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FUNCTIONALS ON TOROIDAL SURFACES

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Abstract. We show that the torus in $\mathbb{R}^3$ is a critical point of a sequence of functionals $\mathcal{F}_n$ ($n = 1, 2, 3, \ldots$) defined over compact surfaces (closed membranes) in $\mathbb{R}^3$. When the Lagrange function $\mathcal{E}$ is a polynomial of degree n of the mean curvature H of the torus, the radii (a, r) of the torus are constrained to satisfy $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n \geq 2$. A simple generalization of torus in $\mathbb{R}^3$ is a tube of radius r along a curve α which we call it toroidal surface (TS). We show that toroidal surfaces with non-circular curve α do not provide minimal energy surfaces of the functionals $\mathcal{F}_n$ ($n = 2, 3$) on closed surfaces. We discuss possible applications of the functionals discussed in this work on cell membranes.

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1. Introduction

In the history of differential geometry there are some special subclasses of surfaces in $\mathbb{R}^3$, such as surfaces of constant Gaussian curvature, surfaces of constant mean curvature, minimal surfaces and the Willmore surfaces. These surfaces arise in many different branches of sciences. In particular, in various parts of theoretical physics (string theory, general theory of relativity), cell-biology and differential geometry [2,4–8,11–15,19–25]. All these special surfaces constitute critical points of certain functionals. Euler-Lagrange equations of these functionals are very complicated and difficult. There are some techniques developed to solve them, such as using the deformation of the Lax equations of the integrable equations so that it is possible to construct surfaces in $\mathbb{R}^3$ [4, 6, 7, 18] solving the Euler-Lagrange equations.

The main objective in our work is to investigate surfaces derivable from a variational principle, such as the minimal and Willmore surfaces [23–25] and surfaces solving the shape equation [11–14, 19–22]. All these surfaces are critical points of a functional where the Lagrange function is a polynomial of degree less or equal to two in the mean curvature of the surface. It is natural to ask whether there are surfaces solving the Euler-Lagrange equations corresponding to more general Lagrange functions depending on the mean and Gaussian curvatures of the surface [20–22]. It is the purpose of this work to give an answer to such a question.

The quadratic Helfrich functional [8] for a theoretical model of a closed cell-membrane is

$$\mathcal{F} = \frac{1}{2}\int_S (k_c\,(2\,H + c_0)^2 + 2\,w)\,\mathrm{d}A + p\int_V \mathrm{d}V \tag{1}$$

where k_c is the elasticity constant, H and c_0 are the mean and the spontaneous curvatures, w is the surface tension and p is the pressure difference between in and out of the surface. First variation of the above functional gives the shape equation

$$2\,k_c\,\nabla^2\,H + k_c\,(2H^2 - c_0H - 2K)(2H + c_0) + p - 2wH = 0 \tag{2}$$

Sphere with an arbitrary radius is an exact solution of this equation. The radius of the sphere is related to the model parameters k_c, c_0, w and p through the shape equation (2). Stability of this solution has been studied in [12]. A special torus, called the Clifford torus, is also an exact solution of (2) (see [20]).

In this work we consider a generalization of the Helfrich's functional (1). Let $\mathcal{F}_n = \int_S \mathcal{E}_n\,\mathrm{d}A + p\int_V \mathrm{d}V$, $n \geq 2$

$$\mathcal{E}_n = \sum_{k=0}^{n} a_{n+1-k}\,H^k = a_1\,H^n + a_2\,H^{n-1} + \cdots + a_n\,H + a_{n+1} \tag{3}$$

where $n = 1, 2, \ldots$ and a_k's are constants which can be considered to be related to the parameters of a cell-membrane model. Hence it is worthwhile to study such functionals and search for possible critical points. In this work we consider only surfaces which are diffeomorphic to torus. We call such surfaces as toroidal surfaces. The first example we consider is the torus itself. The second example is the tube of radius r about a closed curve α.

In Section 2 we introduce the first and second variations of a functional $\mathcal{F} = \int_S \mathcal{E}\mathrm{d}A + p\int_V \mathrm{d}V$ where the Lagrange function $\mathcal{E}$ is an arbitrary function of the mean and Gauss curvatures of the surface S [20]. We introduce the torus in this section. In Section 3 we consider the Lagrange functions $\mathcal{E}$ depends only on the the mean curvature H. Letting $\mathcal{E}_n$ be a polynomial of H of degree n, we then show that torus T_n is a critical point of the corresponding functional $\mathcal{F}_n = \int_S \mathcal{E}_n\mathrm{d}A + p\int_V \mathrm{d}V$ with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n = 2, 3, 4, \ldots$. When $n = 2$ the corresponding

torus T_2 is the Clifford torus. In Section 4 we take tubes or toroidal surfaces about a closed curve α in $\mathbb{R}^3$. We show that among these toroidal surfaces the torus T_n ($n = 2, 3 \ldots$) is the only one solving the general shape equation. In concluding remarks we discuss possible applications of our results on cell membranes.

2. Surfaces From a Variational Principle

Let S be a regular closed surface in $\mathbb{R}^3$ with Gaussian (K) and mean (H) curvatures. A functional $\mathcal{F}$ is defined by

$$\mathcal{F} = \int_S \mathcal{E}(H, K) \mathrm{d}A + p \int_V \mathrm{d}V \tag{4}$$

where $\mathcal{E}$ is the Lagrange function depending on H and K. The above functional is also called curvature energy or shape energy. Here p is a constant which play the role of Lagrange multiplier and V is the volume enclosed within the surface S. We obtain the Euler-Lagrange equations corresponding to the above functional from its first variation. Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the first variation of $\mathcal{F}$ is given by

$$\delta \mathcal{F} = \int_S E(\mathcal{E})\, \Omega \, \mathrm{d}A \tag{5}$$

where Ω is an arbitrary smooth function on S. Then the Euler-Lagrange equation $E(\mathcal{E}) = 0$ reduces [20–22] to

$$E(\mathcal{E}) = (\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0. \tag{6}$$

Here and in what follows we will use the notation $\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right)$ and $\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} K h^{ij} \frac{\partial}{\partial x^j} \right)$, $g = \det(g_{ij})$, g^{ij} and h^{ij} are inverse components of the first and second fundamental forms, $x^i = (u, v)$ and we assume Einstein's summation convention on repeated indices over their ranges.

Weingarten surfaces are the surfaces where the mean and Gauss curvatures satisfy certain algebraic relations. Surfaces are called linear Weingarten surfaces if $\alpha H + \beta K + \gamma = 0$ relation holds for any constants α, β and γ. Here we have a nice theorem on linear Weingarten surfaces.

Theorem 1. *Let S be a linear Weingarten surface ,i.e., $\alpha H + \beta K + \gamma = 0$, where α, β and γ are constants. Then S is a critical point of the functional $\mathcal{F}$ with a Lagrange function $\mathcal{E} = \frac{\beta}{2} H + \frac{\alpha}{2}$ and $p = -\gamma$*

Proof: Inserting $\mathcal{E} = \frac{\beta}{2} H + \frac{\alpha}{2}$ into the Euler-Lagrange equation (6) we simply obtain the linear Weingarten relation with $\gamma = -p$. ■

For the second variation of the functional we assume that $\mathcal{E}$ depends only on H. In this case the expression is much simpler [20]

$$\begin{aligned}\delta^2 \mathcal{F} \;=\; \int_S \Big(& \mathcal{E}_1\, \Omega^2 \,+ \mathcal{E}_2\, \Omega\, \nabla^2 \Omega - 2\frac{\partial \mathcal{E}}{\partial H}\, \Omega \nabla \cdot \tilde{\nabla}\, \Omega + \frac{1}{4}\, \frac{\partial^2 \mathcal{E}}{\partial H^2}\, (\nabla^2 \Omega)^2 \\ & + \frac{\partial \mathcal{E}}{\partial H}\, \big(\nabla(H\Omega) \cdot \nabla\Omega - \nabla\Omega \cdot \tilde{\nabla}\Omega \big) \Big)\, \mathrm{d}A \end{aligned} \tag{7}$$

in which

$$\begin{aligned} \mathcal{E}_1 \;&=\; (2H^2 - K)^2\, \frac{\partial^2 \mathcal{E}}{\partial H^2} - 2HK \frac{\partial \mathcal{E}}{\partial H} + 2K\mathcal{E} - 2Hp \\ \mathcal{E}_2 \;&=\; (2H^2 - K)\, \frac{\partial^2 \mathcal{E}}{\partial H^2} + 2H \frac{\partial \mathcal{E}}{\partial H} - \mathcal{E} \end{aligned} \tag{8}$$

and where Ω is an arbitrary function over the closed surface. To have minimal energy solutions of (6) it is expected that the second variation $\delta^2 \mathcal{F} > 0$.

We have the following classical examples:

i) Minimal surfaces: $\mathcal{E} = 1,\ p = 0$ $(H = 0)$.
ii) Constant mean curvature surfaces: $\mathcal{E} = 1$ $(H = \frac{p}{2})$.
iii) Constant Gauss curvature surfaces: $\mathcal{E} = \alpha H$ $(K = \frac{p}{\alpha})$
iv) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants, $aH + 2bK - b = 0$.
v) Willmore surfaces: $\mathcal{E} = H^2$ [23–25].
vi) Surfaces solving the shape equation (2) of lipid bilayer cell membranes: $\mathcal{E} = \frac{1}{2}\, k_c\, (2H + c_0)^2 + w$, where k_c, c_0 and w are constants [12, 22].

The sphere in $\mathbb{R}^3$ has constant mean and Gaussian curvatures. Hence the sphere is a critical point of the most general functional (4). Equation (6) gives a relation between p, radius of the sphere and other parameters in model.

Another compact surface in $\mathbb{R}^3$ is the torus, T. It has been shown [11] that a special kind of torus, known as the Clifford torus, solves the shape equation (example v) above). In this work we shall show that, T is not only a critical point of quadratic functional but it is also critical point of functional with Lagrange function $\mathcal{E}$ is any polynomial function of the mean curvature H, provided that the radii of the torus satisfies certain relations.

Definition 2 (The Torus)**.** *Torus T in $\mathbb{R}^3$ is defined as the surface $X : U \to \mathbb{R}^3$* (cf. [3])

$$\begin{aligned} X(u, v) \;&=\; ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u) \\ & \qquad 0 < u < 2\pi, \qquad 0 < v < 2\pi. \end{aligned} \tag{9}$$

The first and second fundamental forms of T are

$$\begin{aligned} \mathrm{d}s_1^2 &= g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = r^2\,\mathrm{d}u^2 + (a + r\cos u)^2\,\mathrm{d}v^2 \\ \mathrm{d}s_2^2 &= h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = r\,\mathrm{d}u^2 + (a + r\cos u)\cos u\mathrm{d}v^2 \end{aligned} \tag{10}$$

The Gaussian, K, and the mean, H, curvatures of T are

$$K = \frac{\cos u}{r(a + r\cos u)}, \qquad H = \frac{1}{2}\left(\frac{1}{r} + \frac{\cos u}{(a + r\cos u)}\right) \tag{11}$$

where a and r $(a > r)$ are the radii of the torus.

It is interesting that K and H satisfy the linear equation $r^2K - 2rH + 1 = 0$. Hence the torus T is a linear Weingarten surface.

3. Functionals with Mean Curvature

In this section we shall consider the Lagrange function $\mathcal{E}$ depending only on the mean curvature H of the surface. Furthermore se shall assume that $\mathcal{E}$ is a polynomial function of H of the type specified in (3) where a_k, $(k = 1, 2, \ldots)$ are constants to be determined. Assuming that the torus is a critical point of the functional $\mathcal{F}$ we shall determine the coefficients a_i of the polynomial expansion of $\mathcal{E}$ and p in terms of the torus radii a and r. We shall give three examples here in this section. In all examples in this section S is the torus and H is the mean curvature of the torus.

Example 1 (First Order Functional)**.** *Since the torus T is a linear Weingarten surfaces then by* Theorem 1 *it is a critical point of the functional where the Lagrange function is a linear function of H, i.e., $\mathcal{E}_1 = a_1\,H + a_2$. The Euler-Lagrange equation* (6) *can be solved exactly, provided*

$$p = -\frac{a_1}{r^2}, \qquad a_2 = -\frac{a_1}{r}\cdot \tag{12}$$

There is no restriction on the radii a and r. The Torus T is a critical point of the functional where $\mathcal{E}_1 = a_1\,H + a_2$ for all values of r and a.

Example 2 (Second Order Functional)**.** *Lagrange function is a quadratic function of H, i.e., $\mathcal{E}_2 = a_1\,H^2 + a_2\,H + a_3$. Euler-Lagrange equation* (6) *is exactly solvable, provided*

$$p = -\frac{a_2}{r^2}, \qquad a_3 = -\frac{a_2}{r}, \qquad a^2 = 2r^2. \tag{13}$$

This is the Clifford Torus. We find that (for $p = 0$)

$$\mathcal{F}_2 = \int_T \mathcal{E}_2 \mathrm{d}A = 2\pi^2\, a_1. \tag{14}$$

Hence the torus with $a^2 = 2r^2$ minimizes the functional $\int_S \mathcal{E}_2 dS$ and the minimum energy is given in (14). Willmore conjecture [23], [25] states that ($a_1 \neq 0$)

$$\frac{1}{a_1} \int_S \mathcal{E}_2 \mathrm{d}A \geq 2\pi^2 \tag{15}$$

for all compact surfaces S with genus $g > 0$. The proof of this conjecture has been given very recently [9]. In terms of the Helfrich's functional (1) we have $a_1 = 2\,k_c$, $a_2 = 2k_c c_0$ and $a_3 = \frac{1}{2}\, k_c c_0^2 + w$. The parameters p, w, c_0 must satisfy

$$p = \frac{2k_c c_0}{r^2}, \qquad w = p\, r\, (1 + \frac{1}{4} r c_0). \tag{16}$$

Remark 3. *Since K is a topological invariant and total curvature for torus is zero, there will be no contribution by adding linear K terms to $\mathcal{E}$.*

Example 3 (Third Order Functional)**.** *Lagrange function is a polynomial of H of degree three, i.e., $\mathcal{E}_3 = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4$. The Euler-Lagrange equation* (6) *can be solved exactly, provided that*

$$p = -\frac{3a_1 - a_3 r^2}{r^4}, \quad a_4 = \frac{2a_1 - a_3 r^2}{r^3}, \quad a_2 = \frac{15 a_1}{2r}, \quad a^2 = (6/5) r^2. \tag{17}$$

We also find that (for $p = 0$)

$$\mathcal{F}_3 = \int_T \mathcal{E}_3 \mathrm{d}A = 9\sqrt{5}\, \pi^2\, (a_1/r) \tag{18}$$

the torus with $a^2 = (6/5) r^2$ minimizes the functional $\int_S \mathcal{E}_3 \mathrm{d}A$ and the minimum energy is given in (18). We expect that ($a_1 \neq 0$)

$$(a_1/r)^{-1} \int_S \mathcal{E}_3 \mathrm{d}A \geq 9\sqrt{5}\, \pi^2 \tag{19}$$

for all compact surfaces S with genus $g > 0$. This can be considered as the Willmore conjecture for $n = 3$.

Definition 4 (Torii T_n)**.** *The tori T with radii (a, r) satisfying the relation*

$$\frac{a^2}{r^2} = \frac{n^2 - n}{n^2 - n - 1}, \qquad n = 2, 3, 4, \ldots \tag{20}$$

are special and denoted by T_n. For all these surfaces $1 < \frac{a^2}{r^2} \leq 2$.

It is possible to continue on finding critical points of higher order functionals with $\mathcal{E}_n$, $(n = 4, 5, 6, \ldots)$. The following theorem asserts that critical points of functionals $\mathcal{F}_n$, in general, are the special torii T_n $(n = 2, 3, \ldots)$.

Theorem 5. *A torus T is a critical point of a functional $\mathcal{F}_n = \int_S \mathcal{E}_n \, \mathrm{d}A + p \int_V \mathrm{d}V$ where $\mathcal{E}_n$ is the n^{th} degree polynomial* (3) *of the mean curvature H of the surface T if and only if $T = T_n$ for all $n \geq 2$.*

Proof: The shape equation with $\mathcal{E} = \mathcal{E}(H)$ is given by

$$E(\mathcal{E}) = (\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} - 4H\mathcal{E} + 2p = 0. \tag{21}$$

Using equations (10)-(11) for the torus we obtain

$$\begin{aligned} \nabla^2 H &= \frac{1}{a^2 r^3}\,(4r^3(-a^2+r^2)H^3 + 2r^2(5a^2-6r^2)H^2 \\ &\quad +4r(-2a^2+3r^2)H + 2(a^2-2r^2)) \\ (\nabla H)\cdot(\nabla H) &= \frac{1}{a^2 r^4}\,(4r^4(-a^2+r^2)\,H^4 + 4r^3(3a^2-4r^2)\,H^3 \\ &\quad +r^2(-13a^2+24r^2)\,H^2 + 2r(3a^2-8r^2)\,H + 4r^2). \end{aligned} \tag{22}$$

With the aid of these equations for the torus and for all $n \geq 2$ we get the identity

$$\nabla^2 H^n = \frac{4n^2}{a^2}(-a^2+r^2)H^{n+2} + \frac{2}{a^2 r}((6n^2-n)a^2 - (8n^2-2n)r^2)H^{n+1} + \cdots \tag{23}$$

Hence inserting

$$\mathcal{E}_n = a_1\, H^n + a_2\, H^{n-1} + \cdots + a_n\, H + a_{n+1} \tag{24}$$

into the general shape equation (21) we get

$$\begin{aligned} E(\mathcal{E}) &= (na_1\nabla^2 H^{n-1} + (n-1)a_2\nabla^2 H^{n-2} + \cdots) + (4(n-1)a_1 H^{n+1} \\ &\quad +4(n-2)a_2 H^n + \cdots) - \frac{2}{r^2}(2rH-1)(na_1 H^{n-1} \\ &\quad +(n-1)a_2 H^{n-2} + \cdots) = 0. \end{aligned} \tag{25}$$

Using the identity (23) for the torus and collecting the coefficients of the powers of H we get equations for a_i's. The coefficient of the highest power H^{n+1} in $E(\mathcal{E})$ can be calculated exactly and since $E(\mathcal{E}) = 0$ we get

$$a_1\left(\frac{4n(n-1)^2\,(-a^2+r^2)}{a^2} + 4(n-1)\right)H^{n+1} + \cdots = 0 \tag{26}$$

where $a_1 \neq 0$. Then coefficient of H^{n+1} must vanish which leads to the constraint equations

$$\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$$

for all $n \geq 2$. The remaining $n+1$ number of equations are linear algebraic equations for a_i, $(i = 1, 2, \ldots, n+1)$ and p. In general one can solve them in terms of one arbitrary parameter, for instance a_1. ■

In Examples 1-3 above, the solutions contain two arbitrary coefficients. This means that one of the remaining equations in (26) vanishes identically. This is due to the following property

Theorem 6. *Critical points T_n of the functional $\mathcal{F}_n$. T_n are left invariant under the change of the Lagrange function $\bar{\mathcal{E}}_n = \mathcal{E}_n + b_1 H + b_0$ where b_1 and b_0 are constants satisfying*

$$\bar{p} = p - \frac{b_1}{r^2}, \qquad b_0 = -\frac{b_1}{r}. \tag{27}$$

Proof: It is straightforward to show that

$$E(\bar{\mathcal{E}}_n) = E(\mathcal{E}_n) + 2\bar{p} - 2p - 4(b_0 + \frac{b_1}{r})H + \frac{2b_1}{r^2}. \tag{28}$$

Since $E(\bar{\mathcal{E}}_n) = E(\mathcal{E}_n) = 0$ we obtain the equations in (27). ■

Here b_1 is left arbitrary. This is the reason why the coefficients of the linear H terms are arbitrary in the first, second and third order functionals studied in Examples 1-3. It is left arbitrary in all $\mathcal{E}_n$.

4. Toroidal Surfaces

In this section we will give a generalization of torus in $\mathbb{R}^3$. This is a regular surface which is diffeomorphic to torus.

Definition 7 (Toridal Surfaces)**.** *Let $\alpha(v)$ be a simple and a regular closed plane curve in $\mathbb{R}^3$ with the unit tangent vector $\mathbf{t}(v)$, the unit normal vector $\mathbf{n}(v)$ and the bi-normal vector $\mathbf{b}(v)$. Here $v \in I = [v_1, v_2]$ is the arclength parameter of the curve. A parametrization $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of this surface is given as*

$$X(u, v) = a\,\alpha(v) + r(-\cos u\,\mathbf{n}(v) + \sin u\,\mathbf{b}(v)) \tag{29}$$

where a and r are constants. Here $u \in [0, 2\pi]$ and $v \in I$ such that $\alpha(v_1) = \alpha(v_2)$. This is a tube of radius r around the closed curve α. The radius r is so chosen that tube has no self intersections. We call these surfaces as toroidal surface. The first and second fundamental forms of this surface are

$$\begin{aligned} \mathrm{d}s_1^2 &= g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ &= r^2\,\mathrm{d}u^2 - 2r^2\,\tau(v)\,\mathrm{d}u\mathrm{d}v + ((a+rk(v)\,\cos u)^2 + (\tau(v))^2\,r^2)\,\mathrm{d}v^2 \\ \mathrm{d}s_2^2 &= h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ &= r\,\mathrm{d}u^2 - 2r\tau(v)\mathrm{d}u\mathrm{d}v + ((a+rk(v)\,\cos u)\,k(v)\,\cos u + r(\tau(v))^2)\,\mathrm{d}v^2. \end{aligned} \tag{30}$$

The Gaussian and the mean curvatures are

$$K = \frac{k(v)\,\cos u}{r(a+rk(v)\,\cos u)}, \qquad H = \frac{1}{2}\left(\frac{1}{r} + \frac{k(v)\,\cos u}{(a+rk(v)\,\cos u)}\right) \tag{31}$$

where $k(v)$ and $\tau(v)$ are the curvature and torsion of the closed curve α.

It is simple to show that K and H satisfy the linear equation $r^2K - 2rH + 1 = 0$. Hence TS is also a linear Weingarten surface. When $k = 1$ this surface becomes the the Torus we discussed in Section 2. Below we shall assume that the Lagrange function $\mathcal{E}$ is a polynomial of the mean curvature H.

Example 4 (Linear functional)**.** *Since the toroidal surface* TS *is a linear Weingarten surface then by* Theorem 1 *it is a critical point of the functional with the Lagrange function $\mathcal{E}_1 = a_1\,H + a_2$, provided that*

$$p = -\frac{a_1}{r^2}\,, \qquad a_2 = -\frac{a_1}{r}. \tag{32}$$

Hence any toroidal surface with arbitrary closed curve $\alpha(v)$ in $\mathbb{R}^3$ is a critical point of the corresponding functional $\mathcal{F}_1$.

Example 5 (Quadratic functional)**.** *The Lagrange function is a quadratic function of H, i.e., $\mathcal{E}_2 = a_1\,H^2 + a_2\,H + a_3$. The toroidal surface* TS *is a critical point of the corresponding functional $\mathcal{F}_2$ provided that $\tau = 0$, $k = k_0$ a constant and*

$$p = -\frac{a_2}{r^2}\,, \qquad a_3 = -\frac{a_2}{r}, \qquad a^2 = 2k_0^2r^2. \tag{33}$$

Without loosing any generality we take $k_0 = 1$. Hence TS *is the torus T_2, i.e., Clifford Torus.*

Example 6 (Qubic functional)**.** *The Lagrange function is a cubic polynomial of H, i.e., $\mathcal{E}_3 = a_1\,H^3 + a_2\,H^2 + a_3\,H + a_4$. The toroidal surface* TS *is a critical point of the corresponding functional $\mathcal{F}_3$ provided that $\tau = 0$, $k = k_0$ a constant and*

$$p = \frac{3a_1 - a_3}{r^4}\,, \quad a_2 = \frac{15a_1}{2r}, \quad a_4 = -\frac{2a_1 - a_3r^2}{r^3}, \quad a^2 = \frac{6}{5}k_0^2\,r^2. \tag{34}$$

Again we take $k_0 = 1$. Hence TS *is the torus T_3.*

We may continue finding solutions of the Euler-Lagrange equations for $\mathcal{E}_n$ with $n \geq 4$. We observe that, except $n = 1$, for all $n \geq 2$ toroidal surfaces reduce to torus. We claim that this is true in general. Toroidal surface TS with non-vanishing torsion τ, non-constant curvature k is not a critical point of the functional $\mathcal{F}_n$ ($n = 2, 3, \ldots$) where the Lagrange function $\mathcal{E}_n$ is a polynomial of the mean curvature H.

Theorem 8. *Let S be a toroidal surface* TS *given in* Definition 7. *Let the Lagrange function function $\mathcal{E}$ be a polynomial of the mean curvature H of degree $n \geq 2$ given in* (3). *Then the critical points of the functional* (4) *on* TS *are the surfaces T_n, $n \geq 2$.*

Proof: The Euler-Lagrange equation for the Lagrange function (3) takes the form

$$\begin{aligned} E(\mathcal{E}) \;=\; & (na_1\nabla^2 H^{n-1} + (n-1)a_2\nabla^2 H^{n-2} + \cdots) \\ & +(4(n-1)a_1 H^{n+1} + 4(n-2)a_2 H^n + \cdots) \\ & -\frac{2}{r^2}(2rH-1)(na_1 H^{n-1} + (n-1)a_2 H^{n-2} + \cdots) = 0. \end{aligned} \tag{35}$$

We can write $\nabla^2 H^{n-1}$, $\nabla^2 H^{n-2}$, $\cdots$ by using $\nabla^2 H$ and $\nabla H \cdot \nabla H$ given in the Appendix. We collect the terms $\sin u\, H^n$ and H^n for all $n = 0, 1, 2, \ldots$ and equate the coefficients of these terms to zero. It is clear from the expression of $\nabla^2 H$ the highest order term $\sin u H^{n+4}$ gives $\tau = 0$. This simplifies the remaining equations considerably. Equating the coefficient of the highest order factor H^{n+4} to zero in the remaining equations we get $k' = 0$. With this result α reduces to a plane curve with constant curvature. Since it is a closed curve then α is the circle with $k = 1$ and TS is a torus. From the theorem 3 we know that critical points of functional (4) on TS are T_n ($n \geq 2$).

■

5. Concluding Remarks

1. Clifford torus is a critical point of the Willmore and also Helfrich's functionals where the Lagrange function is a quadratic polynomial in the mean curvature of a closed surface in $\mathbb{R}^3$. One of the main contributions of this work is that the sequence torus surfaces $\{T_n\}$ where radii a and r restricted to satisfy $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ for all $n \geq 2$ are the critical points of the functionals $\mathcal{F}_n$ where the Lagrange function $\mathcal{E}_n$ are polynomial of degree n in the mean curvature H of the surface. We have given three examples $n = 1, 2, 3$ and proved this assertion in Section 3.

2. A simple generalization of the torus in $\mathbb{R}^3$ is the tube around a closed planar curve α. We call such surfaces as toroidal surfaces which are topologically diffeomorphic to torus. Except the linear case we showed that these surfaces with non-vanishing torsion τ and nonconstant curvature k are not critical points of the functionals $\mathcal{F}_n$. Euler-Lagrange equations force the torsion of the curve α to vanish and the curvature be a constant.

3. In Section 3, for each solution with $n = 2, 3$ and $p = 0$ we have calculated the curvature energy $\mathcal{F}_n$. As in the case of the Willmore energy functional ($n = 2$) it is expected the torus surfaces for $n \geq 3$ with the constraints are minimal energy surfaces. To support this assertion, second variation of the functionals on these surfaces must be nonnegative. Another point to be examined is the stability of these minimal energy surfaces. These points will be clarified in a forthcoming communication.

4. The constraints $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ can be utilized to select the correct functional for the toroidal fluid membranes. These functionals are used to minimize the energy of the lipid membranes. The ratio a/r of toroidal configuration can be measured experimentally. Comparing the measured value of this ratio with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ we can identify the degree of the polynomial function $\mathcal{E}_n$ from (20), hence finding the functional for the corresponding closed membrane. As an example, for vesicle membranes such a measurement had been done by Mutz and Bensimon [10]. They measured the value of this ratio approximately as $\frac{a}{r} = 1.43$, or $\frac{a^2}{r^2} = 2.04$. Hence for vesicle membranes the correct functional should be the quadratic one which was first introduced by Helfrich [8] several years ago. For other closed fluid membranes the functionals might be different.

5. In his study of vesicles with toroidal topology Seifert [16, 17] claimed that circular toroidal configurations exist for any $0 < v < 1$ where v is the reduced volume parameter defined by

$$v = \frac{V}{\frac{4\pi}{3}(\frac{A}{4\pi})^{3/2}}. \tag{36}$$

Here V is the volume and A is the surface are of the toroidal membranes. In terms of the torus radii we have

$$\frac{a^2}{r^2} = \frac{1}{1.94v^4}. \tag{37}$$

Seifert claims also that the membranes with $v \geq 0.84$ are not stable. This corresponds to $\frac{a^2}{r^2} \leq 1.188$. This stability analysis seems to save the quadratic and cubic functionals we have given above. One must be careful about discarding higher order functionals because the stability analysis mentioned by Seifert is done

by using the quadratic functional. To investigate the stability of toroidal configurations with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$, $n = 2, 3, 4, \ldots$ one must use the functionals $\mathcal{F}_n$, $(n = 3, 4, 5, 6, \ldots)$ respectively.

6. Appendix

In this Appendix we give the Laplacian and the norm of the gradient of H for toroidal surfaces in Section 4. We factorize the results with respect to $\sin u$ and $\cos u$. To get simpler expressions we expressed $\cos u$ in terms of H, i.e.,

$$\begin{aligned}
\nabla^2 H \;=\; & \frac{1}{k^2 a^4 r^3}\,[96\sin u H^5 h' a k^2 r^8 \tau + 8\sin u H^4 a k^2 r^7(-56k'\tau - \tau' k) \\
& +32\sin u H^3 a k^2 r^6(26k'\tau + \tau' k) + 48\sin u H^2 a k^2 r^5(-16k'\tau - \tau' k) \\
& +32\sin u H a k^2 r^4(11k'\tau + \tau' k) + 8\sin u a k^2 r^3(-8k'\tau - \tau' k) \\
& +48H^5 r^7(a^2 k'^2 - a^2 k^2\tau^2 + k^4 r^2\tau^2) + 8H^4 r^6(-a^2 k k'' - 24k'^2 a^2 \\
& +25a^2 k^2\tau^2 - 3k^4 r^2\tau^2) + 4r^3 H^3(7a^2 r^2 k k'' + 75k'^2 a^2 r^2 \\
& -a^4 r^2 + a^2 k^2 r^2 - 82a^2 r^2 k^2\tau^2 + 120k^4 r^4\tau^2) + 2r^2 H^2(-18a^2 r^2 k k'' \\
& -114k'^2 a^2 r^2 + 5a^4 k^2 - 6a^2 k^4 r^2 + 132a^2 k^2 r^2\tau^2 - 240k^4 r^4\tau^2) \\
& +4rH(5a^2 r^2 k k'' + 21k'^2 a^2 r^2 - 2a^4 k^2 + 3a^2 k^4 r^2 \\
& -26a^2 k^2 r^2\tau^2 60k^4 r^4\tau^2) + 2(-2a^2 r^2 k k'' - 6k'^2 a^2 r^2 + a^4 k^2 \\
& -2a^2 k^4 r^2 + 8a^2 k^2 r^2\tau^2 - 24k^4 r^2\tau^2)]
\end{aligned}$$

$$\begin{aligned}
\nabla H \cdot \nabla H \;=\; & \frac{1}{a^4 r^4 k^2}\,[16r^8 H^6(k'^2 a^2\tau + k'^2 a^2 - a^2 k^2\tau^2 - a^2 k^2\tau + k^4 r^2\tau^2 \\
& +k^4 r^2\tau) + 16r^7 H^5(-5k'^2 a^2\tau - 5k'^2 a^2 + 5a^2 k^2\tau^2 + 5a^2 k^2\tau \\
& -6k^4 r^2\tau^2 - 6k^4 r^2\tau) + 4r^4 H^4(41k'^2 a^2 r^2\tau + 41k'^2 a^2 r^2 \\
& -a^4 k^2 + a^2 k^4 r^2 - 41a^2 k^2 r^2\tau^2 - 41a^2 k^2 r^2\tau + 60k^4 r^4\tau^2 \\
& +60k^4 r^4\tau) + 4r^3 H^3(-44k'^2 a^2 r^2\tau - 44k'^2 a^2 r^2 + 3a^4 k^2 \\
& -4a^2 k^4 r^2 + 44a^2 k^2 r^2\tau^2 + 44r^2 k^2 a^2\tau - 80k^4 r^4\tau^2 - 80k^4 r^4\tau) \\
& +r^2 H^2(104k'^2 a^2 r^2\tau + 104k'^2 a^2 r^2 - 13a^4 k^2 + 24a^2 k^4 r^2 \\
& -104a^2 k^2 r^2\tau^2 - 104r^2 k^2 a^2\tau + 240k^4 r^4\tau^2 + 240k^4 r^4\tau) \\
& +2rH(-16k'^2 a^2 r^2\tau - 16k'^2 a^2 r^2 + 3a^4 k^2 - 24a^2 k^4 r^2 \\
& +16a^2 k^2 r^2\tau^2 + 16r^2 k^2 a^2\tau - 48k^4 r^4\tau^2 - 48k^4 r^4\tau) \\
& +4k'^2 a^2 r^2\tau + 4k'^2 a^2 r^2 - a^4 k^2 + 4a^2 k^4 r^2 \\
& -4a^2 k^2 r^2\tau^2 - 4r^2 k^2 a^2\tau + 16k^4 r^4\tau^2 + 16k^4 r^4\tau].
\end{aligned}$$

When $\tau = 0$ and k is a constant then the above equations reduce to those (22).

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