ Left Inverse.

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a, b]z \in G$ such that

G3) $a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$ Left Gyroassociative Law.

If $\mathrm{gyr}[a, b]$ denotes the map $\mathrm{gyr}[a, b] : G \to G$ given by $z \mapsto \mathrm{gyr}[a, b]z$ then

G4) $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ Gyroautomorphism

and $\mathrm{gyr}[a, b]$ is called the **Thomas gyration**, or the **gyroautomorphism** of G, generated by $a, b \in G$. The operation $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the **gyrooperation** of G. Finally, the gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in G$ satisfies

G5) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ Left Loop Property.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. The definition of gyrocommutativity in gyrogroups follows.

Definition 2 (Gyrocommutative Gyrogroups). The gyrogroup $(G, \oplus)$ is **gyrocommutative** if for all $a, b \in G$

G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ Gyrocommutative Law.

Definition 3 (Inner Product Gyrovector Spaces). A(n inner product) gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that admits:

1) Inner product '$\cdot$' i) which gives rise to a positive definite norm $\|\mathbf{v}\|$, that is, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$, $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$, $|\mathbf{u}\cdot\mathbf{v}\,| \leq \|\mathbf{u}\|\|\mathbf{v}\|$;

and ii) which is invariant under gyroautomorphisms, that is,

$$\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$$

for all gyrovectors $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

2) Scalar multiplication, $\otimes$, satisfying the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all gyrovectors $\mathbf{v} \in G$:

V1)	$1\otimes\mathbf{v} = \mathbf{v}$	
V2)	$(r_1 + r_2)\otimes\mathbf{v} = r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}$	Scalar Distributive Law
V3)	$(r_1 r_2)\otimes\mathbf{v} = r_1\otimes(r_2\otimes\mathbf{v})$	Scalar Associative Law
V4)	$\dfrac{\lvert r\rvert\otimes\mathbf{v}}{\lVert r\otimes\mathbf{v}\rVert} = \dfrac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$	Scaling Property
V5)	$\mathrm{gyr}[\mathbf{a},\mathbf{b}](r\otimes\mathbf{v}) = r\otimes \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}$	Gyroautomorphism Property
V6)	$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$	Identity Automorphism.

3) Real vector space structure $(\lVert G\rVert, \oplus, \otimes)$ for the set $\lVert G\rVert$ of one-dimensional "vectors"

$$\lVert G\rVert = \{\pm\lVert\mathbf{v}\rVert;\ \mathbf{v} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{u},\mathbf{v} \in G$,

V7)	$\lVert r\otimes\mathbf{v}\rVert = \lvert r\rvert\otimes\lVert\mathbf{v}\rVert$	Homogeneity Property
V8)	$\lVert\mathbf{u}\oplus\mathbf{v}\rVert \le \lVert\mathbf{u}\rVert\oplus\lVert\mathbf{v}\rVert$	Gyrotriangle Inequality.

A gyrovector space $G = (G, \oplus, \otimes)$ is a gyrometric space given by the distance function

$$\rho(\mathbf{u},\mathbf{v}) = \lVert\ominus\mathbf{u}\oplus\mathbf{v}\rVert = \lVert\mathbf{v}\ominus\mathbf{u}\rVert \tag{1}$$

satisfying the gyrotriangle inequality

$$\lVert\ominus\mathbf{u}\oplus\mathbf{w}\rVert \le \lVert\ominus\mathbf{u}\oplus\mathbf{v}\rVert\oplus\lVert\ominus\mathbf{v}\oplus\mathbf{w}\rVert \tag{2}$$

verified below.

By a gyrogroup identity we have

$$\ominus\mathbf{u}\oplus\mathbf{w} = (\ominus\mathbf{u}\oplus\mathbf{v})\oplus \mathrm{gyr}[\mathbf{u},\ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w}). \tag{3}$$

Hence, by the gyrotriangle inequality V8) we have

$$\begin{aligned}\lVert\ominus\mathbf{u}\oplus\mathbf{w}\rVert &= \lVert(\ominus\mathbf{u}\oplus\mathbf{v})\oplus \mathrm{gyr}[\mathbf{u},\ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w})\rVert \\ &\le \lVert\ominus\mathbf{u}\oplus\mathbf{v}\rVert\oplus\lVert \mathrm{gyr}[\mathbf{u},\ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{w})\rVert \\ &= \lVert\ominus\mathbf{u}\oplus\mathbf{v}\rVert\oplus\lVert\ominus\mathbf{v}\oplus\mathbf{w}\rVert.\end{aligned} \tag{4}$$

Our ambiguous use of $\oplus$ and $\otimes$, Definition 3, as operations in the gyrovector space $(G, \oplus, \otimes)$ and in the vector space $(\|G\|, \oplus, \otimes)$ should raise no confusion, since the sets in which these operations operate are always clear from the context. These operations in the former are nonassociative-nondistributive gyrovector space operations, and in the latter are associative-distributive vector space operations. Additionally, the gyro-addition $\oplus$ is gyrocommutative in the former and commutative in the latter.

An inner product gyrovector space possesses a weak form of a distributive law,

$$r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) = r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_2\otimes\mathbf{v}) \tag{5}$$

called the **monodistributive law**, which follows from V2) and V3)

$$\begin{aligned} r\otimes(r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v}) &= r\otimes\{(r_1+r_2)\otimes\mathbf{v}\} \\ &= (r(r_1+r_2))\otimes\mathbf{v} \\ &= (rr_1+rr_2)\otimes\mathbf{v} \\ &= (rr_1)\otimes\mathbf{v}\oplus(rr_1)\otimes\mathbf{v} \\ &= r\otimes(r_1\otimes\mathbf{v})\oplus r\otimes(r_1\otimes\mathbf{v}). \end{aligned} \tag{6}$$

3. Möbius Gyrovector Spaces

Definition 4 (Möbius Addition). Let $\mathbb{V}$ be a real inner product space, and let $\mathbb{B} = \{\mathbf{v} \in \mathbb{V};\ \|\mathbf{v}\| < 1\}$ be the open unit ball of $\mathbb{V}$. Möbius addition $\oplus$ in the ball $\mathbb{B}$ is a binary operation in $\mathbb{B}$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{7}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{B}$ inherits from its space $\mathbb{V}$.

To justify calling $\oplus$ in Definition 4 a Möbius addition we note that it is a natural extension of a special Möbius transformation of the complex disc, as explained in [2].

The groupoid $(\mathbb{B}, \oplus)$ is a gyrocommutative gyrogroup, as demonstrated in [2], giving rise to a Möbius gyrogroup. Furthermore, it admits scalar multiplication $\otimes$, turning it into a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$.

Definition 5 (Möbius Scalar Multiplication). Let $(\mathbb{B}, \oplus)$ be a Möbius gyrogroup. The Möbius scalar multiplication $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ in $\mathbb{B}$ is given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= \frac{(1+\|\mathbf{v}\|)^r-(1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r+(1-\|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh(r\ \tanh^{-1}\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{8}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{B}$, $\mathbf{v} \neq \mathbf{0}$ and $r \otimes \mathbf{0} = \mathbf{0}$.

As an example we present the **Möbius half**,

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{9}$$

where $\gamma_{\mathbf{v}} = (1-\|\mathbf{v}\|^2)^{-1/2}$. Then, in accordance with the scalar associative law of gyrovector spaces,

$$2\otimes\left(\tfrac{1}{2}\otimes\mathbf{v}\right) = 2\otimes\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} = \mathbf{v}. \tag{10}$$

4. The Poincaré Ball Model of Hyperbolic Geometry

Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry, as demonstrated in [2], just as vector spaces form the setting for the standard model of Euclidean geometry. Thus, the unique geodesic passing through the points $\mathbf{a}, \mathbf{b} \in \mathbb{B}$ in a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ is given by the equation

$$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{11}$$

with the real parameter $t \in \mathbb{R}$. It passes through the point $\mathbf{a}$ at “time” $t = 0$ and, owing to the left cancellation law of Möbius addition, it passes through the point $\mathbf{b}$ at “time” $t = 1$. The cosine of the hyperbolic angle generated by two geodesics passing, respectively, through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}, \mathbf{c}$ in the Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$, Fig. 1, is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|} \tag{12}$$

in full analogy with its Euclidean counterpart.

Three geodesic segments that form a triangle, and the triangle angles in the Möbius gyrovector plane $\mathbb{B}$ are shown in Fig. 1.

5. Hyperbolic Trigonometry

Theorem 6 (The Hyperbolic Law of Cosines in Möbius gyrovector spaces)**.** Let $\Delta\mathbf{abc}$ be a triangle in a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{B}$, and sides and length sides

$$\begin{aligned} \mathbf{A} &= \ominus\mathbf{b}\oplus\mathbf{c}, & a &= \|\mathbf{A}\| \\ \mathbf{B} &= \ominus\mathbf{c}\oplus\mathbf{a}, & b &= \|\mathbf{B}\| \\ \mathbf{C} &= \ominus\mathbf{a}\oplus\mathbf{b}, & c &= \|\mathbf{C}\| \end{aligned} \tag{13}$$

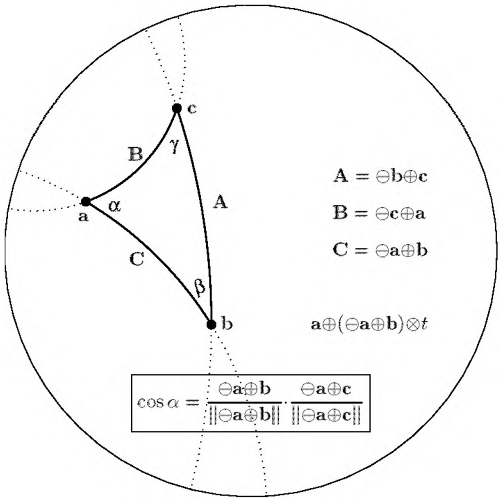

Figure 1. A Möbius triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $(\mathbb{B}, \oplus, \otimes)$ is shown. Its sides are formed by geodesic segments that link its vertices, in full analogy with Euclidean triangles. The cosine of its angles are given by an identity that is fully analogous to their Euclidean counterparts. The Möbius gyrovector plane form the setting for the Poincaré disc model of hyperbolic geometry, allowing hyperbolic geometric properties to be studied analitically obtaining, for instance, the Hyperbolic Pythagorean Theorem in Fig. 2.

and with hyperbolic angles α, β and γ at the vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, Fig. 1. Then

$$\boxed{c^2 = a^2 \oplus b^2 \ominus \frac{2ab\cos\gamma}{(1+a^2)(1+b^2) - 2ab\cos\gamma}.} \tag{14}$$

Proof: The proof of the hyperbolic law of cosines is by straightforward algebra, noting the hyperbolic angle definition of in (12). □

We may note that the Möbius addition $\oplus$ in (13) is a gyrogroup operation in the Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$. In contrast, the Möbius addition $\oplus$ in (14) is a group operation in the Möbius group $(\mathbb{I}, \oplus)$, $\mathbb{I}$ being the open unit interval $\mathbb{I} = (-1, 1)$.

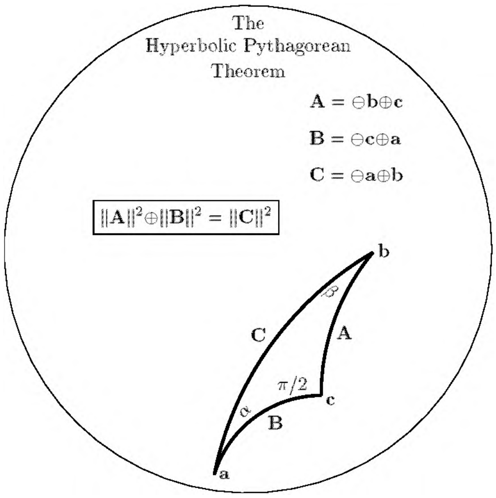

Figure 2. The Hyperbolic Pythagorean Theorem in the Poincaré ball model of hyperbolic geometry and, equivalently, in the Möbius gyrovector plane $(\mathbb{B}, \oplus, \otimes)$.

The hyperbolic law of cosines (14) is an identity in the Möbius vector space $(\mathbb{I}, \oplus, \otimes)$. To solve it for $\cos\gamma$ we use the notation

$$\begin{aligned} P_{abc} &= a^2 \oplus b^2 \ominus c^2 \\ Q_{ab} &= 2ab \\ R_{ab} &= (1+a^2)(1+b^2) \end{aligned} \tag{15}$$

so that (14) can be written as

$$\frac{Q_{ab}\cos\gamma}{R_{ab} - Q_{ab}\cos\gamma} = P_{abc} \tag{16}$$

implying

$$\cos\gamma = \frac{P_{abc}R_{ab}}{(1+P_{abc})Q_{ab}} \tag{17}$$

and, similarly by cyclic permutations,

$$\cos\alpha = \frac{P_{bca}R_{bc}}{(1+P_{bca})Q_{bc}} \tag{18}$$

$$\cos\beta = \frac{P_{cab}R_{ca}}{(1+P_{cab})Q_{ca}}. \tag{19}$$

In the special case of which $\gamma = \pi/2$, corresponding to a hyperbolic right angled triangle, Fig. 3, the hyperbolic law of cosines is of particular interest, giving rise to the hyperbolic Pythagorean theorem in the Poincaré ball model of hyperbolic geometry.

Theorem 7 (The Möbius Hyperbolic Pythagorean Theorem). Let $\Delta\mathbf{abc}$ be a triangle in a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{B}$, and sides and length sides

$$\begin{aligned} \mathbf{A} &= \ominus\mathbf{b}\oplus\mathbf{c}, & a &= \|\mathbf{A}\| \\ \mathbf{B} &= \ominus\mathbf{c}\oplus\mathbf{a}, & b &= \|\mathbf{B}\| \\ \mathbf{C} &= \ominus\mathbf{a}\oplus\mathbf{b}, & c &= \|\mathbf{C}\| \end{aligned} \tag{20}$$

and with hyperbolic angles α, β and γ at the vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. If $\gamma = \pi/2$, Fig. 3, then

$$\boxed{c^2 = a^2 \oplus b^2.} \tag{21}$$

Proof: The hyperbolic Pythagorean identity (21) follows from the hyperbolic law of cosines (14) with $\gamma = \pi/2$. □

Two equivalent versions of the Möbius hyperbolic Pythagorean identity (21), which involve ordinary rather than Möbius addition, are presented in (27).

We use the notation

$$\begin{aligned} \mathbf{a}_{\mathrm{M}} &= \gamma_{\mathbf{a}}^2 \mathbf{a} = \frac{\mathbf{a}}{1 - \|\mathbf{a}\|^2} \\ \mathbf{a}_{\mathrm{P}} &= \beta_{\mathbf{a}}^2 \mathbf{a} = \frac{\mathbf{a}}{1 + \|\mathbf{a}\|^2} \end{aligned} \tag{22}$$

for $\mathbf{a} \in \mathbb{B}$, where $g_{a,b}$ and $\beta_{\mathbf{a}}$ are the gamma and the beta factors given by the equations

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \qquad \text{and} \qquad \beta_{\mathbf{v}} = \frac{1}{\sqrt{1 + \|\mathbf{v}\|^2}} \tag{23}$$

for any $\mathbf{v} \in \mathbb{B}$. We call $\mathbf{a}_{\mathrm{M}}$ and $\mathbf{a}_{\mathrm{P}}$, respectively, the M(inus) and the P(lus) corrections of $\mathbf{a}$.

Taking magnitudes in (22), we have

$$\begin{aligned} \|\mathbf{a}\|_{\mathrm{M}} &= \gamma_{\mathbf{a}}^2 \|\mathbf{a}\| = \frac{\|\mathbf{a}\|}{1 - \|\mathbf{a}\|^2} \\ \|\mathbf{a}\|_{\mathrm{P}} &= \beta_{\mathbf{a}}^2 \|\mathbf{a}\| = \frac{\|\mathbf{a}\|}{1 + \|\mathbf{a}\|^2} \end{aligned} \tag{24}$$

for $\mathbf{a} \in \mathbb{B}$, calling $\|\mathbf{a}\|_{\mathrm{M}}$ and $\|\mathbf{a}\|_{\mathrm{P}}$, respectively, the M(inus) and the P(lus) corrections of $\|\mathbf{a}\|$. Clearly, $\|\mathbf{a}\|_{\mathrm{M}} \in [0, \infty)$ and $\|\mathbf{a}\|_{\mathrm{P}} \in [0, 1/2)$.

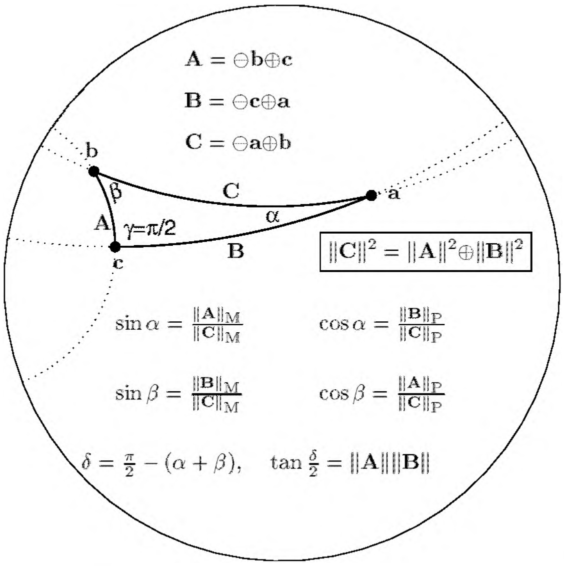

Figure 3. The Möbius Hyperbolic Pythagorean Theorem II. A Möbius right angled triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ is shown for the special case of the Möbius gyrovector plane. i) Its sides, formed by the geodesic segments $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that link its vertices, satisfy the Möbius hyperbolic Pythagorean identity (21), and ii) its acute angles α and β satisfy the hyperbolic trigonometric identities (28) and (29). The right angled triangle defect δ is calculated in (35), giving rise to the remarkably elegant result $\tan(\delta/2) = \|\mathbf{A}\|\,\|\mathbf{B}\|$.

Inverting (22) and (24) we have,

$$\begin{aligned} \mathbf{a} &= \frac{2\mathbf{a}_{\mathrm{M}}}{1+\sqrt{1+(2\|\mathbf{a}\|_{\mathrm{M}})^2}} \\ \mathbf{a} &= \frac{2\mathbf{a}_{\mathrm{P}}}{1+\sqrt{1-(2\|\mathbf{a}\|_{\mathrm{P}})^2}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \|\mathbf{a}\| &= \frac{2\|\mathbf{a}\|_{\mathrm{M}}}{1+\sqrt{1+(2\|\mathbf{a}\|_{\mathrm{M}})^2}} \\ \|\mathbf{a}\| &= \frac{2\|\mathbf{a}\|_{\mathrm{P}}}{1+\sqrt{1-(2\|\mathbf{a}\|_{\mathrm{P}})^2}}. \end{aligned} \tag{26}$$

It follows from the Möbius hyperbolic Pythagorean Identity (21), by straightforward algebra, that in the notation of Fig. 3 for a hyperbolic right angled triangle we have

$$\begin{aligned} \left(\frac{\|\mathbf{A}\|_{\mathrm{M}}}{\|\mathbf{C}\|_{\mathrm{M}}}\right)^2 + \left(\frac{\|\mathbf{B}\|_{\mathrm{P}}}{\|\mathbf{C}\|_{\mathrm{P}}}\right)^2 &= 1 \\ \left(\frac{\|\mathbf{A}\|_{\mathrm{P}}}{\|\mathbf{C}\|_{\mathrm{P}}}\right)^2 + \left(\frac{\|\mathbf{B}\|_{\mathrm{M}}}{\|\mathbf{C}\|_{\mathrm{M}}}\right)^2 &= 1. \end{aligned} \tag{27}$$

The two identities in (27) are equivalent to each other and may be considered as an equivalent version of the hyperbolic Pythagorean theorem in the Poincaré ball model of hyperbolic geometry.

In the special case in which $\gamma = \pi/2$, corresponding to a hyperbolic right angled triangle $\Delta\mathbf{abc}$, Fig. 3, it follows from (18)–(19) that

$$\begin{aligned} \cos\alpha &= \frac{\|\mathbf{B}\|_{\mathrm{P}}}{\|\mathbf{C}\|_{\mathrm{P}}} = \frac{b_P}{c_P} \\ \cos\beta &= \frac{\|\mathbf{A}\|_{\mathrm{P}}}{\|\mathbf{C}\|_{\mathrm{P}}} = \frac{a_P}{c_P}. \end{aligned} \tag{28}$$

Furthermore, it follows from (28) and (27) that

$$\begin{aligned} \sin\alpha &= \frac{\|\mathbf{A}\|_{\mathrm{M}}}{\|\mathbf{C}\|_{\mathrm{M}}} = \frac{a_M}{c_M} \\ \sin\beta &= \frac{\|\mathbf{B}\|_{\mathrm{M}}}{\|\mathbf{C}\|_{\mathrm{M}}} = \frac{b_M}{c_M}. \end{aligned} \tag{29}$$

The latter, in turn, gives the following

Theorem 8 (The Möbius Hyperbolic Law of Sines)**.** Let $\Delta\mathbf{abc}$ be a triangle in a Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{B}$, and sides and side lengths

$$\begin{aligned} \mathbf{A} &= \ominus\mathbf{b}\oplus\mathbf{c}, & a &= \|\mathbf{A}\| \\ \mathbf{B} &= \ominus\mathbf{c}\oplus\mathbf{a}, & b &= \|\mathbf{B}\| \\ \mathbf{C} &= \ominus\mathbf{a}\oplus\mathbf{b}, & c &= \|\mathbf{C}\| \end{aligned} \tag{30}$$

and with hyperbolic angles α, β and γ at the vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, Fig. 1. Then

$$\boxed{\frac{a_M}{\sin\alpha} = \frac{b_M}{\sin\beta} = \frac{c_M}{\sin\gamma}.} \tag{31}$$

6. The Angular Defect of the Hyperbolic Right Angled Triangle

The sum of the angles α and β of the right angled triangle $\Delta\mathbf{abc}$ in Fig. 2 is smaller than $\frac{\pi}{2}$ so that it possesses a positive **angular defect** $\delta = \frac{\pi}{2} - (\alpha + \beta)$. The cosine and the sine of the angular defect δ of triangle $\Delta\mathbf{abc}$ are

$$\begin{aligned}\cos\delta &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)\\ &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{a_M}{c_M}\frac{a_P}{c_P} + \frac{b_P}{c_P}\frac{b_M}{c_M}\end{aligned} \tag{32}$$

and

$$\begin{aligned}\sin\delta &= \sin\left(\frac{\pi}{2} - (\alpha+\beta)\right)\\ &= \cos(\alpha+\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \frac{a_P b_P}{c_P^2} - \frac{a_M b_M}{c_M^2}.\end{aligned} \tag{33}$$

Interestingly, the tangent $\tan(\delta/2)$ of the half angular defect $\delta/2$ is particularly simple and elegant. It follows from (32) and (33),

$$\tan\delta = \frac{2ab}{1-a^2b^2} \tag{34}$$

so that the M and P corrections disappear, and

$$\boxed{\tan\frac{\delta}{2} = ab.} \tag{35}$$

The hyperbolic right angled triangle angular defect δ in (35) was first calculated in [2], using algebraic methods of hyperbolic trigonometry as shown in this article. Later, but independently, it was calculated by Hartshorne in [1, Fig. 5] using geometric methods.

7. The Angular Defect of the Hyperbolic Triangle

Let α, β and γ be the hyperbolic angles of the hyperbolic triangle $\Delta\mathbf{abc}$ in Fig. 1. The cosines of these angles are calculated by means of the Hyperbolic Law of Cosines. The sines of these angles are related to each other by the Hyperbolic Law of Sines, enabling us to calculate $\cos\delta$ where $\delta = \pi - (\alpha+\beta+\gamma)$ is the triangle defect. The latter, in turn, is employed to calculate $\tan(\delta/2)$ by means of the trigonometric identity $\tan^2(\phi/2) = (1 - \cos\phi)/(1 + \cos\phi)$, obtaining

$$\tan\frac{\delta}{2} = \frac{\sqrt{a+b+c+abc}\sqrt{-a+b+c-abc}\sqrt{a-b+c-abc}\sqrt{a+b-c-abc}}{2+a^2b^2c^2-a^2-b^2-c^2}. \tag{36}$$

In the special case when the triangle is right angled and the lengths of its perpendicular sides are a and b, the triangle side lengths are related by the hyperbolic Pythagorean identity (21). Under this condition it can be shown that (36) reduces to (35).

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[2] Ungar A., *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession. The Theory of Gyrogroups and Gyrovector Spaces.* Kluwer, Dordrecht, 2001.