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EXTENDED HARMONIC MAPPINGS AND EULER-LAGRANGE EQUATIONS

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Abstract. Via the Lagrangian formalism, an example of extended harmonic CMC immersion and conservation laws are obtained.

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1. Introduction

We consider the Lagrangian formalism, where Lagrangians have the potential functions. In the previous papers [1–3], the periodicity of some families of S^1-equivariant CMC (constant mean curvature) surfaces in the Berger sphere or the hyperbolic three-space was proved by making use of the conservation laws, in particular, we find that the potential functions of Lagrangians which correspond to S^1-equivariant CMC-H surfaces contain the constant mean curvature H itself (see §3 and also [4]). Throughout the paper, we consider some extended harmonic mappings via Euler-Lagrange equations (Propositions 1 and 5). The extended harmonic mapping can be considered as a natural extension of harmonic mapping, since the potential function of corresponding Lagrangian to harmonic mapping is vanishing. We give examples of extended harmonic mapping and extended harmonic CMC-H immersion (§3). By using the conservation laws (Theorems 2, 3 and §6) with respect to the Hamiltonians, we investigate a certain geometric relationship between an extended harmonic mapping and a smooth mapping with vanishing tension field (Theorems 4 and 6).

2. Euler-Lagrange Equations

Let $\phi : (M, g) \to (N, h)$ be a smooth mapping, where (M, g) and (N, h) are Riemannian manifolds of dimension two and three with Riemannian metrics g and h, respectively. Then we consider the following Lagrangian of ϕ

$$L_\phi = \frac{1}{2} \sum_{i,j=1}^{2} \sum_{\alpha,\beta=1}^{3} g^{ij} \partial_i \phi^\alpha \partial_j \phi^\beta h_{\alpha\beta}(\phi) - G(\phi) \tag{1}$$

where $\phi^\alpha := y^\alpha \circ \phi$, $\alpha = 1, 2, 3$, and (x^1, x^2), (y^1, y^2, y^3) are local coordinate systems on M, respectively N, $\partial_1 \phi^\alpha$ and $\partial_2 \phi^\alpha$ denote the partial derivatives $\frac{\partial}{\partial x^1}\phi^\alpha$ and $\frac{\partial}{\partial x^2}\phi^\alpha$, and we will make use of the notation

$$g := \sum_{i,j=1}^{2} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

$$h(\phi)(x) := \sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(\phi(x))(\mathrm{d}y^\alpha)_{\phi(x)} \otimes (\mathrm{d}y^\beta)_{\phi(x)}$$

$$G(\phi) := G \circ \phi, \qquad G \in C^\infty(N).$$

The formula (1) does not depend on the way we choose local coordinates systems of M and N, since the first term of the right hand side of (1) means $\frac{1}{2}\mathrm{trace}_g(\phi^* h)$. Then we can define the generalized (canonical) momenta

$$p^i_\alpha := \frac{\partial L_\phi}{\partial(\partial_i \phi^\alpha)}, \qquad i = 1, 2, \qquad \alpha = 1, 2, 3.$$

Here p^i_α and $\partial_i \phi^\alpha$ can be regarded as the components of tensor fields. Under the transformations of the local coordinate systems: $(x^1, x^2) \to (\tilde{x^1}, \tilde{x^2})$ and $(y^1, y^2, y^3) \to (\tilde{y^1}, \tilde{y^2}, \tilde{y^3})$, we have

$$\tilde{\partial}_j \tilde{\phi}^\alpha = \sum_{i=1}^{2} \sum_{\beta=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{y}^\alpha}{\partial y^\beta}(\phi)\, \partial_i \phi^\beta, \qquad \tilde{p}^i_\alpha = \sum_{j=1}^{2} \sum_{\beta=1}^{3} \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial y^\beta}{\partial \tilde{y}^\alpha}(\phi)\, p^j_\beta.$$

Thus we have the tensor fields $\mathrm{d}\phi$ ([5]) and p

$$(\mathrm{d}\phi)(x) = \sum_{1=1}^{2} \sum_{\alpha=1}^{3} \partial_i \phi^\alpha (\mathrm{d}x^i)_x \otimes (\frac{\partial}{\partial y^\alpha})_{\phi(x)}$$

$$p(x) = \sum_{i=1}^{2} \sum_{\alpha=1}^{3} p^i_\alpha (\frac{\partial}{\partial x^i})_x \otimes (\mathrm{d}y^\alpha)_{\phi(x)}, \qquad x \in M.$$

Then the Lagrangian (1) of ϕ implies that

$$p_\alpha^i = \sum_{j=1}^{2}\sum_{\beta=1}^{3} g^{ij}\partial_j\phi^\beta h_{\alpha\beta}(\phi).$$

Proposition 1. *Let (M, g) be the Euclidean plane $(\mathbb{R}^2, g_0)$, where g_0 is the standard metric on $\mathbb{R}^2$. Then, under the Lagrangian (1) of $\phi : (\mathbb{R}^2, g_0) \to (N, h)$, the statements a) and b) below are equivalent*

a) *Euler-Lagrange equations*

$$\sum_{i=1}^{2}\partial_i p_\alpha^i - \frac{\partial L_\phi}{\partial \phi^\alpha} = 0, \qquad \alpha = 1, 2, 3 \tag{2}$$

b)

$$\tau_\phi = -\operatorname{grad}_h G(\phi) \tag{3}$$

where τ_ϕ stands for the tension field of ϕ ([6]) and

$$\operatorname{grad}_h G(\phi)(x) = \sum_{\alpha,\beta=1}^{3} h^{\alpha\beta}(\phi(x))(\frac{\partial G(\phi)}{\partial\phi^\alpha})(x)(\frac{\partial}{\partial y^\beta})_{\phi(x)}, \qquad x \in \mathbb{R}^2.$$

Proof: The formula (1) of the Lagrangian of ϕ implies that L_ϕ can be expressed as $L_\phi = L_\phi(\,\phi,\, \mathrm{d}\phi)$, then we have

$$\begin{aligned}\sum_{i=1}^{2}\partial_i p_\gamma^i &= \sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i(\partial_i\phi^\alpha h_{\alpha\gamma}(\phi)) \\ &= \sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i^2\phi^\alpha h_{\alpha\gamma}(\phi) + \sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\partial_i\phi^\alpha\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^\beta}\partial_i\phi^\beta\end{aligned}$$

and

$$\frac{\partial L_\phi}{\partial\phi^\gamma} = \frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\partial_i\phi^\alpha\partial_i\phi^\beta\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^\gamma} - \frac{\partial G(\phi)}{\partial\phi^\gamma}.$$

Then we have

$$\begin{aligned}&\sum_{i=1}^{2}\partial_i p_\gamma^i - \frac{\partial L_\phi}{\partial\phi^\gamma} \\ &= \sum_{i=1}^{2}\sum_{\alpha=1}^{3}\partial_i^2\phi^\alpha h_{\alpha\gamma}(\phi) + \sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}(\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^\beta} - \frac{1}{2}\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^\gamma})\partial_i\phi^\alpha\partial_i\phi^\beta + \frac{\partial G(\phi)}{\partial\phi^\gamma}.\end{aligned}$$

On the other hand, we have as well

$$\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^\alpha\partial_i\phi^\beta=\sum_{i=1}^{2}\sum_{\alpha,\beta,\gamma=1}^{3}h^{\mu\gamma}(\phi)(\frac{\partial h_{\alpha\gamma}(\phi)}{\partial\phi^\beta}-\frac{1}{2}\frac{\partial h_{\alpha\beta}(\phi)}{\partial\phi^\gamma})\partial_i\phi^\alpha\partial_i\phi^\beta$$

where $\Gamma^{\mu}_{\alpha\beta}$ denote the coefficients of Levi-Civita connection of (N,h). As a consequence we have

$$\sum_{\gamma=1}^{3}(\sum_{i=1}^{2}\partial_i p^i_\gamma-\frac{\partial L_\phi}{\partial\phi^\gamma})h^{\mu\gamma}(\phi)$$
$$=\sum_{i=1}^{2}\partial_i^2\phi^\mu+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^\alpha\partial_i\phi^\beta+\sum_{\gamma=1}^{3}h^{\gamma\mu}(\phi)\frac{\partial G(\phi)}{\partial\phi^\gamma}\cdot$$

Finally, since

$$\tau_\phi=\sum_{\mu=1}^{3}(\sum_{i=1}^{2}\partial_i^2\phi^\mu+\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3}\Gamma^{\mu}_{\alpha\beta}(\phi)\partial_i\phi^\alpha\partial_i\phi^\beta)(\frac{\partial}{\partial y^\mu})_\phi$$

and

$$\operatorname{grad}_h G(\phi)=\sum_{\gamma,\mu=1}^{3}h^{\gamma\mu}(\phi)\frac{\partial G(\phi)}{\partial\phi^\gamma}(\frac{\partial}{\partial y^\mu})_\phi$$

we obtain

$$\tau_\phi+\operatorname{grad}_h G(\phi)=\sum_{\gamma,\mu}^{3}(\sum_{i=1}^{2}\partial_i p^i_\gamma-\frac{\partial L_\phi}{\partial\phi^\gamma})h^{\mu\gamma}(\phi)(\frac{\partial}{\partial y^\mu})_\phi$$

from which, it is proved that $a)$ and $b)$ are equivalent. ■

Let ϕ be as in Proposition 1. In this paper, if the tension field τ_ϕ of ϕ is given by the formula (3) for some $G\in C^\infty(N)$, then such a smooth mapping ϕ is called an extended harmonic mapping and $G(\phi)$ the potential function associated with ϕ. When we give an extended harmonic mapping ϕ such that the associated potential function is $G(\phi)=G\circ\phi$, we always consider the Lagrangian (1) and the corresponding Euler-Lagrange equations (2) throughout the paper. In particular, ϕ is called an extended harmonic immersion, if ϕ is an extended harmonic mapping and an immersion.

3. Extended Harmonic Mapping

Let $\phi:(\mathbb{R}^2,g_0)\to(H^3(-1),h)$ be an extended harmonic mapping with the associated potential function $G(\phi)$, where h stands for the following Riemannian metric on the hyperbolic three-space $H^3(-1)$ of constant curvature -1

$$\sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta = \mathrm{d}y^1 \otimes \mathrm{d}y^1 + \cosh^2 y^1\, \mathrm{d}y^2 \otimes \mathrm{d}y^2 + \cosh^2 y^1 \cosh^2 y^2\, \mathrm{d}y^3 \otimes \mathrm{d}y^3$$

under a suitable parameterization of $H^3(-1)$.
Then, from (1), we have

$$\begin{aligned} L_\phi = & \frac{1}{2}(((\partial_1\theta)^2 + (\partial_2\theta)^2 + ((\partial_1\varphi)^2 + (\partial_2\varphi)^2)\cosh^2\theta \\ & +((\partial_1\psi)^2 + (\partial_2\psi)^2)\cosh^2\theta\cosh^2\varphi) - G(\phi) \end{aligned} \tag{4}$$

where $\theta = \phi^1$, $\varphi = \phi^2$, $\psi = \phi^3$, $\partial_1\theta = \frac{\partial}{\partial x^1}\theta$, $\partial_2\theta = \frac{\partial}{\partial x^2}\theta$, etc.
By using Lagrangian (4) and Euler-Lagrange equations (2), we have

$$\begin{aligned} \frac{\partial G(\phi)}{\partial\theta} = & -\Delta\theta + \frac{1}{2}g_0(\mathrm{grad}\ \varphi, \mathrm{grad}\ \varphi)\sinh 2\theta \\ & +\frac{1}{2}g_0(\mathrm{grad}\ \psi, \mathrm{grad}\ \psi)\sinh 2\theta\cosh^2\varphi \\ \frac{\partial G(\phi)}{\partial\varphi} = & -\cosh^2\theta\Delta\varphi + \frac{1}{2}g_0(\mathrm{grad}\ \psi, \mathrm{grad}\ \psi)\cosh^2\theta\sinh 2\varphi \\ & -g_0(\mathrm{grad}\ \theta, \mathrm{grad}\ \varphi)\sinh 2\theta \\ \frac{\partial G(\phi)}{\partial\psi} = & -\cosh^2\theta\cosh^2\varphi\Delta\psi - g_0(\mathrm{grad}\ \theta, \mathrm{grad}\ \psi)\sinh 2\theta\cosh^2\varphi \\ & -g_0(\mathrm{grad}\ \varphi, \mathrm{grad}\ \psi)\sinh 2\varphi\cosh^2\theta \end{aligned}$$

where Δ and grad stand for the Laplacian and the gradient operator on $(\mathbb{R}^2, g_0)$, respectively.
As usually θ, φ and ψ are called cyclic coordinates with respect to the Lagrangian (4), if

$$\frac{\partial L_\phi}{\partial\theta} = \frac{\partial L_\phi}{\partial\varphi} = \frac{\partial L_\phi}{\partial\psi} = 0$$

which does not depend on the way to choose a local coordinate system on $H^3(-1)$.
If θ, φ and ψ are cyclic coordinates, then we have

$$\begin{aligned} \frac{\partial G(\phi)}{\partial\theta} = & \frac{1}{2}(g_0(\mathrm{grad}\ \varphi, \mathrm{grad}\ \varphi) + g_0(\mathrm{grad}\ \psi, \mathrm{grad}\ \psi)\cosh^2\varphi)\sinh 2\theta \\ \frac{\partial G(\phi)}{\partial\varphi} = & \frac{1}{2}g_0(\mathrm{grad}\ \psi, \mathrm{grad}\ \psi)\cosh^2\theta\sinh 2\varphi, \qquad \frac{\partial G(\phi)}{\partial\psi} = 0. \end{aligned} \tag{5}$$

Assume that θ, φ and ψ are cyclic coordinates with respect to L_ϕ. Then, since $\tau_\phi = -\operatorname{grad}_h G(\phi)$, from (5), we have

$$\begin{aligned}\tau_\phi = &-\frac{1}{2}(g_0(\operatorname{grad}\varphi, \operatorname{grad}\varphi) + g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\cosh^2\varphi)\sinh 2\theta\,(\frac{\partial}{\partial y^1})_\phi \\ &-\frac{1}{2}g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\sinh 2\varphi\,(\frac{\partial}{\partial y^2})_\phi.\end{aligned} \tag{6}$$

Let $\phi : (\mathbb{R}^2, g_0) \to (H^3(-1), h)$ be an extended harmonic CMC (constant mean curvature) - H immersion with associated potential function $G(\phi) = G \circ \phi$, where H stands for the constant mean curvature of ϕ and we assume that H is a positive constant. Then $h(\phi)(\tau_\phi, \tau_\phi) = 4H^2$. By using (6), we have

$$\begin{aligned}h(\phi)(\tau_\phi, \tau_\phi) = &\frac{1}{4}(g_0(\operatorname{grad}\varphi, \operatorname{grad}\varphi) + g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\cosh^2\varphi)^2\sinh^2 2\theta \\ &+\frac{1}{4}g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)^2\sinh^2 2\varphi\cosh^2\theta\end{aligned}$$

from which, we have

$$\begin{aligned}&(g_0(\operatorname{grad}\varphi, \operatorname{grad}\varphi) + g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\cosh^2\varphi)^2\sinh^2 2\theta \\ &\qquad + g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)^2\sinh^2 2\varphi\cosh^2\theta = 16H^2.\end{aligned}$$

Hence we can take the parameter function $\rho = \rho(x^1, x^2)$ such that

$$(g_0(\operatorname{grad}\varphi, \operatorname{grad}\varphi) + g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\cosh^2\varphi)\sinh 2\theta = 4H\cos\rho$$

and

$$g_0(\operatorname{grad}\psi, \operatorname{grad}\psi)\sinh 2\varphi\cosh\theta = 4H\sin\rho.$$

Then, under the assumption of cyclic coordinates, we can choose the associated potential function $G(\phi)$ as follows

$$G(\phi) = 2H(\int\cos\rho\,\mathrm{d}\theta + \int\cosh\theta\sin\rho\,\mathrm{d}\varphi).$$

Consequently, the associated potential function $G(\phi)$ contains the constant mean curvature H itself.

4. Hamiltonians and Conservation Laws

Let $\phi : (\mathbb{R}^2, g_0) \to (N, h)$ $(\dim N = 3)$ be an extended harmonic mapping with the associated potential function $G(\phi)$. Then we define the Hamiltonians $H_\phi^{(1)}$ and

$H_\phi^{(2)}$ with respect to ϕ

$$H_\phi^{(1)} := \sum_{\alpha=1}^{3} \partial_1\phi^\alpha p_\alpha^1 - L_\phi(\phi, \mathrm{d}\phi)$$

$$H_\phi^{(2)} := \sum_{\alpha=1}^{3} \partial_2\phi^\alpha p_\alpha^2 - L_\phi(\phi, \mathrm{d}\phi)$$

where $L_\phi = L_\phi(\phi, \mathrm{d}\phi)$ is given by the Lagrangian

$$L_\phi = \frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{3} \partial_i\phi^\alpha\partial_i\phi^\beta h_{\alpha\beta}(\phi) - G(\phi)$$

Then we have

$$\partial_1 H_\phi^{(1)} = \sum_{\alpha=1}^{3} \partial_1^2\phi^\alpha p_\alpha^1 + \sum_{\alpha=1}^{3} \partial_1\phi^\alpha\partial_1 p_\alpha^1 - \sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial\phi^\alpha}\partial_1\phi^\alpha - \sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial(\partial_1\phi^\alpha)}\partial_1^2\phi^\alpha - \sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial(\partial_2\phi^\alpha)}\partial_1\partial_2\phi^\alpha.$$

Hence, by using Euler-Lagrange equations (2) and the formula

$$p_\alpha^i = \sum_{\beta=1}^{3} \partial_i\phi^\beta h_{\alpha\beta}(\phi), \qquad i = 1, 2, \qquad \alpha = 1, 2, 3$$

we have

$$\begin{aligned} \partial_1 H_\phi^{(1)} &= \sum_{\alpha=1}^{3}\left(\frac{\partial L_\phi}{\partial\phi^\alpha} - \partial_2 p_\alpha^2\right)\partial_1\phi^\alpha - \sum_{\alpha=1}^{3}\frac{\partial L_\phi}{\partial\phi^\alpha}\partial_1\phi^\alpha - \sum_{\alpha=1}^{3} p_\alpha^2\partial_1\partial_2\phi^\alpha \\ &= -\sum_{\alpha=1}^{3}\partial_2(\partial_1\phi^\alpha p_\alpha^2) = -\partial_2\left(h(\phi)\left(\phi_*\left(\frac{\partial}{\partial x^1}\right), \phi_*\left(\frac{\partial}{\partial x^2}\right)\right)\right). \end{aligned} \tag{7}$$

Similarly we have

$$\partial_2 H_\phi^{(2)} = -\partial_1\left(h(\phi)\left(\phi_*\left(\frac{\partial}{\partial x^2}\right), \phi_*\left(\frac{\partial}{\partial x^1}\right)\right)\right). \tag{8}$$

Thus we have

Theorem 2 (Conservation laws)**.** *Let $\phi : (\mathbb{R}^2, g_0) \to (N, h)$ be an extended harmonic mapping with associated potential function $G(\phi)$ and assume additionally that $h(\phi)(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2}))$ is constant as a smooth function on $\mathbb{R}^2$. Then*

$$\partial_1 H_\phi^{(1)} = \partial_2 H_\phi^{(2)} = 0.$$

On the other hand, the direct computation implies that

$$H_\phi^{(1)} = \frac{1}{2}(h(\phi)(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1})) - h(\phi)(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2}))) + G(\phi)$$

$$H_\phi^{(2)} = \frac{1}{2}(h(\phi)(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2})) - h(\phi)(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1}))) + G(\phi).$$

Furthermore, if ϕ is conformal as a smooth mapping between Riemannian manifolds, then there exists a positive smooth function σ on $\mathbb{R}^2$ such that

$$h(\phi)(\phi_*(\frac{\partial}{\partial x^i}), \phi_*(\frac{\partial}{\partial x^j})) = \sigma g_0(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \qquad 1 = 1, 2.$$

Hence, by using (7), (8) (Theorem 2), we have

Theorem 3. *Let the extended harmonic mapping* $\phi : (\mathbb{R}^2, g_0) \to (N, h)$ *be such that the associated potential function* $G(\phi)$ *is conformal as a smooth mapping between Riemannian manifolds. Then*

a) (*conservation laws*)

$$\partial_1 H_\phi^{(1)} = \partial_2 H_\phi^{(2)} = 0$$

b)

$$H_\phi^{(1)} = H_\phi^{(2)} = G(\phi).$$

Under the assumption of Theorem 3, we have

$$0 = \sum_{i=1}^{2} \partial_i G(\phi) \frac{\partial}{\partial x^i} = \sum_{i=1}^{2} \sum_{\alpha=1}^{3} \frac{\partial G(\phi)}{\partial \phi^\alpha} \partial_i \phi^\alpha \frac{\partial}{\partial x^i} = \sum_{\alpha=1}^{3} \frac{\partial G(\phi)}{\partial \phi^\alpha} \operatorname{grad} \phi^\alpha$$

and

$$\tau_\phi = - \operatorname{grad}_h G(\phi) = - \sum_{\alpha,\beta=1}^{3} h^{\alpha\beta}(\phi) \frac{\partial G(\phi)}{\partial \phi^\alpha} (\frac{\partial}{\partial y^\beta})_\phi.$$

Consequently, we have

Theorem 4. *Let* (N, h) *be a Riemannian manifold* $(\dim N = 3)$ *and* $\phi : (\mathbb{R}^2, g_0) \to (N, h)$ *be an extended harmonic mapping with associated potential function* $G(\phi) = G \circ \phi$ *and assume that* ϕ *is conformal as a mapping between Riemannian manifolds. If* $\operatorname{grad} \phi^1$, $\operatorname{grad} \phi^2$ *and* $\operatorname{grad} \phi^3$ *are linearly independent at each point on* $(\mathbb{R}^2, g_0)$, *where this linear independence does not depend on the way we choose a local coordinate system on* N, *the tension field* τ_ϕ *vanishes.*

5. Complex Lagrangian

Let $\phi : (\mathbb{C}, g_0) \to (N, h)$ be a holomorphic mapping, where $(\mathbb{C}, g_0)$ is the 1-dimensional complex Euclidean space ([6]) with metric $g_0 := \mathrm{Re}(\mathrm{d}z \otimes \mathrm{d}\bar{z}) = \sum_{i=1}^{2} \mathrm{d}x^i \otimes \mathrm{d}x^i$, where $z = x^1 + \sqrt{-1}x^2$ stands for the standard coordinate of $\mathbb{C}$ and (N, h) is an n-dimensional complex manifold with Hermitian metric h, respectively.

We consider the following Lagrangian of ϕ

$$L_\phi = \sum_{i=1}^{2} \sum_{\alpha,\beta=1}^{n} \partial_i \phi^\alpha \partial_i \bar{\phi}^\beta h_{\alpha\bar{\beta}}(\phi) - G(\phi) \tag{9}$$

where $\phi^\alpha := \zeta^\alpha \circ \phi$, $\bar{\phi}^\alpha := \bar{\zeta}^\alpha \circ \phi$, $\alpha = 1, ..., n$ and $(\zeta^1, ..., \zeta^n)$ is a complex local coordinate system on N, and $G(\phi) = G \circ \phi$, G is a complex valued smooth function on N.

Note that we can represent a complex vector field $\phi_*(\frac{\partial}{\partial x^i})$, $i = 1, 2$ as follows

$$\phi_*(\frac{\partial}{\partial x^i}) = \sum_{\alpha=1}^{n} \partial_i \phi^\alpha (\frac{\partial}{\partial \zeta^\alpha})_\phi + \sum_{\alpha=1}^{n} \partial_i \bar{\phi}^\alpha (\frac{\partial}{\partial \bar{\zeta}^\alpha})_\phi, \qquad i = 1, 2.$$

We define

$$\mathrm{d}\phi^\alpha := \sum_{i=1}^{2} \partial_i \phi^\alpha \mathrm{d}x^i, \qquad \mathrm{d}\bar{\phi}^\alpha := \sum_{i=1}^{2} \partial_i \bar{\phi}^\alpha \mathrm{d}x^i, \qquad \alpha = 1, ..., n.$$

Moreover, we can define the generalized momenta

$$p^i_\gamma := \frac{\partial L_\phi}{\partial(\partial_i \phi^\gamma)}, \qquad \bar{p}^i_\gamma := \frac{\partial L_\phi}{\partial(\partial_i \bar{\phi}^\gamma)}, \qquad i = 1, 2, \qquad \gamma = 1, ..., n.$$

Then we have

$$p^i_\gamma = \sum_{\alpha=1}^{n} \partial_i \bar{\phi}^\alpha h_{\gamma\bar{\alpha}}(\phi), \qquad \bar{p}^i_\gamma = \sum_{\alpha=1}^{n} \partial_i \phi^\alpha h_{\alpha\bar{\gamma}}(\phi)$$

$$\sum_{\gamma,\mu=1}^{n} (\sum_{i=1}^{2} \partial_i p^i_\gamma - \frac{\partial L_\phi}{\partial \phi^\gamma}) h^{\gamma\bar{\mu}} (\frac{\partial}{\partial \bar{\zeta}^\mu})_\phi = \tau^{(-)}_\phi + \mathrm{grad}^{(-)}_h G(\phi)$$

$$\sum_{\gamma,\mu=1}^{n} (\sum_{i=1}^{2} \partial_i \bar{p}^i_\gamma - \frac{\partial L_\phi}{\partial \bar{\phi}^\gamma}) h^{\bar{\gamma}\mu} (\frac{\partial}{\partial \zeta^\mu})_\phi = \tau^{(+)}_\phi + \mathrm{grad}^{(+)}_h G(\phi)$$

where

$$\mathrm{grad}^{(+)}_h G(\phi) := \sum_{\gamma,\mu=1}^{n} h^{\gamma\bar{\mu}} \frac{\partial G(\phi)}{\partial \bar{\phi}^\mu} (\frac{\partial}{\partial \zeta^\gamma})_\phi$$

$$\operatorname{grad}_h^{(-)} G(\phi) := \sum_{\gamma.\mu=1}^{n} h^{\gamma\bar{\mu}} \frac{\partial G(\phi)}{\partial \phi^\gamma} (\frac{\partial}{\partial \bar{\zeta}^\mu})_\phi$$

and

$$\tau_\phi^{(+)} := \sum_{i=1}^{2}\sum_{\gamma=1}^{n}(\partial_i^2\phi^\gamma + \sum_{\alpha,\beta=1}^{n}\Gamma_{\alpha\beta}^{\gamma}(\phi)\partial_i\phi^\alpha\partial_i\phi^\beta + 2\sum_{\alpha,\beta=1}^{n}\Gamma_{\alpha\bar{\beta}}^{\gamma}(\phi)\partial_i\phi^\alpha\partial_i\bar{\phi}^\beta)(\frac{\partial}{\partial\zeta^\gamma})_\phi$$

$$\tau_\phi^{(-)} := \sum_{i=1}^{2}\sum_{\gamma=1}^{n}(\partial_i^2\bar{\phi}^\gamma + \sum_{\alpha,\beta=1}^{n}\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}(\phi)\partial_i\bar{\phi}^\alpha\partial_i\bar{\phi}^\beta + 2\sum_{\alpha,\beta=1}^{n}\Gamma_{\alpha\bar{\beta}}^{\bar{\gamma}}(\phi)\partial_i\phi^\alpha\partial_i\bar{\phi}^\beta)(\frac{\partial}{\partial\bar{\zeta}^\gamma})_\phi$$

where $\Gamma_{..}^{\cdot}$ stands for the coefficients of torsion-free affine connection of (N, h), and the tension field τ_ϕ of ϕ is defined as follows

$$\tau_\phi = \sum_{i,j=1}^{2} g_0^{ij}\hat{\nabla}_{\frac{\partial}{\partial x^i}}\phi_*(\frac{\partial}{\partial x^j})$$

since g_0 is the flat metric, where $\hat{\nabla}$ stands for the induced connection on the induced bundle $\phi^{-1}TN$ ([6]). Then $\tau_\phi = \tau_\phi^{(+)} + \tau_\phi^{(-)}$.

Proposition 5. *The following conditions* $a)$ *and* $b)$ *are equivalent.*

a)

$$\sum_{i=1}^{2}\partial_i p_\gamma^i - \frac{\partial L_\phi}{\partial \phi^\gamma} = 0, \qquad \sum_{i=1}^{2}\partial_i \bar{p}_\gamma^i - \frac{\partial L_\phi}{\partial \bar{\phi}^\gamma} = 0, \qquad \gamma = 1, ..., n.$$

b)

$$\tau_\phi^{(-)} = -\operatorname{grad}_h^{(-)} G(\phi), \qquad \tau_\phi^{(+)} = -\operatorname{grad}_h^{(+)} G(\phi).$$

6. Complex Hamiltonians and Conservation Laws

The Euler-Lagrange equations $a)$ as in Proposition 5 are equivalent to

$$\tau_\phi = -\operatorname{grad}_h G(\phi)$$

where

$$\operatorname{grad}_h G(\phi) := \operatorname{grad}_h^{(+)} G(\phi) + \operatorname{grad}_h^{(-)} G(\phi)$$

such a ϕ is called an extended harmonic mapping.

In the following, we define the Hamiltonians of ϕ

$$H_\phi^{(i)} := \sum_{\alpha=1}^{n}\partial_i\phi^\alpha p_\alpha^i + \sum_{\alpha=1}^{n}\partial_i\bar{\phi}^\alpha\bar{p}_\alpha^i - L_\phi, \qquad i = 1, 2$$

where L_ϕ is given by the formula (9).

Then, from the direct computation, we have

$$\partial_1 H_\phi^{(1)} = -\,\partial_2\, h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2}))$$

$$\partial_2 H_\phi^{(2)} = -\,\partial_1\, h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2}))$$

furthermore, from the definitions of Hamiltonians, we obtain

$$H_\phi^{(1)} = \frac{1}{2}(h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1})) - h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2}))) + G(\phi)$$

$$H_\phi^{(2)} = \frac{1}{2}(h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2})) - h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1}))) + G(\phi).$$

Consequently, if ϕ has the conformal properties such as

$$h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^2})) = 0$$

and

$$h_\phi(\phi_*(\frac{\partial}{\partial x^1}), \phi_*(\frac{\partial}{\partial x^1})) = h_\phi(\phi_*(\frac{\partial}{\partial x^2}), \phi_*(\frac{\partial}{\partial x^2})) \tag{10}$$

then we have the conservation laws:

$$\partial_1 H_\phi^{(1)} = \partial_2 H_\phi^{(2)} = 0, \qquad H_\phi^{(1)} = H_\phi^{(2)} = G(\phi).$$

Then, under the assumption of the following theorem, we have

$$0 = \sum_{i=1}^{2} \partial_i G(\phi) dx^i = \sum_{i=1}^{2}\sum_{\alpha=1}^{n}(\frac{\partial G(\phi)}{\partial \phi^\alpha}\partial_i \phi^\alpha \mathrm{d}x^i + \frac{\partial G(\phi)}{\partial \bar{\phi}^\alpha}\partial_i \bar{\phi}^\alpha \mathrm{d}x^i)$$

then

$$\sum_{\alpha=1}^{n} \frac{\partial G(\phi)}{\partial \phi^\alpha}\mathrm{d}\phi^\alpha + \sum_{\alpha=1}^{n} \frac{\partial G(\phi)}{\partial \bar{\phi}^\alpha}\mathrm{d}\bar{\phi}^\alpha = 0$$

and

$$\tau_\phi = -\ \mathrm{grad}_h\, G(\phi) = -\sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}}(\phi)(\frac{\partial G(\phi)}{\partial \bar{\phi}^\beta}(\frac{\partial}{\partial \zeta^\alpha})_\phi + \frac{\partial G(\phi)}{\partial \phi^\alpha}(\frac{\partial}{\partial \bar{\zeta}^\beta})_\phi).$$

Hence, we have

Theorem 6. *Assume that $\phi : (\mathbb{C}, g_0) \to (N, h)$ is an extended harmonic, holomorphic mapping equipped with potential function $G(\phi) = G \circ \phi$ with respect to the Lagrangian* (9)*, and assume ϕ has the conformal properties* (10)*. If $\mathrm{d}\phi^1$,..., $\mathrm{d}\phi^n$, $\mathrm{d}\bar{\phi}^1$,...,$\mathrm{d}\bar{\phi}^n$ are linearly independent over $\mathbb{C}$ (where this linearly independency does not depend on the way to choose a complex local coordinate system on N), then the tension field τ_ϕ of ϕ vanishes.*

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