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INTEGRABLE DYNAMICS OF KNOTTED VORTEX FILAMENTS

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Abstract. The dynamics of vortex filaments has provided for almost a century one of the most interesting connections between differential geometry and soliton equations, and an example in which knotted curves arise as solutions of differential equations possessing an infinite family of symmetries and a remarkably rich geometrical structure. These lectures discuss several aspects of the integrable dynamics of closed vortex filaments in an Eulerian fluid, including its Hamiltonian formulation, the construction of a large class of special solutions, and the role of the Floquet spectrum in characterizing the geometric and topological properties of the evolving curves.

Introduction

Most of the well known soliton equations in one space dimension have been shown to describe integrable curve evolutions: among them the Vortex Filament Equation (or localized induction equation) [28, 34]; the mKdV equation modelling the dynamics of boundaries of vortex patches [23]; the sine-Gordon equation which describes constant torsion curves generating pseudospherical surfaces [16, 15]; and their higher dimensional generalizations [17, 37].

The understanding of connections between curve geometry and integrability has proceeded along several directions. Many of the fundamental properties of soliton equations have been given a geometrical realization: in the case of the Vortex Filament Equation (VFE), its bihamiltonian formulation and recursion operator, its hierarchy of constants of motion, and its relation to the nonlinear Schrödinger equation possess natural geometric interpretations [36, 11]. For equations, such as the VFE, which describe curve dynamics in three-dimensional space, periodic boundary conditions give rise to closed and, in many cases, knotted curves. It is then natural to ask whether the infinitely many symmetries and the associated sequence of integral and spectral invariants are connected with the knot types of the evolving curves, as well as with their special geometric features.

This series of lectures addresses connections between integrability (namely the Floquet spectrum of a given solution), geometry and topology of closed curve solutions of the Vortex Filament Equation. We begin with a derivation of the self-induced dynamics of a vortex filament in an ideal fluid, and the Hamiltonian formulation of the resulting evolution equation for closed vortex filaments in an appropriate infinite-dimensional phase space. We then introduce a transformation discovered by Hasimoto which converts solutions of the Vortex Filament Equation to solutions of the cubic Nonlinear Schrödinger (NLS) equation, thus unveiling its complete integrability. We conclude the first lecture with a geometric interpretation of the Hasimoto map in terms of the horizontal lifting of the curve to its frame bundle with respect to the canonical connection, and introduce its natural frame. In Lecture 2, after discussing some of the universal properties of integrability, we derive the inverse of the Hasimoto map: a reconstruction formula due to Sym and Pohlmeyer, which generates a solution of the VFE from a given NLS potential. Closure conditions for the reconstructed curve can be formulated in terms of a distinguished point in the Floquet spectrum of the associated NLS potential: we discuss this result by Grinevich and Schmidt as the first hint of the important role of the spectral invariants in the characterization of the properties of the curve. After illustrating these ideas for the simplest case of multiply covered circles associated with planar wave NLS potentials, in Lecture 3 we discuss the explicit algebro-geometric construction of N-phase solutions of the Vortex Filament Equation, following Krichever's use of the Baker-Akhiezer function. These are the periodic analogues of solitons on infinite domains and realize, for $N = 2$, 3, curves of interesting topology, such as torus knots and cable knots. The final lecture contains a description of recent investigations by the author and collaborator T. Ivey of a possible relationship between the geometry and the knot types of closed VFE solutions coming from N-phase NLS potentials and their associated Floquet spectra. We describe different approaches to answering this question: the use of exact formulas for N-phase solutions corresponding to curves with certain symmetries, perturbation methods, and the theory of isoperiodic deformations for investigating the topology of curves in neighborhoods of multiply covered circles.

Lecture 1

The Physical Model

The first mathematical model of the evolution of an Eulerian vortex filament, an approximately one-dimensional region of the fluid where the velocity distribution has a rotational component, was derived in 1906 by Luigi Da Rios [49], a student of Tullio Levi-Civita. A careful derivation of the self-induced dynamics of a vortex filament can be found in G. Batchelor [5]; a brief description of its set-up is given below (see also [8]).

Let $\mathbf{u}$ be the velocity distribution of an incompressible $(\mathrm{div}(\mathbf{u}) = 0)$ fluid filling an unbounded simply connected region in space, and assume that its vorticity $\mathbf{w} = \mathrm{curl}(\mathbf{u})$ is concentrated on a smooth arclength parametrized curved γ of length L. We write

$$\mathbf{w}(\mathbf{x},t) = \Gamma \int_0^L \delta(\mathbf{x} - \gamma(s,t)) \frac{\partial \gamma}{\partial s} \,\mathrm{d}s \tag{1}$$

where $\Gamma = \oint \mathbf{u} \cdot \mathrm{d}\mathbf{l}$ is the (finite) circulation of $\mathbf{u}$ around a closed circuit threaded by the vortex filament, and δ is the δ-function in $\mathbb{R}^3$.

The Biot-Savart law gives the following representation of the velocity field in terms of the vorticity of the fluid:

$$\mathbf{u} = -\frac{1}{4\pi} \int \mathrm{d}^3\mathbf{x}' \frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{w}(\mathbf{x}')}{||\mathbf{x} - \mathbf{x}'||^3} = -\frac{\Gamma}{4\pi} \int \frac{\mathbf{x} - \gamma(s,t)}{||\mathbf{x} - \gamma(s,t)||^3} \times \frac{\partial \gamma}{\partial s} \,\mathrm{d}s. \tag{2}$$

Since $\dfrac{\partial \gamma}{\partial t}(s',t) = \mathbf{u}(\gamma(s',t))$, we obtain the divergent integral

$$\frac{\partial \gamma}{\partial t}(s',t) = -\frac{\Gamma}{4\pi} \int \frac{\gamma(s',t) - \gamma(s,t)}{||\gamma(s',t) - \gamma(s,t)||^3} \times \frac{\partial \gamma}{\partial s} \,\mathrm{d}s$$

which can be reduced to the following expression using a Taylor's expansion about $s = s'$

$$\frac{\partial \gamma}{\partial t} = \frac{\Gamma}{4\pi} \left[\frac{\partial \gamma}{\partial s'} \times \frac{\partial^2 \gamma}{\partial s'^2} \int \frac{\mathrm{d}s}{|s - s'|} + \mathcal{O}(1) \right].$$

This is the self-induction approximation: only parts of the curve "close" to a given point determine the velocity field at that point. By introducing the "cut-off" $|s - s'| > \epsilon$ and rescaling the time variable as $t \to \ln(\epsilon^{-1}) \dfrac{\Gamma}{4\pi} t$ in order to absorb the logarithmic singularity, one obtains (renaming $s' \to s$) the **Vortex Filament Equation** (VFE)

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2}. \tag{3}$$

Equation (3) can be rewritten in more transparent form by defining the Frenet-Serret frame of the curve: $\mathbf{T} = \dfrac{\partial \gamma}{\partial s}$, $\dfrac{\partial \mathbf{T}}{\partial s} = \kappa \mathbf{N}$, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, with $\kappa(s)$ the curvature function of γ. At a point of non-vanishing curvature the VFE can be rewritten as

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{B} \tag{4}$$

where the vectorfield is directed like the binormal to the curve, and is non-zero only where the filament has non-vanishing curvature (there is no self-induced motion of a straight line vortex).

The Hamiltonian Formulation

The Vortex Filament Equation can be derived as a Hamiltonian system on a suitable infinite dimensional symplectic manifold. Following the treatment in J.-L. Brylinsky's [9], we restrict consideration to closed smooth curves in three-dimensional Euclidean space.

Let $\mathcal{X} = \{\gamma : S^1 \to \mathbb{R}^3;\ \gamma \in C^\infty(S^1, \mathbb{R}^3)\}$ be the loop space of $\mathbb{R}^3$. The infinite-dimensional group $\mathrm{Diff}^+(S^1)$ of orientation preserving diffeomorphisms of the circle acts on $\mathcal{X}$ by reparametrization. The quotient space $\mathcal{Y} = \mathcal{X}/\mathrm{Diff}^+(S^1)$ of unparametrized oriented loops is a singular space, as it contains curves with an infinite number of self-intersections or self-tangencies of infinite order, and these possess a non-trivial isotropy group. If $\widehat{\mathcal{X}}$ is the subspace of immersions with finitely many self-intersections and with contacts of finite order, then $\mathrm{Diff}^+(S^1)$ acts freely on $\widehat{\mathcal{X}}$:

Theorem 1 ([9]). *The space of oriented singular knots*

$$\widehat{\mathcal{Y}} = \widehat{\mathcal{X}}/\mathrm{Diff}^+(S^1)$$

is a smooth infinite-dimensional manifold modelled on $C^\infty(S^1, \mathbb{R}^2)$.

A tangent vector at a point $\gamma \in \widehat{\mathcal{Y}}$ is a smooth choice of normal vectorfield along the curve γ, therefore the tangent space $T\widehat{\mathcal{Y}}_\gamma$ can be identified with the space of smooth sections of the normal bundle of γ.

The following 2-form on $T\widehat{\mathcal{Y}}$ was introduced by J. Marsden and A. Weinstein in their study of vortex dynamics in incompressible fluids [44]:

$$\omega_\gamma(\mathbf{u}, \mathbf{v}) = \int_0^{2\pi} \left(\frac{\mathrm{d}\gamma}{\mathrm{d}s} \times \mathbf{u} \cdot \mathbf{v} \right) \mathrm{d}s = \int_0^{2\pi} \nu\left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}, \mathbf{u}, \mathbf{v}\right) \mathrm{d}s \tag{5}$$

where $\mathbf{u}, \mathbf{v}$ are arbitrary tangent vectors at γ, s is the arclength parameter, and ν is the standard volume form in $\mathbb{R}^3$.

The closure of ω is an immediate consequence of the fact that the exterior derivative of the volume form ν in $\mathbb{R}^3$ is identically zero. Nondegeneracy also follows easily.

A symplectic form ω on a finite-dimensional manifold M determines a natural isomorphism

$$\begin{array}{ccc} TM & \longrightarrow & T^*M \\ X & \longrightarrow & \omega(X, \cdot) \end{array}$$

between tangent vectors and one-forms (see for example [2]). In particular, ω associates to any smooth function H on M (a Hamiltonian) a unique vectorfield X_H (and thus a Hamiltonian flow on M) defined by

$$\omega(X_H, \cdot) = \mathrm{d}H(\cdot). \tag{6}$$

In the infinite-dimensional setting the correspondence $TM \to T^*M$ may fail to be surjective, i.e. not every Hamiltonian functional has an associated Hamiltonian vectorfield. However, the inverse map $T^*M \to TM$ exists and is well-defined for a class of functionals (see [9] for a discussion of this issue) which include the **total length** functional

$$\mathcal{L}(\gamma) = \int_0^{2\pi} \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}s} \right\| \mathrm{d}s \tag{7}$$

defined on arbitrary smooth closed curves. The Frechét derivative of $\mathcal{L}(\gamma)$ restricted to the arclength parametrized representative of γ is

$$\mathrm{d}\mathcal{L}(\gamma) = -\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2}.$$

Using the Marsden-Weinstein symplectic form one easily computes the associated Hamiltonian vectorfield

$$X_{\mathcal{L}} = \frac{\mathrm{d}\gamma}{\mathrm{d}s} \times \frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2}$$

which coincides with the right-hand side of the VFE.

An immediate consequence of the Hamiltonian formulation of the VFE is that the total length of the filament is a conserved functional. A short calculation shows that local arclength is also preserved during the evolution, i.e. $\partial_t \|\gamma_s\| = 0$ (the vortex filament moves without stretching), and that therefore s and t are independent variables: we will see how the ability to "commute mixed partials" is at the root of the most interesting properties of the Vortex Filament Equation.

The Hasimoto Map

In 1972, R. Hasimoto [28] introduced the complex function $q = \kappa \exp(\mathrm{i} \int^s \tau \, ds')$ of the curvature κ and the torsion τ of a space curve, and showed that if the curve evolves according to the VFE (3), then q solves the focussing cubic nonlinear Schrödinger equation

$$\mathrm{i}q_t + q_{ss} + \frac{1}{2}|q|^2 q = 0. \tag{8}$$

To prove this result we introduce the orthonormal frame $(\mathbf{T}, \mathbf{U}, \mathbf{V})$, where

$$\begin{aligned}
\mathbf{U} &= \cos\Big(\int^s \tau \, \mathrm{d}s'\Big)\mathbf{N} - \sin\Big(\int^s \tau \, \mathrm{d}s'\Big)\mathbf{B} \\
\mathbf{V} &= \sin\Big(\int^s \tau \, \mathrm{d}s'\Big)\mathbf{N} + \cos\Big(\int^s \tau \, \mathrm{d}s'\Big)\mathbf{B}.
\end{aligned}$$

Using the Darboux equations for the new frame

$$\mathbf{T}_s = \kappa_1 \mathbf{U} + \kappa_2 \mathbf{V}, \qquad \mathbf{U}_s = -\kappa_1 \mathbf{T}, \qquad \mathbf{V}_s = -\kappa_2 \mathbf{T} \tag{9}$$

with $\kappa_1 = \kappa \cos(\int^s \tau \, \mathrm{d}s')$ and $\kappa_2 = \kappa \sin(\int^s \tau \, \mathrm{d}s')$, we rewrite the VFE as

$$\boldsymbol{\gamma}_t = -\kappa_2 \mathbf{U} + \kappa_1 \mathbf{V}. \tag{10}$$

Differentiating (10) twice with respect to s, we obtain

$$\boldsymbol{\gamma}_{tss} = -\kappa_{2ss} \mathbf{U} + \kappa_{1ss} \mathbf{V} + (\kappa_1 \kappa_{2s} - \kappa_2 \kappa_{1s}) \mathbf{T}. \tag{11}$$

On the other hand $\boldsymbol{\gamma}_{ss} = \kappa_1 \mathbf{U} + \kappa_2 \mathbf{V}$, from which we derive

$$\boldsymbol{\gamma}_{sst} = \kappa_{1t} \mathbf{U} + \kappa_{2t} \mathbf{V} + \kappa_1 \mathbf{U}_t + \kappa_2 \mathbf{V}_t. \tag{12}$$

In order to compare equations (11) and (12), we observe that $\mathbf{T}_t = -\kappa_{2s} \mathbf{U} + \kappa_{1s} \mathbf{V}$ (this follows directly from differentiating (10) with respect to s), and that $\mathbf{U}_t \cdot \mathbf{V} = -\mathbf{U} \cdot \mathbf{V}_t$. Using the Darboux equations (9) we compute

$$\begin{aligned} (\mathbf{U}_t \cdot \mathbf{V})_s &= \mathbf{U}_{st} \cdot \mathbf{V} + \mathbf{U}_t \cdot \mathbf{V}_s = -\kappa_1 \kappa_{1s} - \kappa_2 \mathbf{U}_t \cdot \mathbf{T} \\ &= -\kappa_1 \kappa_{1s} + \kappa_2 \mathbf{U} \cdot \mathbf{T}_t = -\kappa_1 \kappa_{1s} - \kappa_2 \kappa_{2s}. \end{aligned}$$

It follows that $\mathbf{U}_t \cdot \mathbf{V} = -(k_1^2 + k_2^2)/2 + A(t)$, where $A(t)$ is an arbitrary s-independent function. Equating the right-hand sides of (11) and (12), we obtain the following system of equations

$$\begin{aligned} \kappa_{1t} + \left[\frac{1}{2}(\kappa_1^2 + \kappa_2^2) - A(t) \right] \kappa_2 &= -\kappa_{2ss} \\ \kappa_{2t} - \left[\frac{1}{2}(\kappa_1^2 + \kappa_2^2) - A(t) \right] k_1 &= \kappa_{1ss} \end{aligned}$$

which reduces to

$$\mathrm{i}\psi_t + \psi_{ss} + \left[\frac{1}{2} |\psi|^2 \psi - A(t) \right] \psi = 0 \tag{13}$$

for the complex-valued function $\psi = \kappa_1 + \mathrm{i}\kappa_2$, and becomes the focussing nonlinear Schrödinger equation (8) if ψ is rescaled to $\exp(-\mathrm{i} \int^t A(t') \, \mathrm{d}t') \psi$.

A Geometric Interpretation of the Hasimoto Map

Given a smooth arclength parametrized space curve $\boldsymbol{\gamma}$, its unit tangent vector $\mathbf{T}$ describes a curve in the unit sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3 ; \, x_1^2 + x_2^2 + x_3^2 = 1\}$. Then, an orthonormal frame of the curve $\boldsymbol{\gamma}$ can be identified with a unit tangent vectorfield to S^2 at $\boldsymbol{\gamma}$.

Introduce the **circle bundle** of S^2

$$\mathcal{T}_1 S^2 = \left\{ (\mathbf{x}, \mathbf{v}) \, ; \, \mathbf{x} \in S^2, \ \mathbf{v} \in T_{\mathbf{x}} S^2, \ \|\mathbf{v}\| = 1 \right\}$$

together with the projection map $\pi : T_1S^2 \to S^2, \pi(\mathbf{x}, \mathbf{v}) = \mathbf{x}$, and fiber $\pi^{-1}(\mathbf{x}) = S^1$, where $\|\cdot\|$ is the inherited Euclidean norm from $\mathbb{R}^3$.

Remarks:

1. The group S^1 acts freely on T_1S^2 by fixing the base point $\mathbf{x}$ and rotating $\mathbf{v}$ within the tangent plane to S^2 at $\mathbf{x}$:

$$\begin{array}{ccc} S^1 \times T_1S^2 & \longrightarrow & T_1S^2 \\ h \cdot (\mathbf{x}, \mathbf{v}) & \longrightarrow & (\mathbf{x}, h\mathbf{v}). \end{array}$$

2. We can construct a local cross section, i.e. a smooth map $\phi^{-1} : \pi^{-1}(U) \to U \times S^1$ for some neighbourhood U of each point $\mathbf{x} \in S^2$. To do so, we choose a smooth unit vectorfield $\mathbf{e}(U)$ (for example, we can take the vectorfield $\partial_u/\|\partial_u\|$ in a local coordinate system (u, v)) and define $\phi(\mathbf{x}, h) = (\mathbf{x}, h\mathbf{e}(\mathbf{x}))$. Since the S^1-action is free and $\mathbf{e}(\mathbf{x})$ is smooth and never zero in U, ϕ is smooth and invertible and so is ϕ^{-1}.

3. The properties described above define T_1S^2 as a principal fibre bundle with fibre S^1 and base S^2.

4. The action of the orthonormal frame group $SO(3, \mathbb{R})$ on T_1S^2 given by $g \cdot (\mathbf{x}, \mathbf{v}) = (g\mathbf{x}, g\mathbf{v})$, $g \in SO(3, \mathbb{R})$, is transitive and free. We can then identify $SO(3, \mathbb{R}) \cong T_1S^2$.

We now summarize the notion of a *connection* in a *principal fibre bundle* and the construction of the canonical invariant connection for T_1S^2 (see [31] for a comprehensive treatment and [50] for a discussion of the circle bundle of a 2-dimensional Riemannian manifold). A choice of a connection in a manifold prescribes a way of parallel translating tangent vectors and of intrinsically defining a directional derivative. In the case of a principal bundle P with structure group G over a manifold M

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \Big\downarrow \pi \\ & & M \end{array}$$

we best explain the role of a connection when lifting a vectorfield $v \in TM$ to a vectorfield $\tilde{v} \in TP$ in a unique way. For each $p \in P$, let G_p be the subspace of T_pP consisting of all the vectors tangent to the vertical fibre. The lifting of v is unique if we require $\tilde{v}(p)$ to lie in a subspace of T_pP complementary to G_p. A smooth and G-invariant choice of the complementary subspace is called a **connection** on P. More precisely,

Definition 1. *A connection on a principal bundle P is a smooth assignment of a subspace $H_p \subset T_pP$, $\forall p \in P$ such that:*

1. $TP_p = G_p \oplus H_p$
2. $H_{gp} = (\mathcal{L}_g)_* H_p, \quad \forall g \in G, \quad \mathcal{L}_g$ *is the left-translation in* G.

Given a connection, the horizontal subspace H_p is mapped isomorphically by $d\pi$ onto $T_{\pi(p)}M$. Therefore the lifting of v is the unique horizontal $\tilde{v}$ which projects onto v.

An equivalent way of assigning a connection is by means of a Lie algebra valued 1-form ϕ (the connection form). If $A \in \mathfrak{g}$ (the Lie algebra of G), let A^* be the vectorfield on P induced by the action of the 1-parameter subgroup e^{tA}. Since G maps each fibre into itself, then A^* is tangent to the vertical fibre at each point. For each $X \in T_pP$, let $\phi(X)$ be the unique $A \in \mathfrak{g}$ such that A^* is equal to the vertical component of X. Then $\phi(X) = 0$ if and only if X is horizontal.

Proposition 1. *A connection* 1*-form* ϕ *has the following properties:*

1. $\phi(A^*) = A$
2. $(\mathcal{L}_g)^*\phi = Ad_g\phi, \quad \forall g \in G, \quad Ad$ *is the adjoint representation of* G.

The first property follows immediately from the definition of connection 1-form; for a proof of the second property we refer to [31].

We are now ready to construct an invariant connection on T_1S^2. Define the involution σ on the group $SO(3,\mathbb{R})$ by $\sigma(g) = T^{-1}gT$, $g \in SO(3,\mathbb{R})$, where T is the matrix $T = \begin{pmatrix} \mathrm{I}_2 & \\ & -1 \end{pmatrix}$. The identity component of the set of elements $h \in G$ invariant under the action of σ is the subgroup of elements of the form $h = \begin{pmatrix} * & \\ & 1 \end{pmatrix}$, which can be identified with the group $SO(2,\mathbb{R})$. The resulting symmetric homogeneous space $SO(3,\mathbb{R})/SO(2,\mathbb{R})$ is naturally diffeomorphic to the 2-dimensional sphere. This can be shown, for example, by constructing a suitable transitive action of $SO(3,\mathbb{R})$ on S^2.

The group involution σ induces a Lie algebra automorphism σ' on $\mathfrak{g} = \mathfrak{so}(3,R)$, which inherits the direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$$

where $\mathfrak{h} = \{X \in \mathfrak{g}\,;\, \sigma'X = X\}$ is the subalgebra of the invariant subgroup $H = SO(2,\mathbb{R})$ and $\mathfrak{k} = \{X \in \mathfrak{g}\,;\, \sigma'X = -X\}$.

Let θ be the canonical 1-form of $SO(3,\mathbb{R})$, i.e. the left-invariant $\mathfrak{g}$-valued 1-form defined by

$$\theta(A) = A, \qquad A \in \mathfrak{g}.$$

Then

Theorem 2. *The* $\mathfrak{h}$*-component* ϕ *of the canonical* 1*-form* θ *of* $SO(3,\mathbb{R})$ *defines a left-invariant connection on* $\mathcal{T}_1 S^2$.

Let now (V, E_1, E_2) be the canonical basis for the Lie algebra $\mathfrak{g}$ such that V spans the vertical space $\mathfrak{h}$ and (E_1, E_2) span the horizontal subspace $\mathfrak{k}$. We have

$$[V, E_1] = E_2, \qquad [V, E_2] = -E_1, \qquad [E_1, E_2] = V.$$

Setting $\theta = \phi V + \omega_1 E_1 + \omega_2 E_2$ and using the Maurer-Cartan equation

$$\mathrm{d}\theta(X, Y) = -\frac{1}{2}\theta([X, Y]), \qquad X, Y \in \mathfrak{g}$$

we obtain the structure equations for the dual basis $(\phi, \omega_1, \omega_2)$

$$\mathrm{d}\phi = -\omega_1 \wedge \omega_2, \qquad \mathrm{d}\omega_1 = \phi \wedge \omega_2, \qquad \mathrm{d}\omega_2 = -\phi \wedge \omega_1.$$

Remarks:

1. If the Riemannian metric on S^2 is the restriction of the Euclidean metric on $\mathbb{R}^3$, then the invariant connection constructed above coincides with the Riemannian connection on $\mathcal{T}_1 S^2$ (the unique connection which leaves the metric invariant and has zero torsion).

2. The 1-forms ω_1, ω_2 can be explicitly defined in terms of the Riemannian structure and the isomorphism $\mathrm{d}\pi : H_{(\mathbf{x},\mathbf{v})} \to T_{\mathbf{x}} S^2$ between the horizontal subspace at $(\mathbf{x}, \mathbf{v})$ and the tangent space to S^2 at $\mathbf{x}$. The horizontal basis vectors (E_1, E_2) can be identified with the orthonormal basis $(\mathbf{e}_1, \mathrm{i}\mathbf{e}_1)$ of $T_{\mathbf{o}} S^2$, where $\mathrm{i}\mathbf{e}_1$ is the $90°$ rotation of $\mathbf{e}_1$ within the tangent plane. Then for $t \in T_{(\mathbf{o},\mathbf{e}_1)} \mathcal{T}_1 S^2$, $\omega_1|_{(\mathbf{o},\mathbf{e}_1)}(t)$ and $\omega_2|_{(\mathbf{o},\mathbf{e}_1)}(t)$ are the components of $\mathrm{d}\pi(t)$ with respect to $\mathbf{e}_1$ and $\mathrm{i}\mathbf{e}_1$ (e.g., $\omega_1|_{(\mathbf{o},\mathbf{e}_1)}(t) = \langle \mathrm{d}\pi(t), \mathbf{e}_1 \rangle$, where $\langle\ ,\ \rangle$ denotes the metric on S^2). Since ω_1, ω_2 are invariant under the circle action, we can define them on all tangent vectors $\mathbf{v} = h\mathbf{e}_1$ at $\mathbf{o}$, $h \in S^1$. The left-invariance of the metric under the full group action defines them everywhere:

$$\omega_1|_{(\mathbf{x},\mathbf{v})}(t) = \langle \mathrm{d}\pi(t), \mathbf{v} \rangle, \qquad \omega_2|_{(\mathbf{x},\mathbf{v})}(t) = \langle \mathrm{d}\pi(t), \mathrm{i}\mathbf{v} \rangle, \qquad t \in T_{(\mathbf{x},\mathbf{v})} \mathcal{T}_1 S^2.$$

Given the curve $\mathbf{c}(s) = \mathbf{T}(s)$ described by the tangent indicatrix of γ in S^2, we construct the unique horizontal lifting $\tilde{c}_o = (\mathbf{T}, \mathbf{v})$, with $\mathbf{v}(0) = \mathbf{N}(0)$ (i.e. we require that $\tilde{c}_o$ agrees with the Frenet frame at $s = 0$). In order to compute the tangent vector field to $\tilde{c}_o$, we work in a local coordinate patch $U \subset S^2$ and use the map $\phi : U \times S^1 \longrightarrow \pi^{-1}(U)$ to identify $\pi^{-1}(U)$ with $U \times S^1$. In a local patch, there exist smooth real-valued functions $\beta(s)$, $\delta(s)$ for which $(\mathbf{T}, \mathbf{N}) \cong (\mathbf{T}, \mathrm{e}^{\mathrm{i}\beta(s)})$ and

$$\tilde{c}_o(s) \cong (\mathbf{T}(s), h(s)), \qquad \text{with } h(s) = \mathrm{e}^{\mathrm{i}(\beta(s)+\delta(s))} \in S^1.$$

If ∂_α denotes the unit tangent vector field to S^1 and $r : \mathbb{R} \rightarrow S^1$ the exponential map $r(\alpha) = \mathrm{e}^{\mathrm{i}\alpha}$, then $\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{\mathrm{d}(\beta+\delta)}{\mathrm{d}s}\mathrm{d}r\left(\frac{\mathrm{d}}{\mathrm{d}s}\right) = \frac{\mathrm{d}(\beta+\delta)}{\mathrm{d}s}\partial_\alpha$. Therefore the velocity field of $\tilde{c}_o$ is

$$\frac{\mathrm{d}\tilde{c}_o}{\mathrm{d}s} = \left(k\mathbf{N}, \left(\frac{\mathrm{d}\beta}{\mathrm{d}s} + \frac{\mathrm{d}\delta}{\mathrm{d}s}\right)\partial_\alpha\right).$$

The lifting $\tilde{c}_o$ is horizontal if and only if the component of its tangent vectorfield along the vertical fibre vanishes, i.e. when $\frac{\mathrm{d}\beta}{\mathrm{d}s} = -\frac{\mathrm{d}\delta}{\mathrm{d}s}$. Since $\frac{\mathrm{d}\beta}{\mathrm{d}s} = \tau$ (in fact, the covariant derivative of the unit normal vector satisfies $\nabla_{\mathbf{T}}\mathbf{N} = \tau\mathbf{B} = \tau\partial_\alpha$), then $\delta(s) = -\int_0^s \tau(u)\mathrm{d}u$ and we obtain the following expression for the horizontal lifting

$$\tilde{c}_h(s) = \left(\mathbf{T}(s), \mathrm{e}^{-\mathrm{i}\int_0^s \tau(u)\mathrm{d}u}\mathbf{N}\right) = \left(\mathbf{T}, \cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{N} - \sin\left(\int_0^s \tau\mathrm{d}u\right)\mathbf{B}\right)$$

where we define the binormal vector $\mathbf{B} = \mathrm{i}\mathbf{N}$.

The orthonormal frame

$$\mathbf{U} = \cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{N} - \sin\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{B},$$
$$\mathbf{V} = \sin\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{N} + \cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{B}$$

given by the horizontal lifting is known as the *natural frame* of the curve γ (see [7] for some history and further discussion), and satisfies the following Darboux equations

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} &= \kappa\cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{U} + \kappa\sin\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{V} \\
\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}s} &= -\kappa\cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{T} \\
\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}s} &= -\kappa\sin\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{T}.
\end{aligned} \tag{14}$$

Correspondingly, the components of the projection of the velocity field of the horizontal lifting with respect to $\mathbf{U}$ and $\mathbf{V}$ are called the **natural curvatures**:

$$\kappa_1 = \omega_1|_{\tilde{c}_h}\left(\frac{\mathrm{d}\tilde{c}_h}{\mathrm{d}s}\right) = \langle\kappa\mathbf{N}, \cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{N} - \sin\left(\int_0^s \tau\mathrm{d}u\right)\mathrm{i}\mathbf{N}\rangle = \kappa\cos\left(\int_0^s \tau\,\mathrm{d}u\right)$$
$$\kappa_2 = \omega_2|_{\tilde{c}_h}\left(\frac{\mathrm{d}\tilde{c}_h}{\mathrm{d}s}\right) = \langle\kappa\mathbf{N}, \cos\left(\int_0^s \tau\,\mathrm{d}u\right)\mathrm{i}\mathbf{N} + \sin\left(\int_0^s \tau\,\mathrm{d}u\right)\mathbf{N}\rangle = \kappa\sin\left(\int_0^s \tau\,\mathrm{d}u\right).$$

We summarize this discussion by giving the following interpretation of the Hasimoto map $\mathcal{H} : \gamma(s) \longrightarrow q(s) = k(s)\exp\left(\mathrm{i}\int_0^s \tau\,\mathrm{d}u\right)$.

Proposition 2 (A. Calini).

$$\mathcal{H}(\gamma) = \mathrm{d}\pi \left(\frac{\mathrm{d}\tilde{c}_h}{\mathrm{d}s} \right).$$

Given the parallel transport of the Frenet frame of γ with respect to the canonical connection on $\mathcal{T}_1 S^2$, the image of the Hasimoto map is the projection of its velocity field onto the tangent space to S^2 at the curve described by the unit tangent vector of γ, $T_{\mathbf{c}} S^2 \cong \mathbb{C}$.

Remarks:

1. A natural frame (a choice of a parallel vectorfield) exists at every point along the curve, while the Frenet frame is not defined where the curvature $\kappa(s)$ vanishes. In fact, once the vector $\mathbf{v}(0)$ is chosen, its horizontal lifting is unique.
2. The Frenet lifting of a closed curve is closed, however the horizontal lifting needs not be. The condition for a closed lifting is the following quantization condition for the total torsion: $\oint \tau \, \mathrm{d}u = 2\pi j$, $j \in \mathbb{Z}$. Since $q(2\pi) = q(0) \exp(\mathrm{i} \oint \tau \, \mathrm{d}u)$, periodic solutions of the VFE are in general mapped to quasi-periodic solutions of the NLS equation.

Lecture 2

Some Consequences of Integrability

In 1972 Zakharov and Shabat [53] proved the complete integrability of the cubic nonlinear Schrödinger equation, adding it to the growing list of soliton equations. This special class of nonlinear partial differential equations gained its importance in the mid-sixties with M. Kruskal and N. Zabuski's observation of solitary wave solutions of the Korteweg-de Vries equation (KdV), and with the development of the inverse scattering method for the KdV equation by Gardner, Green, Kruskal and Miura. Several features shared by these nonlinear equations make them the infinite-dimensional counterparts of completely integrable Hamiltonian systems in finite dimensions.

We summarize some of the most important properties of integrable PDE's. A good overview of the subject can be found in the book by A. Newell [45] and a very clear exposition is contained in the book by G. Lamb [34]; we also refer to M. Ablowitz and H. Segur's monograph [1].

Integrable PDE's possess a class of special solutions widely known as solitons. These are solitary waves in the form of pulses whose behavior is particle-like. During their evolution, solitons propagate without change of shape and with no energy loss. When two or more solitons with different propagation speeds collide, the pulses emerge with the same initial shapes (but with a shift in the phases), and no energy is lost in radiation processes during the nonlinear interaction. The

1- and 2-soliton solutions for the Vortex Filament Equations were first computed by H. Hasimoto [28] and S. Kida [30]; the periodic analogs of solitons (the N-phase solutions) were derived by A. Sym [52] for the case $N = 2$, and the explicit formulas for general N will be discussed in Lecture 3 (see also [11]).

Soliton equations possess an infinite sequence $\{I_k\}_{k=1}^{\infty}$ of constants of motion (or integral invariants), whose gradients are linearly independent and whose associated Hamiltonian flows pairwise commute (i.e., the I_k's are in involution with respect to a given Poisson structure). For example, the focussing cubic nonlinear Schrödinger equation

$$\mathrm{i}q_t + q_{ss} + \frac{1}{2}|q|^2\psi = 0$$

can be rewritten in Hamiltonian form

$$q_t = \{H, q\} = \mathcal{J}\frac{\delta H}{\delta q}$$

with Hamiltonian functional

$$H[q] = \int_{\mathcal{D}} (|q_s|^2 - |q|^4)\,\mathrm{d}s$$

with respect to the inner product

$$\langle f, g\rangle = \frac{1}{2}\Re \int_{\mathcal{D}} f(s)\bar{g}(s)\,\mathrm{d}s$$

and the Poisson bracket

$$\{F, G\} = \left\langle \mathcal{J}\frac{\delta F}{\delta q}, \frac{\delta G}{\delta q}\right\rangle = \mathrm{i}\int_{\mathcal{D}} \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta \bar{q}} - \frac{\delta F}{\delta \bar{q}}\frac{\delta G}{\delta q}\right)\mathrm{d}s.$$

$\mathcal{D}$ denotes the domain of solutions, $\delta F/\delta q$ the Frechét derivative of the functional $F(q, \bar{q})$, and $\mathcal{J} = \mathrm{i}$ is the standard symplectic operator of the NLS equation.

The first few NLS invariants are

$$I_1 = \int_{\mathcal{D}} |q|^2\,\mathrm{d}s, \quad I_2 = \frac{1}{2\mathrm{i}}\int_{\mathcal{D}} (q_s\bar{q} - \bar{q}_s q)\,\mathrm{d}s, \quad I_3 = \frac{1}{2}\int_{\mathcal{D}} \left(|q_s|^2 - |q|^4\right)\mathrm{d}s, \ldots.$$

They satisfy $\{I_k, I_j\} = 0, \forall k, j$ and their gradients are linearly independent. The first few conserved functionals for the VFE are listed below:

$$J_{-1} = \int_{\mathcal{D}} \|\gamma_s\|\,ds, \qquad J_0 = \int_{\mathcal{D}} \tau\,\mathrm{d}s$$

$$J_1 = \int_{\mathcal{D}} \frac{1}{2}\kappa^2\,\mathrm{d}s, \quad J_2 = \int_{\mathcal{D}} \kappa^2\tau\,\mathrm{d}s, \quad J_3 = \int_{\mathcal{D}} \left[(\kappa_s)^2 + \kappa^2\tau^2 - \kappa^4\right]\mathrm{d}s, \ldots.$$

The first two invariants (total length and total torsion of γ) do not have a corresponding NLS invariant, while the remaining conserved quantities are associated with NLS invariants via the Hasimoto map.

In his seminal 1978 article, F. Magri [42] discovered the existence of a second symplectic operator $\mathcal{K}$ for the NLS equation, compatible with $\mathcal{J}$ and such that the NLS equation can be rewritten as a Hamiltonian system with respect to two different symplectic structures and two different Hamiltonians:

$$q_t = \mathcal{J}\frac{\delta H}{\delta q} = \mathcal{K}\frac{\delta H'}{\delta q}.$$

Given two compatible symplectic operators, one constructs a **recursion operator** $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$ which generates an infinite sequence of Hamiltonian vectorfields $X_n = \mathcal{R}^n X_0$, and an associated family of Poisson structures $\{F, G\}^n = \left\langle \mathcal{R}^n J\frac{\delta F}{\delta q}, \frac{\delta G}{\delta q}\right\rangle$.

For asymptotically linear boundary conditions, J. Langer and R. Perline [36] studied the bihamiltonian structure of the Vortex Filament Equation and showed that the VFE and NLS equations can be regarded as the same Hamiltonian system written with respect to two different Poisson brackets. More precisely, the Hasimoto transformation is a Poisson map satisfying $\{f \circ \mathcal{H}, g \circ \mathcal{H}\}_{\mathrm{VFE}}(\gamma) = \{f, g\}^4_{\mathrm{NLS}}(\mathcal{H}(\gamma))$, i.e., $\mathcal{H}$ maps the Marsden-Weinstein Poisson bracket to the fourth NLS Poisson structure.

As a consequence of the existence of a hierarchy $\{I_k\}_{k=1}^{\infty}$ of invariants, the phase trajectories are restricted to lie on the infinite codimensional intersection of the level sets $I_k = c_k$, $k = 1, \ldots, \infty$. In the case of a finite-dimensional phase space, the preimage of a regular value $\mathbf{c}$ of the momentum map $\mathbf{I} = (I_1, \ldots, I_n)$ is diffeomorphic to a product of circles and lines [2], and the dynamical system can be described in terms of the linear evolution of a collection of action-angle variables. For completely integrable PDE's, the inverse scattering method explicitly constructs a nonlinear change of variables that linearizes the flow. The KdV equation was the first soliton equation for which the inverse scattering method was developed: in this case, the analogues of action-angle variables are the scattering data of a related linear Hamiltonian recursion operator. The initial value problem for the KdV equation can then be solved exactly by mapping the initial condition to its scattering data, evolving the scattering data according to a linear evolution up to time t, and using the inverse transform to reconstruct the solution of the original PDE at time t.

At the heart of the inverse scattering method for the KdV equation is an underlying linear operator $\mathcal{L}$, whose spectrum does not vary under the KdV evolution [39]. If the spectrum of $\mathcal{L}$ is independent of time, then its evolution can be written as

$$\mathcal{L}(t) = \mathcal{U}(t)\mathcal{L}(0)\mathcal{U}^{-1}(t) \tag{15}$$

for some time-dependent unitary operator $\mathcal{U}(t)$. It follows directly from equation (15) that, if some function ϕ solves the eigenvalue problem $\mathcal{L}\phi = \lambda\phi$, then ϕ is also a solution of the linear system $\phi_t = \mathcal{B}\phi$, where $\mathcal{B} = \mathcal{U}_t\mathcal{U}^{-1}$. Differentiating equation (15) with respect to time gives Lax form of the KdV equation

$$\mathcal{L}_t = [\mathcal{B}, \mathcal{L}]$$

where the operators $\mathcal{L}$ and $\mathcal{B}$ are called a **Lax pair** for the soliton equation. Lax also showed the existence of an infinite sequence of operators $\mathcal{B}_k$'s that give rise to spectrum preserving evolutions of $\mathcal{L}$: the corresponding hierarchy of Lax equations coincide with the infinite sequence of commuting Hamiltonian KdV flows mentioned above.

An alternative and more common representation of the NLS equation (8) is as the solvability condition of the following pair of linear systems [22]:

$$\begin{aligned} \mathbf{F}_s &= U\mathbf{F} \\ \mathbf{F}_t &= V\mathbf{F} \end{aligned} \tag{16}$$

where $\mathbf{F}$ is an auxiliary complex vector-valued function,

$$U = \mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & \frac{\mathrm{i}}{2}\bar{q} \\ \frac{\mathrm{i}}{2}q & 0 \end{pmatrix}, V = (2\mathrm{i}\lambda^2 - \frac{\mathrm{i}}{4}|q|^2)\sigma_3 + \begin{pmatrix} 0 & 2\mathrm{i}\lambda\bar{q} + \frac{1}{2}\bar{q}_s \\ 2\mathrm{i}\lambda q - \frac{1}{2}q_s & 0 \end{pmatrix}$$

are matrices whose entries depend on s and t through the complex-valued function q, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The parameter λ is called the **spectral parameter**. Differentiating the first equation of the overdetermined system (16) with respect to t, and the second equation with respect to s and equating the mixed partial derivatives, one obtains

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial s} + [U, V] = 0 \tag{17}$$

which is equivalent to the NLS equation (8) for the complex potential q. System (16) can be interpreted as the equations of parallel transport in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$, where the vector function $\mathbf{F}$ takes values in the fibre $\mathbb{C}^2$ and the matrices U and V are local connection coefficients. Then, equations (16) express the fact that the covariant derivative of $\mathbf{F}$ is zero and equation (17) expresses the flatness of the (U, V)-connection on $\mathbb{R}^2 \times \mathbb{C}^2$. For this reason, linear systems of the form (16) are also called the **zero curvature** formulation of the associated soliton equation.

The Inverse of the Hasimoto Map

The skew-hermitian matrices

$$E_1 = -\mathrm{i}\sigma_3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \; E_2 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; E_3 = -\mathrm{i}\sigma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

where the σ_i's are the Pauli matrices form a right-handed orthonormal basis for the Lie algebra $\mathfrak{su}(2)$ with respect to the inner product $\langle A, B \rangle = -\frac{1}{2}\,\mathrm{trace}(AB)$,

$A, B \in \mathfrak{su}(2)$. Then, the map

$$\begin{array}{ccc} (\mathbb{R}^3, \cdot) & \longrightarrow & (\mathfrak{su}(2), \langle\, , \,\rangle) \\ \mathbf{x} & \longrightarrow & \displaystyle\sum_{k=1}^{3} x_k E_k \end{array} \tag{18}$$

is an isometry. Let (T, U, V) be the image of the natural frame $(\mathbf{T}, \mathbf{U}, \mathbf{V})$ via the isometry (18). Since the group $SU(2)$ acts transitively on the space of orthonormal triples, there exists $\Omega \in SU(2)$ which conjugates the matrices of the "natural frame" to the basis (E_1, E_2, E_3):

$$T = \Omega^{-1} E_1 \Omega, \qquad U = \Omega^{-1} E_2 \Omega, \qquad V = \Omega^{-1} E_3 \Omega.$$

The Darboux equations (14) for the natural frame can then be written in the following form

$$\left[E_1, \frac{\mathrm{d}\Omega}{\mathrm{d}s}\Omega^{-1}\right] = \kappa_1 E_2 + \kappa_2 E_3, \; \left[E_2, \frac{\mathrm{d}\Omega}{\mathrm{d}s}\Omega^{-1}\right] = -\kappa_1 E_1, \; \left[E_3, \frac{\mathrm{d}\Omega}{\mathrm{d}s}\Omega^{-1}\right] = -\kappa_2 E_1.$$

We obtain $\dfrac{\mathrm{d}\Omega}{\mathrm{d}s}\Omega^{-1} = -\dfrac{\kappa_1}{2} E_3 + \dfrac{\kappa_2}{2} E_1$, which coincides with the spatial part of the NLS linear system (16) at $\lambda = 0$:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}s} = \begin{pmatrix} 0 & \mathrm{i}\bar{q}/2 \\ \mathrm{i}q/2 & 0 \end{pmatrix} \Omega. \tag{19}$$

The above discussion rederives a result originally due to A. Sym [51] and K. Pohlmeyer [47] which allows the recostruction of the curve $\gamma = \mathcal{H}^{-1}[q]$ associated with an NLS potential q via the Hasimoto map:

Proposition 3 (A. Sym [51] and K. Pohlmeyer [47]). *Let Φ be a matrix of independent solutions of the NLS linear system* (16) *with $\Phi|_{s=0} \in SU(2)$, then*

1. *the skew-hermitian matrix representing the unit tangent vector of the associated solution γ of the VFE is*

$$T = \left.(\Phi^{-1} E_1 \Phi)\right|_{\lambda=0} \tag{20}$$

2. *the skew-hermitian matrix Γ which represents the position vector of γ is given by*

$$\Gamma(s,t) = -\left.\Phi^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\right|_{\lambda=0} + \Gamma_0 \tag{21}$$

where Γ_0 is a constant matrix representing the initial position vector at $s = 0$.

Remarks:

1. The reconstruction formula (21) is obtained by differentiating the spatial part of the NLS linear system (16) with respect to the parameter λ and evaluating the

resulting expression at $\lambda = 0$. One obtains $T = \frac{\mathrm{d}}{\mathrm{d}s}(-\Phi^{-1}\Phi_\lambda)\Big|_{\lambda=0}$, and computes the expression for the matrix Γ by taking an antiderivative with respect to s.

2. The described procedure is equivalent to reconstructing the curve by solving its Frenet equations for given curvature κ and torsion τ. However, for several classes of NLS potential, including multi-solitons and N-phase solutions, the Sym-Pohlmeyer reconstruction formula produces many explicit solutions of the VFE.

3. Formula (21) evaluated at $\lambda = \lambda_0 \neq 0$ produces a curve of the same curvature as γ and torsion $\tau + \lambda_0$.

Closure Conditions

A generic periodic NLS potential $q(s+L,t) = q(s,t)$ does not produce a closed curve $\gamma = \mathcal{H}^{-1}(q)$, nor a closed Frenet frame, unless

$$\begin{aligned} \mathbf{T}(s+L,t) &= \mathbf{T}(s,t) \quad \text{(periodicity of the tangent vector)} \\ \int_0^L \mathbf{T}(s,t)\,\mathrm{d}s &= \gamma(L) - \gamma(0) = 0 \quad \text{(closure of the curve).} \end{aligned}$$

Since the closure of the initial curve is preserved by the VFE, we drop the t-dependence in following discussion, which summarizes P. Grinevich and M. Schmidt's [26] reformulation of the closure conditions of γ as conditions on the Floquet spectrum of the associated NLS potential q.

Consider the simultaneous eigenvalue problems

$$\begin{aligned} \mathcal{L}\phi &= \lambda\phi \\ \mathcal{S}\phi &= w\phi \end{aligned} \tag{22}$$

for the operators $\mathcal{L} = -\mathrm{i}\sigma_3\frac{\mathrm{d}}{\mathrm{d}s} + \begin{pmatrix} 0 & \mathrm{i}\bar{q}/2 \\ \mathrm{i}q/2 & 0 \end{pmatrix}$ (the spatial part of the NLS linear system (16)), and $\mathcal{S} : (\mathcal{S}\phi)(s) = \phi(s+L)$ (the shift operator). Since $\mathcal{L}$ and $\mathcal{S}$ are commuting linear operators, system (22) generically admits non-trivial solutions (known as **Bloch eigenfunctions**) and one introduces the **spectral curve** of q:

$$\mathcal{C}(q) = \Big\{(\lambda, w) \in \mathbb{C}^2\,;\ \exists \text{ a non trivial Bloch eigenfunction } \phi\Big\}.$$

For general $\lambda \in \mathbb{C}$, there exist two linearly independent eigenfunctions ϕ_1, ϕ_2 with distinct multipliers $w_1(\lambda) \neq w_2(\lambda)$ satisfying $|w_1 w_2| = 1$. Therefore $\mathcal{C}(q)$ is a double cover of the λ-plane (i.e., a hyperelliptic Riemann surface), branched over the λ's at which the multipliers coincide ($w_1(\lambda) = w_2(\lambda)$) and the Bloch eigenfunctions fail to be linearly independent ($\phi_1 = c\phi_2$). The **branch points** of $\mathcal{C}(q)$ are the endpoints of the continuous spectrum (known as **Floquet spectrum** for linear operators with periodic coefficients) of $\mathcal{L}(q)$

$$\sigma(q) = \{\lambda \in \mathbf{C}\,;\ |w(\lambda)| = 1\}.$$

An equivalent definition of the Floquet spectrum of $\mathcal{L}(q)$ can be given in terms of the **Floquet discriminant** Δ : $\{$periodic functions$\} \times \mathbb{C} \to \mathbb{C}$ defined below. (See, e.g., [21, 41] for discussion and proofs of the properties of the Floquet discriminant functional.)

Let $\Phi(s;\lambda)$ be the fundamental solution matrix of $\mathcal{L}\phi = \lambda\phi$ (i.e., satifying $\Phi(0;\lambda) = \mathrm{I}$). We introduce the **transfer matrix** $\Phi(L;\lambda)$ across a period L. It is easy to check that $\Phi(L;\lambda)$ has determinant 1, and as a consequence its eigenvalues are completely characterized in terms of its trace

$$\Delta(q;\lambda) = \mathrm{trace}\,[\Phi(L;\lambda l)]\,. \tag{23}$$

The function $\Delta(q;\lambda)$ is called the Floquet discriminant of the linear operator $\mathcal{L}(q)$. Then, the Floquet spectrum of $\mathcal{L}(q)$ is the set of eigenvalues for which the associated eigenfunction is bounded for all values of s:

$$\sigma(q) = \{\lambda \in \mathbb{C};\ \Delta(q;\lambda) \in \mathbb{R} \text{ and } -2 \leq \Delta(q;\lambda) \leq 2\}\,.$$

The Floquet discriminant has the following properties (property 1 follows from the analytic dependence on λ of the fundamental solution matrix Φ, the verification of properties 2 and 3 involve simple computations):

Proposition 4. *If q is a solution of the NLS equation, then*

1. $\Delta(q;\lambda)$ *is an analytic function of* λ.
2. $\Delta(q;\lambda)$ *is invariant under the NLS evolution, i.e.*
$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta(q(t),\lambda) = 0.$$
3. *The multipliers w_1, w_2 are the eigenvalues of* $\Phi(L;\lambda)$ *and satisfy*
$$w_{1,2} = \frac{\Delta \pm \mathrm{i}\sqrt{4-\Delta^2}}{2}.$$

Remarks:

1. Since the Floquet discriminant is a constant of motion $\forall\lambda \in \mathbb{C}$, then Δ is a generating function for the NLS conserved functionals. In fact, the usual hierarchy of constant of motions can be extracted from the coefficients of the asymptotic expansion of the discriminant at a distinguished value of λ [22].

2. Figure 1 shows the Floquet spectrum of a generic NLS potential. Because of the symmetry $\overline{\Phi(s,\lambda)}^T = \Phi^{-1}(s,\bar{\lambda})$ of solutions of the eigenvalue problem (16), the spectrum possesses the symmetry $\lambda \to \bar{\lambda}$. As a consequence, the entire real axis is part of the continuous spectrum.

3. Potentials with a finite number of branch points (that is, a finite number of complex bands of continuous spectrum) are called **finite-gap** or $\boldsymbol{N}$**-phase** potentials. We will discuss them more in depth in the next lecture.

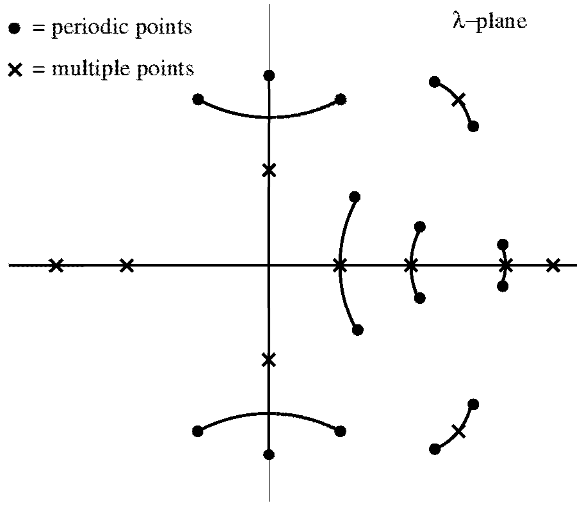

Figure 1. The Floquet spectrum of a typical NLS potential.

4. At a critical point λ_c, such that $\mathrm{d}\Delta/\mathrm{d}\lambda|_{\lambda_c} = 0$, the Floquet multipliers coincide $w_1(\lambda_c) = w_2(\lambda_c)$. It can be shown that two bands of continuous spectrum can have a common endpoint λ_m if λ_m is a critical point. Such critical points, which satisfy the additional condition $\Delta(q, \lambda_m) = \pm 2$, are called **multiple points**.

We now state the main result by P. Grinevich and M. Schmidt on the closure conditions for $\gamma = \mathcal{H}^{-1}[q]$, where $q(s)$ is a smooth periodic NLS potential of period L. Let $\lambda_0 \in \mathbb{R}$, and let $\mathbf{F}(s; \lambda_0)$ be the NLS Bloch eigenfunction normalized as follows: $F_1(0, \lambda_0)\bar{F}_2(0, \lambda_0) + F_2(0, \lambda_0)\bar{F}_1(0, \lambda_0) = 1$. One easily shows that the associated fundamental solution matrix

$$\Omega = \begin{pmatrix} F_1 & -\bar{F}_2 \\ F_2 & \bar{F}_1 \end{pmatrix} \quad \text{satisfies} \quad \Omega(s+L) = \Omega(s) \begin{pmatrix} w_1(\lambda_0) & 0 \\ 0 & w_2(\lambda_0) \end{pmatrix}.$$

Then, $T(s+L) = T(s)$ if and only if (see equation (20)) $w_1(\lambda_0) = \pm 1$:

C1: *The unit tangent vector of $\gamma = \mathcal{H}^{-1}[q]$ is periodic of period L if and only if λ_0 is a real multiple point.*

Compute [26]

$$\int_0^L T(s)\,\mathrm{d}s = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \text{where} \quad a = -\mathrm{i}\,\frac{w'(\lambda)}{w(\lambda)}\bigg|_{\lambda=\lambda_0} = -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda_0} \ln w$$

and introduce $w(\lambda) = \mathrm{e}^{2\pi \mathrm{i}\theta(\lambda)}$, where

$$\mathrm{d}\theta = \frac{1}{2\pi \mathrm{i}} \frac{w'(\lambda)}{w(\lambda)} \mathrm{d}\lambda \tag{24}$$

is the **quasimomentum differential**.

C2: *The curve γ is periodic of period L if and only if λ_0 is a real multiple point and a zero of the quasimomentum differential.*

Lecture 3

Plane Waves to Circles: An Example

The simplest family of NLS solutions consists of the spatially independent plane wave potentials

$$q_a(t) = a\mathrm{e}^{\frac{\mathrm{i}}{2}a^2 t}, \qquad a > 0.$$

For a fixed $t = 0$, one easily computes the fundamental solution matrix of the spatial part of the NLS linear system (16),

$$\Omega(s;\lambda) = \begin{pmatrix} \cos(\omega s) - \frac{\mathrm{i}\lambda}{\omega}\sin(\omega s) & \frac{\mathrm{i}a}{2\omega}\sin(\omega s) \\ \frac{\mathrm{i}a}{2\omega}\sin(\omega s) & \cos(\omega s) + \frac{\mathrm{i}\lambda}{\omega}\sin(\omega s) \end{pmatrix}$$

where $\omega = \sqrt{\lambda^2 + \frac{a^2}{4}}$ and $s \in [0, L]$. The expression for the Floquet discriminant is

$$\Delta(a;\lambda) = \mathrm{trace}(\Omega(L;\lambda)) = 2\cos(\omega L).$$

It follows that the Floquet spectrum is the set (see Figure 2)

$$\sigma(a) = \{\lambda \in \mathbb{C};\ \Delta \in \mathbb{R},\ -2 \le \Delta \le 2\} = \mathbb{R} \cup \left[-\frac{\mathrm{i}a}{2}, \frac{\mathrm{i}a}{2}\right].$$

The critical points are the solutions of

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta = -\sin(\omega L)\frac{2\lambda L}{\sqrt{4\lambda^2 + a^2}} = 0.$$

All of them are double points except for $\lambda = 0$ which is a critical point of multiplicity 4. There is an infinite number of real double points

$$\lambda_n^r = \sqrt{\left(\frac{n\pi}{L}\right)^2 - \frac{a^2}{2}}, \qquad n > \left[\frac{aL}{2\pi}\right]$$

and a finite number of complex double points (in this case they are all pure imaginary) given by

$$\lambda_m^c = \pm \mathrm{i}\sqrt{\frac{a^2}{2} - \left(\frac{m\pi}{L}\right)^2}, \qquad m = 1, 2, \ldots, \left[\frac{aL}{2\pi}\right]$$

where $[x]$ denotes the integer part of x.

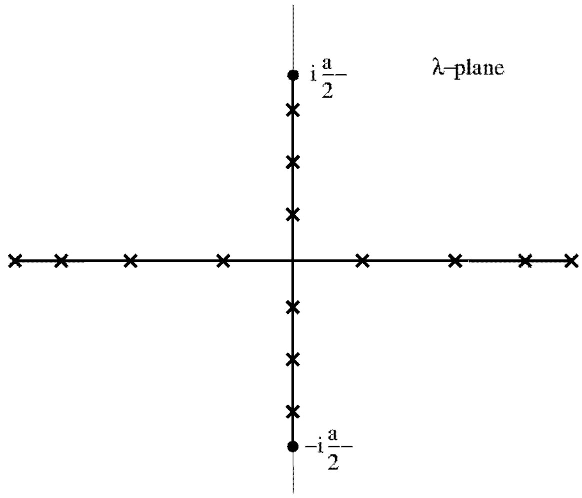

Figure 2. The Floquet spectrum of a plane wave potential with three imaginary double points. The corresponding curve is a four-fold circle.

From the expression of the quasimomentum differential

$$\mathrm{d}\theta = \frac{1}{4\pi}\frac{\Delta'}{\sqrt{4-\Delta^2}}\,\mathrm{d}\lambda = \frac{L}{\pi}\frac{\lambda}{\sqrt{4\lambda^2 + a^2}}\,\mathrm{d}\lambda$$

one finds that its only zero is $\lambda = 0$, which is a multiple point (i.e. also satisfying the condition $\Delta(a; 0) = \pm 2$) if and only if $a = \dfrac{2\pi M}{L}$ (where $M - 1$ is the number of imaginary double points). The associated curve obtained by using the reconstruction formula (21) at $\lambda_0 = 0$ is the following M-covered circle of curvature $\kappa_M = \dfrac{2\pi M}{L}$

$$\gamma(s) = \frac{L}{2\pi M}\left[-\sin\left(\frac{2\pi M}{L}s\right)\mathbf{e}_1 + \cos\left(\frac{2\pi M}{L}\right)\mathbf{e}_2\right] + \gamma(0)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard unit vectors in the (x, y)-plane in $\mathbb{R}^3$.

Remarks:

1. Reconstructing the curve at a different real $\lambda_0 \neq 0$ produces a family of helices of curvature $\kappa_M = \dfrac{2\pi M}{L}$ and torsion λ_0.

2. The following result is proved using the representation of the quasimomentum θ in terms of the Floquet discriminant and the analyticity of Δ as a function of λ:

Lemma 1. *Real zeros of the quasimomentum differential* $\mathrm{d}\theta$ *(if any exists) are located at the points intersection of two bands (one of them the real axis) of continuous spectrum.*

N-phase Solutions of the Vortex Filament Equations

Soliton equations, when considered on periodic domains, admit large classes of special solutions which are the analogues of solitary waves for rapidly decreasing initial data on an infinite domain. An N-phase quasiperiodic potential depends on s and t through a finite number of linear phases

$$\theta_i(s,t) = k_i s + \omega_i t, \qquad i = 1, \dots, N.$$

The potential $q(s,t) = q(\boldsymbol{\theta}(s,t))$ is periodic of finite period in each phase, $q(\theta_1, \dots, \theta_i + 2\pi, \dots, \theta_N) = q(\theta_1, \dots, \theta_i, \dots, \theta_N)$. If $\boldsymbol{\theta}$ is a real-valued vector, then $\mathbf{k}$ is called the vector of spatial frequencies, and $\boldsymbol{\omega}$ the vector of time frequencies.

N-phase solutions of soliton equations such as KdV, NLS and general AKNS systems were first constructed by S. Novikov, B. Dubrovin and I. Krichever [32, 18]. Their method is a rediscovery of classical Riemann surface techniques developed by J. Burchnall and T. Chaundy [10] and by H. Baker [4] for classifying commutative algebras of scalar differential operators.

Peter Lax gave a different characterization of N-phase solutions (in particularly for the KdV equation) [40] as critical points of linear combinations of the conserved functionals, and proved that the isospectral set of a given N-phase solution is an N-dimensional torus in the space of all solutions.

A. Marcenko and I. Ovstrovskii [43] (for the KdV equation), and P. Grinevich [25] (for the the NLS equation and the VFE) showed that any sufficiently smooth periodic potential can be approximated by N-phase solutions of the same period to an arbitrary degree of precision.

In this lecture we sketch the construction of explicit formulas for N-phase solutions of the VFE following Krichever's approach [32] and E. Previato's treatment of the reality conditions for N-phase NLS potentials [48], and making use of the Sym-Pohlmeyer reconstruction formula (21).

Every N-phase solution is associated to a set of data on a Riemann surface. We start with a hyperelliptic Riemann surface Σ of genus g described by the equation

$$\mu^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i). \tag{25}$$

We mark the two points ∞_+ and ∞_-, which are permuted by the holomorphic involution $\tau(\lambda, \mu) = (\lambda, -\mu)$ exchanging the two sheets. We also choose a set of $g+1$ distinct points $\mathcal{D} = P_1 + \cdots + P_{g+1}$ placed in a generic position (a *non-special divisor*) and not containing $\infty_\pm$.

Let $\lambda : \Sigma \to \mathbb{C} \cup \{\infty\}$ be the hyperelliptic projection; this is a meromorphic function on Σ whose pole divisor is $\infty_+ + \infty_-$ (i.e. whose poles are the preimages of ∞ via the map λ). In neighborhoods of $\infty_\pm$ we choose the local parameters $k_\pm$ such that $(k_\pm)^{-1} = (\lambda(P))^{-1}$.

Krichever's main idea is to construct a function $\psi(P)$ on the Riemann surface Σ which is uniquely defined by a prescribed set of singularities and by a prescribed asymptotic behavior near $\infty_\pm$.

Definition 2. *A* ***Baker-Akhiezer function*** *associated to* $(\Sigma, \mathcal{D}, \infty_\pm)$ *is a function* ψ *on* Σ *which:*

- *is meromorphic everywhere except at* $\infty_\pm$*, and has pole divisor in* $\mathcal{D}$
- *has essential singularities at* $\infty_\pm$ *that locally are of the form* $\psi(k_\pm) \sim Ce^{p(k_\pm)}$*, where* C *is a constant and* $p(k)$ *an arbitrary polynomial with complex coefficients.*

The singular structure of ψ and its normalization at the essential singularities define it uniquely as described in the following

Proposition 5 (I. Krichever [32]). *Suppose that the following technical condition holds:*

Condition 1. *The divisor* $P_1 + \cdots + P_{g+1} - \infty_+ - \infty_-$ *is not linearly equivalent to a positive divisor.*

Then, if $p(k) = \mathrm{i}ks + \mathrm{i}Q(k)t$*, where* s *and* t *are complex parameters with* $|s|$*,* $|t|$ *sufficiently small, and* $Q(k)$ *is a given polynomial, the linear vector space of Baker-Akhiezer functions associated with* $(\Sigma, \mathcal{D}, \infty_\pm)$ *is* 2*-dimensional and it has a unique basis* ψ^1*,* ψ^2 *with the following normalized expansions at* $\infty_\pm$*:*

$$\psi^{j,\pm}(s,t;k_\pm) = \mathrm{e}^{\mathrm{i}k_\pm s + iQ(k_\pm)t} \left(\sum_{n=0}^{\infty} \zeta_n^{j\pm}(s,t) k_\pm^{-n} \right), \quad j = 1,2 \tag{26}$$

where $\zeta_n^{j\pm}$ *are functions of the parameters* s *and* t *and* $\zeta_0^{1+} = 1$*,* $\zeta_0^{1-} = 0$*,* $\zeta_0^{2+} = 0$*,* $\zeta_0^{2-} = 1$.

Remarks:

1. Condition 1 can be rephrased as: there exists no non-constant meromorphic function with pole divisor $P_1 + \cdots + P_{g+1}$ which vanishes simultaneously at ∞_+ and at ∞_-.

2. The proof of the uniqueness property makes use of the Riemann-Roch Theorem (see for example [24]), stating that the dimension $h^0(\mathcal{D})$ of the linear space of meromorphic functions on Σ whose pole divisor is a non-special divisor $\mathcal{D}$ is

$$h^0(\mathcal{D}) = \begin{cases} 1, & d \leq g \\ d - g + 1, & d > g. \end{cases} \tag{27}$$

We refer the reader to [48] for more details.

The existence of the Baker-Akhiezer eigenfunction is shown by explicitly constructing its components in terms of the Riemann Theta function of Σ. We summarize the main steps of this algebro-geometric construction (see [18, 6] for a comprehensive discussion and application of this method to several other soliton equations). Let

$$a_1, \ldots, a_g, \quad b_1, \ldots, b_g$$

be a canonical homology basis for Σ, such that $a_i \cdot a_j = 0$, $b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$, and let

$$\omega_1, \ldots, \omega_g, \qquad \text{with} \qquad \oint_{a_i} \omega_j = \delta_{ij}, \qquad i, j = 1, \ldots, g$$

be g normalized holomorphic differentials. We introduce the period matrix B of entries

$$B_{ij} = \oint_{b_i} \omega_j, \qquad i, j = 1, \ldots, g$$

and construct the associated **Riemann theta function**

$$\theta(\mathbf{z}) = \sum_{\mathbf{n} \in \mathcal{Z}^g} \exp\left(\mathrm{i}\pi \langle \mathbf{n}, B\mathbf{n} \rangle + 2\langle \mathbf{n}, \mathbf{z} \rangle\right), \quad \mathbf{z} \in \mathbb{C}^g. \tag{28}$$

The series (28) is absolutely convergent; this follows from the fact that $\Im(B)$ is a positive definite matrix. Moreover, if $\mathbf{e}_k$'s are the standard basis vectors of $\mathbb{C}^g$ and $\mathbf{f}_k = B\mathbf{e}_k$, $k = 1, \ldots, g$, then

$$\theta(\mathbf{z} + \mathbf{e}_j) = \theta(\mathbf{z}), \qquad \theta(\mathbf{z} + \mathbf{f}_j) = \mathrm{e}^{-\mathrm{i}\pi B_{jj} + 2\mathrm{i}\pi z_j} \theta(\mathbf{z}), \quad j = 1, \ldots, g,$$

i.e., θ is a quasi-periodic function. (The $\mathbf{e}_k$'s are called **periods** and the $\mathbf{f}_k$'s the **quasiperiods** of θ.) Let Λ be the $2g$-dimensional lattice spanned by the columns of the matrix $(\mathrm{I} \mid B)$, then the complex torus $Jac(\Sigma) = \mathbb{C}^g/\Lambda$ is called the **Jacobian** of the Riemann surface.

The essential singularity of ψ at ∞ is introduced by means of the unique differentials of the second kind η and ζ with a simple pole at ∞ and local expansions

$\eta \sim \mathrm{id}k$, $\zeta \sim \mathrm{id}Q(k)$ (dictated by the polynomial dependence of the exponent on the local parameter k), and normalized so that their integrals along the a-cycles vanish.

The components of the Baker-Akhiezer eigenfunction ψ^1 are now constructed in terms of rations of theta functions as follows:

$$\begin{aligned}\psi^{1,\pm}(P) = &\exp\left[s\left(\int_{P_0}^{P}\eta - \eta_{\pm}^{\infty}\right) + t\left(\int_{P_0}^{P}\zeta - \zeta_{\pm}^{\infty}\right)\right] \\ &\times \frac{\theta(\mathcal{A}(P) + Us + Wt - \mathcal{A}(\mathcal{D}_{\pm}) - K)}{\theta(\mathcal{A}(P) - \mathcal{A}(\mathcal{D}_{\pm}) - K)} \\ &\times \frac{\theta(\mathcal{A}(\infty_{\pm}) - \mathcal{A}(\mathcal{D}_{\pm}) - K)}{\theta(\mathcal{A}(\infty_{\pm}) + Us + Wt - \mathcal{A}(\mathcal{D}_{\pm}) - K)} \times \frac{g_{\pm}(P)}{g_{\pm}(\infty_{\pm})}\end{aligned} \tag{29}$$

where $P_0 \in \Sigma$ is a base point, $\mathcal{D}_{\pm}$ are the unique positive divisors linearly equivalent to $\mathcal{D} - \infty_{\pm}$, and K is the vector of Riemann constants [18] (such that $\theta(\mathcal{A}(P) - \mathcal{A}(\mathcal{D}_{\pm}) - K)$ has zeros precisely in $\mathcal{D}_{\pm}$). The Abel map

$$\begin{aligned}\mathcal{A} : \Sigma^{\bigotimes_g} &\longrightarrow \mathbb{C}^g/\Lambda \\ \sum_{k=1}^{g} Q_k &\longrightarrow \sum_{k=1}^{g}\int_{P_0}^{Q_k}\boldsymbol{\omega}\end{aligned} \tag{30}$$

($\Sigma^{\bigotimes_g}$ is the set of unordered g-tuples of points on Σ and $\boldsymbol{\omega}$ is the vector of holomorphic differentials) associates to a divisor $\sum_{k=1}^{g} Q_k$ a point on $Jac(\Sigma)$.

The **frequency vectors** U and W are introduced to make ψ a well-defined function on Σ. In fact, if the path of integration is modified by adding an integer combination of homology cycles, ψ changes by the factor

$$\exp\left(\sum_{k=1}^{g}\left[m_k\left(s\oint_{b_k}\eta + t\oint_{b_k}\zeta\right) - m_k\left(sU_k + tW_k\right)\right]\right)$$

which equals 1 if we define the components of U and W as

$$U_k = \frac{1}{2\pi\mathrm{i}}\oint_{b_k}\eta, \qquad \frac{1}{2\pi\mathrm{i}}W_k = \oint_{b_k}\zeta.$$

The constant terms $\eta_{\pm}^{\infty}$ and $\zeta_{\pm}^{\infty}$ at the exponent of expression (29) are subtracted to make the leading coefficient of the meromorphic part of the eigenfunction matrix be the identity. Finally, in order to make the pole divisor be $\mathcal{D}$, $\psi^{\pm}$ is multiplied by a meromorphic function $g_{\pm}(P)$ whose zeros lie in $\mathcal{D}_{\pm} + \infty_{\mp}$ and whose poles lie in the original divisor $\mathcal{D}$.

An explicit computation using the asymptotic expansions of the eigenfunctions ψ^j's at $\infty_{\pm}$ (see, e.g. [48]) shows that, for $Q(k) = 2k^2$, they are linearly independent solutions of the pair of NLS linear systems (16), and that the matrix

$\Psi(s,t;P) = \begin{pmatrix} \psi^{1,+}(P) & \psi^{1,+}(\tau(P)) \\ \psi^{1,-}(P) & \psi^{1,-}(\tau(P)) \end{pmatrix}$ is a fundamental solution matrix. In general, $Q(k)$ and the normalization of ψ at the essential singularity prescribe which soliton equation is encoded in the coefficients of the asymptotic expansion of the Baker-Akhiezer eigenfunction and thus, its explicit construction provides one with both the soliton equation, a large class of initial conditions and the associated solutions.

We now derive a formula for the corresponding N-phase solution of the VFE, making use of the explicit formula for the fundamental solution matrix Ψ and of the Sym-Pohlmeyer reconstruction formula (21):

Proposition 6 (A. Calini). *The components $(\gamma_1, \gamma_2, \gamma_3)$ of the position vector $\boldsymbol{\gamma}$ of an N-phase solution of the Vortex Filament Equation are given by the following expressions*

$$\begin{aligned} \gamma_1 + \mathrm{i}\gamma_2 &= \left.\frac{\partial}{\partial P}\right|_{P=\lambda_{0+}} \log\theta(a(P) + Us + Wt) + \eta(\lambda_{0+})s \\ \gamma_3 &= \left.\frac{\partial}{\partial P}\right|_{P=\lambda_{-0}} \log\theta(a(P) + Us + Wt) + \eta(\lambda_{-0})s \end{aligned} \tag{31}$$

where $\pi(\lambda_{0\pm}) = \lambda_0$, $\lambda_0 \in \mathbb{R}$, (U, W) are the vectors of spatial and temporal frequencies, $a(P)$ is a vector of "phases" depending on the divisor and the branch points of Σ, and $\eta(\lambda)\,\mathrm{d}\lambda = \mathrm{d}\theta(\lambda)$ is the quasimomentum differential.

Remarks:

1. The conditions for the components of $\boldsymbol{\gamma}$ to be real-valued are established in the work by E. Previato [48] for the focussing NLS equation, and require the Floquet spectrum to possess the additional symmetry $\lambda \to \bar{\lambda}$, and the divisor to satisfy certain additional conditions (see also [6]).

2. We recover Grinevich and Schmidt's closure condition from the explicit formula: the N-phase curve (31) is closed if and only if λ_0 is a real zero of the quasimomentum differential.

In the case $g = 1$, we can express the family of 2-phase solutions in terms of elliptic integrals, since the associated Riemann surface is an elliptic curve. The Floquet spectrum of a generic 2-phase solution is shown in Figure 3; the intersections α_1 and α_2 of the complex bands of spectrum with the real axis are the real zeros of the quasimomentum differential

$$\mathrm{d}\theta(\lambda) = -\frac{1}{2}\frac{\lambda^2 + c_0\lambda + c_1}{\sqrt{\prod_{j=1}^{4}(\lambda - \lambda_j)}}\,\mathrm{d}\lambda.$$

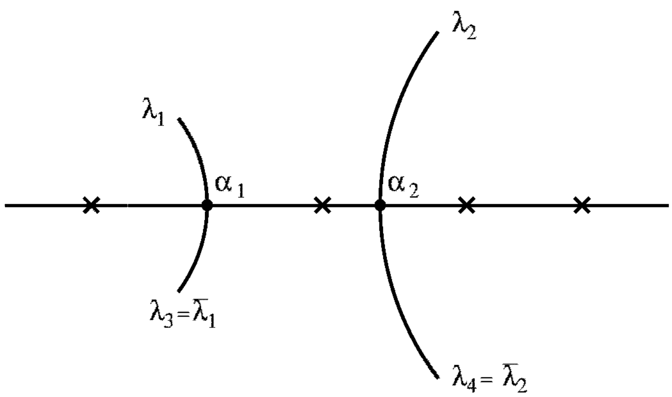

Figure 3. The Floquet spectrum of a generic 2-phase solution

Closed curves are obtained by selecting the branch points λ_j's so that the complex **spines** intersect the real axis at double points and reconstructing the curve by means of formula (21) evaluated at one such point.

These curves are interesting objects from a geometric and topological point of view as they are extremals of the following linear combination of global geometric invariants

$$\int_0^L \kappa^2 \, \mathrm{d}s + a_0 \int_0^L \tau \, \mathrm{d}s + a_{-1} \int_0^L 1 \, \mathrm{d}s.$$

The solutions of the corresponding constrained variational problem form the family of centerlines of Kirchhoff elastic rods (extremizing the **bending energy** $\int_0^L \kappa^2 \, \mathrm{d}s$ for constant total torsion and length) which includes the classical Euler elastic curves ($a_0 = 0$) and the free elastica ($a_0 = a_{-1} = 0$). Several of them are shown in Figure 4 (in which the curves have been thickened for better viewing).

The curves in Figure 4 are symmetric presentations of torus knots (knots that lie on the surface of a torus without self-intersections); moreover 2-phase solutions exhaust all possible torus knot types, as proved in [29] (see also [38]).

Proposition 7 (T. Ivey and D. Singer). *Every torus knot has a smooth closed elastic rod centerline representative.*

Remark: 2-phase solutions of the Vortex Filament Equation evolve by a combination of translation and screw motion [38], and thus move rigidly without change of topology. This observations makes the associated Floquet spectra suitable candidates for topological invariants.

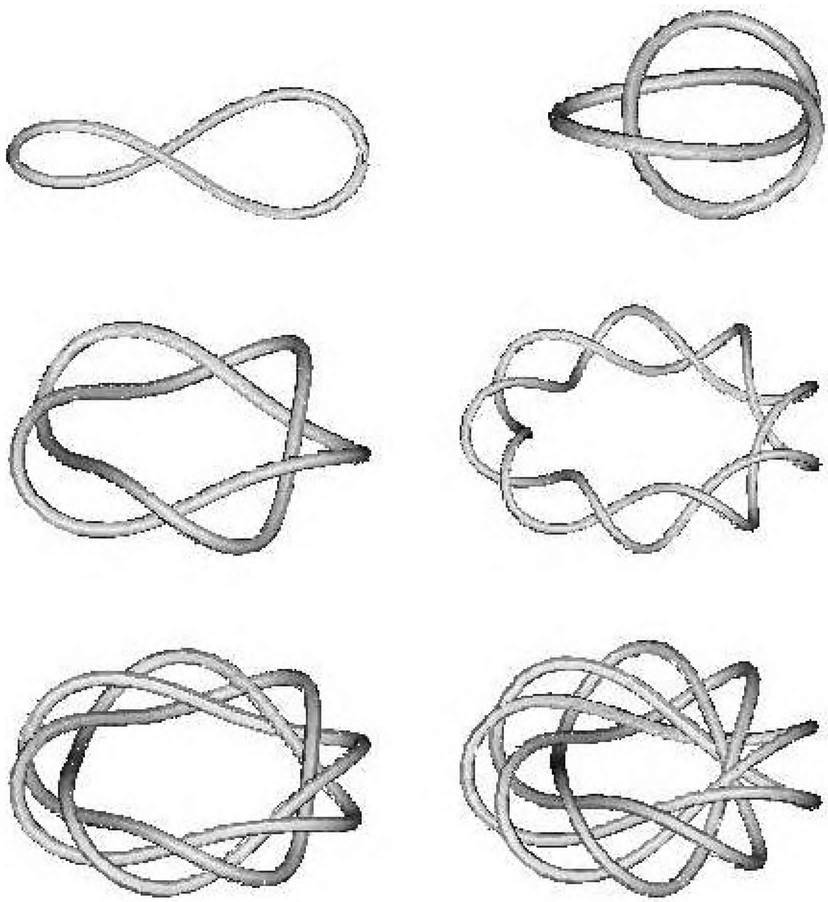

Figure 4. A gallery of 2-phase solutions of the Vortex Filament Equation, including an unknot, a trifoil knot (a (2,3)-torus knot), (2,5)-, (2,9)-, (3,8)- and (4,9)-torus knots.

Lecture 4

This lecture describes various recent results and work in progress by the author and Thomas A. Ivey on investigating connections between the Floquet spectra of N-phase solutions of the Vortex Filament Equations and their geometric and topological properties. A brief description of three different approaches is given below and we refer the reader to the cited references for further detail.

Use of Exact Formulas for N-phase Solutions

The exact formula (31) for N-phase solutions of the VFE in terms of Riemann theta functions are ultimately necessary for a detailed description of their geometric properties however, effective implementation of these formulas is difficult (see [6] for a discussion of the effectivization problem, [3] for reduction to lower genus formulas in the case of algebraic curves with symmetries, and [46] for a numerical investigation of N-phase NLS solutions in the defocussing case).

When the Riemann surface Σ possesses the additional involution $(\lambda, \mu) \to (-\lambda, (-1)^{g+1}\mu)$, one has a nearly complete classification of 2- and 3-phase solutions as described in the following theorem

Theorem 3 (T. Ivey). *If the spectrum of an N-phase NLS potential with $N \leq 4$ possesses the symmetry $\lambda \to -\lambda$, then the associated VFE solution is planar at one time. This implies that, for $N \leq 3$ the curve γ has self-intersections for all times.*

Figure 5 shows a self-intersecting curve with its corresponding symmetric spectrum.

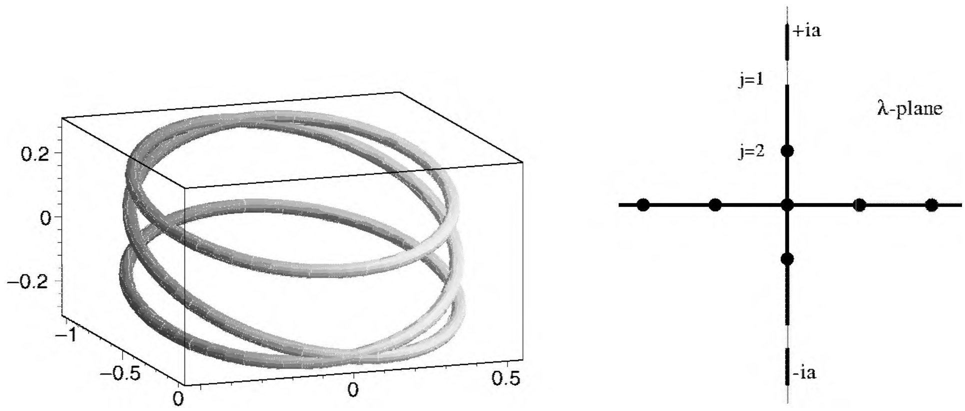

Figure 5. A self-intersecting curve arising from an even perturbation of a triply covered circle.

Perturbations of Multiply-covered Circles

Another seemingly universal feature of soliton equations is the existence of Bäcklund transformations. Bäcklund formulas produce new and more complex solutions of the nonlinear equation from a given one and from the Bloch eigenfunctions of the associated pair of linear systems.

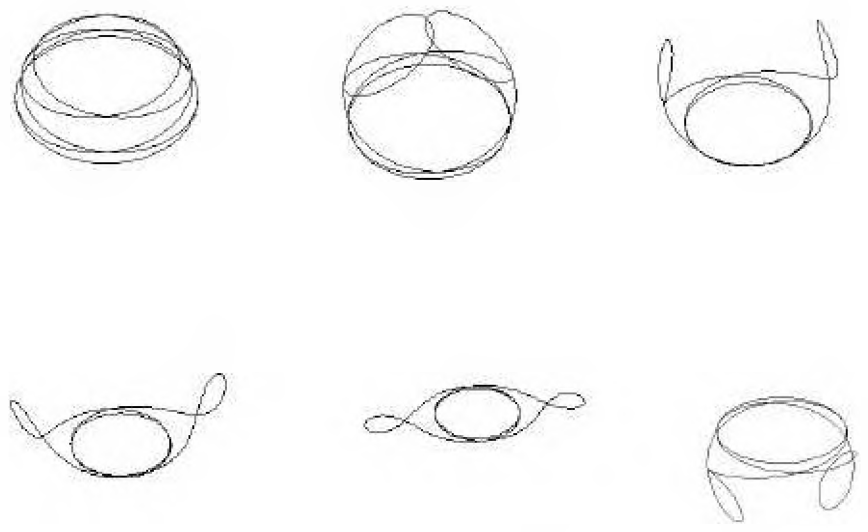

Figure 6. Time evolution frames of the Bäcklund transformation of a 5-fold planar circle.

Bäcklund transformations are commonly used to generate homoclinic orbits q_h of linearly unstable potentials q_0, i.e. solutions which converge to q_0 in forward and backward time, leaving and returning to the level set of q_0 along its stable and unstable manifolds. The importance of understanding the homoclinic manifolds of unstable N-phase solutions is advocated in a series of articles by N. Ercolani, G. Forest and D. McLaughlin [19, 20, 21]. A finite number of Bäcklund transformations produce an explicit parametrization of the homoclinic manifold and thus a labelling of the underlying level sets. Moreover, using Bäcklund transformations, one is able to construct both the tangent and normal vectorfields to the level set of a given solution and to generate explicitly the entire isospectral set.

The author derived the Bäcklund transformation for the Vortex Filament Equation in [12], and used the resulting formulas to generate the homoclinic orbits of linearly unstable circles, the simplest VFE solutions possessing linear instabilities. Figure 2 illustrates the Floquet spectrum of one of these solutions. A simple Fourier series expansion argument shows that the imaginary double points are associated with the linear instabilities of the potential (e.g. a four-fold circle has three-dimensional stable and unstable manifolds). Correspondingly, Bäcklund formulas implemented at different complex double points produce homoclinic orbits which converge to the original potential for $t \to \pm\infty$ along different directions within its stable and unstable linear eigenspaces.

Figure 6 shows a sequence of time frames of the evolution of the homoclinic orbit of a 5-fold circle, computed using a Bäcklund transformation at the second imaginary double point. The new curve possesses two distinguished points of self-intersection which persist when the evolving curve undergoes a significant excursion away from the original circle. It appears to be a general feature of these singular curves that the number of such "stable" self-intersections equals the ordering of the complex double point at which the Bäcklund transformation is computed.

An interesting question is whether the presence of self-intersections has a topological meaning. A more general related question is whether the curves produced by the Bäcklund transformation play a role in distinguishing different knot classes of N-phase solutions, and whether the Floquet spectrum can be used to characterize their knot types.

In order to partially answer these questions, the perturbative approach used in [14, 13] introduces a complex periodic perturbation of the plane wave potential $q_a = \dfrac{2\pi M}{L}$:

$$q(s) = a + \epsilon[\mathrm{e}^{\mathrm{i}\theta_1}\cos(\mu s) + r\mathrm{e}^{\mathrm{i}\theta_2}\sin(\mu s)] = q_0 + \epsilon q_1 \tag{32}$$

with $r, \theta_1, \theta_2 \in \mathbb{R}$.

When $\mu = \mu_j = \dfrac{2\pi j}{L}$, $1 \le j \le M-1$, the perturbation causes the j-th double point to split into a *gap* or *cross* configuration as shown in Figure 7.

Since the solution of NLS linear system (16) depends analytically on the potential q, we assume the following perturbation expansion of the fundamental solution matrix

$$\Psi = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \dots.$$

By computing the coefficient matrices Ψ_k recursively, the trace of Ψ, and the associated curve via equation (21), one arrives at the following result.

Lemma 2. 1. *The Floquet discriminant of the perturbed potential satisfies*

$$\Delta(a;\lambda) = 2\cos\left(\sqrt{\lambda^2 + \frac{a^2}{4}}L\right) + \mathcal{O}(\epsilon^2).$$

2. *The associated curve*

$$\gamma(s) = \gamma_0(s) + \epsilon\gamma_1(s) + \mathcal{O}(\epsilon^2)$$

is closed up to order ϵ^2.

Therefore, the Floquet spectrum of $q_0 + \epsilon q_1$ can be used to characterize the properties of the closed curve $\gamma_0 + \epsilon\gamma_1$.

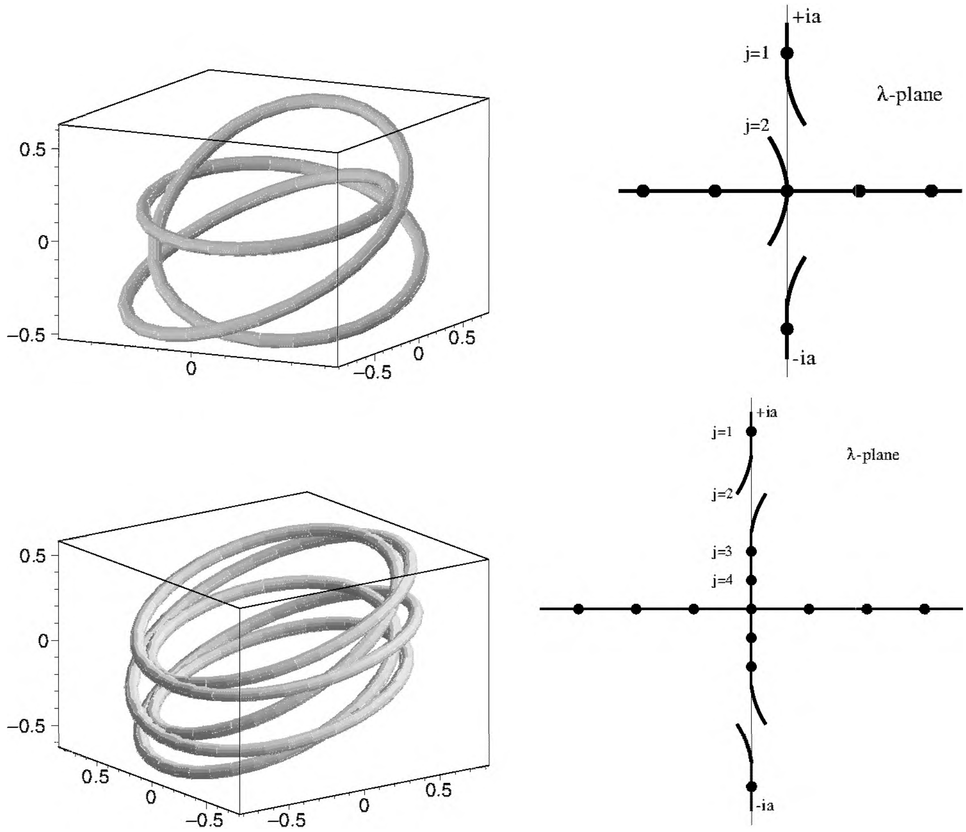

Figure 7. Above: a left-handed trefoil and its spectrum ($M = 3$, $j = 2$, $\rho = 0.2$, $\phi = \pi/4$, $\epsilon = 0.02$). Below: a right-handed $(5,2)$-torus knot and its spectrum ($M = 5$, $j = 2$, $\rho = -0.2$, $\phi = \pi/4$, $\epsilon = 0.1$)

The perturbation (32) with $r \neq 0$, $\theta_1 \neq \theta_2, \theta_2 + \pi$ and $\mu = \mu_j = \dfrac{2\pi j}{L}$, $1 \leq j \leq M - 1$ causes the j-th complex double point to split asymmetrically. In this case we obtain the following result

Proposition 8. *For $\epsilon > 0$ sufficiently small, the curve $\gamma_0 + \epsilon\gamma_1$ is a torus knot of type $(M, \pm j)$.*

Remarks:

1. The proof involves computing the exact expression of $\gamma_0 + \epsilon\gamma_1$ in terms of the Frenet frame of γ_0 [14].

2. Figure 7 (top) shows an asymmetric perturbation splitting the second imaginary double point of a triply covered circle ($M = 3$), and its associated left-handed trefoil knot (a $(3, -2)$-torus knot).

3. Figure 7 (bottom) shows an asymmetric perturbation splitting the second imaginary double point of a 5-covered circle ($M = 5$), and its associated right-handed $(5, 2)$-torus knot. Observe that this curve is close to the long-time asymptotics of the Bäcklund transformation of a 5-fold circle shown in Figure 6, this suggests that homoclinic orbits indeed separates knots of different topologies (in this case, torus knots of opposite handedness) and that the Floquet spectrum may be used for classifying N-phase solutions close to multiply covered circles.

Knots of increasing complexity can be generated using the perturbation procedure outlined above. For example, a periodic perturbation of the form

$$q_1(s) = \omega_j[\cos(\mu_j s) + \alpha_j \sin(\mu_j s)] + \omega_k[\cos(\mu_k s) + \alpha_k \sin(\mu_k s)] \qquad (33)$$

with $|\omega_j| = |\omega_k| = 1$, $\alpha_j, \alpha_k \in \mathbb{R}$ and $\mu_j = \dfrac{2\pi j}{L}$, $\mu_j = \dfrac{2\pi j}{L}$, $1 \leq j, k \leq M - 1$, causes the j-th and k-th double point to split, and we obtain the following result:

Proposition 9. *Under certain non-resonance conditions on μ_j, μ_j, $\frac{2\pi}{L}$, the curve $\gamma = \gamma_0 + \epsilon\gamma_1$ is a* cable knot *with companion an $\left(\frac{M}{d}, \pm\frac{j}{d}\right)$-torus knot, patterned on a $(d, \pm k)$-torus knot, where $d =$ g.c.d. (j, M).*

Remark: For a given cable knot, the *pattern* is an embedding of a loop into a solid torus, and the **companion knot** is the given embedding of the solid torus in three-space. For example, in Figure 8 the pattern is a $(2, 5)$-torus knot, and the companion knot is a left-handed trefoil.

Isoperiodic Deformations of NLS Potentials

In a 1998 article [29], T. Ivey and D. Singer proved the following results:

Theorem 4 (T. Ivey and D. Singer)**.** 1. *Every torus knot type is realized by a smooth closed elastic rod centerline (i.e. by an initial condition for a 2-phase solution of the Vortex Filament Equation).*

2. *There is a countable family of regular homotopies of closed elastic rod centerlines between the m-covered circle and the m-covered circle for every k, m relatively prime (such family was observed in numerical experiments by Li and Maddocks* [35]*). Each homotopy family contains exactly one elastic curve, one self-intersecting curve and one constant torsion curve.*

It is natural to ask whether the Floquet spectra associated to the distinguished curves within each homotopy family have features which reflect their special geometric or topological properties. As the homotopy transforms a multiply-covered circle to a different multiply-covered circle (the Floquet spectra of which are explicitly known), one needs to choose a deformation of the branch points such that

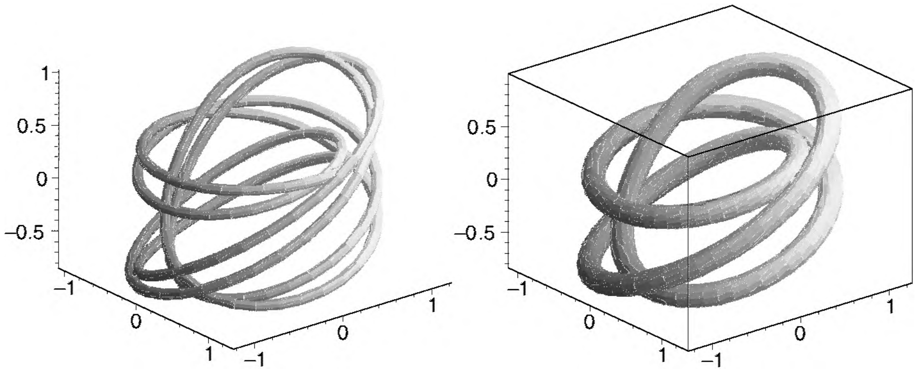

Figure 8. A cable knot, together with its companion torus knot, obtained using perturbation with $M = 6$, $j = 4$, $k = 5$, $w_j = 1$, $w_k = 0.1$, $\phi_j = \phi_k = \pi/2$, $\rho_j = 0.2$, $\rho_k = -0.3$ and $\epsilon = 0.02$.

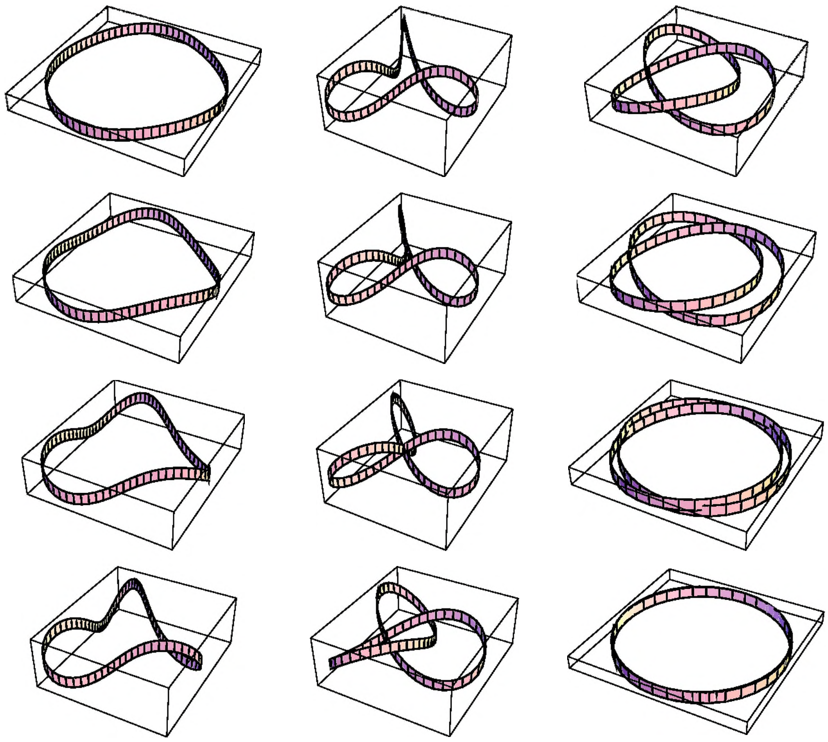

Figure 9. A family of homotopy deformations of elastic rod centerlines between a once-covered circle and a doubly-covered circle.

the curve is closed at each value of the deformation parameter. This involves transcendental conditions that require the spatial frequencies U_k's to be commensurate, and the quasi-momentum differential $\mathrm{d}\theta$ and its real zeros to be preserved. These conditions are expressed in terms of elliptic integrals for 2-phase solutions (and hyperelliptic integrals for general N-phase solutions), and are very difficult to solve in terms of the branch points λ_k's.

Consider the following deformation, and observe how the complex spine of spectrum not containing the zero of the quasimomentum at which the curve is reconstructed converges to a real double point as the curve approaches the limiting circle.

Interestingly, such spectral deformation as the one illustrated in Figure 10 can be achieved by preserving the frequencies and the quasimomentum differential. Isoperiodic deformations have been studied by I. Krichever [33] and P. Grinevich and M. Schimdt [27] and can be written as the following system of ordinary differential equations for the $2g+2$ branch points λ_k's and the $g+1$ zero's α_k's of the quasimomentum differential:

$$\begin{aligned} \frac{\partial \lambda_i}{\partial \xi} &= -\sum_{k=1}^{g+1} \frac{c_k}{\lambda_i - \alpha_k} \\ \frac{\partial \alpha_l}{\partial \xi} &= \sum_{k \neq l} \frac{c_k - c_l}{\alpha_k - \alpha_l} - \frac{1}{2} \sum_i \frac{c_l}{\lambda_i - \alpha_l} \end{aligned} \tag{34}$$

where the c_k's are arbitrary functions (the *controls* of the system) of the deformation parameter ξ, and are subject to the same reality condition as the corresponding λ_k's (e.g., $c_2 = \bar{c}_1$). If one reconstructs the curve at $\lambda = \alpha_0$, then the associated control c_0 to must equal zero, in order for α_0 to remain a double points throughout the deformation [26]. Then, if $g = 1$, there only a one-parameter family of deformations which preserve closure of the associated curve.

By choosing as initial condition of system (34) the spectrum of a multiply covered circle and *opening up* a real double point (i.e., by reversing the deformation shown in Figure 10), one generates closed 2-phase solutions which belong to the family of homotopy deformations described in Theorem 4. Figure 11 shows several interesting curves along the homotopy deformation (α_0 is the reconstruction point for each of the curves):

1. In the Floquet spectra of the right-handed $(2,5)$-torus knot and the left-handed $(3,5)$-torus knots, the locations of the second real zero of the quasimomentum differential satisfy prescribed resonance relations with the period of the curve.

2. Constant torsion is achieved when the complex bands of spectrum are symmetrically placed with respect to the midpoint of the real zeros of the quasimomentum.

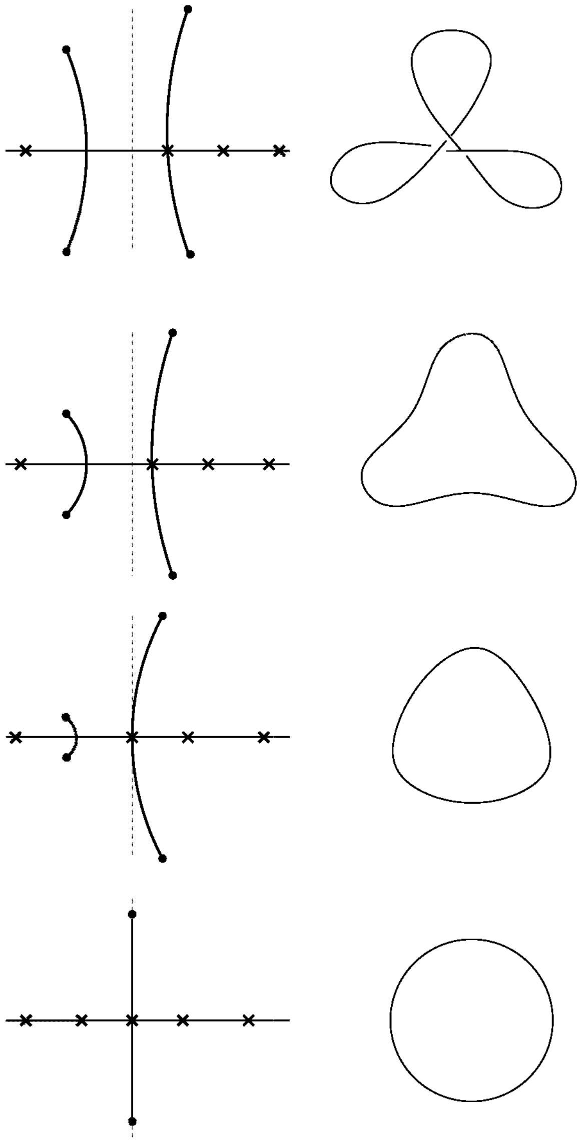

Figure 10. A family of elastic rod centerlines deformations and their spectra: from a trefoil knot to a circle

3. The Euler elastic curve is realized when the two real zeros of the quasimomentum coalesce.

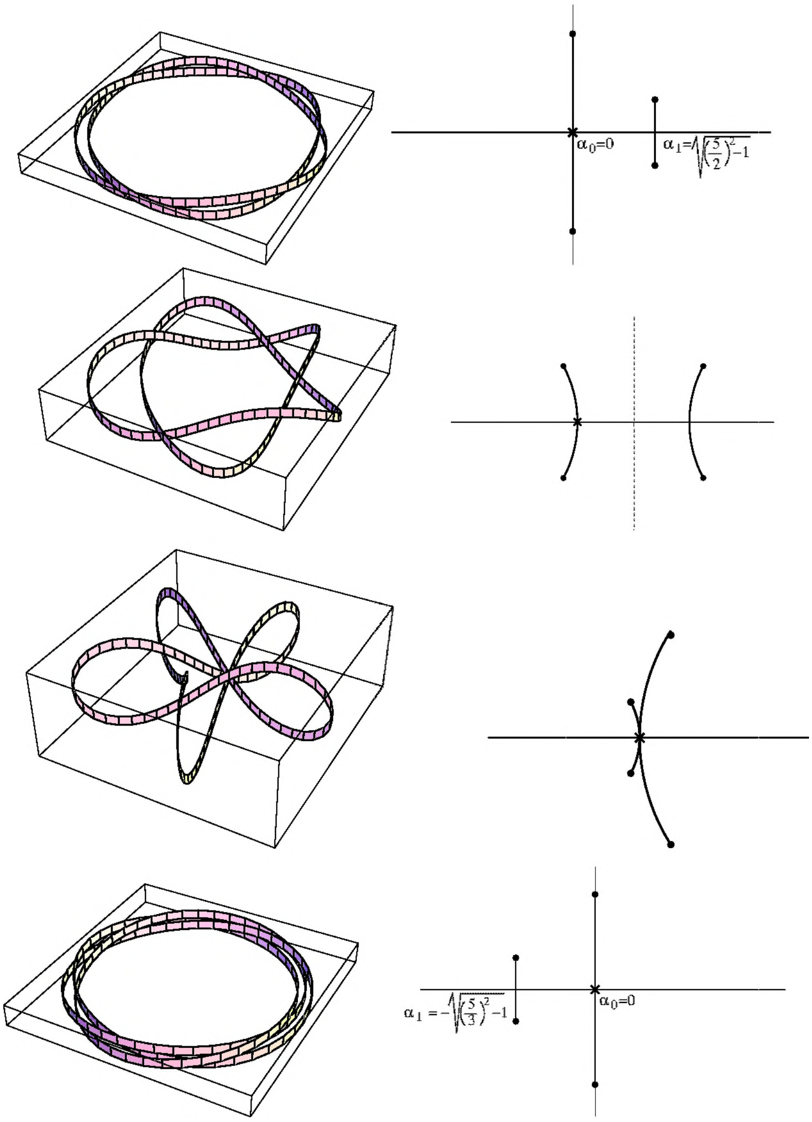

Figure 11. Distinguished curves along the homotopy deformation between a doubly- and a triply-covered circle and their spectra. From the top down: a $(2,5)$-torus knot, a curve of constant torsion, an Euler elastic curve, and a $(-3,5)$-torus knot.

Remarks:

1. It is not yet known what special spectral configuration, if any, is associated with self-intersections.

2. The perturbative deformation scheme based on the theory of isoperiodic deformation is easily adaptable to a systematic investigation of higher phase solutions (obtained by successively opening up several real double points), which can be shown to realize all possible knot types in a neighborhood of multiply covered circles. For example, one can produce cabling of a given torus knots and a labelling of its knot type in terms of the location of the relevant double points. The role played by the choice of divisor for higher phase solutions is currently under study.

3. These observations provide strong evidence of an important role of the Floquet spectrum in determining both the geometric and topological properties of N-phase VFE solutions "close" to multiply covered circles. One hopes that the Floquet discriminant functional is a topological invariant for a restricted class of VFE solutions large enough to contain representatives of many knot families.

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