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SYMMETRY, PHASES AND QUANTISATION

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Abstract. We explain rich geometric structures that appear in the quantisation of linear bosonic and fermionic systems. By contrasting with the quantisation of general curved phase spaces, we focus on results that shed light on one of the most basic problems in quantisation: the dependence of the quantum Hilbert space on auxiliary data such as the choice of polarisations that is necessary to define a quantum Hilbert space.

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Le plus court chemin entre deux vérités dans le domaine réel
passe par le domiane complexe.
Jacques Hadamard

CONTENTS

1. Introduction

Geometric quantisation [23,37] is a rich subject that originates from the need from physics to associate to a classical phase space (or a symplectic manifold) a Hilbert space of quantum states. Though there have been tremendous progress and many fruitful applications to physics and representation theory, a number of fundamental problems remain. In this paper, we concentrate on one of them: the dependence of the quantum Hilbert space on the choice of auxiliary data such as polarisations, and we describe the way to resolve this problem in concrete examples.

The paper is organised as follows. In Section 2, we review the construction of the quantum Hilbert space from a symplectic manifold with the choice of a polarisation. Roughly, a polarisation is a maximal set of commuting variables which the wave functions depend on. The Hilbert spaces then form a bundle over the space of polarisations. In Section 3, we show that for a symplectic vector space, the bundle of quantum Hilbert spaces has a projectively flat connection if the polarisations are restricted to linear compatible complex structures. In this way, the projectivised Hilbert spaces from all these polarisations can be naturally identified. The curvature of the bundle is proportional to the standard Kähler form on the set of such polarisations as a non-compact Hermitian symmetric space. In Section 4, we interpret the standard Segal-Bargmann and Fourier transforms as parallel transports under the projectively flat connection along the geodesics in the space of complex structures. We show that the holonomy of the connection is related to Kashiwara's triple Maslov index, which is thus expressed as an integration of the Kähler form on a geodesic triangle. In Section 5, starting with a general discussion on symmetry, symmetry breaking and anomaly in classical and quantum physics, we propose an anomaly-free condition for Hamiltonian group actions on symplectic manifolds. In Section 6, we study the quantisation of fermions whose phase space is a fermionic copy of an even dimensional Euclidean space. Appearance of projectively flat connections enriches our understanding of the representation theory of Clifford algebras. In Section 7, we consider the quantisation of fermions with an odd dimensional phase space. This is an example in which the operator algebra has two inequivalent irreducible representations.

Most of this article (except Sections 5 and 7) is a survey of existing work on the quantisation of linear bosonic and fermionic systems. We refer the reader to [21,49,50] and references therein for details. See also [29,36,38] for survey articles on related topics in the same proceeding series. Our presentation is in the conventional framework of geometric quantisation. Although our primary examples are linear phase spaces, we will discuss the quantisation of general symplectic manifolds from time to time for comparison and for conceptual clarifications. For monographs and textbooks on geometric quantisation, see [3, 14, 16, 20, 34, 35, 42, 45].

We did not explore from the viewpoint of deformation quantisation [4], which is a topic we plan to explore in the future.

2. The Quantisation Problem

Quantisation means finding irreducible representations of the algebra of observables in a Hilbert space: vectors (or rays of vectors) in the Hilbert space are quantum stares and the observables are operators on it. The algebra of observables has a conjugation action and the the real elements in the algebra act as self-adjoint operators. Irreducibility comes from the requirement that quantum states can be distinguished completely by measurements; on a reducible representation, it would be possible to find different quantum states with the same expectation values for all observables. For a particle on the configuration space $\mathbb{R}^n$, the operator algebra is the Heisenberg algebra, generated by position operators $\hat{q}^i$ and momentum operators $\hat{p}_i, i = 1, \cdots, n$ subject to the canonical commutation relations $[\hat{q}^i, \hat{p}_j] = \sqrt{-1}\delta^i_j\hbar, i, j = 1, \cdots, n$, where $\hbar$ is the Planck constant which will be set to one throughout. The Stone-von Neumann theorem [30] states that there is a unique (up to unitary equivalence) irreducible representation of the Heisenberg algebra. By irreducibility, any two unitary equivalences differ by a phase.

Despite its uniqueness, there are many ways to construct the representation. In the coordinate picture, the Hilbert space is the set of L^2-functions of the (classical) coordinates $q^i, i = 1, \cdots, n$, and $\hat{q}^i$ acts by multiplication of q^i whereas $\hat{p}_i$ acts by $-\sqrt{-1}\frac{\partial}{\partial \hat{q}^i}$ (the standard Schrödinger representation). In the momentum picture, the Hilbert space contains L^2-functions of the momenta $p_i, i = 1, \cdots, n$ instead, but it is unitarily equivalent to the Hilbert space in the coordinate picture by the standard Fourier transform. In fact, $q^i, p_i, i = 1, \cdots, n$ are the linear coordinates of a symplectic vector space $V \cong \mathbb{R}^{2n}$ with the standard symplectic form $\omega = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$, and any Lagrangian subspace L determines a real picture (or a real polarisation) because it selects a maximal set of commuting variables. The quantum Hilbert space $\mathcal{H}_L$ is roughly the space of L^2-functions on the linear quotient space V/L. For two transverse Lagrangian subspaces L and L', there is a general Fourier transform $\mathcal{F}_{L'L} \colon \mathcal{H}_L \to \mathcal{H}_{L'}$ (see for example [26]). When $n = 1$, any real line in $\mathbb{R}^2$ through the origin is a Lagrangian subspace and the collection of these lines is $\mathbb{RP}^1$ or a circle. So the usual Fourier transform (between the coordinate and momentum pictures) is in a family of unitary transforms parametrized by an angle [8].

In addition to the above real polarisations, there are also complex polarisations and the Hilbert space is in the coherent state picture. Generally, given a symmetric $n \times n$ complex matrix $\Omega = \Omega_1 + \sqrt{-1}\Omega_2$ with positive definite imaginary part Ω_2, we can introduce complex coordinates $z_i^\Omega = (1/\sqrt{2\Omega_2})_{ij}(q^j - \bar{\Omega}^{jk}p_k)$ on

$V \cong \mathbb{C}^n$. The set of such matrices is known as the Siegel upper half space $\mathfrak{H}_n$, which is a non-compact Hermitian symmetric space of type $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ and of complex dimension $\frac{1}{2}n(n+1)$. In the complex coordinates z_i^{Ω}, the symplectic form is $\omega = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z_i^{\Omega} \wedge \mathrm{d}\bar{z}_i^{\Omega}$ and the quantum Hilbert space $\mathcal{H}_{\Omega}$ consists of L^2-finite 'wave functions' $\psi(z^{\Omega},\bar{z}^{\Omega}) = \phi(z^{\Omega})\,\mathrm{e}^{-|z^{\Omega}|^2/2}$, where ϕ is holomorphic in z_i^{Ω} (so that it depends on only half of the variables on V). When $n=1$, Ω is a complex number $\tau = \tau_1 + \sqrt{-1}\tau_2$ ($\tau_2 > 0$) in the upper half plane $\mathfrak{H}_1$, and the complex coordinate on $V \cong \mathbb{R}^2$ is $z^{\tau} = (q - \bar{\tau}p)/\sqrt{2\tau_2}$. The Hilbert space $\mathcal{H}_{\tau}$ has an orthogonal basis $\{(z^{\tau})^k\,\mathrm{e}^{-|z^{\tau}|^2/2}\}_{k\geq 0}$. For $\tau = \sqrt{-1}$, $z = (q+\sqrt{-1}p)/\sqrt{2}$ is the usual coherent state coordinate. More intrinsically, $\Omega \in \mathfrak{H}_n$ determines a linear complex structure J on V compatible with ω, that is, $\omega(J\cdot,J\cdot) = \omega(\cdot,\cdot)$ and $\omega(\cdot,J\cdot)$ is positive definite. The set $\mathcal{J}$ of such complex structures can be identified with $\mathfrak{H}_n$ and we can write the Hilbert space $\mathcal{H}_{\Omega}$ as $\mathcal{H}_J$.

In general, the phase space is a symplectic manifold (M,ω) and a (positive) complex polarisation is an almost complex structure J compatible with the symplectic form ω. The space $\mathcal{J}$ of such J is an infinite dimensional contractible space. We shall also assume the existence of a prequantum line bundle ℓ over M, that is, ℓ has a unitary connection whose curvature is $\omega/\sqrt{-1}$. Then the Hilbert space $\mathcal{H}_J$ is the space of L^2-section of ℓ that are covariantly constant along the anti-holomorphic directions $T_J^{0,1}M$. (In the case $M=V$ above, the exponential factor in ψ is due to the identification of sections of ℓ as functions on V in a particular trivialisation of ℓ.) In this way, we obtain a bundle of Hilbert spaces $\mathcal{H} = \sqcup_{J\in\mathcal{J}}\mathcal{H}_J$ over $\mathcal{J}$. Since it is a subbundle of the trivial bundle $\mathcal{J}\times L^2(M,L)$, there is a unitary connection $\nabla^{\mathcal{H}}$ on $\mathcal{H}$ [2]. For any path γ in $\mathcal{J}$ from J to J', the parallel transport $U_{\gamma}\colon \mathcal{H}_J \to \mathcal{H}_{J'}$, which is a unitary isomorphism, identifies the two Hilbert spaces and the operators acting on them. However, the identification is not unique because U_{γ} depends on the path γ, not just the initial and final points J and J'.

We can also choose a real polarisation L, which is a Lagrangian distribution, that is, for each $x\in M$, L_x is a Lagrangian subspace of (T_xM,ω_x). The wave functions are required to be covariantly constant along L. More generally, we can use a mixed polarisation P, which is a complex Lagrangian distribution such that $P\cap\bar{P}$ is a (real) distribution of constant rank. The wave functions are required to be covariantly constant along $\bar{P}$. There are topological and geometrical obstructions to the existence of polarisations P such that the rank of $P\cap\bar{P}$ is positive. For example, the Euler characteristic of M must vanish and M can not be a complex manifold with positive sectional curvature [39]. On the other hand, there is always a compatible almost complex structure J on (M,ω), leading to a strictly complex polarisation $P = T^{1,0}M$. If J is integrable, then M is Kähler and $P = T^{1,0}M$ is called a Kähler polarisation. There are symplectic manifolds which admit real but not Kähler polarisations [10]. There are also symplectic manifolds which admit neither real nor Kähler polarisations [13].

3. Projectively Flat Connections

The problem of the non-uniqueness of quantisation would be settled if the connection $\nabla^{\mathcal{H}}$ described above on the bundle $\mathcal{H}$ of quantum Hilbert spaces is projectively flat. The curvature of a projectively flat connection is a scalar valued two-form times the identity operator on the fibre, and parallel transports along two paths with the same end points differ only by a phase. So if $\nabla^{\mathcal{H}}$ is projectively flat, the rays in the Hilbert spaces $\mathcal{H}_J$ and $\mathcal{H}_{J'}$ are naturally identified. This is sufficient because a quantum state is in fact a ray in the Hilbert space. In addition, a projectively flat connection on $\mathcal{H}$ induces a flat connection on $\mathrm{End}(\mathcal{H})$, and hence the operators on $\mathcal{H}_J$ for all J are naturally identified. Unfortunately, if $\mathcal{J}$ is the space of all compatible (almost) complex structures, the connection $\nabla^{\mathcal{H}}$ fails to be projectively flat even for simplest symplectic manifolds such as S^2 [11]. Thus we restrict ourselves to a special class of almost complex structures, often those invariant under the symmetries of the system. For example, if $(M,\omega)=(V,\omega)$ is a symplectic vector space, $\mathcal{J}$ is the space of linear complex structures compatible with ω as above. Then the connection on the Hilbert space bundle $\mathcal{H}\to\mathcal{J}$ is indeed projectively flat, with curvature $F^{\mathcal{H}}=-\frac{1}{8}\,\mathrm{tr}_{V_J^{1,0}}(\mathrm{d}J\wedge\mathrm{d}J)\ \mathrm{Id}_{\mathcal{H}}$ [2]. Since $\mathcal{J}$ is the Hermitian symmetric space $\mathfrak{H}_n$, it has a standard Kähler σ_ω. Then the curvature of the projectively flat connection $\nabla^{\mathcal{H}}$ is $F^{\mathcal{H}}=\sigma_\omega/2\sqrt{-1}$ [21].

We now study the parallel transport under $\nabla^{\mathcal{H}}$. The simplest curves in $\mathcal{J}$ are the geodesics in the standard Kähler metric. For simplicity, we describe the results for $n=1$ but refer the reader to [21] (and [49, 50] in the coordinate-free language) for the general cases. Writing the complex coordinate on the upper half plane $\mathfrak{H}_1$ as $\tau=\tau_1+\sqrt{-1}\tau_2$, $\tau_2>0$, the Kähler form $\sigma_\omega=(2\tau_2^2)^{-1}\mathrm{d}\tau_1\wedge\mathrm{d}\tau_2$ is from the Poincaré metric. By the action of $\mathrm{Sp}(V,\omega)=\mathrm{SL}(2,\mathbb{R})$, any two complex structures J and J' on $V=\mathbb{R}^2$ can be brought to $\tau=\sqrt{-1}$ and $\tau'=\sqrt{-1}\,e^{2t}$ for some $t\in\mathbb{R}$. Since the action of $\mathrm{Sp}(V,\omega)$ on $\mathcal{J}$ lifts to the bundle $\mathcal{H}$ preserving the connection $\nabla^{\mathcal{H}}$, we can assume, without loss of generality, that J and J' are given by τ and τ'. Then the (unique) geodesic joining J and J' becomes the vertical line segment between τ and τ'. Let $z=(q+\sqrt{-1}\,p)/\sqrt{2}$ and $z'=(\mathrm{e}^{-t}q+\sqrt{-1}\,\mathrm{e}^{t}p)/\sqrt{2}$ be the respective complex coordinates on $V\cong\mathbb{R}^2$. On the linear bases of $\mathcal{H}_J$ and $\mathcal{H}_{J'}$, the parallel transport $U_{J'J}\colon\mathcal{H}_J\to\mathcal{H}_{J'}$ along the geodesic from J to J' is the unitary operator [21]

$$U_{J'J}\colon \left\{\begin{matrix} 1 \\ z \\ z^2 \\ \vdots \end{matrix}\right\} \mathrm{e}^{-\frac{1}{2}|z|^2} \longmapsto \sqrt{\operatorname{sech} t}\, \left\{\begin{matrix} 1 \\ z' \operatorname{sech} t \\ z'^2 \operatorname{sech}^2 t + \tanh t \\ \vdots \end{matrix}\right\} \mathrm{e}^{-\frac{1}{2}z'^2 \tanh t - \frac{1}{2}|z'|^2}.$$

On the coherent state $\mathrm{e}^{\bar\alpha z - |z|^2/2}$ ($\alpha \in \mathbb{C}$) which is the generating function of the linear basis, the transformation is

$$U_{J'J}\colon \mathrm{e}^{\bar\alpha z - \frac{1}{2}|z|^2} \longmapsto \sqrt{\operatorname{sech} t}\, \mathrm{e}^{\bar\alpha z' \operatorname{sech} t + \frac{1}{2}(\bar\alpha^2 - z'^2)\tanh t - \frac{1}{2}|z'|^2}.$$

The creation and annihilation operators on $\mathcal{H}_J$ go to the Bogoliubov transformations [5] of those on $\mathcal{H}_{J'}$.

In geometric quantisation, the actual quantum Hilbert space is the one with the half-form (metaplectic) or half-density correction. Let J be a compatible almost complex structure on a symplectic manifold (M,ω) of dimension $2n$. The canonical line bundle $\kappa_J = \wedge^n (T_J^{1,0}M)^*$ over M is equipped with a partial connection along $T_J^{0,1}M$. The half-form (or metaplectically) corrected Hilbert space $\hat{\mathcal{H}}_J$ is the space of L^2-sections of the bundle $\ell \otimes \kappa_J^{-1/2}$ that are covariantly constant along $T_J^{0,1}M$. Here the prequantum line bundle ℓ or the square root $\kappa_J^{1/2}$ need not exist separately, but the tensor product $\ell \otimes \kappa_J^{-1/2}$ does. We can form a Hilbert space bundle $\hat{\mathcal{H}} = \sqcup_{J\in\mathcal{J}} \hat{\mathcal{H}}_J$ over $\mathcal{J}$ by collecting the metaplectically corrected Hilbert spaces. There is a natural BKS pairing between two Hilbert spaces $\hat{\mathcal{H}}_J$ and $\hat{\mathcal{H}}_{J'}$ and, in particular, there is an inner product on each $\hat{\mathcal{H}}_J$. The operator $\hat{\mathcal{H}}_J \to \hat{\mathcal{H}}_{J'}$ induced by the BKS pairing is not necessarily unitary when J and J' are finitely apart, but it becomes so in the limit when J' approaches J. In this way, we get an unitary connection $\nabla^{\hat{\mathcal{H}}}$ on the bundle $\hat{\mathcal{H}}$. Unlike the map from the BKS pairing, the parallel transport $\hat U_\gamma\colon \hat{\mathcal{H}}_J \to \hat{\mathcal{H}}_{J'}$ along a path γ from J to J' is always unitary. The half-density corrected Hilbert space $\tilde{\mathcal{H}}_J$ consists of L^2-sections of $\ell \otimes |\kappa_J|^{-1/2}$ that are constant along $T_J^{0,1}M$. Here $|\kappa_J|$ is a real line bundle whose transition functions are the norm of those of κ_J. We get a Hilbert space bundle $\tilde{\mathcal{H}} \to \mathcal{J}$ with a unitary connection $\nabla^{\tilde{\mathcal{H}}}$ for the same reason. In particular, given a path γ in $\mathcal{J}$ from J to J', the parallel transport $\tilde U_\gamma\colon \tilde{\mathcal{H}}_J \to \tilde{\mathcal{H}}_{J'}$ is a unitary isomorphism.

Whereas neither $\nabla^{\hat{\mathcal{H}}}$ nor $\nabla^{\tilde{\mathcal{H}}}$ is expected to be projectively flat in general, there are significant simplifications when the phase space is a symplectic vector space (V,ω) and $\mathcal{J}$ is the set of linear compatible complex structures. We can regard κ_J as the complex line $\wedge^n (V_J^{1,0})^*$ independent of points on V. These lines κ_J form a complex line bundle $\mathcal{K} = \sqcup_{J\in\mathcal{J}} \kappa_J$ over $\mathcal{J}$. Being a subbundle of the product bundle $\mathcal{J} \times \wedge^n(V^{\mathbb{C}})^*$, $\mathcal{K}$ has an induced unitary connection $\nabla^{\mathcal{K}}$ whose curvature turns out to be $F^{\mathcal{K}} = \sigma_\omega/\sqrt{-1}$. The metaplectically corrected Hilbert space $\hat{\mathcal{H}}_J$ is simply the tensor product $\mathcal{H}_J \otimes \kappa_J^{-1/2}$, and $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}^{-1/2}$ as bundles over $\mathcal{J}$. Here the bundle $\mathcal{K}^{-1/2}$ with its connection is uniquely determined by $\mathcal{K}$ since the base space $\mathcal{J}$ is contractible. The connection $\nabla^{\hat{\mathcal{H}}}$ on $\hat{\mathcal{H}}$ is induced by the tensor product, and the curvature is $F^{\hat{\mathcal{H}}} = F^{\mathcal{H}} - \frac{1}{2}F^{\mathcal{K}} = 0$. So $\hat{\mathcal{H}} \to \mathcal{J}$ is flat, not just projectively flat, and hence it has a natural but somewhat non-trivial global trivialisation, making it possible to identify $\hat{\mathcal{H}}_J$ for all $J \in \mathcal{J}$ with no phase ambiguity. This identification is also given by the BKS pairing, which is unitary in this case even for two polarisations that are of a finite distance apart. Similarly, $|\kappa_J|$ is the same real line over all points on V, and they form the real line bundle $|\mathcal{K}|$ over $\mathcal{J}$ such that $\tilde{\mathcal{H}} = \mathcal{H} \otimes |\mathcal{K}|^{-1/2}$. The connection on $|\mathcal{K}|$ is clearly flat, and hence $F^{\tilde{\mathcal{H}}} = F^{\mathcal{H}} = \sigma_\omega/2\sqrt{-1}$. Despite having the same curvature as $\mathcal{H}$, there is an advantage of $\tilde{\mathcal{H}}$ over $\mathcal{H}$ which will be explained below.

There are few examples in which we obtain a projectively flat connection on the bundle of quantum Hilbert spaces. A celebrated example is the quantisation of the moduli space $\mathcal{M}(C, G)$ of flat G-connections on a compact orientable surface C, where G is a compact Lie group. The smooth part of $\mathcal{M}(C, G)$ (which we denote by the same notation) has a natural symplectic form [1, 12] and it is the phase space of the Chern-Simons gauge theory [43]. A complex structure on C induces one on $\mathcal{M}(C, G)$, making it a Kähler space. Quantisation under the Kähler polarisations leads to the space of non-Abelian theta functions whose dimension is given by the Verlinde formula [40]. As the complex structure on C varies, the Hilbert spaces do form a projectively flat bundle [2, 17]. The moduli space $\mathcal{M}(C, G)$ can also be quantised using real polarisations [19]. In a forthcoming work [22], we show that when the phase space is the cotangent bundle of a compact Lie group, there exists a family of complex structures on which the (metaplectically corrected) Hilbert space bundle is flat. Parallel transport under this connection is related to the Segal-Bargmann transform on Lie groups [15] at one end and to the Peter-Weyl theorem [31] at the other.

4. Geometric Phases and Integral Transforms

Given a symplectic vector space (V, ω) of dimension $2n$, the set $\mathcal{J}$ of compatible linear complex structures on V is a non-compact Hermitian symmetric space identifiable with the Siegel upper half space $\mathfrak{H}_n$, and it can be holomorphically embedded as a bounded domain in a complex vector space. For example, when $n = 1$, the upper half plane $\mathfrak{H}_1$ is biholomorphic to the open unit disk via the Cayley transform $\tau \mapsto (\tau - \sqrt{-1})(\tau + \sqrt{-1})^{-1}$ sending $\tau = \sqrt{-1}$ to the origin and the boundary real axis to the unit circle. Points on the boundary are not complex structures but are limits of them. For $n = 1$, when $\tau \to \tau_1$ becomes real, the complex line $\mathbb{C}(\partial/\partial\bar{z}^\tau)$ become the real line $q = \tau_1 p$. So the unit circle at the boundary parametrises real Lagrangian subspaces of V. For $n > 1$, not all points on the topological boundary of the bounded domain are real Lagrangian subspaces, but those on the Shilov boundary are, while points on the rest of the boundary are mixed polarisations that are partly real and partly complex. The Shilov boundary is a compact manifold diffeomorphic to $\mathrm{U}(n)/\mathrm{SO}(n)$ of real dimension $\frac{1}{2}n(n+1)$.

We have identified the set $\mathcal{L}$ of Lagrangian subspaces in (V,ω) as the Shilov boundary of $\mathcal{J}$. Though $\mathcal{L}$ is compact, it is best to think two different points of $\mathcal{L}$ as infinitely apart, connected by geodesics in $\mathcal{J}$. In fact, any geodesic in $\mathcal{J}$ approaches a point in $\mathcal{L}$ at infinity, and for two transverse Lagrangian subspaces $L, L' \in \mathcal{L}$, there is a geodesic $\{J_s\}_{s\in\mathbb{R}}$ in $\mathcal{J}$ that approaches L and L' at the two ends [21], i.e.,

$$\lim_{s\to-\infty} J_s = L, \qquad \lim_{s\to+\infty} J_s = L'.$$

Since $\mathcal{J}$ is non-positively curved, there is a unique geodesic joining any two point $J, J' \in \mathcal{J}$ in the interior, and we can denote the parallel transports along this geodesic in the Hilbert space bundles $\mathcal{H}$, $\tilde{\mathcal{H}}$, $\hat{\mathcal{H}}$ by $U_{J',J}$, $\tilde{U}_{J',J}$, $\hat{U}_{J',J}$, respectively. It turns out that along any geodesic $\{J_s\}_{s\in\mathbb{R}}$ in $\mathcal{J}$ with $J_0 = J \in \mathcal{J}$ and joining L and L' on the Shilov boundary, the limits

$$\tilde{U}_{J,L} = \lim_{s\to-\infty} \tilde{U}_{J_0,J_s}, \qquad \hat{U}_{J,L} = \lim_{s\to-\infty} \hat{U}_{J_0,J_s}$$

exist and are equal to the Segal-Bargmann transform $\tilde{\mathcal{H}}_L \to \tilde{\mathcal{H}}_J$ and $\hat{\mathcal{H}}_L \to \hat{\mathcal{H}}_J$. Similarly, the limits

$$\tilde{U}_{L',L} = \lim_{s,s'\to+\infty} \tilde{U}_{J_{s'},J_{-s}} \qquad \hat{U}_{L',L} = \lim_{s,s'\to+\infty} \hat{U}_{J_{s'},J_{-s}}$$

exist and are equal to the Fourier transform $\tilde{\mathcal{H}}_L \to \tilde{\mathcal{H}}_{L'}$ and $\hat{\mathcal{H}}_L \to \hat{\mathcal{H}}_{L'}$ [21] (see [49] for $\tilde{U}$). In this way, we obtain a geometric interpretation of these well known integral transforms. The limits do not exist without the half-density or half-form correction. Unless $n = 1$, the geodesics in $\mathcal{J}$ joining $L, L' \in \mathcal{L}$ are not unique, but they can be deformed from one another and they all lie on a flat strip in $\mathcal{J}$ on which the Kähler form σ_ω, and hence the curvature $F^{\tilde{\mathcal{H}}}$, vanish. So $\tilde{U}_{L',L}$, and certainly $\hat{U}_{L',L}$, do not depend on the geodesic joining L and L'.

Given three mutually transverse Lagrangian subspaces $L, L', L \in \mathcal{L}$ of (V,ω), we have three Fourier transforms

$$\tilde{U}_{L',L} \colon \tilde{\mathcal{H}}_L \to \tilde{\mathcal{H}}_{L'}, \qquad \tilde{U}_{L'',L'} \colon \tilde{\mathcal{H}}_{L'} \to \tilde{\mathcal{H}}_{L''}, \qquad \tilde{U}_{L,L''} \colon \tilde{\mathcal{H}}_{L''} \to \tilde{\mathcal{H}}_L.$$

Their composition, which is an operator on $\tilde{\mathcal{H}}_L$, is a phase because it commutes with action of the Heisenberg algebra. In fact, the phase is an eighth root of unity, i.e.,

$$\tilde{U}_{L,L''} \circ \tilde{U}_{L'',L'} \circ \tilde{U}_{L',L} = \mathrm{e}^{\frac{\sqrt{-1}\pi}{4}\mu(L,L',L'')}\ \mathrm{Id}_{\tilde{\mathcal{H}}_L}$$

where $\mu(L, L', L'') \in \mathbb{Z}$ is Kashiwara's Maslov triple index (cf. [26]) defined as the signature of the quadratic form $q(x, x', x'') = \omega(x, x') + \omega(x', x'') + \omega(x'', x)$ on $L \oplus L' \oplus L''$. Since $\tilde{U}_{L',L}$, $\tilde{U}_{L'',L'}$, $\tilde{U}_{L,L''}$ are the holonomies of the projectively flat connection $\nabla^{\tilde{\mathcal{H}}}$ along the geodesics joining L and L', L' and L'', L'' and L, their composition can be calculated by integrating the curvature $F^{\tilde{\mathcal{H}}} = \sigma_\omega/2\sqrt{-1}$ on a surface Δ bounded by the three geodesics. (See [49] for a justification of integrating σ_ω over Δ that extends to infinity in $\mathcal{J}$.) So at least modulo 8, we have

$$\mu(L, L', L'') = \frac{2}{\pi}\int_\Delta \sigma_\omega.$$

That this identity in fact holds in $\mathbb{Z}$ can be verified either by an induction on n or by using a generalisation of the Maslov index explained below. In this way, the triple Maslov index has a geometric interpretation as an integration of the standard Kähler form on a geodesic triangle [49]. This appeared in [47] and was generalised to other tube domains [48]. If $n = 1$, Lagrangian subspaces are described by $\tau \in \mathbb{R} \cup \{\infty\}$, and two Lagrangian subspaces are transverse if the τ's are different. For three mutually different $\tau, \tau', \tau'' \in \mathbb{R} \cup \{\infty\}$, the triple Maslov index is 1 if their Cayley transforms are counterclockwise on the unit circle and -1 if they are clockwise (see also [7]).

There is a generalisation $\mu(J, J', J'') \in \mathbb{R}$ of the triple Maslov index defined for three interior point $J, J', J'' \in \mathcal{J}$ that goes to $\mu(L, L', L'') \in \mathbb{Z}$ when J, J', J'' go to boundary points $L, L', L'' \in \mathcal{L}$ [27]. Furthermore, for two $J, J' \in \mathcal{J}$, there is an integral transform [27] from $\tilde{\mathcal{H}}_J$ to $\tilde{\mathcal{H}}_{J'}$ which we denote by $\tilde{U}_{J',J}$ because it coincides with the parallel transport along the geodesic in $\mathcal{J}$ joining J and J' [49]. For $J, J', J'' \in \mathcal{J}$, the integral transforms satisfy [27]

$$\tilde{U}_{J,J''} \circ \tilde{U}_{J'',J'} \circ \tilde{U}_{J',J} = \mathrm{e}^{\frac{\sqrt{-1}\pi}{4}\mu(J,J',J'')}\ \mathrm{Id}_{\tilde{\mathcal{H}}_J}.$$

But with the interpretation of the integral transforms as parallel transports, the right hand side is the holonomy of the projectively flat connection $\nabla^{\tilde{\mathcal{H}}}$ which can be calculated by integrating the curvature $F^{\tilde{\mathcal{H}}}$ over a surface Δ in $\mathcal{J}$ bounded by the three geodesics joining J, J', J''. So at least modulo eight, that is, in $\mathbb{R}/8\mathbb{Z}$, we have

$$\mu(J, J', J'') = \frac{2}{\pi}\int_\Delta \sigma_\omega.$$

The above equality holds in $\mathbb{R}$ because both sides are continuous in $J, J', J'' \in \mathcal{J}$ and vanish when any two of them coincide [48,49]. Taking the limit $(J, J', J'') \to$

(L, L', L''), where $L, L', L'' \in \mathcal{L}$ are three mutually transverse Lagrangian subspaces, we obtain the above geometric formula in $\mathbb{Z}$ for Kashiwara's triple Maslov index $\mu(L, L', L'')$.

5. Symmetry, Symmetry Breaking and Anomaly

Finding a representation of the quantum operator algebra is quantisation: the quantum Hilbert space is an irreducible representation of the algebra. Finding a representation of the symmetry group (or algebra) of the system on the same Hilbert space is an entirely different problem. Quantum mechanically, a symmetry is an automorphism of the operator algebra preserving the Hamiltonian. If an automorphism is outer, it brings one irreducible representation to another, and thus does not act on a particular quantum Hilbert space. This is called symmetry breaking, and it is what happens when quantising a scalar field with a rotationally invariant Mexican hat potential: the vacuum state in the Hilbert space is not invariant under rotations. For systems of finite degrees of freedom, we will encounter an example of symmetry breaking in the quantisation of a fermionic system with an odd dimensional phase space. For quantum mechanics of a bosonic particle with finite degrees of freedom and moving on a flat space, the possibility of symmetry breaking can be safely ruled out, as the Stone-von Neumann theorem guarantees that Heisenberg algebra has a unique irreducible representation up to unitary equivalence. Physically, the degeneracy of classical vacua due to symmetry is lifted at the quantum level by the tunneling effect, leading to a unique quantum vacuum. The representation of the symmetry group can be projective as the unitary equivalence between two irreducible representations is determined with a phase ambiguity.

There is however another possibility that an expected symmetry is lost upon quantisation for a more fundamental reason. A classical symmetry is a transformation on the dynamical variables preserving the action (in the Lagrangian approach) or a canonical transformation on the phase space preserving the Hamiltonian (in the Hamiltonian approach). If no quantisation scheme is compatible with the full classical symmetry, the quantum operator algebra is has to be constructed at the cost of abandoning some classical symmetries as its automorphisms. In this case, though the quantum theory itself is still defined, the symmetry is absent at the quantum level, and we say that the classical symmetry is anomalous. Anomalies usually occur in quantum field theories where there is no symmetry invariant regularisation procedure to avoid infinities. They can also occur in geometric quantisation because there may not exist a polarisation respected by the full symmetry group. But if the phase space is a symplectic vector space, the problem goes away again even though no polarisation is compatible with the full symplectic group.

Let (V, ω) be a finite dimensional symplectic vector space. Instead of the infinite dimensional group of symplectic diffeomorphisms, we focus on the linear symplectic group $\mathrm{Sp}(V, \omega)$. A Hamiltonian function on V will further restrict the symmetry group to a subgroup of $\mathrm{Sp}(V, \omega)$, but let us examine if there is a representation of the full group $\mathrm{Sp}(V, \omega)$ on the quantum Hilbert space. At first sight, this could be problematic, as there is no complex or real polarisation that is invariant under $\mathrm{Sp}(V, \omega)$. In fact, a compatible linear complex structure J on V reduces $\mathrm{Sp}(V, \omega)$ to a unitary subgroup, which is necessarily compact. Nevertheless, $\mathrm{Sp}(V, \omega)$ does act as automorphisms of the Heisenberg algebra, and by the above reasoning using the Stone-von Neumann theorem, there should be a (possibly projective) representation of $\mathrm{Sp}(V, \omega)$ on the unique irreducible representation of the Heisenberg algebra. Indeed, a double cover $\mathrm{Mp}(V, \omega)$ of $\mathrm{Sp}(V, \omega)$ known as the metaplectic group acts honestly (i.e., without phase ambiquity) on the Hilbert space so that $\mathrm{Sp}(V, \omega)$ acts projectively. This representation of $\mathrm{Mp}(V, \omega)$ is known as the Segal-Shale-Weil representation [32, 33, 41]. Alternatively, the action of the Lie algebra of $\mathrm{Sp}(V, \omega)$ on the Heisenberg algebra is given by inner derivations of various components on the moment map, which are quadratic on V and for which the ordering ambiguity does not arise. So a covering group of $\mathrm{Sp}(V, \omega)$, in this case its double cover $\mathrm{Mp}(V, \omega)$, acts as inner automorphisms of the Heisenberg algebra and hence acts on its representation space. Yet another explanation by embedding $\mathrm{Mp}(V, \omega)$ into the Heisenberg algebra, the symplectic analogue of the Clifford algebra [9, 24].

Next, we present a geometric construction of the Segal-Shale-Weil representation to which (projective) flatness is the key. For simplicity, we use the metaplectically corrected Hilbert space bundle $\hat{\mathcal{H}} \to \mathcal{J}$, whose connection $\nabla^{\hat{\mathcal{H}}}$ is flat. The symplectic group $\mathrm{Sp}(V, \omega)$ acts on the space $\mathcal{J}$ of compatible linear complex structures. This action can be lifted to the Hilbert space bundle $\mathcal{H} \to \mathcal{J}$ preserving the projectively flat connection $\nabla^{\mathcal{H}}$. On the other hand, though $\mathrm{Sp}(V, \omega)$ also acts on the canonical bundle $\mathcal{K} \to \mathcal{J}$, only its double cover $\mathrm{Mp}(V, \omega)$ acts honestly on the square root bundle $\mathcal{K}^{1/2}$. Therefore we have an action of $\mathrm{Mp}(V, \omega)$ on $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}^{-1/2}$ preserving the flat connection $\nabla^{\hat{\mathcal{H}}}$ and covering the action of $\mathrm{Sp}(V, \omega)$ on $\mathcal{J}$. For $g \in \mathrm{Mp}(V, \omega)$, let $\hat{g}\colon \hat{\mathcal{H}}_J \to \hat{\mathcal{H}}_{g\cdot J}$ be the action on $\hat{\mathcal{H}}$. If our symmetry is a compact subgroup of $\mathrm{Sp}(V, \omega)$, then we can choose an invariant $J \in \mathcal{J}$, and the symmetry group (or its double cover) acts on $\hat{\mathcal{H}}_J$. But for the whole group $\mathrm{Sp}(V, \omega)$, the space $\mathcal{J}$ is a single orbit. Fix a $J \in \mathcal{J}$. The parallel transport $\hat{U}_{J',J}\colon \hat{\mathcal{H}}_J \to \hat{\mathcal{H}}_{J'}$ from J to another point $J' \in \mathcal{J}$ is independent of the path joining J and J'. Invariance of the connection $\nabla^{\hat{\mathcal{H}}}$ means that $\hat{g} \circ \hat{U}_{J',J} = \hat{U}_{g\cdot J', g\cdot J} \circ \hat{g}$ for all $g \in \mathrm{Mp}(V, \omega)$. We define the representation $\hat{\varrho}$ of $\mathrm{Mp}(V, \omega)$ on $\hat{\mathcal{H}}_J$ by $\hat{\varrho}(g) = \hat{U}_{J, g\cdot J} \circ \hat{g}$; we verify that $\hat{\varrho}(e) = \mathrm{Id}_{\hat{\mathcal{H}}}$ and that for all $g, h \in \mathrm{Mp}(V, \omega)$

$$\hat{\varrho}(gh) = \hat{U}_{J,(gh)\cdot J} \circ \widehat{gh} = \hat{U}_{J,g\cdot J} \circ \hat{U}_{g\cdot J, g\cdot(h\cdot J)} \circ \hat{g} \circ \hat{h}$$
$$= \hat{U}_{J,g\cdot J} \circ \hat{g} \circ \hat{U}_{J,h\cdot J} \circ \hat{h} = \hat{\varrho}(g) \circ \hat{\varrho}(h).$$

Finally, let (M, ω) be a general quantisable symplectic manifold and let $\mathcal{J}$ be the space of all compatible almost complex structures. Suppose a Lie group G acts Hamiltonianly and effectively on (M, ω). Then the G-action can be lifted to the action of a covering group on the prequantum line bundle. A G-invariant polarisation on M is preserved by the Hamiltonian flows of various components of the moment map. So the Lie algebra of G acts on the quantum Hilbert space and G acts on it projectively; a covering group of G acts honestly. Furthermore, G acts honestly on the set of operators on the Hilbert space, giving rise to automorphisms of the operator algebra. If G is compact, an invariant almost complex structure compatible with ω can be constructed by using a G-invariant Riemannian metric. Whether there exists an integrable G-invariant complex structure is a much much harder problem. Conversely, if G fixes a $J \in \mathcal{J}$, then G consists of isometries of the Riemannian metric $\omega(\cdot, J\cdot)$ and must be compact if M is compact. So the Hamiltonian action of a compact group G is always anomaly free, that is, there is a quantisation scheme (by selecting a G-invariant polarisation) such that G acts on the Hilbert space, at least projectively, and G acts on the operators honestly. If G is non-compact, then the orbit $G \cdot J$ in $\mathcal{J}$ is no longer a point. Consider the Hilbert space bundles $\mathcal{H}$, $\tilde{\mathcal{H}}$ or $\hat{\mathcal{H}}$ over $\mathcal{J}$. The anomaly-free condition for Hamiltonian group actions is that $\mathcal{H}$, $\tilde{\mathcal{H}}$ or $\hat{\mathcal{H}}$ is flat or projectively flat over the orbit $G \cdot J$ for some $J \in \mathcal{J}$. If, for example, $\tilde{\mathcal{H}}$ is projectively flat on $G \cdot J$, we follow the previous paragraph to construct a projective action of G on $\tilde{\mathcal{H}}_J$. For any $g \in G$, the parallel transport $\tilde{U}_{g\cdot J,J}$ from J to $g \cdot J$ is defined up to a phase, and so is $\tilde{\varrho}(g) = \tilde{U}_{J,g\cdot J} \circ \tilde{g}$, where $\tilde{g}$ is the action of g on $\tilde{\mathcal{H}}$. Then $\tilde{\varrho}(e) = \mathrm{Id}_{\tilde{\mathcal{H}}}$ and the same derivation shows that for $g, h \in G$, $\tilde{\varrho}(gh) = \tilde{\varrho}(g) \circ \tilde{\varrho}(h)$ up to a phase.

6. Quantisation of Fermionic Systems

Quantisation of a fermionic system means finding an irreducible representation of the algebra with the canonical anti-commutation algebra $\{\hat{\psi}_i, \hat{\psi}_j\} = \frac{1}{2} b_{ij}$, where b_{ij} is symmetric. There is a major difference between the quantisation of bosons and that of fermions [44]. Two requirements of quantisation are unitarity and that the energy is bonded from below. Unitarity means that the quantum Hilbert space has a positive definite Hermitian product and that real elements (invariant under the conjugation) in the operator algebra are represented as self-adjoint operators. For a bosonic system, the sign of the symplectic form in unimportant for unitarity because both $\pm\sqrt{-1}\partial/\partial q$ are self-adjoint. The condition of positivity is on the polarisation: if J is compatible with the symplectic form, the energy $\frac{1}{2}\omega(\cdot, J\cdot)$ of the harmonic oscillator is non-negative. In geometric quantisation, the cohomology of the prequantum line bundle (or its half-form, half-density correction) is expected to concentrate at degree zero. For a fermionic system, energy is always bounded from below because of the Dirac sea. The positivity requirement is instead for preserving unitarity: the representation can not be unitary unless b is positive definite. The phase space of the system is a fermionic copy of a Euclidean space (V, b): it does not exist as a set of points, but the ring of functions is the exterior algebra $\wedge V^*$. In this sense, we say that the phase space has fermionic coordinates. Fermionic integration requires an orientation on V. Similarly, the prequantum line bundle over V does not exist, but the set of its sections, upon a 'trivialisation', is the space $\wedge(V^*)^{\mathbb{C}}$.

We assume that $\dim V = 2n$ is even. The operator algebra is the Clifford algebra $\mathrm{Cl}(V, b)$, and it has a unique irreducible representation, of dimension 2^n, called the spinor representation [6]. This uniqueness is the fermionic counterpart of the Stone-von Neumann theorem. The standard construction is to choose a complex structure J on V such that $b(J\cdot, J\cdot) = b(\cdot,\cdot)$ and compatible with the orientation of V. Then the representation space is $\wedge(V_J^{1,0})^*$. On the other hand, prequantisation [25] and quantisation [46] can be performed following the analogy of the bosonic theory. In doing so, we also need to choose a complex structure J with the same properties so that we can pick the sections that are covariantly constant along $V_J^{0,1}$ as the 'wave functions'. The novelty is that the quantum Hilbert space is not merely $\wedge(V_J^{1,0})^*$ but $\mathcal{H}_J = \mathrm{e}^{-\frac{1}{2}b(\theta,\bar{\theta})}\wedge(V_J^{1,0})^*$, with an additional fermionic Gaussian factor $\mathrm{e}^{-\frac{1}{2}b(\theta,\bar{\theta})}$, where $\theta = (\theta_1, \ldots, \theta_n) \in V_J^{1,0}$ contains the complex fermionic coordinates on (V, J). Thus similar to the bosonic case, an element in $\mathcal{H}_J$ is of the form $\psi(\theta, \bar{\theta}) = \phi(\theta)\,\mathrm{e}^{-\frac{1}{2}b(\theta,\bar{\theta})}$, where ϕ depends only on θ or equivalently, $\phi \in \wedge(V_J^{1,0})^*$.

The space $\mathcal{J}$ of these allowed complex structures is a compact Hermitian symmetric space of type $\mathrm{SO}(2n)/\mathrm{U}(n)$ and of complex dimension $\frac{1}{2}n(n-1)$. Let σ_b be the standard Kähler form. We form a (finite rank) Hilbert space bundle $\mathcal{H}$ over $\mathcal{J}$ whose fibre over $J \in \mathcal{J}$ is $\mathcal{H}_J$. Since $\mathcal{H}$ is a subbundle of the product bundle $\mathcal{J} \times \wedge(V^*)^{\mathbb{C}}$, there is a unitary connection $\nabla^{\mathcal{H}}$ on $\mathcal{H}$. It turns out that $\nabla^{\mathcal{H}}$ is also projectively flat, and the curvature is $F^{\mathcal{H}} = \sigma_b/2\sqrt{-1}$ [50]. The fermionic exponential factor is crucial for projective flatness as it affects the subspaces $\mathcal{H}_J$ in $\wedge(V^*)^{\mathbb{C}}$. The identification of spinor representations $\mathcal{H}_J$ and $\mathcal{H}_{J'}$ can be accomplished geometrically by the parallel transport along a geodesic joining J and J'. The space $\mathcal{J}$ is compact and without boundary, and there is no real polarisation. But complication occurs if J and J' are conjugation points. If J' is not on the cut locus of J, then there is a unique length-minimising geodesic jointing J and J', and the parallel transport along it can be denoted by $U_{J',J}\colon \mathcal{H}_J \to \mathcal{H}_{J'}$. If $J, J', J'' \in \mathcal{J}$ are mutually away from the cut locus of each other, then [49]

$$U_{J,J''} \circ U_{J'',J'} \circ U_{J',J} = \mathrm{e}^{\frac{\sqrt{-1}}{2}\int_\Delta \sigma_b}\ \mathrm{Id}_{\mathcal{H}_J}$$

where Δ is a surfaces bounded by three length-minimising geodesics joining J, J', J''. Since the period of σ_b is in $4\pi\mathbb{Z}$, the phase factor does not depend on the choice of Δ. The exponent is related to the orthogonal analogue of the Maslov index introduced in [28].

If $n = 1$, then $\mathcal{J}$ is a point. The first non-trivial case is when $n = 2$ and $\mathcal{J}$ is the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. At the complex structure J corresponding to $0 \in \mathbb{C}$, let (θ_1, θ_2) be the complex fermionic coordinates on $V_J^{1,0}$. The Hilbert space $\mathcal{H}_J$ is spanned by four quantum states $\theta_1^{k_1}\theta_2^{k_2}\,\mathrm{e}^{-\frac{1}{2}\theta\bar\theta}$, where $\theta\bar\theta = \theta_1\bar\theta_1 + \theta_2\bar\theta_2$ and k_1, k_2 take values $0, 1$. Along the geodesic in $\mathcal{J}$ parametrised by $\tan t$, the parallel transport is

$$U_{J',J}\colon \begin{Bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ \theta_1\theta_2 \end{Bmatrix} \mathrm{e}^{-\frac{1}{2}\theta\bar\theta} \longmapsto \begin{Bmatrix} \cos t + \theta_1'\theta_2' \sin t \\ \theta_1 \\ \theta_2 \\ \theta_1'\theta_2' \sec t - \sin t \end{Bmatrix} \mathrm{e}^{-\frac{1}{2}\theta'\bar\theta'}$$

where $\theta' = (\theta_1', \theta_2')$ are the complex fermionic coordinates in the complex structure J' corresponding to $\tan t$. The map $U_{J',J}$ on the fermionic coherent state $\mathrm{e}^{\theta\bar\alpha - \frac{1}{2}\theta\bar\theta}$ is [50]

$$U_{J',J}\colon \mathrm{e}^{\theta\bar\alpha-\frac{1}{2}\theta\bar\theta} \longmapsto (\cos t)\,\mathrm{e}^{\theta'\bar\alpha\sec t + (\bar\alpha_1\bar\alpha_2 + \theta_1'\theta_2')\tan t - \frac{1}{2}\theta'\bar\theta'}$$

where $\alpha = (\alpha_1, \alpha_2)$ are complex fermionic constants. These formulae hold when $|t| < \pi/2$, or before the geodesic reaches the cut locus of J. The effect of parallel transport on the fermionic creation and annihilation operators are the fermionic Bogoliubov transformations.

The canonical lines $\kappa_J = \wedge^n(V_J^{1,0})^*$, $J \in \mathcal{J}$, form a complex line bundle $\mathcal{K} = \bigsqcup_{J\in\mathcal{J}} \kappa_J$ over $\mathcal{J}$ with a unitary connection $\nabla^{\mathcal{K}}$ whose curvature is $F^{\mathcal{K}} = -\sigma_b/\sqrt{-1}$. Since $c_1(\mathcal{K})$ is divisible by two and since $\mathcal{J}$ is simply connected, there is a unique square root bundle $\mathcal{K}^{1/2}$ with a uniquely determined connection. Notice that the curvature has an opposite sign in comparison with the bosonic case. On the other hand, since the fermionic and bosonic volume forms transform in opposite manner, the half-form (or metaplectic) correction to the quantum Hilbert space is $\hat{\mathcal{H}}_J = \mathcal{H}_J \otimes \kappa_J^{1/2}$ [50]. The spaces $\hat{\mathcal{H}}_J$ form a bundle $\hat{\mathcal{H}} = \bigsqcup_{J\in\mathcal{J}} \hat{\mathcal{H}}_J = \mathcal{H} \otimes \mathcal{K}^{1/2}$ over $\mathcal{J}$. The induced connection $\nabla^{\hat{\mathcal{H}}}$ on $\hat{\mathcal{H}}$ is flat because $F^{\hat{\mathcal{H}}} = F^{\mathcal{H}} + \frac{1}{2}F^{\mathcal{K}} = 0$. Since $\mathcal{J}$ is simply connected, there is a natural trivialisation of the bundle $\hat{\mathcal{H}}$ identifying the spinor representations $\hat{\mathcal{H}}_J$ of the Clifford algebra for all $J\mathcal{J}$. There are other descriptions of this identification. The pairing between $\nu^{1/2}$ and $\nu'^{1/2}$ for

$\nu \in \kappa_J^{-1}$ and $\nu' \in \kappa_{J'}^{-1}$ is the ratio of $\bar{\nu} \wedge \nu'$ and the unit volume element of V. This gives rise to the fermionic analogue of the BKS pairing between $\hat{\mathcal{H}}_J$ and $\hat{\mathcal{H}}_{J'}$ is non-degenerate when J' is not in the cut locus of J and it induces a unitary isomorphism identifying $\hat{\mathcal{H}}_J$ and $\hat{\mathcal{H}}_{J'}$ [50]. Finally, the half-density corrected Hilbert space is $\tilde{\mathcal{H}}_J = \mathcal{H}_J \otimes |\kappa_J|^{1/2}$ [49]. The half-density corrected Hilbert space bundle $\tilde{\mathcal{H}} = \mathcal{H} \otimes |\mathcal{K}|^{1/2}$ has a projectively flat connection $\nabla^{\tilde{\mathcal{H}}}$ whose curvature is $F^{\tilde{\mathcal{H}}} = F^{\mathcal{H}} = \sigma_b/2\sqrt{-1}$ [49].

The group $\mathrm{SO}(V, b)$ acts on the Clifford algebra $\mathrm{Cl}(V, b)$ by automorphisms. When $\dim V$ is even, by the uniqueness of the spinor representation, there is a projective representation of $\mathrm{SO}(V, b)$ and its double cover $\mathrm{Spin}(V, b)$ acts honestly. Alternatively, $\mathrm{Spin}(V, b)$ can be embedded in $\mathrm{Cl}(V, b)$ and hence acts on the spinor representation. Finally, since $\mathrm{Spin}(V, b)$ acts on the square root bundle $\mathcal{K}^{1/2}$, its representation on $\hat{\mathcal{H}}_J$ for a fixed $J \in \mathcal{J}$ can be constructed by the same geometric method using the flat bundle $\hat{\mathcal{H}}$ as in the bosonic case.

7. Odd Dimensional Fermionic Phase Space

Since the phase space of a fermionic system is the fermionic copy of a Euclidean space (V, b), it can be odd dimensional. Quantisation of odd dimensional fermionic phase spaces was considered in [18]. Here we summarise the results in a coordinate-free language. Recall that a (linear) polarisation on V is a (possibly complex) subspace $P \subset V^{\mathbb{C}}$ containing a maximal set of commuting (in the bosonic case) or anti-commuting (in the fermionic case) variables. When $\dim V$ is even, the subspace P can be chosen of half the dimension of V and such that the restriction to P of the symplectic form or the Euclidean metric is zero. If $\dim V = 2n + 1$, then there is no subspace of half the dimension of V, but the maximal dimension of the complex subspace P on which the Euclidean metric b is zero is n. The fermionic wave functions can be consistently required to be covariantly constant along P. Such a subspace P determines a complex structure J on a real subspace V_J of codimension 1 in V such that $b(J\cdot, J\cdot) = b(\cdot, \cdot)$ on V_J. The relation between P and (V_J, J) is that P is the $(1, 0)$-subspace of (V_J, J). Since J determines an orientation on V_J and since we also require an orientation on V, we get a unit vector $v_J \in V$ orthogonal to V_J. The set $\mathcal{J}$ of such J is a symmetric space $\mathrm{SO}(2n + 1)/\mathrm{U}(n)$ of real dimension $n(n + 1)$. It is the total space of a fibration over $\mathrm{SO}(2n + 1)/\mathrm{SO}(2n) = S^{2n}$, mapping $J \in \mathcal{J}$ to the unit vector v_J, and the fibres are $\mathrm{SO}(2n)/\mathrm{U}(n)$, the space of complex structures on V_J compatible with the restriction of b.

To quantise the fermionic phase space given by the Euclidean space (V, b) of dimension $2n+1$, we use a Euclidean space $(V^\sharp, b^\sharp)$ of one dimension higher, where $V^\sharp = \mathbb{R}v_0 \oplus V$ and $b^\sharp$ is the Euclidean metric which restricts to b on V and such that

v_0 is a unit vector orthogonal to V. The space $\mathcal{J}^\sharp$ of complex structures compatible with the Euclidean metric $b^\sharp$ and the induced orientation on $V^\sharp$ is the compact Hermitian symmetric space $\mathrm{SO}(2n+2)/\mathrm{U}(n+1)$ of complex dimension $\frac{1}{2}n(n+1)$. We show that $\mathcal{J}^\sharp$ can be identified with the set $\mathcal{J}$ defined above from the odd dimensional Euclidean space (V,b). First, $J\in\mathcal{J}$ can be extended to $J^\sharp$ such that $J^\sharp=J$ on V_J, $J^\sharp(v_0)=v_J$ and $J^\sharp(v_J)=-v_0$. Conversely, $J^\sharp\in\mathcal{J}^\sharp$ determines $v_J=J^\sharp(v_0)$, $V_J=(\mathbb{R}v_J)^\perp$ and J as the restriction of $J^\sharp$ to V_J. The complex subspaces $P=V_J^{1,0}$ and $P^\sharp=(V^\sharp_{J^\sharp})^{1,0}$ are related by $P^\sharp=P+\mathbb{C}(v_J+\sqrt{-1}v_0)$ and $P=P^\sharp\cap V^{\mathbb{C}}$. Given $J\in\mathcal{J}=\mathcal{J}^\sharp$, let $\mathcal{H}^\sharp_J$ be the quantum Hilbert space of the fermionic phase space $(V^\sharp,b^\sharp)$ and let $\tilde{\mathcal{H}}^\sharp_J$, $\hat{\mathcal{H}}^\sharp_J$ be its half-density, half-form corrections. They form, respectively, the vector bundles $\mathcal{H}^\sharp$, $\tilde{\mathcal{H}}^\sharp$, $\hat{\mathcal{H}}^\sharp$ of rank 2^{n+1} over $\mathcal{J}^\sharp=\mathcal{J}$. The connections $\nabla^{\mathcal{H}^\sharp}$ and $\nabla^{\tilde{\mathcal{H}}^\sharp}$ on $\mathcal{H}^\sharp$ and $\tilde{\mathcal{H}}^\sharp$ are projectively flat, and their curvatures are $\sigma_{b^\sharp}/2\sqrt{-1}$ times the identity operator, whereas the connection $\nabla^{\hat{\mathcal{H}}^\sharp}$ on $\hat{\mathcal{H}}^\sharp$ is flat.

The Clifford algebra $\mathrm{Cl}(V^\sharp,b^\sharp)$ contains $\mathrm{Cl}(V,B)$ and an additional generator e_0 which graded-commutes with $\mathrm{Cl}(V,b)$ and which can be normalised so that $e_0^2=1$. The centre of $\mathrm{Cl}(V,b)$ is generated by 1 and e_V, where e_V is again normalised so that $e_V^2=1$. The spinor representation of $\mathrm{Cl}(V^\sharp,b^\sharp)$, for example on $\mathcal{H}^\sharp_J$ for some $J\in\mathcal{J}$, is irreducible, but it decomposes according to $\mathcal{H}^\sharp_J=\mathcal{H}^+_J\oplus\mathcal{H}^-_J$ as representations of $\mathrm{Cl}(V,b)$. Here $\mathcal{H}^\pm_J$ are the subspaces on which $e_V=\pm1$, respectively, and they are the two (inequivalent) irreducible representations of $\mathrm{Cl}(V,b)$, each of dimension 2^n. Alternatively, $\mathcal{H}^\sharp_J$ splits into subspaces of spinors of positive and negative chirality according to the eigenvalues ±1 of the chirality operator $e_{V^\sharp}=e_0e_V$, which also satisfies $e_{V^\sharp}^2=1$. Then $\mathcal{H}^+_J$ is the subspace of $\mathcal{H}^\sharp_J$ on which either $e_{V^\sharp}=1$, $e_0=1$ or $e_{V^\sharp}=-1$, $e_0=-1$, whereas $\mathcal{H}^-_J$ is the subspace on which either $e_{V^\sharp}=1$, $e_0=-1$ or $e_{V^\sharp}=-1$, $e_0=1$. As $J\in\mathcal{J}$ varies, we have a splitting $\mathcal{H}^\sharp=\mathcal{H}^+\oplus\mathcal{H}^-$ as vector bundles over $\mathcal{J}$. Since e_0, e_V and e_V came from quantisation of observables on $V^\sharp$, they are parallel sections of $\mathrm{End}(\mathcal{H}^\sharp)$. Therefore by restricting $\nabla^{\mathcal{H}^\sharp}$ to $\mathcal{H}^\pm$, we obtain projectively flat connections $\nabla^{\mathcal{H}^\pm}$ on the bundles $\mathcal{H}^\pm$.

We now explain the half-density and half-form corrections when $\dim V=2n+1$. For any $J\in\mathcal{J}$, $\dim_{\mathbb{C}}V_J^{1,0}=n$ and let $\kappa_J^{-1}=\wedge^nV_J^{1,0}$. The metaplectically corrected Hilbert spaces are $\hat{\mathcal{H}}^\pm_J=\mathcal{H}^\pm_J\otimes\kappa_J^{-1/2}$. The lines κ_J^{-1} form a line bundle $\mathcal{K}^{-1}$ over $\mathcal{J}$ and thus we have the corrected Hilbert space bundles $\hat{\mathcal{H}}^\pm=\mathcal{H}^\pm\otimes\mathcal{K}^{-1/2}$. Under the identification of $\mathcal{J}$ and $\mathcal{J}^\sharp$, the line bundle $\mathcal{K}^{-1}$, together with their respective connections, can be identified with the bundle $(\mathcal{K}^\sharp)^{-1}$ constructed from the even dimensional space $(V^\sharp,b^\sharp)$ [18]. Thus $\hat{\mathcal{H}}^\sharp=\hat{\mathcal{H}}^+\oplus\hat{\mathcal{H}}^-$ as bundles with connections, and both $\hat{\mathcal{H}}^\pm$ have trivial connections enabling the identification of $\hat{\mathcal{H}}_J^{\pm}$ for all $J \in \mathcal{J}$. Alternatively, the pairing between $\nu^{-1/2} \in \kappa_J^{-1/2}$ and $\nu'^{-1/2} \in \kappa_{J'}^{-1/2}$ is the ratio of $(v_J + v_{J'}) \wedge \bar{\nu} \wedge \nu'$ and the unit volume element of V [18]. This gives BKS pairings between $\hat{\mathcal{H}}_J^{\pm}$ and $\hat{\mathcal{H}}_{J'}^{\pm}$ which are non-degenerate when J, J' are not on the cut locus of each other, and they define the same unitary maps between these spaces. Finally, the half-density corrected Hilbert spaces $\tilde{\mathcal{H}}_J^{\pm}$ form, as J varies, Hilbert space bundles $\tilde{\mathcal{H}}^{\pm}$ over $\mathcal{J}$ which admit projectively flat connections of the same curvature as $\tilde{\mathcal{H}}^{\sharp}$.

When $\dim V$ is odd, the symmetry group $\mathrm{SO}(V, b)$ acts as automorphisms of the Clifford algebra $\mathrm{Cl}(V, b)$, and its double cover $\mathrm{Spin}(V, b)$ acts on the representation spaces $\hat{\mathcal{H}}_J^{\pm}$ (for a fixed J) of $\mathrm{Cl}(V, b)$ using the same construction when $\dim V$ is even. Moreover, $\mathrm{O}(V, b)$ also acts as automorphisms of $\mathrm{Cl}(V, b)$. When $\dim V$ is even, $\mathrm{O}(V, b)$ acts as inner automorphisms and its double cover, $\mathrm{Pin}(V, b)$, acts on the representation space $\hat{\mathcal{H}}_J$. However, when $\dim V$ is odd, only the identity component $\mathrm{SO}(V, b)$ acts as inner automorphisms, and the other component maps $\hat{\mathcal{H}}_J^{+}$ to $\hat{\mathcal{H}}_J^{-}$ and vice versa. So we say that the $\mathrm{O}(V, b)$ symmetry is broken to $\mathrm{SO}(V, b)$.

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