Eighteenth International Conference on
Geometry, Integrability and Quantization
June 03–08, 2016, Varna, Bulgaria
Ivaïlo M. Mladenov, Guowu Meng
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2017, pp 105–116
doi: 10.7546/giq-18-2017-105-116

GENERALIZED EULER ANGLES VIEWED AS SPHERICAL COORDINATES

DANAIL S. BREZOV, CLEMENTINA D. MLADENOVA AND IVAÏLO M. MLADENOV

University of Architecture, Civil Engineering and Geodesy, 1 Hristo Smirnenski Blvd. 1046 Sofia, Bulgaria
† *Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bllock 4, 1113 Sofia, Bulgaria*
‡ *Institute of Biophysics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Block 21, 1113 Sofia, Bulgaria*

Abstract. Here we develop a specific factorization technique for rotations in $\mathbb{R}^3$ into five factors about two or three fixed axes. Although not always providing the most efficient solution, the method allows for avoiding gimbal lock singularities and decouples the dependence on the invariant axis $\mathbf{n}$ and the angle ϕ of the compound rotation. In particular, the solutions in the classical Euler setting are given directly by the angle of rotation ϕ and the coordinates of the unit vector $\mathbf{n}$ without additional calculations. The immediate implementations in rigid body kinematics are also discussed and some generalizations and potential applications in other branches of science and technology are pointed out as well.
MSC: 53A10, 52C99
Keywords: Cayley map, Lie algebras, Lie groups, rigid body kinematics, Rodrigues rotation formula, vector-parameterization

Introduction

Euler angles and their variants are well-studied tool for the representation of the three-dimensional rotation group and its spin cover $\mathrm{SU}(2) \cong \mathbb{S}^3$, which find numerous applications in both classical and quantum mechanics. There are twelve known configurations, in which the compound rotation is decomposed into three factors, whose unit invariant vectors $\{\hat{\boldsymbol{c}}_k\}$ constitute an orthonormal frame. Moreover, one may generalize this construction by demanding only the axes of two successive rotations to be perpendicular, i.e., $\hat{\boldsymbol{c}}_2 \perp \hat{\boldsymbol{c}}_{1,3}$, while no restriction is imposed on the angle between $\hat{\boldsymbol{c}}_1$ and $\hat{\boldsymbol{c}}_3$. This is the so-called Davenport setting, which still guarantees the decomposability of any rotation $\mathcal{R} \in \mathrm{SO}(3)$. Besides, one encounters non-trivial necessary and sufficient conditions [4] that need to be satisfied in order to guarantee the generalized Euler decomposition [2]. Such generalization is also desirable for the applications, where orthogonal axes are not always available (or preferable). Here we use our previous results on this problem and consider a different type of factorization, which resembles the well-known Wigner decomposition of the Lorentz group [13]. For an arbitrary rotation $\mathcal{R}(\boldsymbol{n}, \phi) \in \mathrm{SO}(3)$, where $\boldsymbol{n}(\theta, \phi) \in \mathbb{S}^2$ denotes the unit invariant vector given in spherical coordinates and $\phi \in [0, 2\pi)$ is the corresponding angle, we have a representation in the form

$$\mathcal{R}(\boldsymbol{n}, \phi) = \mathcal{R}_z(\varphi)\mathcal{R}_y(\theta)\mathcal{R}_z(\phi)\mathcal{R}_y^t(\theta)\mathcal{R}_z^t(\varphi) \tag{1}$$

where the subscript indicates the corresponding invariant axis. To see this, we note that the above is actually equivalent to saying that rotating the unit vector along the z-axis first by the azimuthal angle θ about OY and then, by the polar angle φ about OZ, we obtain exactly $\boldsymbol{n}$, which is the very definition of spherical coordinates. The radial variable ϕ yields the parameter set (for the so-called rotation vector) as an open ball of radius 2π. Periodicity, however, demands its boundary to be glued to the origin, which is not surprising considering the twisted topology of $\mathrm{SO}(3) \cong \mathbb{RP}^3$.

This article is organized as follows: in the first section we introduce the vector-parameter technique, which appears to be very convenient in the description of three-dimensional rotations. Then, we generalize formula (1) for arbitrary axes studying also the inevitable restrictions and some specific cases. In the third and final section we briefly discuss this type of factorizations from the perspective of kinematics.

The Vector-Parameter Technique

We begin by introducing the notion of the *vector-parameter* defined as $\boldsymbol{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0}$, where $(\zeta_0, \boldsymbol{\zeta}) = \zeta_0 + \zeta_1\boldsymbol{i} + \zeta_2\boldsymbol{j} + \zeta_3\boldsymbol{k} \in \mathbb{H}$ is usually chosen with unit norm, i.e., an element of the spin covering group $\mathrm{SU}(2) \cong \mathbb{S}^3$, so that $\boldsymbol{c}$ appears as a natural parameter in the projective space $\mathbb{RP}^3 \cong \mathrm{SO}(3)$. It may also be obtained via the Euler trigonometric substitution as $\boldsymbol{c} = \tau\boldsymbol{n}$ where $\tau = \tan\dfrac{\phi}{2} \in \mathbb{RP}^1$ is the *scalar parameter* (ϕ being the rotation angle) and $\boldsymbol{n} \in \mathbb{S}^2$ stands for the unit vector along the invariant axis given by Euler's theorem. Applied to the classical Rodrigues' formula describing three dimensional rotations in terms of their axes and angles, the above substitution allows for the alternative parameterization

$$\mathcal{R}(\boldsymbol{c}) = \frac{(1-\boldsymbol{c}^2)\,\mathcal{I} + 2\boldsymbol{c}\otimes\boldsymbol{c}^t + 2\,\boldsymbol{c}^\times}{1+\boldsymbol{c}^2} \in \mathrm{SO}(3) \tag{2}$$

where $\mathcal{I}$ stands for the identity, $\otimes$ denotes the tensor product and $\boldsymbol{c}^\times$ is an extension to the Hodge duality to vector-parameters, which gives the cross product in $\mathbb{RP}^3$, i.e., $\boldsymbol{c}^\times\tilde{\boldsymbol{c}} = \boldsymbol{c}\times\tilde{\boldsymbol{c}}$. Note that formula (2) may be written equivalently as

$$\mathrm{Cay}\circ{}^\times \,:\, \boldsymbol{c} \,\in\, \mathbb{RP}^3 \qquad \longrightarrow \qquad \mathrm{Cay}(\boldsymbol{c}^\times) = \frac{\mathcal{I}+\boldsymbol{c}^\times}{\mathcal{I}-\boldsymbol{c}^\times} = \mathcal{R}(\boldsymbol{c}) \,\in\, \mathrm{SO}(3) \tag{3}$$

that is the famous Cayley representation of rotations $\mathcal{R}(\boldsymbol{c}) = \mathrm{Cay}(\boldsymbol{c}^\times)$. Apart from being elegant and convenient for analytic treatment, the expressions (2) and (3) yield exact values for the matrix entries since unlike the Rodrigues' rotation formula they do not involve transcendent functions.

Another feature of vectorial parameterization, which we consider even more important, is that it allows for a compact composition law substituting the matrix equality $\mathcal{R}(\boldsymbol{c}_2)\,\mathcal{R}(\boldsymbol{c}_1) = \mathcal{R}(\boldsymbol{c})$, namely as

$$\boldsymbol{c} = \langle\, \boldsymbol{c}_2, \boldsymbol{c}_1 \rangle = \frac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2\times\boldsymbol{c}_1}{1-(\boldsymbol{c}_2,\boldsymbol{c}_1)} \tag{4}$$

that is just a projective version of the quaternion multiplication rule. However, composing rotations in this manner is much more efficient (in terms of computational complexity) compared to the usual matrix multiplication [6].

With the aid of these two simple formulas we have resolved (see [2]) the generalized Euler decomposition problem $\mathcal{R}(\boldsymbol{c}) = \mathcal{R}(\boldsymbol{c}_3)\mathcal{R}(\boldsymbol{c}_2)\mathcal{R}(\boldsymbol{c}_1)$, where $\boldsymbol{c}_i = \tau_i\hat{\boldsymbol{c}}_i$ is the vector-parameter of the i-th rotation in the decomposition, i.e., $\tau_i = \tan\dfrac{\phi_i}{2}$ and $\hat{\boldsymbol{c}}_i \in \mathbb{S}^2$. The solutions are particularly simple if $r_{ij} = g_{ij}$ holds for some $i > j$, where we use the notation

$$g_{ij} = (\hat{\boldsymbol{c}}_i, \hat{\boldsymbol{c}}_j), \qquad r_{ij} = (\hat{\boldsymbol{c}}_i, \mathcal{R}(\boldsymbol{c})\,\hat{\boldsymbol{c}}_j).$$

In this case one may decompose $\mathcal{R}(\boldsymbol{c})$ into a pair of rotations about the i-th and the j-th axis, i.e., $\mathcal{R}(\boldsymbol{c}) = \mathcal{R}(\tau_j\,\hat{\boldsymbol{c}}_j\,)\mathcal{R}(\tau_i\hat{\boldsymbol{c}}_i)$, with scalar parameters given as

$$\tau_i = \frac{\varepsilon_{ij\,k}\tilde{\upsilon}_k}{g_{i[j}\,\upsilon_{i]}}, \qquad \tau_j = \frac{\varepsilon_{ij\,k}\tilde{\upsilon}_k}{g_{j\,[\,i}\upsilon_{j\,]}}. \tag{5}$$

Here $\varepsilon_{ij\,k}$ denotes the Levi-Civita symbol and we make use of the standard notation for the (anti-)symmetrization of indices

$$a_{(i}b_{j\,)} = a_ib_j + a_j\,b_i, \qquad a_{[i}b_{j\,]} = a_ib_j - a_j\,b_i$$

as well as (here i, j and k take different values and no summation is assumed)

$$\upsilon_k = (\boldsymbol{n}, \hat{\boldsymbol{c}}_k), \qquad \varepsilon_{ij\,k}\tilde{\upsilon}_k = (\boldsymbol{n}, \hat{\boldsymbol{c}}_i \times \hat{\boldsymbol{c}}_j\,).$$

Decomposition via Conjugation

Consider the decomposition

$$\mathcal{R} = \mathcal{R}_1^{-1}\,\mathcal{R}_2^{-1}\,\mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 \iff \boldsymbol{c}_3 = \mathcal{R}_2\mathcal{R}_1\,\boldsymbol{c} \tag{6}$$

which leads to $\tau_3 = \pm\tau$ (for simplicity we work only with the positive sign) together with the expression

$$\tilde{\boldsymbol{c}} = \langle \boldsymbol{c}_2, \boldsymbol{c}_1 \rangle = \frac{1}{1+\upsilon_3}\left[\,\boldsymbol{n}\times\hat{\boldsymbol{c}}_3 + \lambda(\boldsymbol{n}+\hat{\boldsymbol{c}}_3)\,\right]. \tag{7}$$

This relation follows from the fact that the adjoint action $\tilde{\mathcal{R}}\,\mathcal{R}\,\tilde{\mathcal{R}}^{-1}$ rotates the axis of $\mathcal{R}$ with $\tilde{\mathcal{R}}$, while for the latter we note that the invariant axis of the conjugating rotation $\mathcal{R}(\tilde{\boldsymbol{c}})\,:\,\boldsymbol{n} \to \hat{\boldsymbol{c}}_3$ belongs to the plane of mirror symmetry of the unit vectors $\boldsymbol{n}$ and $\hat{\boldsymbol{c}}_3$. Therefore, one has the expression

$$\tilde{\boldsymbol{c}} = \mu\,(\boldsymbol{n}\times\hat{\boldsymbol{c}}_3) + \nu\,(\boldsymbol{n}+\hat{\boldsymbol{c}}_3), \qquad \mu,\,\nu \in \mathbb{RP}^1$$

for the vector-parameter $\tilde{\boldsymbol{c}}$ and formula (3) yields

$$(1+\tilde{\boldsymbol{c}}^{\times})\,\boldsymbol{n} = (1-\tilde{\boldsymbol{c}}^{\times})\,\hat{\boldsymbol{c}}_3$$

from which one finds $\mu = (1+\upsilon_3)^{-1}$ but fails to determine ν, replaced here for convenience with the parameter $\lambda = (1+\upsilon_3)\,\nu \in \mathbb{RP}^1$. With this formula (7) follows directly.

Furthermore, the necessary and sufficient condition for the factorization $\mathcal{R}(\tilde{\boldsymbol{c}}) = \mathcal{R}_2\mathcal{R}_1$ may be written as (see [2,4])

$$(\hat{\boldsymbol{c}}_2, \mathcal{R}(\tilde{\boldsymbol{c}})\,\hat{\boldsymbol{c}}_1) = (\hat{\boldsymbol{c}}_2, \hat{\boldsymbol{c}}_1) \tag{8}$$

and expanding formula (3) in a power series one obtains

$$(\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_2, \sum_{n=0}^{\infty}(\tilde{\boldsymbol{c}}^{\times})^n\hat{\boldsymbol{c}}_1) = \frac{1}{1+\tilde{\boldsymbol{c}}^2}\left[(\hat{\boldsymbol{c}}_1,\,\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_2) + (\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_1,\,\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_2)\right] = 0$$

that finally yields (8) in the equivalent form

$$(\tilde{\boldsymbol{c}},\,\hat{\boldsymbol{c}}_1\times\hat{\boldsymbol{c}}_2) = (\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_1,\,\tilde{\boldsymbol{c}}\times\hat{\boldsymbol{c}}_2). \tag{9}$$

Substituting the expression (7) for $\tilde{\boldsymbol{c}}$ in the above formula, we obtain for the undetermined parameter $\lambda \in \mathbb{RP}^1$ a quadratic equation with discriminant

$$\begin{aligned}\big(\upsilon_1\upsilon_2 - g_{(1[2)}\upsilon_{3]} + g_{13}g_{23} - 2g_{12}\big)\big(\upsilon_1\upsilon_2 - 2g_{23}\upsilon_1\upsilon_3 + g_{3[1}\upsilon_{2]} + g_{13}g_{23}\big)&\\ + \left[(\upsilon_1 + g_{13})\,\tilde{\upsilon}_1 - (1+\upsilon_3)\,\omega\right]^2 = \Delta \geq 0 &\end{aligned} \tag{10}$$

where $\omega = (\hat{\boldsymbol{c}}_1 \times \hat{\boldsymbol{c}}_2, \hat{\boldsymbol{c}}_3)$ denotes the volume spanned by the unit vectors $\{\hat{\boldsymbol{c}}_i\}$. Provided that (10) holds, the values of λ are real and explicitly given as

$$\lambda^{\pm} = \frac{(\upsilon_1 + g_{13})\,\tilde{\upsilon}_1 - (1 + \upsilon_3)\,\omega \pm \sqrt{\Delta}}{\upsilon_1 \upsilon_2 - g_{(1[2)}\upsilon_{3]} + g_{13} g_{23} - 2 g_{12}} \cdot \tag{11}$$

Substituted in (7), these values ensure that $\tilde{\boldsymbol{c}}$ may be decomposed with τ_{12} determined by equation (5) as

$$\tau_1^{\pm} = \frac{(\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \tilde{\boldsymbol{c}}^{\pm})}{g_{1[2}(\hat{\boldsymbol{c}}_{1]}, \tilde{\boldsymbol{c}}^{\pm})}, \qquad \tau_2^{\pm} = \frac{(\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \tilde{\boldsymbol{c}}^{\pm})}{g_{2[1}(\hat{\boldsymbol{c}}_{2]}, \tilde{\boldsymbol{c}}^{\pm})}$$

explicitly given as linear-fractional functions of $\lambda^{\pm}$, namely

$$\begin{aligned} \tau_1^{\pm} &= \frac{\upsilon_{[1} g_{2]3} + \lambda^{\pm}(\tilde{\upsilon}_3 + \omega)}{\lambda^{\pm}(g_{1[2} g_{1]3} + g_{1[2}\upsilon_{1]}) + \tilde{\upsilon}_1 + g_{12}\tilde{\upsilon}_2} \\ \tau_2^{\pm} &= \frac{\upsilon_{[1} g_{2]3} + \lambda^{\pm}(\tilde{\upsilon}_3 + \omega)}{\lambda^{\pm}(g_{2[1} g_{2]3} + g_{2[1}\upsilon_{2]}) - g_{12}\tilde{\upsilon}_1 - \tilde{\upsilon}_2} \cdot \end{aligned} \tag{12}$$

An identical argument holds for the case $\tau_3 = -\tau$ as well and the corresponding expressions may be obtained directly by changing the sign of $\hat{\boldsymbol{c}}_3$ in equation (7).

Never mind how complicated the formula (10) might initially appear as it has the remarkable property of depending only on the relative angles $\gamma_{ij} = \arccos|g_{ij}| = \measuredangle(\hat{\boldsymbol{c}}_i, \hat{\boldsymbol{c}}_j)$ and $\beta_i = \arccos|\upsilon_i| = \measuredangle(\boldsymbol{n}, \hat{\boldsymbol{c}}_i)$, but not on the compound rotation's angle ϕ. Hence, the expressions (11) for the parameter λ and the consequent solutions (12) share this property. The only dependence on ϕ is in $\tau_3 = \tan\frac{\phi}{2}\cdot$ We find this quite convenient in numerous considerations, which makes the decomposition (6) useful and applicable even though it is not always optimal. For example, in some cases, such as the orthogonal axes setting, there might exist decompositions into three or four factors that are usually more economical.

In the Davenport setting $g_{12} = g_{23} = 0$ formula (10) yields

$$\Delta_D = [(\upsilon_1 + g_{13})\,\tilde{\upsilon}_1 - (1 + \upsilon_3)\,\omega]^2 + \upsilon_2^2(\upsilon_1 + g_{13})^2 \geq 0$$

that is always satisfied and the solutions for λ are given by

$$\lambda_D^{\pm} = \frac{(\upsilon_1 + g_{13})\,\tilde{\upsilon}_1 - (1 + \upsilon_3)\,\omega \pm \sqrt{\Delta_D}}{\upsilon_2(\upsilon_1 + g_{13})} \tag{13}$$

while equation (12) for the scalar parameters reduces to

$$\tau_1^{\pm} = \frac{g_{13}\upsilon_2 - \lambda_D^{\pm}(\tilde{\upsilon}_3 + \omega)}{\lambda_D^{\pm}\upsilon_2 - \tilde{\upsilon}_1}, \qquad \tau_2^{\pm} = \frac{g_{13}\upsilon_2 - \lambda_D^{\pm}(\tilde{\upsilon}_3 + \omega)}{\lambda_D^{\pm}(\upsilon_1 + g_{13}) + \tilde{\upsilon}_2} \cdot \tag{14}$$

One example is the classical Bryan XYZ decomposition with $g_{ij} = \delta_{ij}$, $\omega = 1$ and (here and below n_i denote the components of the unit vector $\boldsymbol{n}$ in the standard basis) $\upsilon_i = \tilde{\upsilon}_i = n_i$, in which the condition (10) takes the simple form

$$\Delta_B = (n_1^2 - n_3 - 1)^2 + n_1^2 n_2^2 \geq 0$$

and the corresponding solutions for λ are given as

$$\lambda_B^{\pm} = \frac{n_1^2 - n_3 - 1 \pm \sqrt{(n_1^2 - n_3 - 1)^2 + n_1^2 n_2^2}}{n_1 n_2}.$$

Moreover, the expressions for the scalar parameters (14) assume the form

$$\tau_1^{\pm} = \frac{\lambda_B^{\pm}(1 + n_3)}{n_1 - n_2\lambda_B^{\pm}}, \qquad \tau_2^{\pm} = -\frac{\lambda_B^{\pm}(1 + n_3)}{n_2 + n_1\lambda_B^{\pm}}. \tag{15}$$

Next, we discuss the symmetric setting $\hat{\boldsymbol{c}}_3 = \hat{\boldsymbol{c}}_1$, in which the calculations also simplify greatly. For example, the discriminant condition (10) reduces to

$$\Delta_S = (1 + \upsilon_1)^2[\tilde{\upsilon}_3^2 + (\upsilon_2 - g_{12})(\upsilon_2 + g_{12} - 2g_{12}\upsilon_1)] \geq 0$$

which has one solution for $\upsilon_2 \geq g_{12}$, i.e., $\gamma \geq \beta_2$ that guarantees a decomposition into three factors in the reverse order (cf. [4]) and is thus not optimal. The other one, in the form $\upsilon_2 \leq g_{12}(2\upsilon_1 - 1)$, is useful for small angles γ. In particular, the limit $\gamma \to 0$ yields $\beta_1 = \beta_2 = 0$, which we already know. Finally, formula (12) reduces in the symmetric case to

$$\tau_1^{\pm} = \frac{g_{12}\upsilon_1 - \upsilon_2 + \lambda_S^{\pm}\tilde{\upsilon}_3}{\lambda_S^{\pm}(g_{12}\upsilon_1 - \upsilon_2) + \tilde{\upsilon}_3}, \qquad \tau_2^{\pm} = \frac{g_{12}\upsilon_1 - \upsilon_2 + \lambda_S^{\pm}\tilde{\upsilon}_3}{\lambda_S^{\pm}(g_{12}\upsilon_2 - \upsilon_1 + g_{12}^2 - 1) - \tilde{\upsilon}_3} \tag{16}$$

with

$$\lambda_S^{\pm} = \frac{\tilde{\upsilon}_3}{\upsilon_2 - g_{12}} \pm \sqrt{\left(\frac{\tilde{\upsilon}_3}{\upsilon_2 - g_{12}}\right)^2 + \frac{g_{12} + \upsilon_2 - 2g_{12}\upsilon_1}{\upsilon_2 - g_{12}}}.$$

Note that there is a more general sufficient condition for decomposition into five factors in the form $\beta_1 \leq 2\gamma$ (see [4]), to which the above is a particular case. Combining $\beta_2 \leq \gamma$ with the triangle inequality $\beta_1 \leq \beta_2 + \gamma$, we obtain precisely $\beta_1 \leq 2\gamma$. However, $\upsilon_2 \leq g_{12}(2\upsilon_1 - 1)$ is not implied this way.

The Euler Setting

In the classical Euler ZYZ decomposition setting we have $g_{2i} = \omega = 0$, $\upsilon_1 = \upsilon_3 = n_3$, $\upsilon_2 = n_2$, $\tilde{\upsilon}_1 = -\tilde{\upsilon}_3 = n_1$ and $\tilde{\upsilon}_2 = 0$. Thus, the crucial condition (10), which in this case is reduced to the very simple form

$$\Delta_E = (1 + n_3)^2(1 - n_3^2) \geq 0$$

holds always and we have quite useful and compact expression for $\lambda^{\pm}$, namely

$$\lambda_E^{\pm} = \frac{n_1 \pm \sqrt{1-n_3^2}}{n_2} \,. \tag{17}$$

Substituting (17) in (14) or (16), we easily obtain the solutions in the form

$$\tau_1^{\pm} = \frac{n_1 \pm \sqrt{1-n_3^2}}{n_2} = \lambda_E^{\pm}, \qquad \tau_2^{\pm} = \pm\sqrt{\frac{1-n_3}{1+n_3}}, \qquad \tau_3^{\pm} = \tau.$$

Note that the angles ϕ_j do not need to be calculated, since by construction

$$\phi_1^{+} = \pi - \varphi, \qquad \phi_1^{-} = -\varphi, \qquad \phi_2^{\pm} = \pm\,\theta, \qquad \phi_3^{\pm} = \phi \tag{18}$$

where $\varphi = \operatorname{atan_2}\dfrac{n_2}{n_1}$ and $\theta = \arccos n_3$ are the spherical coordinates (polar and azimuthal angle, respectively) of the compound rotation's invariant axis

$$\boldsymbol{n} = (\,\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta\,)^t \,.$$

The simplicity of the above expressions may be explained with a geometric argument using the idea of conjugation. Namely, the transformation that sends the unit vector $\boldsymbol{n} \in \mathbb{S}^2$ to the north pole (the positive z-direction) is a composition of two successive rotations - first, by an angle $-\varphi$ about OZ and then, by an angle $-\theta$ about OY. In the classical Euler setting such simplicity is present only for half-turns, i.e., $\phi = \pi$, more precisely (cf. [2])

$$\phi_1^{+} = \pi - \varphi, \qquad \phi_1^{-} = -\varphi, \qquad \phi_2^{\pm} = \pm 2\theta \qquad \phi_3^{+} = \varphi, \qquad \phi_3^{-} = \varphi - \pi.$$

In any other case, even as simple as quarter-turns ($\tau = 1$), the dependence on the spherical coordinates θ, φ and the rotation angle ϕ is coupled and the expressions are far more complicated. The same applies to the Bryan setting as well.

Note that the inversion of $\hat{\boldsymbol{c}}_3$, i.e., the transition from $\tau_3 = \tau$ to $\tau_3 = -\tau$ yields different solutions in the generic case. Particularly in the Euler and Bryan settings this transition corresponds to the simple symmetry $\lambda \to -\lambda$.

Specific Cases

There are several cases, in which our construction simplifies significantly. The first one is when the compound rotation's invariant axis is parallel to $\hat{\boldsymbol{c}}_1$, which yields $\boldsymbol{c}_3 = \mathcal{R}_2\,\boldsymbol{c}$ and if we apply formula (7) with $\tilde{\boldsymbol{c}} = \boldsymbol{c}_2$ taking into account that $\tau_3 = \pm\tau$, we see that one has either $\hat{\boldsymbol{c}}_2 \in \operatorname{Span}\{\boldsymbol{n}\times\hat{\boldsymbol{c}}_3,\, \boldsymbol{n}+\hat{\boldsymbol{c}}_3\}$ or $\hat{\boldsymbol{c}}_2 \in \operatorname{Span}\{\boldsymbol{n}\times\hat{\boldsymbol{c}}_3,\, \hat{\boldsymbol{c}}_3 - \boldsymbol{n}\}$. Thus, as long as $\upsilon_3 \neq \pm 1$, we have the non-trivial condition $g_{23} \pm \upsilon_2 = 0 \Rightarrow \gamma_{23}^{\pm} = \beta_2, \pi - \beta_2$. Suppose the first one holds, i.e., $\hat{\boldsymbol{c}}_2 = \mu\,\boldsymbol{n}\times\hat{\boldsymbol{c}}_3 + \nu\,(\boldsymbol{n}+\hat{\boldsymbol{c}}_3)$ with $\mu^2 + \nu^2 = 4 - (1-\upsilon_3)^2$. Then, it is not hard to find $\mu = -\tilde{\upsilon}_1(1-\upsilon_3^2)^{-\frac{1}{2}}$ and

$\nu = \sqrt{2}g_{23}(1+\upsilon_3)^{-\frac{1}{2}}$, which allows for obtaining directly the scalar parameter τ_2 using formula (7) with $\tilde{\boldsymbol{c}} = \boldsymbol{c}_2$ ($\boldsymbol{c}_1 = 0$) in the form (the case $\tau_3 = -\tau$ is analogical and the superscript $\pm$ here denotes the solution corresponding to $\tau_3 = \pm\tau$, respectively)

$$\tau_2^{\pm} = \mp\frac{1}{\tilde{\upsilon}_1}\sqrt{\frac{1 \mp \upsilon_3}{1 \pm \upsilon_3}}\,\cdot \tag{19}$$

Similarly, for $\boldsymbol{n} = \pm\hat{\boldsymbol{c}}_3$, apart from the trivial solution $\tau_3 = \pm\tau$, $\tau_{1,2} = 0$, there is one more, involving a decomposition of a half-turn. Namely, for $\upsilon_3 = -1$

$$\mathcal{R}_2\mathcal{R}_1 = 2\,\tilde{\boldsymbol{n}}\otimes\tilde{\boldsymbol{n}}^t - \mathcal{I}, \qquad \tilde{\boldsymbol{n}} \perp \boldsymbol{n}$$

and we may use the composition law (4) to obtain

$$1 - g_{12}\tau_1\tau_2 = 0, \qquad \upsilon_1\tau_1 + \upsilon_2\tau_2 - \tilde{\upsilon}_3\tau_1\tau_2 = 0$$

which yield in the generic case ($\upsilon_1 \neq 0$) the two solutions

$$\tau_1^{\pm} = \frac{\tilde{\upsilon}_3 \pm \sqrt{\tilde{\upsilon}_3^2 - 4g_{12}\upsilon_1\upsilon_2}}{2g_{12}\upsilon_1}, \qquad \tau_2^{\pm} = \frac{2\upsilon_1}{\tilde{\upsilon}_3 \pm \sqrt{\tilde{\upsilon}_3^2 - 4g_{12}\upsilon_1\upsilon_2}}\,\cdot \tag{20}$$

If $\upsilon_1 = 0$, on the other hand, one may simply choose $\tilde{\boldsymbol{n}} = \hat{\boldsymbol{c}}_1$, thus $\tau_1 = \infty$, $\tau_2 = 0$ and the non-trivial solution has the form

$$\tau_1 = \frac{\upsilon_2}{\tilde{\upsilon}_3}, \qquad \tau_2 = \frac{\tilde{\upsilon}_3}{g_{12}\upsilon_2}\,\cdot \tag{21}$$

Similarly, one may consider formula (20) with $\upsilon_2 = 0$.

Kinematics

In this section we discuss possible applications of the above described technique to actual problems appearing in attitude kinematics, navigation, quantum mechanics, virtual reality and many other areas of research [5, 6]. One major advantage of our approach is that unlike the classical Euler-type decomposition it allows for avoiding gimbal lock singularities. Moreover, note that even when the minimal number of rotations in a given factorization is five (cf. [4, 7]), finding the optimal solution is a non-trivial problem, which goes far beyond the scope of the present work. The conjugated factorization introduced above, on the other hand, does not involve additional parameters and cumbersome optimization procedures. Thus, it may naturally be preferred for its relative simplicity.

Finally, we take advantage of the decoupling property of the above construction in relation to kinematical problems. Recall that the total derivative has a longitudinal and normal component, i.e., $\dot{\boldsymbol{c}} = \dot{\tau}\boldsymbol{n} + \tau\dot{\boldsymbol{n}}$, where $(\dot{\boldsymbol{n}}, \boldsymbol{n}) = 0$. Then, for the former we have $\dot{\tau} = \dot{\tau}_3$, while the latter involves partial derivatives - namely, using spherical coordinates

$$\boldsymbol{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)^t \ \in\ \mathbb{S}^2$$

for the unit vector $\boldsymbol{n}$, a straightforward differentiation yields

$$\dot{\boldsymbol{n}}\left(\theta,\varphi\right)=\dot{\theta}\,\frac{\partial\boldsymbol{n}}{\partial\theta}+\dot{\varphi}\frac{\partial\boldsymbol{n}}{\partial\varphi}=\begin{bmatrix}\dot{\theta}\cos\theta\cos\varphi-\dot{\varphi}\,\sin\theta\sin\varphi\\ \dot{\theta}\cos\theta\sin\varphi+\dot{\varphi}\,\sin\theta\cos\varphi\\ -\,\dot{\theta}\sin\theta\end{bmatrix}.$$

Now, let $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ denote the angular velocities in the body and the inertial frame, respectively, i.e.,

$$\boldsymbol{\Omega}^{\times}=\dot{\mathcal{R}}\,\mathcal{R}^{t},\qquad \boldsymbol{\omega}^{\times}=\mathcal{R}^{t}\dot{\mathcal{R}}.$$

The Cayley representation (3) yields the expressions [8]

$$\boldsymbol{\Omega}=\frac{2}{1+\boldsymbol{c}^{2}}\left(\mathcal{I}+\boldsymbol{c}^{\times}\right)\dot{\boldsymbol{c}},\qquad \boldsymbol{\omega}=\frac{2}{1+\boldsymbol{c}^{2}}\left(\mathcal{I}-\boldsymbol{c}^{\times}\right)\dot{\boldsymbol{c}}\tag{22}$$

and in the reverse direction one has [8, 9]

$$\dot{\boldsymbol{c}}=\frac{1}{2}(\boldsymbol{\Omega}+(\boldsymbol{c},\boldsymbol{\Omega})\,\boldsymbol{c}-\boldsymbol{c}\times\boldsymbol{\Omega})\qquad \dot{\boldsymbol{c}}\;=\;\frac{1}{2}(\boldsymbol{\omega}+(\boldsymbol{c},\boldsymbol{\omega})\,\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{\omega}).\tag{23}$$

The above expressions allow for deriving relations like

$$\left(\boldsymbol{c}^{2}\mathcal{P}_{\boldsymbol{n}}^{\perp}+\boldsymbol{c}^{\times}\right)\boldsymbol{\Omega}=2\,\boldsymbol{c}\times\dot{\boldsymbol{c}},\qquad \mathcal{P}_{\boldsymbol{n}}^{\perp}=\mathcal{I}-\boldsymbol{n}\otimes\boldsymbol{n}^{t}$$

in which $\mathcal{P}_{\boldsymbol{n}}^{\perp}$ denotes the projection operator onto the plane orthogonal to $\boldsymbol{n}$. With the aid of formula (22) it is not hard to obtain for the angular velocity in the body system

$$\boldsymbol{\Omega}=\dot{\phi}\,\boldsymbol{n}+\sin\phi\,\dot{\boldsymbol{n}}+(1-\cos\phi)\,\boldsymbol{n}\times\dot{\boldsymbol{n}}\tag{24}$$

which may be written in spherical coordinates as

$$\begin{aligned}\boldsymbol{\Omega}\;=\;&\dot{\phi}\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix}+\dot{\theta}\begin{bmatrix}\sin\phi\cos\theta\cos\varphi-(1-\cos\phi)\sin\varphi\\ \sin\phi\cos\theta\sin\varphi+(1-\cos\phi)\cos\varphi\\ -\sin\phi\sin\theta\end{bmatrix}\\ &+\dot{\varphi}\sin\theta\begin{bmatrix}-\sin\phi\sin\varphi-(1-\cos\phi)\cos\theta\cos\varphi\\ \sin\phi\cos\varphi+(1-\cos\phi)\cos\theta\sin\varphi\\ (1-\cos\phi)\sin\theta\end{bmatrix}.\end{aligned}\tag{25}$$

Thus, the angular velocity is expressed explicitly in terms of the angles of the decomposition $\phi_1=-\varphi$, $\phi_2=-\theta$, $\phi_3=\phi$. On the other hand, formula (23) yields a relation of the type

$$\mathcal{A}\left(\dot{\phi},\,\dot{\theta},\,\dot{\varphi}\right)^{t}=\mathcal{B}\,\boldsymbol{\Omega}\tag{26}$$

where $\mathcal{A}\left(\dot{\phi},\,\dot{\theta},\,\dot{\varphi}\right)^{t}=\dot{\phi}\,\boldsymbol{n}+\sin\phi\,\dot{\boldsymbol{n}}$ and we denote

$$2\mathcal{B}=(1+\cos\phi)\,\mathcal{I}+(1-\cos\phi)\,\boldsymbol{n}\otimes\boldsymbol{n}^{t}-\sin\phi\,\boldsymbol{n}^{\times}.$$

Since $\det \mathcal{A} = \sin^2\phi \sin\theta$, for $\phi, \theta \neq k\pi$ one has

$$\mathcal{A}^{-1} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \csc\phi\cos\theta\cos\varphi & \csc\phi\cos\theta\sin\varphi & -\sin\theta \\ -\csc\phi\csc\theta\sin\varphi & \csc\phi\csc\theta\cos\varphi & 0 \end{bmatrix}.$$

The time derivatives of ϕ, θ and φ may then be expressed as linear functions of the angular velocity components, namely

$$\begin{aligned} \dot\phi &= \boldsymbol{n}\,.\,\boldsymbol{\Omega} \\ \dot\theta &= \frac{1}{2}\left(\cot\frac{\phi}{2}\,\frac{\partial\boldsymbol{n}}{\partial\theta} + \csc\theta\frac{\partial\boldsymbol{n}}{\partial\varphi}\right).\,\boldsymbol{\Omega} \\ \dot\varphi &= \frac{1}{2}\left(\cot\frac{\phi}{2}\,\csc^2\theta\,\frac{\partial\boldsymbol{n}}{\partial\varphi} - \csc\theta\frac{\partial\boldsymbol{n}}{\partial\theta}\right).\,\boldsymbol{\Omega} \end{aligned} \tag{27}$$

which yields in particular for the collinear case $\boldsymbol{\Omega} \parallel \boldsymbol{n}$ a constant axis of rotation and $\phi(t) = \int \Omega(t)\,\mathrm{d}t$ with $\Omega = \boldsymbol{n}\,.\,\boldsymbol{\Omega}$ that can be easily derived from formula (24). Similarly, when $\boldsymbol{\Omega} \perp \boldsymbol{n}$ one has a varying axis $\boldsymbol{n}$ and a constant angle ϕ of the compound rotation. Although the above kinematic equations are still rather complicated in the generic case, they provide one major advantage compared to the standard Euler decomposition - namely, the intuitive physical interpretation of the unknown parameters as spherical coordinates of the invariant unit vector $\boldsymbol{n} \in \mathbb{S}^2$ and the compound angle ϕ of the rotation. Furthermore, one may proceed directly with the Euler-Lagrange dynamical equations. However, we restrain from such temptation leaving this matter for a future study.

Next, we note that for a decomposition into n factors

$$\mathcal{R}(\boldsymbol{c}) = \mathcal{R}(\boldsymbol{c}_n)\mathcal{R}(\boldsymbol{c}_{n-1})\ldots\mathcal{R}(\boldsymbol{c}_1)$$

the Leibnitz rule yields a recursive formula

$$\boldsymbol{\Lambda}^{(k)} = \mathcal{R}(\boldsymbol{c}_k)\boldsymbol{\Lambda}^{(k-1)} + \boldsymbol{\Omega}_k, \qquad \boldsymbol{\Lambda}^{(1)} = \boldsymbol{\Omega}_1, \qquad \boldsymbol{\Omega} = \boldsymbol{\Lambda}^{(n)} \tag{28}$$

where $\boldsymbol{\Omega}_k$ denotes the angular velocity with respect to $\hat{\boldsymbol{c}}_k$ in the body system. Similarly, in the inertial one we end up with the expression (see [10] for details)

$$\boldsymbol{\lambda}^{(k)} = \mathcal{R}(-\boldsymbol{c}_k)\boldsymbol{\lambda}^{(k+1)} + \boldsymbol{\omega}_k, \qquad \boldsymbol{\lambda}^{(n)} = \boldsymbol{\omega}_n, \qquad \boldsymbol{\omega} = \boldsymbol{\lambda}^{(1)}.$$

On the other hand, in the decomposition considered in this article, one has $\boldsymbol{c}_5 = -\boldsymbol{c}_1$ and $\boldsymbol{c}_4 = -\boldsymbol{c}_2$ that yields according to the above considerations

$$\boldsymbol{\Omega}_5 = -\boldsymbol{\omega}_1 = -\mathcal{R}_1^t\boldsymbol{\Omega}_1, \qquad \boldsymbol{\Omega}_4 = -\boldsymbol{\omega}_2 = -\mathcal{R}_2^t\boldsymbol{\Omega}_2$$

which we substitute in formula (28) in order to obtain

$$\boldsymbol{\Omega} = \mathcal{R}_1^t\left(\mathcal{R}_2^t\left(\mathcal{R}_3\left(\mathcal{R}_2\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2\right) + \boldsymbol{\Omega}_3 - \boldsymbol{\Omega}_2\right) - \boldsymbol{\Omega}_1\right).$$

In particular, using spherical coordinates, we derive for the ZYZ setting the five angular velocities associated with the decomposition (6) in the form

$$\begin{aligned}
\boldsymbol{\Omega}_1 &= -\begin{bmatrix} \dot{\theta}\sin\varphi \\ \dot{\theta}\,(1-\cos\varphi) \\ \dot{\varphi} \end{bmatrix}, \qquad \boldsymbol{\Omega}_2 = \begin{bmatrix} \dot{\varphi}\sin\theta - \dot{\theta}\,(1-\cos\theta) \\ -\,\dot{\theta} \\ \dot{\varphi}\,(1-\cos\theta) + \dot{\theta}\sin\theta \end{bmatrix} \\
\boldsymbol{\Omega}_3 &= \begin{bmatrix} \dot{\theta}\cos\varphi\sin\phi - \dot{\theta}\sin\varphi\,(1-\cos\theta) \\ \dot{\theta}\sin\varphi\sin\phi + \dot{\theta}\cos\varphi\,(1-\cos\theta) \\ \dot{\phi} \end{bmatrix} \qquad\qquad (29) \\
\boldsymbol{\Omega}_4 &= \begin{bmatrix} \dot{\varphi}\sin\theta + \dot{\theta}\,(1-\cos\varphi) \\ -\,\dot{\theta} \\ \dot{\varphi}\,(1-\cos\varphi) + \dot{\theta}\sin\theta \end{bmatrix}, \qquad \boldsymbol{\Omega}_5 = \begin{bmatrix} \dot{\theta}\,(\sin 2\varphi - \sin\varphi) \\ \dot{\theta}\,(\cos\varphi - \cos 2\varphi) \\ \dot{\varphi} \end{bmatrix}.
\end{aligned}$$

Final Remarks

Above in the text we put the emphasis on the peculiar fact that the unknown parameters in the decomposition in our case coincide (up to a sign) with the spherical coordinates of the invariant vector and the angle of the compound rotation. Apart from the obvious technical convenience our approach suggests, there are some conceptual advantages. In particular, it allows to decompose a specific class of rational rotations (for which both the components of the unit invariant vector and the scalar parameter are rational numbers) into products of such rotations that is not possible in the classical Euler setting unless we impose some additional, highly non-trivial conditions. For a thorough discussion of such rational decompositions we refer to [11]. Note also that in [12] we have considered quite analogous construction for the dual group of the hyperbolic pseudo-rotations $\mathrm{SO}(2,1)$ and its spin cover $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$. On the other side, the extension to the standard Lorentz group $\mathrm{SO}(3,1)$ is not so straightforward due to the invariant axis problem in the Minkowski space $\mathbb{R}^{3,1}$. One possible generalization is directly related to the well-known Wigner decomposition [13]. Analogous techniques can be applied also to the groups $\mathrm{SO}(4)$, $\mathrm{SO}(2,2)$ and $\mathrm{SO}^*(4)$, for which the splitting of the corresponding Lie algebras, namely $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, $\mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ and $\mathfrak{so}^*(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2,1)$ will provide similar parameterizations that deserve further exploration.

References

[1] Chen C., *Application of Algebra of Rotations in Robot Kinematics*, Mech. Mach. Theory **22** (1987) 77-83.

[2] Brezov D., Mladenova C. and Mladenov I., *A Decoupled Solution to the Generalized Euler Decomposition Problem in* $\mathbb{R}^3$ *and* $\mathbb{R}^{2,1}$, J. Geom. Symmetry Phys. **33** (2014) 47-78, doi: 10.7546/jgsp-33-2014-47-78.

[3] Piña E., *A New Parametrization of the Rotation Matrix*, Am. J. Phys. **51** (1983) 375-379.

[4] Brezov D., Mladenova C. and Mladenov I., *On the Necessary and Sufficient Condition for the Decomposition of Orthogonal Transformations*, AIP Conf. Proc. **1690** (2015) 050003-6pp., doi: 10.1063/1.4936733.

[5] Kuipers J., *Quaternions and Rotation Sequences*, Geom. Integrability & Quantization **1** (2000) 127-143, doi:10.7546/giq-1-2000-127-143.

[6] Mladenova C., *Group-Theoretical Methods in Manipulator Kinematics and Symbolic Computations*, J. Intell. Robot. Syst. **8** (1993) 21-34.

[7] Hamada M., *The Minimum Number of Rotations About Two Axes for Constructing an Arbitrarily Fixed Rotation*, R. Soc. Open Sci. **1** (2014) 140-145.

[8] Mladenova C., *An Approach to Description of a Rigid Body Motion*, C. R. Acad. Sci. Bulg. **38** (1985) 1657-1660.

[9] Piña E., *Rotations with Rodrigues' Vector*, Eur. J. Phys. **32** (2011) 1171-1178.

[10] Henderson D., *Recursive Relationships for Body Axis Rotation Rates*, Journal of Guidance, Control and Dynamics **6** (1983) 318-320.

[11] Brezov D., Mladenova C. and Mladenov I., *The Geometry of Pythagorean Quadruples and Rational Decomposition of Pseudo-Rotations*, In: Mathematics in Industry, Cambridge Scholars Publishing, Newcastle upon Tyne 2014, pp 176-197, doi: 10.13140/RG.2.1.4784.4884.

[12] Brezov D., Mladenova C. and Mladenov I., *Factorizations in Special Relativity and Quantum Scattering on the Line*, In: Advanced Computing in Industrial Mathematics, Springer, Berlin 2016, (in press).

[13] Wigner E., *On Unitary Representations of the Inhomogeneous Lorentz Group* Ann. Math. **40** (1939) 149-204.