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VECTOR-PARAMETER FORMS OF SU(1,1), SL(2, $\mathbb{R}$) AND THEIR CONNECTION TO SO(2,1)

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Abstract. The *Cayley* maps for the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$ converting them into the corresponding Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}(2,1)$ along their natural vector-parameterizations are examined. Using the isomorphism between $\mathrm{SU}(1,1)$ and SL(2, $\mathbb{R}$), the vector-parameterization of the latter is also established. The explicit form of the covering map $\mathrm{SU}(1,1) \to \mathrm{SO}(2,1)$ and its sections are presented. Using the so developed vector-parameter formalism, the composition law of $\mathrm{SO}(2,1)$ in vector-parameter form is extended so that it covers compositions of all kinds of elements including also those that can not be parameterized properly by regular $\mathrm{SO}(2,1)$ vector-parameters. The latter are characterized and it is shown that they can be represented by $\mathrm{SU}(1,1)$ vector parameters with pseudo length equal to minus four. In all cases of compositions inside $\mathrm{SO}(2,1)$, criteria for determination of their type (elliptic, parabolic, hyperbolic) have been presented. On the base of the vector-parameter formalism the problem of taking a square root in SO(2,1) is solved explicitly. Also, an analogue of Cartan's theorem about the decomposition of orthogonal matrix of order n into product of at most n reflections is formulated and proved for the subset of hyperbolic elements of the group of pseudo-orthogonal matrices from $\mathrm{SO}(2,1)$.

Notation and Nomenclature

$\mathbb{R}^*$	the set of nonzero real numbers
$\alpha_1, \alpha_2,\ \beta_1,\ \beta_2, x, y, z$	real numbers
$\alpha = \alpha_1 + \mathrm{i}\alpha_2,\ \ \beta = \beta_1 + \mathrm{i}\beta_2$	complex numbers
θ	angle (rapidity)
$\mathcal{I}$	the unit matrix of respective dimension
$\mathcal{I}_{p,q}$	$\mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$
$\eta_1 = \mathcal{I}_{1,1},\ \ \eta = \mathcal{I}_{2,1}$	flat metrics in $\mathbb{R}^{1,1}$ and $\mathbb{R}^{2,1}$
$\mathbf{x},\ \mathbf{y}$	vectors in $\mathbb{R}^3$
$\mathbf{n}$	unit vectors in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$
$(\mathbf{x}, \mathbf{z}) = \mathbf{x}.\mathbf{z},\ \ \ \mathbf{u} \times \mathbf{v}$	scalar and vector products in $\mathbb{R}^3$
$(\mathbf{x}, \eta\,\mathbf{z}) = \mathbf{x}.\eta\mathbf{z},\ \ \mathbf{u} \curlywedge \mathbf{v} = \eta(\mathbf{u} \times \mathbf{v})$	pseudo scalar and vector products in $\mathbb{R}^{2,1}$
$\delta^i_j,\ \ \varepsilon_{ijk}$	*Kronecker* and *Levi-Civita* symbols
$\mathrm{SL}(n, \mathbb{R})$	special real linear group of order n
$\mathcal{L}$	general element of the *Lie* group $\mathrm{SL}(2, \mathbb{R})$
$\mathfrak{so}(2, 1)$	the *Lie* algebra of the *Lie* group $\mathrm{SO}(2, 1)$
$\{P_k\}_{k=1,2,3}$	a basis of the *Lie* algebra $\mathfrak{so}(2, 1)$
$\mathcal{C}(\boldsymbol{c}) = \boldsymbol{c}^{\curlywedge}$	$\mathfrak{so}(2, 1)$ element associated with the vector $\boldsymbol{c}$
$\boldsymbol{c},\ \boldsymbol{c}_k,\ \widetilde{\boldsymbol{c}},\ \boldsymbol{c}_{\pm},\ \boldsymbol{a},\ \boldsymbol{a}_k,\ \widetilde{\boldsymbol{a}}$	vector-parmeters of SU(1,1) elements
$\mathrm{SO}(p, q)$	special orthogonal group of signature (p, q)
H	general element of the *Lie* group $\mathrm{SO}(2, 1)$
$\mathcal{R}_h,\ \mathcal{R}_{hi},\ \mathcal{R}_h(\boldsymbol{c})$	regular pseudo-rotations in $\mathrm{SO}(2, 1)$
$\mathcal{R}_H$	the set of regular pseudo rotations in $\mathrm{SO}(2, 1)$
$\mathcal{O}_h,\ \mathcal{O}_{hi},\ \mathcal{O}_h(\mathbf{m})$	pseudo half-turns in $\mathrm{SO}(2, 1)$
$\mathcal{Q}_h$	pseudo quater-turn in $\mathrm{SO}(2, 1)$
$\mathcal{O}(\mathbf{n})$	half-turn in $\mathrm{SO}(3, \mathbb{R})$
$\mathcal{O}_H$	the set of pseudo half-turns in $\mathrm{SO}(2, 1)$
$\mathfrak{su}(1, 1)$	the *Lie* algebra of the *Lie* group $\mathrm{SU}(1, 1)$
$\{E_k\}_{k=1,2,3}$	a $\mathbb{R}-$basis of the *Lie* algebra $\mathfrak{su}(1, 1)$
$M,\ M_i$	elements of the *Lie* algebra $\mathfrak{su}(1, 1)$
$\mathbf{m},\ \mathbf{m}_k,\ \widetilde{\mathbf{m}},\ \mathbf{a},\ \mathbf{a}_k,\ \widetilde{\mathbf{a}},\ \mathbf{a}_{\pm}$	SO(3, $\mathbb{R}$) and SO(2,1) vector-parmeters
$\mathrm{SU}(p, q)$	special unitary group of signature (p, q)
$\mathcal{M}_h$	general element of the *Lie* group $\mathrm{SU}(1, 1)$
$\mathcal{U}_h,\ \mathcal{U}_{hi}$	$\mathrm{SU}(1, 1)$ generators of regular $\mathrm{SO}(2, 1)$ elements
$\mathcal{W}_h$	$\mathrm{SU}(1, 1)$ generators of $\mathrm{SO}(2, 1)$ pseudo half-turns
$M_{i,j} = M(i, j)$	the element at position (i, j) in the matrix M

CONTENTS

1. Introduction

The explicit parameterizations are used to describe Lie groups in an easier and more intuitive way and have found many applications in modern physics and mathematics [9, 11, 17, 18]. Let G be a finite dimensional Lie group with a Lie algebra $\mathfrak{g}$. A vector-parameterization of G is a map $\mathfrak{g} \to G$, which is diffeomorphic onto its image [15]. We can extend the definition by allowing the vector-parameterization to be defined only on some subset of $\mathfrak{g}$.

Recall especially the subgroups of the pseudo-orthogonal and pseudo-unitary groups

$$\mathrm{SO}(p,q) = \{\, M \in \mathrm{Mat}(p+q,\mathbb{R})\,;\; M^t \mathcal{I}_{p\times q} M = \mathcal{I}_{p\times q}, \det M = 1\,\}$$
$$\mathrm{SU}(p,q) = \{\, M \in \mathrm{Mat}(p+q,\mathbb{C})\,;\; M^\dagger \mathcal{I}_{p\times q} M = \mathcal{I}_{p\times q}, \det M = 1\,\}$$

where $p+q \geq 2$, $\mathcal{I}_{p,q} = \begin{pmatrix} \mathcal{I}_p & 0 \\ 0 & -\mathcal{I}_q \end{pmatrix} \in \mathrm{Mat}(p+q,\mathbb{R})$, and $\mathcal{I}_n$ is the unit matrix of order n. For ease we will use further on the following notation

$$\eta = \mathcal{I}_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \eta_1 = \mathcal{I}_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also, till the end of the paper $\mathcal{I}$ will stand for the unit matrix with dimension consistent with the context.

In particular, SU$(1,1)$ can be described as

$$\mathrm{SU}(1,1) = \left\{ \mathcal{M}_h(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}\,;\; \alpha\overline{\alpha} - \beta\overline{\beta} = 1 \right\}$$

where $\alpha = \alpha_1 + \mathrm{i}\alpha_2,\ \beta = \beta_1 + \mathrm{i}\beta_2$ and $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in \mathbb{R}$.

It is well known fact that SU$(1,1)$ is isomorphic as a group to the special linear group of order two with real entities, i.e., SL$(2,\mathbb{R})$ by the map

$$\begin{aligned} &\psi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SU}(1,1) \\ &\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}\right)^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}, \qquad ad - bc = 1. \end{aligned} \tag{1}$$

The preimage of $\mathcal{M}_h(\alpha,\beta) = \mathcal{M}_h(\alpha_1,\alpha_2,\beta_1,\beta_2)$ in SL$(2,\mathbb{R})$ is given by the formula

$$\mathcal{L} = \mathcal{L}(\alpha,\beta) = \psi^{-1}(\mathcal{M}_h(\alpha,\beta)) = \begin{pmatrix} \alpha_1+\beta_2 & \beta_1+\alpha_2 \\ \beta_1-\alpha_2 & \alpha_1-\beta_2 \end{pmatrix}.$$

Recall that the classification of SL$(2,\mathbb{R})$ (respectively SU(1,1)) elements is made in terms of $\operatorname{tr}\mathcal{L} = \operatorname{tr}\mathcal{M}_h$ [3]. From the above we have that $\operatorname{tr}\mathcal{L}(\alpha_1,\alpha_2,\beta_1,\beta_2) = 2\alpha_1$ and thus

$$\mathcal{L}\ \text{is} \begin{cases} \mathit{hyperbolic} & \text{if } |\operatorname{tr}\mathcal{L}| > 2 \Leftrightarrow |\alpha_1| > 1 \\ \mathit{parabolic} & \text{if } |\operatorname{tr}\mathcal{L}| = 2 \Leftrightarrow |\alpha_1| = 1, \qquad \text{i.e., } \alpha_1 = \pm 1 \\ \mathit{elliptic} & \text{if } |\operatorname{tr}\mathcal{L}| < 2 \Leftrightarrow |\alpha_1| < 1. \end{cases} \tag{2}$$

Consider the map [1, Chapter 9]

$$\psi : \mathrm{SU}(1,1) \longrightarrow \mathrm{SO}(2,1)$$

which sends the matrix $\mathcal{M}_h = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ into $\varphi(\mathcal{M}_h) = H(\alpha,\beta) \in \mathrm{SO}(2,1)$ where

$$\varphi(\mathcal{M}_h) = \begin{pmatrix} -\frac{1}{2}(\beta^2 + \overline{\beta}^2 - \alpha^2 - \overline{\alpha}^2) & \frac{\mathrm{i}}{2}(\overline{\alpha}^2 + \overline{\beta}^2 - \alpha^2 - \beta^2) & \mathrm{i}(\overline{\alpha}\overline{\beta} - \alpha\beta) \\ -\frac{\mathrm{i}}{2}(\beta^2 - \overline{\beta}^2 - \alpha^2 + \overline{\alpha}^2) & \frac{1}{2}(\overline{\alpha}^2 + \overline{\beta}^2 + \alpha^2 + \beta^2) & \alpha\beta + \overline{\alpha}\overline{\beta} \\ \mathrm{i}(\overline{\alpha}\beta - \alpha\overline{\beta}) & \overline{\alpha}\beta + \alpha\overline{\beta} & \alpha\overline{\alpha} + \beta\overline{\beta} \end{pmatrix}. \tag{3}$$

The map φ is a double covering group homomorphism with $\ker\varphi = \{\pm\mathcal{I}\}$, i.e., $\mathrm{SU}(1,1)/\mathbb{Z}_2 \cong \mathrm{SO}(2,1)$. Observe also that $\operatorname{tr}\varphi(\mathcal{M}_h) = 4\alpha_1^2 - 1$. Thus the criteria (2) classifies the $\mathrm{SO}(2,1)$ elements in a natural way, i.e.,

$$H = \varphi(\mathcal{M}_h) \text{ is } \begin{cases} \mathit{hyperbolic} & \text{if} \quad |\operatorname{tr} H| > 3 \;\Leftrightarrow\; |\alpha_1| > 1 \\ \mathit{parabolic} & \text{if} \quad |\operatorname{tr} H| = 3 \;\Leftrightarrow\; |\alpha_1| = 1, \quad \text{i.e., } \alpha_1 = \pm 1 \\ \mathit{elliptic} & \text{if} \quad |\operatorname{tr} H| < 3 \;\Leftrightarrow\; |\alpha_1| < 1. \end{cases} \tag{4}$$

1.1. Vector-Parameter Form of $\mathrm{SU}(2)$ and its Connection to $\mathrm{SO}(3,\mathbb{R})$

Let $\mathcal{R} = \mathcal{R}(\mathbf{n},\theta)$ be the matrix of a proper (i.e., not a half-turn) three-dimensional rotation in the axis-angle formalism. A convenient representation of $\mathcal{R}$ can be made by the vector-parameter $\boldsymbol{c} = \tan\frac{\theta}{2}\mathbf{n}, \theta \neq \frac{\pi}{2}$, i.e.,

$$\mathcal{R}(\boldsymbol{c}) = \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2 & c_1c_2 - c_3 & c_1c_3 + c_2 \\ c_1c_2 + c_3 & 1 + c_2^2 & c_2c_3 - c_1 \\ c_1c_3 - c_2 & c_2c_3 + c_1 & 1 + c_3^2 \end{pmatrix} - \mathcal{I}. \tag{5}$$

However, one has to be careful when half-turns occur because they can not be represented by regular *Gibbs* vectors. If $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ represent the proper rotations $\mathcal{R}(\boldsymbol{c}_1), \mathcal{R}(\boldsymbol{c}_2)$, the composition law in the vector-parameter form is given by the formulas

$$\mathcal{R}(\boldsymbol{c}_3) = \mathcal{R}(\boldsymbol{c}_2)\mathcal{R}(\boldsymbol{c}_1), \qquad \boldsymbol{c}_3 = \boldsymbol{c}_3(\boldsymbol{c}_2,\boldsymbol{c}_1) = \frac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \times \boldsymbol{c}_1}{1 - \boldsymbol{c}_2.\boldsymbol{c}_1}\cdot \tag{6}$$

This composition law is not well-defined either when one of the rotations is a half-turn or when $\boldsymbol{c}_2.\boldsymbol{c}_1 = 1$. The vector-parameterization of the covering group SU(2) and the corresponding composition law is presented in [5]. Also, half-turns are well defined as the composition law there has no singularities. Within this formalism, the natural covering map $\mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ and its sections are written and studied. The technique developed in [5] is used in [6] to *extend* the composition law (6) in the cases when half-turns are involved or the result of the composition is a half-turn, i.e., when $\boldsymbol{c}_2.\boldsymbol{c}_1 = 1$. To do that, a half-turn $\mathcal{R}(\mathbf{n}, \pi) = \mathcal{O}(\mathbf{n})$ is represented as a *ray*, i.e., by the set of all three dimensional non-zero vectors, collinear with the axis of rotation $\mathbf{n}$. This is an alternative description of SO(3, $\mathbb{R}$) and the corresponding composition laws are more intuitive and are computationally cheaper even than the quaternionic formalism [10] when it comes to the composition of rotations [11]. As an application *Cartan*'s theorem is illustrated constructively [6] in the case $n = 3$.

2. Vector-Parameter Form of SO(2,1)

2.1. Description of $\mathfrak{so}(2,1)$

Consider the Lie algebra $\mathfrak{so}(2,1)$ with a basis

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations of these matrices are as follows:

$$[P_1, P_2] = -P_3, \qquad [P_2, P_3] = P_1, \qquad [P_3, P_1] = P_2.$$

Any $\mathcal{C} \in \mathfrak{so}(2,1)$ has a unique representation via some vector $\boldsymbol{c} \in \mathbb{R}^3$

$$\boldsymbol{c} \mapsto \boldsymbol{c}^{\curlywedge} := \mathcal{C}(\boldsymbol{c}) = c_1 P_1 + c_2 P_2 + c_3 P_3 = \boldsymbol{c} \cdot \mathbf{P} = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & -c_1 \\ -c_2 & -c_1 & 0 \end{pmatrix}. \tag{7}$$

Given an arbitrary matrix $\mathcal{C} \in \mathfrak{so}(2,1)$ we can retrieve the vector $\boldsymbol{c} = (c_1, c_2, c_3)$ such that $\mathcal{C} = \boldsymbol{c} \cdot \mathbf{P}$ by the formula

$$\boldsymbol{c}(\mathcal{C}) = \frac{1}{2}(\operatorname{tr} \mathcal{C}.P_1, \operatorname{tr} \mathcal{C}.P_2, -\operatorname{tr} \mathcal{C}.P_3).$$

2.2. The Cayley Map for $\mathfrak{so}(2,1)$

The *Hamilton–Cayley* theorem applied to $\mathcal{C} \equiv \mathcal{C}(\boldsymbol{c})$ from (7) reduces to

$$\mathcal{C}^3 = (1 - \boldsymbol{c}.\eta\boldsymbol{c})\mathcal{C}, \qquad \boldsymbol{c}.\eta\boldsymbol{c} = c_1^2 + c_2^2 - c_3^2 \tag{8}$$

and some direct calculations show that

$$\mathcal{C}^t = -\eta\mathcal{C}\eta = \mathcal{C}(\eta\boldsymbol{c}), \qquad (\mathcal{C}^t)^2 = (\mathcal{C}^2)^t = \eta\mathcal{C}^2\eta. \tag{9}$$

The *Cayley* map applied for $\mathfrak{so}(2,1)$ is

$$\mathcal{R}_h(\boldsymbol{c}) = \mathcal{R}_h = \mathrm{Cay}_{\mathfrak{so}(2,1)}(\mathcal{C}) = (\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = (\mathcal{I}-\mathcal{C})^{-1}(\mathcal{I}+\mathcal{C}). \quad (10)$$

One checks immediately that $\mathcal{I}-\mathcal{C}$ is invertable if and only if $\boldsymbol{c}.\eta\boldsymbol{c} \neq 1$ and in this case (again via *Hamilton–Cayley* theorem) we can obtain

$$(\mathcal{I}-\mathcal{C})^{-1} = \mathcal{I} + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C} + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C}^2. \quad (11)$$

Let us now prove that $\mathcal{R}_h \in \mathrm{O}(2,1)$. Using (8), (9), (11) and the fact that $\mathcal{I}-\mathcal{C}$ and $\mathcal{I}+\mathcal{C}$ commute we obtain

$$\begin{aligned}
\mathcal{R}_h^t\eta\mathcal{R}_h &= ((\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1})^t\eta(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}\\
&= (\mathcal{I}-\mathcal{C}^t)^{-1}(I-\eta\mathcal{C}\eta)\eta(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}\\
&= (\mathcal{I}-\mathcal{C}^t)^{-1}(\eta-\eta\mathcal{C})(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = (\mathcal{I}-\mathcal{C}^t)^{-1}\eta(\mathcal{I}+\mathcal{C})\\
&= \left(\mathcal{I} + \frac{1}{1-(\eta\boldsymbol{c}).(\eta^2\boldsymbol{c})}\mathcal{C}^t + \frac{1}{1-(\eta\boldsymbol{c}).(\eta^2\boldsymbol{c})}(\mathcal{C}^t)^2\right)\eta(\mathcal{I}+\mathcal{C}) \qquad (12)\\
&= \left(\eta^2 - \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\eta\mathcal{C}\eta + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\eta\mathcal{C}^2\eta\right)\eta(\mathcal{I}+\mathcal{C})\\
&= \eta\left(\mathcal{I} - \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\eta\mathcal{C}\eta + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\eta\mathcal{C}^2\eta\right)\eta^2(\mathcal{I}+\mathcal{C})\\
&= \eta(\mathcal{I}+\mathcal{C})^{-1}(\mathcal{I}+\mathcal{C}) = \eta.
\end{aligned}$$

Calculation shows that $\det(\mathcal{I}+\mathcal{C}) = \det(\mathcal{I}-\mathcal{C}) = 1-\boldsymbol{c}.\eta\boldsymbol{c}$ which is equavalent to $\det\mathcal{R}_h = 1$. Thus the *Cayley* map (10) takes values in $\mathrm{SO}(2,1)$. Similar technique has been used in [13]. By making use of (11) we obtain the explicit form of (10), i.e.,

$$\begin{aligned}
\mathcal{R}_h(\boldsymbol{c}) = \mathrm{Cay}_{\mathfrak{so}(2,1)}(\mathcal{C}) &= (\mathcal{I}+\mathcal{C})(\mathcal{I} + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C} + \frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C}^2)\\
&= \mathcal{I} + \frac{2}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C} + \frac{2}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\mathcal{C}^2 \qquad (13)\\
&= \frac{2}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\begin{pmatrix} 1-c_1^2 & c_1c_2+c_3 & -c_1c_3-c_2\\ c_1c_2-c_3 & 1-c_2^2 & c_2c_3-c_1\\ c_1c_3-c_2 & -c_2c_3-c_1 & 1+c_3^2\end{pmatrix} - \mathcal{I}.
\end{aligned}$$

Note we can express the *type* of $\mathcal{R}_h(\boldsymbol{c})$ in terms of $\boldsymbol{c}.\eta\boldsymbol{c}$ in the following way

$$\mathcal{R}_h(\boldsymbol{c}) \text{ is } \begin{cases} \textit{hyperbolic} & \text{if } \ |\operatorname{tr}\mathcal{R}_h| > 3 \ \Leftrightarrow \ 0 < \boldsymbol{c}.\eta\boldsymbol{c} < 1\\ \textit{parabolic} & \text{if } \ |\operatorname{tr}\mathcal{R}_h| = 3 \ \Leftrightarrow \ \boldsymbol{c}.\eta\boldsymbol{c} = 0\\ \textit{elliptic} & \text{if } \ |\operatorname{tr}\mathcal{R}_h| < 3 \ \Leftrightarrow \ \boldsymbol{c}.\eta\boldsymbol{c} < 0. \end{cases} \quad (14)$$

The inverse of the *Cayley* map (10) is

$$\mathcal{C} = (\mathcal{R}_h - \mathcal{I})(\mathcal{R}_h + \mathcal{I})^{-1} \tag{15}$$

and it is well-defined if and only if $\det(\mathcal{R}_h + \mathcal{I}) \neq 0$ which is equivalent to the condition that $\mathcal{R}_h$ does not have an eigenvalue of -1. We will need to characterize the $\mathrm{SO}(2,1)$ elements, that are not in the range of the $Cayley$ map.

Lemma 1. *Let the element* $H = H(\alpha, \beta) = H(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathrm{SO}(2,1)$ *with* $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$ *be such that* $H \neq \mathcal{I}$. *The following are equivalent*

i) H *has as eigenvalue* -1, *i.e.,* $\det(H + \mathcal{I}) = 0$.

ii) $H = H(0, \alpha_2, \beta_1, \beta_2)$.

iii) $\eta H^t = H\eta$.

Proof: i) ⇔ ii). After straight, but tedious calculations one can obtain the equality $\det(H + \mathcal{I}) = 8(1 - \alpha_2^2 + \beta_1^2 + \beta_2^2)$. Thus, $\det(H + \mathcal{I}) = 0$ if and only if $\alpha_2^2 - \beta_1^2 - \beta_2^2 = 1$ but given the fact that $\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1$, this is equivalent to $\alpha_1 = 0$.

ii) ⇔ iii) Let us mention that $H^t = \eta H \eta \Longleftrightarrow \eta H^t = H\eta$. Again, after a straight calculation we obtain

$$H^t - \eta H \eta = 4\alpha_1 \begin{pmatrix} 0 & \alpha_2 & -\beta_2 \\ -\alpha_2 & 0 & -\beta_1 \\ -\beta_2 & -\beta_1 & 0 \end{pmatrix}. \tag{16}$$

Now it is clear that $\alpha_1 = 0 \Rightarrow H^t = \eta H \eta$. Conversely, let $H^t = \eta H \eta$. Then we have either $\alpha_1 = 0$ or $\alpha_2 = \beta_1 = \beta_2 = 0$. If we suppose that the second holds true then from $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$ it follows that $\alpha_1 = \pm 1$. Then $H = \mathcal{I}$, a contradiction. Thus, $\alpha_1 = 0$, which completes the proof. ■

An $\mathrm{SO}(2,1)$ element for which whatever of the conditions from Lemma 1 is fulfilled will be denoted by $\mathcal{O}_h$. Let us denote

$$\mathcal{O}_H = \{H \in \mathrm{SO}(2,1);\, \eta H^t = H\eta\}, \qquad \mathcal{R}_H = \mathrm{SO}(2,1) \backslash \mathcal{O}_H.$$

Any element $\mathcal{O}_h$ of $\mathcal{O}_H$ shall be called a pseudo half-turn, whereas any element $\mathcal{R}_h$ of the set $\mathcal{R}_H$ shall be called a proper pseudo rotation.

Relying on (3) one can check that the arbitrary $\mathcal{O}_H$ element $\mathcal{O}_h = H(0, \alpha_2, \beta_1, \beta_2)$ has the following matrix form

$$\mathcal{O}_h = 2\begin{pmatrix} -\beta_1^2 & \beta_1\beta_2 & \alpha_2\beta_1 \\ \beta_1\beta_2 & -\beta_2^2 & -\alpha_2\beta_2 \\ -\alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2^2 \end{pmatrix} - \mathcal{I}, \qquad \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1. \tag{17}$$

Note that such matrices are elliptic elements in accordance with (14).

Finally, let us invert (10) explicitly. Let $\mathcal{R}_h = \mathcal{R}_h(\alpha_1, \alpha_2, \beta_1, \beta_2)$ be a regular $\mathrm{SO}(2,1)$ element (i.e., such that $\det(\mathcal{R}_h + \mathcal{I}) \neq 0$). Thus from Lemma 1 it follows that $\alpha_1 \neq 0$. Taking into account (13) we obtain

$$\begin{aligned} \operatorname{tr}\mathcal{R}_h &= 4\alpha_1^2 - 1 = 2\frac{3 - \boldsymbol{c}.\eta\boldsymbol{c}}{1 - \boldsymbol{c}.\eta\boldsymbol{c}} - 3 \\ \mathcal{R}_h(2,3) + \mathcal{R}_h(3,2) &= 4\alpha_1\beta_1 = -4\frac{c_1}{1 - \boldsymbol{c}.\eta\boldsymbol{c}} \\ \mathcal{R}_h(1,3) + \mathcal{R}_h(3,1) &= 4\alpha_1\beta_2 = -4\frac{c_2}{1 - \boldsymbol{c}.\eta\boldsymbol{c}} \\ \mathcal{R}_h(1,2) - \mathcal{R}_h(2,1) &= 4\alpha_1\alpha_2 = 4\frac{c_3}{1 - \boldsymbol{c}.\eta\boldsymbol{c}}\cdot \end{aligned} \tag{18}$$

In the last three equations of (18) and further on, $\mathcal{R}_h(i,j)$ means the element at position (i,j) in the matrix $\mathcal{R}_h$. From (18) we obtain

$$\mathrm{Cay}^{-1}_{\mathfrak{so}(2,1)}(\mathcal{R}_h(\alpha_1, \alpha_2, \beta_1, \beta_2)) = \boldsymbol{c}\cdot\mathbf{P}, \qquad \boldsymbol{c} = \frac{1}{\alpha_1}(-\beta_1, -\beta_2, \alpha_2). \tag{19}$$

We have proved

Theorem 2. *The Cayley map is a vector-parameterization*

$$\mathrm{Cay}_{\mathfrak{so}(2,1)} : \mathfrak{so}(2,1)\backslash\{\mathcal{C}(\boldsymbol{c}) \in \mathfrak{so}(2,1)\,;\, \boldsymbol{c}.\eta\boldsymbol{c} \neq 1\} \longrightarrow \mathcal{R}_H. \tag{20}$$

Let $\mathcal{R}_h \in \mathrm{SO}(2,1)$ be a regular pseudo rotation. From (3) and Lemma 1 it follows that $1 + \operatorname{tr}\mathcal{R}_h \neq 0$. Taking into account formulas (18) we can retrieve the vector-parameter $\boldsymbol{c}$ which generates $\mathcal{R}_h = \mathcal{R}_h(\boldsymbol{c})$ by means of the general formula

$$\boldsymbol{c}^{\curlywedge} = \frac{\mathcal{R}_h^t - \eta\mathcal{R}_h\eta}{1 + \operatorname{tr}\mathcal{R}_h} \tag{21}$$

which simplifies in components to

$$\boldsymbol{c} = \frac{-1}{1 + \operatorname{tr}\mathcal{R}_h}(\mathcal{R}_h(2,3) + \mathcal{R}_h(3,2), \mathcal{R}_h(1,3) + \mathcal{R}_h(3,1), \mathcal{R}_h(2,1) - \mathcal{R}_h(1,2)).$$

2.3. The Composition Law in SO(2,1)

We are interested in the composition law of $\mathrm{SO}(2,1)$ in vector-parameter form. Let us introduce the notation

$$\boldsymbol{c}_2 \curlywedge \boldsymbol{c}_1 := \eta(\boldsymbol{c}_2 \times \boldsymbol{c}_1).$$

Theorem 3. *Let* $\boldsymbol{c} = (c_1, c_2, c_3)$ *and* $\boldsymbol{a} = (a_1, a_2, a_3)$ *be such vectors for which* $\boldsymbol{c}.\eta\boldsymbol{c} \neq 1, \boldsymbol{a}.\eta\boldsymbol{a} \neq 1$ *and* $1 + \boldsymbol{a}.\eta\boldsymbol{c} \neq 0$. *Let* $\mathcal{R}_h(\boldsymbol{c}), \mathcal{R}_h(\boldsymbol{c})$ *are two* SO(2,1) *elements represented by* **c** *and* ***a***. *Then*

$$\mathcal{R}_h(\widetilde{\boldsymbol{c}}) = \mathcal{R}_h(\boldsymbol{a})\mathcal{R}_h(\boldsymbol{c}), \qquad \widetilde{\boldsymbol{c}}(\boldsymbol{a}, \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle_{\mathrm{SO}(2,1)} = \frac{\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{a} \curlywedge \boldsymbol{c}}{1 + \boldsymbol{a}.\eta\boldsymbol{c}} \cdot \tag{22}$$

Proof: The straightforward evaluation of the matrix $H = \mathcal{R}_h(\boldsymbol{a})\mathcal{R}_h(\boldsymbol{c})$ has as outcome

$$\det(H + \mathcal{I}) = 8\frac{(1 + \boldsymbol{a}.\eta\boldsymbol{c})^2}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})} \neq 0 \tag{23}$$

and the following equalities

$$\begin{aligned}
\operatorname{tr} H &= 4\frac{(1 + \boldsymbol{a}.\eta\boldsymbol{c})^2}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})} - 1 \\
H_{2,3} + H_{3,2} &= 4\frac{1 + \boldsymbol{a}.\eta\boldsymbol{c}}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})}(a_3c_2 - a_2c_3 - a_1 - c_1) \\
H_{1,3} + H_{3,1} &= 4\frac{1 + \boldsymbol{a}.\eta\boldsymbol{c}}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})}(a_1c_3 - a_3c_1 - a_2 - c_2) \\
H_{1,2} - H_{2,1} &= 4\frac{1 + \boldsymbol{a}.\eta\boldsymbol{c}}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})}(-a_1c_2 + a_2c_1 + a_3 + c_3)
\end{aligned} \tag{24}$$

hold true. From Lemma 1, equation (23) and the conditions imposed in the Theorem it follows that there exists $\widetilde{\boldsymbol{c}}, \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}} \neq 1$ such that $\mathcal{R}_h(\widetilde{\boldsymbol{c}}) = H$. Comparing the equalities (18) (written for $\widetilde{\boldsymbol{c}}$) with (24) we obtain

$$\begin{aligned}
\frac{1}{1 - \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}} &= \frac{(1 + \boldsymbol{a}.\eta\boldsymbol{c})^2}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})} \\
\frac{4}{1 - \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}\widetilde{\boldsymbol{c}} &= 4\frac{(1 + \boldsymbol{a}.\eta\boldsymbol{c})}{(1 - \boldsymbol{a}.\eta\boldsymbol{a})(1 - \boldsymbol{c}.\eta\boldsymbol{c})}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{a} \curlywedge \boldsymbol{c}).
\end{aligned} \tag{25}$$

Now (22) follows easily from (25). Note that this result can also be obtained from equation (21). Let us also mention that from (23) it follows that if $\boldsymbol{a}.(\eta\boldsymbol{c}) + 1 = 0$, and thus the resulting $\mathrm{SO}(2,1)$ element is a *pseudo* half-turn. Latter on we will treat this case more rigorously. ■

Equation (22) is the vector-parameter form of SO(2,1) obtained by the parameterization given by the *Cayley* map. The same result was obtained independently by usage of pseudo-quaternions [3]. We are going to write explicitly the matrix forms of the different types of SO(2,1) elements in terms of (4). We will use $\mathrm{ch}x, \mathrm{sh}x$ and $\mathrm{th}x$ to denote the hyperbolic trigonometric functions $\mathrm{ch}x = \dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$, $\mathrm{sh}x = \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$ and $\mathrm{th}x = \dfrac{\mathrm{sh}x}{\mathrm{ch}x} = \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}\cdot$ We will also use $\mathrm{sc}x$ to denote the *secant* function $\dfrac{1}{\sin x}\cdot$ Till the end of the paper, for typographical reasons, we denote respectively the *tangent* and *cotangent* functions of the variable x as $\mathrm{tg}x$ and $\mathrm{ctg}x$. In the *hyperbolic* case, i.e., when $1 > \boldsymbol{c}.\eta\boldsymbol{c} > 0$ there exists $\theta \in \mathbb{R}, \mathbf{n} \in \mathbb{R}^{2,1}$ such that $\mathbf{n}.(\eta\mathbf{n}) = 1, \boldsymbol{c} = \mathrm{th}\frac{\theta}{2}\mathbf{n}$ and the matrix $\mathcal{R}_h(\boldsymbol{c})$ takes the form

$$\begin{pmatrix} n_1^2(1-\mathrm{ch}\theta)+\mathrm{ch}\theta & n_1n_2(-1+\mathrm{ch}\theta)+n_3\mathrm{sh}\theta & n_1n_3(-1+\mathrm{ch}\theta)-n_2\mathrm{sh}\theta \\ n_1n_2(-1+\mathrm{ch}\theta)-n_3\mathrm{sh}\theta & n_2^2(1-\mathrm{ch}\theta)+\mathrm{ch}\theta & n_2n_3(1-\mathrm{ch}\theta)-n_1\mathrm{ch}\theta \\ n_1n_3(-1+\mathrm{ch}\theta)-n_2\mathrm{sh}\theta & n_2n_3(1-\mathrm{ch}\theta)-n_1\mathrm{ch}\theta & n_3^2(-1+\mathrm{ch}\theta)+\mathrm{ch}\theta \end{pmatrix}.$$

In the *elliptic* case, i.e., $\boldsymbol{c}.\eta\boldsymbol{c} < 0$ there exist $\theta \in [0,\pi), \mathbf{n} \in \mathbb{R}^{2,1}$ such that $\mathbf{n}.(\eta\mathbf{n}) = -1, \boldsymbol{c} = \mathrm{tg}\frac{\theta}{2}\mathbf{n}$ and for the matrix $\mathcal{R}_h(\boldsymbol{c})$ we have that it has the form

$$\begin{pmatrix} n_1^2(1-\mathrm{sc}\theta)+\mathrm{sc}\theta & (n_1n_2\mathrm{tg}\frac{\theta}{2}+n_3)\mathrm{tg}\theta & n_1n_3(1-\mathrm{sc}\theta)-n_2\mathrm{tg}\theta \\ (n_1n_2\mathrm{tg}\frac{\theta}{2}-n_3)\mathrm{tg}\theta & n_2^2(1-\mathrm{ch}\theta)+\mathrm{sc}\theta & n_2n_3(-1+\mathrm{sc}\theta)-n_1\mathrm{tg}\theta \\ n_1n_3(-1+\mathrm{sc}\theta)-n_2\mathrm{tg}\theta & n_2n_3(1-\mathrm{sc}\theta)-n_1\mathrm{tg}\theta & n_3^2(-1+\mathrm{ch}\theta)+\mathrm{sc}\theta \end{pmatrix}.$$

Finally, in the *parabolic* case, i.e., $\boldsymbol{c}.\eta\boldsymbol{c} \equiv 0$ there exist $\mathbf{n} \in \mathbb{R}^{2,1}$ such that $\mathbf{n}.(\eta\mathbf{n}) = 0$ and

$$\mathcal{R}_h(\boldsymbol{c}) = \begin{pmatrix} 1-2n_1^2 & 2(n_1n_2+n_3) & -2(n_2+n_3n_1) \\ 2(n_1n_2-n_3) & 1-2n_2^2 & 2(n_2n_3-n_1) \\ -2(n_2-n_3n_1) & -2(n_2n_3+n_1) & 1+2n_3^2 \end{pmatrix}, \qquad \boldsymbol{c} = \mathbf{n}.$$

3. Vector-Parameter Form of SU(1,1)

3.1. Description of $\mathfrak{su}(1,1)$

Let us consider the Lie algebra $\mathfrak{su}(1,1)$ with $\mathbb{R}$-basis

$$\mathrm{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{e}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \mathrm{e}_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\cdot$$

The matrices E_1, E_2 and E_3 defined by

$$E_1 = \frac{1}{2}\mathrm{e}_1, \qquad E_2 = \frac{1}{2}\mathrm{e}_2, \qquad E_3 = \frac{1}{2}\mathrm{e}_3$$

also form a $\mathbb{R}-$basis of $\mathfrak{su}(1,1)$. Moreover, direct calculations confirm that

$$[E_1, E_2] = -E_3, \qquad [E_2, E_3] = E_1, \qquad [E_3, E_1] = E_2.$$

Denoting $\mathbf{E} = (E_1, E_2, E_3)$ we express the $\mathfrak{su}(1,1)$ algebra in the following way

$$\mathfrak{su}(1,1) = \left\{\mathbf{m}\cdot\mathbf{E} = m_1E_1 + m_2E_2 + m_3E_3\,;\ \mathbf{m}\in\mathbb{R}^3\right\}.$$

The corresponding matrix realization of $\mathbf{m}\cdot\mathbf{E}$ is

$$M = M(\mathbf{m}) = \mathbf{m}\cdot\mathbf{E} = \begin{pmatrix} \mathrm{i}\dfrac{m_3}{2} & \dfrac{m_1}{2} + \mathrm{i}\dfrac{m_2}{2} \\ \dfrac{m_1}{2} - \mathrm{i}\dfrac{m_2}{2} & -\mathrm{i}\dfrac{m_3}{2} \end{pmatrix}. \tag{26}$$

In the opposite direction and for any element $M \in \mathfrak{su}(1,1)$ we have

$$\mathbf{m} = \mathbf{m}(M) = 2(\operatorname{tr} M.E_1, \operatorname{tr} M.E_2, -\operatorname{tr} M.E_3).$$

Obviously, the map

$$m_1E_1 + m_2E_2 + m_3E_3 \longrightarrow m_1P_1 + m_2P_2 + m_3P_3 \tag{27}$$

is a linear isomorphism between the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$. Observe that for $M_1,\ M_2 \in \mathfrak{su}(1,1)$, i.e., $M_1 = \mathbf{m}_1\cdot\mathbf{E}$, $M_2 = \mathbf{m}_2\cdot\mathbf{E}$ one can obtain by direct calculations the useful equality

$$M_2M_1 = \frac{\mathbf{m}_2.\eta\mathbf{m}_1}{4}\mathcal{I} + \frac{\mathbf{m}_2 \curlywedge \mathbf{m}_1}{2}\cdot\mathbf{E}. \tag{28}$$

3.2. The Cayley Map for $\mathfrak{su}(1,1)$

Let $M = \mathbf{m}\cdot\mathbf{E} \in \mathfrak{su}(1,1)$. The *Hamilton–Cayley* theorem applied to M reads as

$$M^2 = \frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2}\mathcal{I}.$$

The *Cayley* map applied to $\mathfrak{su}(1,1)$ element M is

$$\mathcal{U}_h(\mathbf{m}) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(M) = (\mathcal{I}+M)(\mathcal{I}-M)^{-1} = (\mathcal{I}-M)^{-1}(\mathcal{I}+M). \tag{29}$$

Let us define

$$\Delta_{\mathbf{m}} = 1 - \frac{m_1^2 + m_2^2 - m_3^2}{4} = 1 - \frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2}$$

which is exactly $\det(\mathcal{I}-M)$. Thus, $\mathrm{Cay}_{\mathfrak{su}(1,1)}$ is well-defined when $\Delta_{\mathbf{m}} \neq 0$. The solutions of $\Delta_{\mathbf{m}} = 0$ geometrically form a simple (one sheeted) hyperboloid.

It is straightforward to check that $\Delta_{\mathbf{m}} \neq 0 \Rightarrow (\mathcal{I} - M)^{-1} = \dfrac{1}{\Delta_{\mathbf{m}}}(\mathcal{I} + M)$. Thus, we can explicitly calculate

$$\begin{aligned} \mathcal{U}_h(\mathbf{m}) &= \frac{1}{\Delta_{\mathbf{m}}}(\mathcal{I}+M)^2 = \frac{1}{\Delta_{\mathbf{m}}}(\mathcal{I}+2M+M^2) = \frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}}\mathcal{I} + \frac{2}{\Delta_{\mathbf{m}}}M \\ &= \frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}}\mathcal{I} + \frac{2}{\Delta_{\mathbf{m}}}\mathbf{m}\cdot\mathbf{E} = \frac{1}{\Delta_{\mathbf{m}}}\begin{pmatrix} 2-\Delta_{\mathbf{m}}+\mathrm{i}m_3 & m_1+\mathrm{i}m_2 \\ m_1-\mathrm{i}m_2 & 2-\Delta_{\mathbf{m}}-\mathrm{i}m_3 \end{pmatrix}. \end{aligned} \tag{30}$$

A direct calculation shows that $\det \mathcal{U}_h(\mathbf{m}) = 1$ and also $\mathcal{U}_h^\dagger(\mathbf{m})\eta_1\mathcal{U}_h(\mathbf{m}) = \eta_1$ which means that when the *Cayley* map is defined it takes values in $\mathrm{SU}(1,1)$.

To invert the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(1,1)} : \mathfrak{su}(1,1) \to \mathrm{SU}(1,1)$ let us consider an arbitrary $\mathrm{SU}(1,1)$ matrix

$$\mathcal{M}_h(\alpha_1,\alpha_2,\beta_1,\beta_2) = \begin{pmatrix} \alpha_1+\mathrm{i}\alpha_2 & \beta_1+\mathrm{i}\beta_2 \\ \beta_1-\mathrm{i}\beta_2 & \alpha_1-\mathrm{i}\alpha_2 \end{pmatrix}, \qquad \alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2 = 1.$$

The matrix $\mathcal{M}_h \in \mathrm{Im}\,\mathrm{Cay}_{\mathfrak{su}(1,1)}$ if and only if there exist $\mathbf{m} \in \mathbb{R}^3, \Delta_m \neq 0$ such that we have $\mathrm{Cay}_{\mathfrak{su}(1,1)}(M(\mathbf{m})) = \mathcal{M}_h(\mathbf{m})$. This is only possible if and only if $\alpha_1 \neq -1$ and provided that the inversion is

$$\mathbf{m} = \frac{2}{1+\alpha_1}(\beta_1,\beta_2,\alpha_2).$$

Let us note that the $\mathrm{SU}(1,1)$ elements that can not be parametrized are of a parabolic type. We can identify the group $\mathrm{SU}(1,1)$ with the points of $\mathbb{R}^4$ for which $\alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2 = 1$. Now only the cone $\alpha_2^2-\beta_1^2-\beta_2^2 = 0$ is not covered by the *Cayley* map. Let us introduce notation for the *regular* SU(1,1) elements (those which are parametrized by the *Cayley* map)

$$\begin{aligned} \mathcal{U}_H &= \{\mathcal{U}_h = \mathcal{M}_h(\alpha,\beta) \in \mathrm{SU}(1,1);\, \alpha_1 \neq -1, \alpha\overline{\alpha} - \beta\overline{\beta} = 1\} \\ &= \{\mathcal{M}_h \in \mathrm{SU}(1,1);\ \mathrm{tr}\,\mathcal{M}_h \neq -2\}. \end{aligned}$$

We have proved the following

Theorem 4. *The Cayley map is a vector-parameterization*

$$\mathrm{Cay}_{\mathfrak{su}(1,1)} : \{\mathbf{m}\cdot\mathbf{E} \in \mathfrak{su}(1,1);\, \Delta_{\mathbf{m}} \neq 0\} \longrightarrow \mathcal{U}_H. \tag{31}$$

Further, we will use the notation $\mathcal{W}_H$ for the set of $\mathrm{SU}(1,1)\backslash\mathcal{U}_H$ group and $\mathcal{W}_h$ for its elements.

Let $\mathcal{U}_h \in \mathcal{U}_H$. We want to extract the vector-parameter $\mathbf{m}$ from the matrix $\mathcal{U}_h$. Relying on formula (30) we have $\Delta_{\mathbf{m}} = \dfrac{4}{2 + \operatorname{tr}\mathcal{U}_h}$ and therefore

$$\mathbf{m} = \frac{2 + \operatorname{tr}\mathcal{M}_h}{4}\big(\Re(\mathcal{U}_{h1,2}), \Im(\mathcal{U}_{h1,2}), \Im(\mathcal{U}_{h1,1})\big) \tag{32}$$

where $\Re z$ and $\Im z$ denote the real and imaginary part of the complex number z.

3.3. Vector-Parameter Form of $\mathrm{SL}(2, \mathbb{R})$

In order to obtain a vector-parameter form of $\mathrm{SL}(2, \mathbb{R})$ by taking advantage of the already obtained such of $\mathrm{SU}(1, 1)$ we need to invert ψ in vector-parameter form and calculate $\psi^{-1}(\mathcal{U}_h(\mathbf{m}))$ where $\mathbf{m} \in \mathbb{R}^3$, $\Delta_{\mathbf{m}} \neq 0$. Direct calculations shows that

$$\psi(\mathbf{m}) := \psi^{-1}(\mathcal{U}_h(\mathbf{m})) = \mathcal{L}(\mathbf{m}) = \frac{1}{\Delta_{\mathbf{m}}}\begin{pmatrix} 2 - \Delta_{\mathbf{m}} + m_2 & m_1 + m_3 \\ m_1 - m_3 & 2 - \Delta_m - m_2 \end{pmatrix}.$$

The isomorphism (1) is trace-preserving since it is a conjugation. Thus

$$\operatorname{tr}\mathcal{U}_h(\mathbf{m}) = \operatorname{tr}\mathcal{L}(\mathbf{m}) = \frac{4 - 2\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}}, \qquad \Delta_{\mathbf{m}} \neq 0.$$

In a straightforward manner we obtain that if $\Delta_{\mathbf{m}} \neq 0$ then

$$\mathcal{L}(\mathbf{m}) \text{ is } \begin{cases} \text{hyperbolic} & \text{if } \ \Delta_{\mathbf{m}} < 1 \ \Leftrightarrow \ \mathbf{m}.\eta\mathbf{m} > 0 \\ \text{parabolic} & \text{if } \ \Delta_{\mathbf{m}} = 1 \ \Leftrightarrow \ \mathbf{m}.\eta\mathbf{m} = 0 \\ \text{elliptic} & \text{if } \ \Delta_{\mathbf{m}} > 1 \ \Leftrightarrow \ \mathbf{m}.\eta\mathbf{m} < 0. \end{cases} \tag{33}$$

Of course, because the *Cayley* map is not onto the whole $\mathrm{SU}(1, 1)$ we parametrize only a subset of $\mathrm{SL}(2, \mathbb{R})$, i.e., $\psi[\Im[\mathrm{Cay}_{\mathfrak{su}(1,1)}]] \subset \mathrm{SL}(2, \mathbb{R})$. Actually, the $\mathrm{SL}(2, \mathbb{R})$ matrices that can not be parametrized are of the type

$$\begin{pmatrix} 1 + \beta_2 & \alpha_2 + \beta_1 \\ -\alpha_2 + \beta_1 & 1 - \beta_2 \end{pmatrix}, \qquad \alpha_2^2 - \beta_1^2 - \beta_2^2 = 0.$$

Based on the above observations, the Lie algebra $\mathfrak{su}(1, 1)$, can be viewed geometrically as splitting of $\mathbb{R}^3$ into the union of the sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$

$$\begin{aligned} \mathcal{X}_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 - x_3^2 > 0,\ x_1^2 + x_2^2 - x_3^2 \neq 4\} \\ \mathcal{X}_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 - x_3^2 < 0\} \\ \mathcal{X}_3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 - x_3^2 = 0\} \\ \mathcal{X}_4 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 - x_3^2 = 4\}. \end{aligned} \tag{34}$$

The set $\mathcal{X}_1$ consists of the preimages of the hyperbolic elements of $\mathrm{SU}(1,1)$ (respectfully, SL(2, $\mathbb{R}$)) and is open. Geometrically, it consists of the exterior of the cone $x_1^2+x_2^2-x_3^2=0$ except the points lying on the hyperboloid $x_1^2+x_2^2-x_3^2=4$ and thus consists of two disjoint open and connected sets, i.e., $\mathcal{X}_1 = \mathcal{X}_1' \cup \mathcal{X}_2''$, where

$$\begin{aligned}\mathcal{X}_1' &= \{(x_1,x_2,x_3)\in\mathbb{R}^3;\ x_1^2+x_2^2-x_3^2>0,\ x_1^2+x_2^2-x_3^2<4\}\\ \mathcal{X}_2' &= \{(x_1,x_2,x_3)\in\mathbb{R}^3;\ x_1^2+x_2^2-x_3^2>0,\ x_1^2+x_2^2-x_3^2>4\}.\end{aligned} \tag{35}$$

The open and disconnected set $\mathcal{X}_2$ consists of the preimages of the elliptic elements of $\mathrm{SU}(1,1)$ and geometrically is the interior of the cone $x_1^2+x_2^2-x_3^2=0$. The closed and connected set $\mathcal{X}_3$ consists of the preimages of the parabolic elements of $\mathrm{SU}(1,1)$. The set $\mathcal{X}_4$ is where $\mathrm{Cay}_{\mathfrak{su}(1,1)}$ is not defined.

Until the end of the paper we would call not only the $\mathrm{SU}(1,1)$ and SL(2, $\mathbb{R}$) matrices hyperbolic, elliptic and parabolic but also the vector-parameters $\mathbf{m}$ for which $\Delta_{\mathbf{m}} \neq 0$ i.e., the vector-parameters $\mathbf{m}\in\mathbb{R}^3\backslash\mathcal{X}_4$.

3.4. The Composition Law in SU(1,1)

Let us start by introducing for any two vectors $\mathbf{m}_1$ and $\mathbf{m}_2\in\mathbb{R}^3$ the scalar function

$$E(\mathbf{m}_2,\mathbf{m}_1) := (\mathbf{m}_2.\eta\mathbf{m}_2)(\mathbf{m}_1.\eta\mathbf{m}_1)+8\mathbf{m}_2.\eta\mathbf{m}_1+16. \tag{36}$$

Theorem 5. *Let $M_1, M_2\in\mathfrak{su}(1,1)$, i.e., $M_1=\mathbf{m}_1\cdot\mathbf{E}$, $M_2=\mathbf{m}_2\cdot\mathbf{E}$ be such that $\Delta_{\mathbf{m}_i}\neq 0$, $i=1,2$ and $E(\mathbf{m}_2,\mathbf{m}_1)\neq 0$. Let $\mathcal{U}_{hi}(\mathbf{m}_i)=\mathrm{Cay}_{\mathfrak{su}(1,1)}(M_i), i=1,2$ be the images under the Cayley map of M_1 and M_2. Then, if $\mathcal{M}_h=\mathcal{U}_{h2}.\mathcal{U}_{h1}$ is the composition of the images in $\mathrm{SU}(1,1)$ then $\mathcal{M}_h=\mathrm{Cay}_{\mathfrak{su}(1,1)}(\widetilde{M})$ where $\widetilde{M}=\widetilde{\mathbf{m}}\cdot\mathbf{E}$, $\widetilde{\mathbf{m}}=\langle\mathbf{m}_2,\mathbf{m}_1\rangle_{\mathrm{SU}(1,1)}$ and*

$$\widetilde{\mathbf{m}} = \frac{(1+\frac{\mathbf{m}_1}{2}\cdot\eta\frac{\mathbf{m}_1}{2})\mathbf{m}_2+(1+\frac{\mathbf{m}_2}{2}\cdot\eta\frac{\mathbf{m}_2}{2})\mathbf{m}_1+\mathbf{m}_2\curlywedge\mathbf{m}_1}{1+2\frac{\mathbf{m}_2}{2}\cdot(\eta\frac{\mathbf{m}_1}{2})+(\frac{\mathbf{m}_2}{2}\cdot\eta\frac{\mathbf{m}_2}{2})(\frac{\mathbf{m}_1}{2}\cdot\eta\frac{\mathbf{m}_1}{2})}\cdot \tag{37}$$

Proof: Let us calculate

$$\begin{aligned}\mathcal{U}_{h2}(\mathbf{m}_2).\mathcal{U}_{h1}(\mathbf{m}_1) &= \Big(\frac{2-\Delta_{\mathbf{m}_2}}{\Delta_{\mathbf{m}_2}}\mathcal{I}+\frac{2}{\Delta_{\mathbf{m}_2}}\mathbf{m}_2\cdot\mathbf{E}\Big)\Big(\frac{2-\Delta_{\mathbf{m}_1}}{\Delta_{\mathbf{m}_1}}\mathcal{I}+\frac{2}{\Delta_{\mathbf{m}_1}}\mathbf{m}_1\cdot\mathbf{E}\Big)\\ &\overset{(28)}{=} \frac{(2-\Delta_{\mathbf{m}_2})(2-\Delta_{\mathbf{m}_1})+\mathbf{m}_2\cdot(\eta\mathbf{m}_1)}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}\mathcal{I}\\ &\quad+\frac{(2-\Delta_{\mathbf{m}_1})\mathbf{m}_2+(2-\Delta_{\mathbf{m}_2})\mathbf{m}_1+\mathbf{m}_2\curlywedge\mathbf{m}_1}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}\cdot\mathbf{E}.\end{aligned}$$

Now, the condition for existence of $\widetilde{\mathbf{m}}$ is

$$\frac{(2-\Delta_{\mathbf{m}_2})(2-\Delta_{\mathbf{m}_1})+\mathbf{m}_2\cdot(\eta\mathbf{m}_1)}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}} \neq -1$$
$$\iff 1+(1-\Delta_{\mathbf{m}_2})(1-\Delta_{\mathbf{m}_1})+2\frac{\mathbf{m}_2}{2}\cdot\eta\frac{\mathbf{m}_1}{2}\neq 0.$$

From the definition of Δ we have that $1-\Delta_{\mathbf{m}_i}=\frac{\mathbf{m}_i}{2}\cdot\eta\frac{\mathbf{m}_i}{2}$, $i=1,2$ and thus the existence condition is equivalent to $E(\mathbf{m}_2,\mathbf{m}_1)\neq 0$. Thus, $\widetilde{\mathbf{m}}$ exists and is such that

$$\begin{aligned}\frac{2-\Delta_{\widetilde{\mathbf{m}}}}{\Delta_{\widetilde{\mathbf{m}}}} &= \frac{(2-\Delta_{\mathbf{m}_2})(2-\Delta_{\mathbf{m}_1})+\mathbf{m}_2.\eta\mathbf{m}_1}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}\\ \frac{2}{\Delta_{\widetilde{\mathbf{m}}}}\widetilde{\mathbf{m}} &= \frac{(2-\Delta_{\mathbf{m}_1})\mathbf{m}_2+(2-\Delta_{\mathbf{m}_2})\mathbf{m}_1+\mathbf{m}_2\curlywedge\mathbf{m}_1}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}.\end{aligned}\tag{38}$$

From (38) we immediately find

$$\Delta_{\widetilde{\mathbf{m}}}=\frac{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}{1+2\frac{\mathbf{m}_2}{2}\cdot\eta\frac{\mathbf{m}_1}{2}+\left(\frac{\mathbf{m}_2}{2}\cdot\eta\frac{\mathbf{m}_2}{2}\right)\left(\frac{\mathbf{m}_1}{2}\cdot\eta\frac{\mathbf{m}_1}{2}\right)}\neq 0$$

and thus again from (38)

$$\widetilde{\mathbf{m}} = \frac{(2-\Delta_{\mathbf{m}_1})\mathbf{m}_2+(2-\Delta_{\mathbf{m}_2})\mathbf{m}_1+\mathbf{m}_2\curlywedge\mathbf{m}_1}{1+\frac{1}{2}\mathbf{m}_2.\eta\mathbf{m}_1+(1-\Delta_{\mathbf{m}_2})(1-\Delta_{\mathbf{m}_1})}$$

from where follows the equality (37). Finally, note that $E(\mathbf{m}_2,\mathbf{m}_1)$ is exactly sixteen times the denominator of (37). ■

Remark 6. *What happens in the case when* $E(\mathbf{m}_2,\mathbf{m}_1)=0$*? In this case there is no vector* $\mathbf{m}$ *to parameterize the matrix parabollic element* $\mathcal{U}_{h2}(\mathbf{m}_2)\mathcal{U}_{h1}(\mathbf{m}_1)$*. However, the element* $-\mathcal{U}_{h2}(\mathbf{m}_2)\mathcal{U}_{h1}(\mathbf{m}_1)$ *is parameterizable by the vector-parameter*

$$\hat{\mathbf{m}}=-\frac{(2-\Delta_{\mathbf{m}_1})\mathbf{m}_2+(2-\Delta_{\mathbf{m}_2})\mathbf{m}_1+\mathbf{m}_2\curlywedge\mathbf{m}_1}{\Delta_{\mathbf{m}_2}\Delta_{\mathbf{m}_1}}.$$

Of course, both the elements $\mathcal{U}_{h2}(\mathbf{m}_2)\mathcal{U}_{h1}(\mathbf{m}_1)$ *and* $-\mathcal{U}_{h2}(\mathbf{m}_2)\mathcal{U}_{h1}(\mathbf{m}_1)$ *are projected by* (3) *into one and the same* SO(2,1) *element.*

Remark 7. *What about when* $\mathbf{m}_2\equiv\mathbf{m}_1$*? Then it is immediate to see that the condition* $E(\mathbf{m}_2,\mathbf{m}_1)=0$ *is equivalent to* $\mathbf{m}_1.\eta\mathbf{m}_1=-4$*. If this is the case, direct substitution leads to* $\mathcal{U}_{h2}(\mathbf{m}_2)\mathcal{U}_{h1}(\mathbf{m}_1)=\mathcal{I}$*. If* $\mathbf{m}_1.\eta\mathbf{m}_1\neq-4$ *then we obtain* $\widetilde{\mathbf{m}}=\dfrac{2\mathbf{m}_1}{1+\frac{\mathbf{m}_1}{2}\cdot\eta\frac{\mathbf{m}_1}{2}}$.

The condition $E(\mathbf{m}_2, \mathbf{m}_1) \neq 0$ is equivalent to

$$(4 + \mathbf{m}_2.\eta\mathbf{m}_1)^2 - (\mathbf{m}_2 \curlywedge \mathbf{m}_1).\eta(\mathbf{m}_2 \curlywedge \mathbf{m}_1) \neq 0 \tag{39}$$

since for $\cdot$ and $\curlywedge$ operations one can easily check that the equality

$$(\mathbf{x} \curlywedge \mathbf{y}).\eta(\mathbf{x} \curlywedge \mathbf{y}) = (\mathbf{x}.\eta\mathbf{y})^2 - (\mathbf{x}.\eta\mathbf{x})(\mathbf{y}.\eta\mathbf{y})$$

holds for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. It is interesting to compare it with the standard vector and scalar products in $\mathbb{R}^3$

$$(\mathbf{x} \times \mathbf{y})^2 = \mathbf{x}^2\mathbf{y}^2 - (\mathbf{x}.\mathbf{y})^2.$$

Let us calculate $\frac{\widetilde{\mathbf{m}}}{2} \cdot \eta \frac{\widetilde{\mathbf{m}}}{2} = 1 - \Delta_{\widetilde{\mathbf{m}}}$. We obtain that if $\widetilde{\mathbf{m}}$ exists (i.e., when (39) holds), then

$$\begin{aligned}\frac{\widetilde{\mathbf{m}}}{2} \cdot \eta \frac{\widetilde{\mathbf{m}}}{2} &= \frac{\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_2}{2} + 2\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_1}{2} + \frac{\mathbf{m}_1}{2} \cdot \eta \frac{\mathbf{m}_1}{2}}{1 + 2\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_1}{2} + (\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_2}{2})(\frac{\mathbf{m}_1}{2} \cdot \eta \frac{\mathbf{m}_1}{2})} \\ &= 4\frac{(\mathbf{m}_2 + \mathbf{m}_1).\eta(\mathbf{m}_2 + \mathbf{m}_1)}{(4 + (\mathbf{m}_2.\eta\mathbf{m}_1))^2 - (\mathbf{m}_2 \curlywedge \mathbf{m}_1).\eta(\mathbf{m}_2 \curlywedge \mathbf{m}_1)}.\end{aligned} \tag{40}$$

Next, we need the formula

$$1 + \frac{\widetilde{\mathbf{m}}}{2} \cdot \eta \frac{\widetilde{\mathbf{m}}}{2} = \frac{(1 + \frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_2}{2})(1 + \frac{\mathbf{m}_1}{2} \cdot \eta \frac{\mathbf{m}_1}{2}) + 4\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_1}{2}}{1 + 2\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_1}{2} + (\frac{\mathbf{m}_2}{2} \cdot \eta \frac{\mathbf{m}_2}{2})(\frac{\mathbf{m}_1}{2} \cdot \eta \frac{\mathbf{m}_1}{2})}. \tag{41}$$

Now, readily from the definitions (34) of the sets $\mathcal{X}_i, i = 1, 2, 3$ we obtain a conditions for determining the type (hyperbolic, elliptic, parabolic) of the composition of two SU(1,1) vector-parameters.

Corollary 8. *In the terms of the conditions and the notation from* Theorem 5*, for the composition vector* $\widetilde{\mathbf{m}}$ *we have that*

$$M(\widetilde{\mathbf{m}}) \text{ is } \begin{cases} hyperbolic & \text{if} \quad \xi\zeta > 0 \\ parabolic & \text{if} \quad \xi = 0 \quad \text{or} \quad \zeta = 0 \\ elliptic & \text{if} \quad \xi\zeta < 0 \end{cases} \tag{42}$$

where

$$\begin{aligned}\xi &= \xi(\mathbf{m}_2, \mathbf{m}_1) = (\mathbf{m}_2 + \mathbf{m}_1).(\eta(\mathbf{m}_2 + \mathbf{m}_1)) \\ \zeta &= \zeta(\mathbf{m}_2, \mathbf{m}_1) = (4 + \mathbf{m}_2.\eta\mathbf{m}_1)^2 - (\mathbf{m}_2 \curlywedge \mathbf{m}_1).(\eta(\mathbf{m}_2 \curlywedge \mathbf{m}_1)).\end{aligned} \tag{43}$$

3.5. The Double Covering $\mathrm{SU}(1,1) \to \mathrm{SO}(2,1)$

We are interested in the vector-parameter form of φ from (3). Let us consider the $\mathrm{SU}(1,1)$ matrix $\mathcal{M}_h(\mathbf{m}) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathbf{m})$. We have

$$\alpha = \frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}} + \mathrm{i}\frac{m_3}{\Delta_{\mathbf{m}}}, \qquad \beta = \frac{m_1}{\Delta_{\mathbf{m}}} + \mathrm{i}\frac{m_2}{\Delta_{\mathbf{m}}}\cdot \tag{44}$$

Substitution of (44) in (3) leads to the explicit formula for $\varphi(\mathbf{m}) = \mathcal{M}_h(\mathbf{m})$

$$H(\mathbf{m}) = \frac{2}{\Delta_{\mathbf{m}}^2}\begin{pmatrix} m_2^2 - m_3^2 & m_1m_2 + m_3\widetilde{\Delta}_{\mathbf{m}} & m_1m_3 + m_2\widetilde{\Delta}_{\mathbf{m}} \\ m_1m_2 - m_3\widetilde{\Delta}_{\mathbf{m}} & m_1^2 - m_3^2 & -m_2m_3 + m_1\widetilde{\Delta}_{\mathbf{m}} \\ -m_1m_3 + m_2\widetilde{\Delta}_{\mathbf{m}} & m_2m_3 + m_1\widetilde{\Delta}_{\mathbf{m}} & m_1^2 + m_2^2 \end{pmatrix} + \mathcal{I} \tag{45}$$

where $\widetilde{\Delta}_{\mathbf{m}} = 2 - \Delta_{\mathbf{m}}$.

It is important to note that all $\mathrm{SO}(2,1)$ matrices can be parametrized by $\mathbf{m}$ despite the fact that the $\mathrm{SU}(1,1)$ elements of the type $\mathcal{M}_h(-1, \alpha_2, \beta_1, \beta_2)$ can not parametrized, see also Remark 6. Indeed, if $\varphi(\mathcal{M}_h) = H(-1, \alpha_2, \beta_1, \beta_2)$ than also

$$\varphi(\mathcal{M}_h(1, \alpha_2, \beta_1, \beta_2)) = \mathcal{R}_h(-1, \alpha_2, \beta_1, \beta_2) = \mathcal{M}_h((\beta_1, \beta_2, \alpha_2)).$$

The vector-parameter form (45) of (3) will allow us to establish the connection between the vector-parameter in SO(2,1) and its cover SU(1,1).

Theorem 9. *Let $\mathcal{M}_h(\mathbf{m})$ is an $\mathrm{SU}(1,1)$ element, represented by the vector-parameter $\mathbf{m}$ such that $\frac{\mathbf{m}}{2} \cdot (\eta\frac{\mathbf{m}}{2}) \neq -1$. Then in $\mathrm{SO}(2,1)$ this element is represented by the matrix $\mathcal{R}_h(\boldsymbol{c})$ generated by the vector*

$$\boldsymbol{c} = -\frac{\eta\mathbf{m}}{1 + \frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2}}\cdot \tag{46}$$

On the other hand, if the vector $\boldsymbol{c}$ represents the $\mathrm{SO}(2,1)$ element $\mathcal{R}_h(\boldsymbol{c})$, then in $\mathrm{SU}(1,1)$ the elements represented by the vector-parameters

$$\mathbf{m}_+(\boldsymbol{c}) = -2\frac{1+\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}\eta\boldsymbol{c}, \qquad \mathbf{m}_-(\boldsymbol{c}) = -2\frac{1-\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}\eta\boldsymbol{c} \tag{47}$$

produce the same element $\mathcal{R}_h(\boldsymbol{c})$ provided that $\boldsymbol{c}.\eta\boldsymbol{c} \neq 0$, i.e., $\mathcal{R}_h(\boldsymbol{c})$ is not parabolic. If $\boldsymbol{c}.\eta\boldsymbol{c} = 0$ then we have

$$\mathbf{m}(\boldsymbol{c}) = -\eta\boldsymbol{c}. \tag{48}$$

Moreover, the following relations hold

$$\begin{gathered}\mathbf{m}_+.\eta\mathbf{m}_- = 4, \qquad (\mathbf{m}_+.\eta\mathbf{m}_+)(\mathbf{m}_-.(\eta\mathbf{m}_-)) = 16 \\ \mathbf{m}_+ = -\frac{4}{\mathbf{m}_-.\eta\mathbf{m}_-}\mathbf{m}_-, \qquad \mathbf{m}_- = -\frac{4}{\mathbf{m}_+.\eta\mathbf{m}_+}\mathbf{m}_+.\end{gathered} \tag{49}$$

Proof: Let us equate some expressions by using the formulas for $\mathcal{R}_h(\boldsymbol{c})$ (cf. (13)) and $H(\mathbf{m})$ (cf. (45))

$$\begin{aligned} \operatorname{tr}\mathcal{R}_h(\boldsymbol{c}) &= \operatorname{tr} H(\mathbf{m}) \\ \mathcal{R}_h(\boldsymbol{c})_{1,2} - \mathcal{R}_h(\boldsymbol{c})_{2,1} &= H(\mathbf{m})_{1,2} - H(\mathbf{m})_{2,1} \\ \mathcal{R}_h(\boldsymbol{c})_{1,3} + \mathcal{R}_h(\boldsymbol{c})_{3,1} &= H(\mathbf{m})_{1,3} + H(\mathbf{m})_{3,1} \\ \mathcal{R}_h(\boldsymbol{c})_{2,3} + \mathcal{R}_h(\boldsymbol{c})_{3,2} &= H(\mathbf{m})_{2,3} + H(\mathbf{m})_{3,2}. \end{aligned} \tag{50}$$

The first equation in (50) is equivalent to

$$\frac{2}{1-\boldsymbol{c}.\eta\boldsymbol{c}}(3-\boldsymbol{c}.\eta\boldsymbol{c}) - 3 = \frac{4}{\Delta_{\mathbf{m}}^2}\mathbf{m}.\eta\mathbf{m} + 3$$

and after some algebraic manipulation we obtain also

$$\frac{2}{1-\boldsymbol{c}.\eta\boldsymbol{c}} = 2\frac{(2-\Delta_{\mathbf{m}})^2}{\Delta_{\mathbf{m}}^2}. \tag{51}$$

The second equation in (50) tells us that

$$\frac{4}{1-\boldsymbol{c}.\eta\boldsymbol{c}}c_3 = 4\frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}^2}m_3$$

while the last two equations in (50) read as

$$\frac{-4}{1-\boldsymbol{c}.\eta\boldsymbol{c}}c_2 = 4\frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}^2}m_2, \qquad \frac{-4}{1-\boldsymbol{c}.\eta\boldsymbol{c}}c_1 = 4\frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}^2}m_1.$$

It is obvious that all these equalities imply the vector equation

$$\frac{1}{1-\boldsymbol{c}.\eta\boldsymbol{c}}\boldsymbol{c} = -\frac{2-\Delta_{\mathbf{m}}}{\Delta_{\mathbf{m}}^2}\eta\mathbf{m} \tag{52}$$

and thus by substituting (51) into (52) we obtain formula (46) directly.

Our next task is to find the vector-parameter form of the sections of the homomorphism φ, i.e., to invert (46). If $\boldsymbol{c}.\eta\boldsymbol{c} = 0$ we obtain (48) directly. Now let $\boldsymbol{c}.\eta\boldsymbol{c} \neq 0$. Making use of (46) we can write

$$\boldsymbol{c}.\eta\boldsymbol{c} = \frac{-\eta\mathbf{m}}{1+\frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2}}\cdot\eta\frac{-\eta\mathbf{m}}{1+\frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2}} = \frac{\mathbf{m}.\eta\mathbf{m}}{\left(1+\frac{\mathbf{m}.\eta\mathbf{m}}{4}\right)^2}.$$

In essence, this is a quadratic equation for $x := \mathbf{m}.\eta\mathbf{m}$

$$\boldsymbol{c}.\eta\boldsymbol{c}x^2 + 8(\boldsymbol{c}.\eta\boldsymbol{c} - 2)x + 16\boldsymbol{c}.\eta\boldsymbol{c} = 0. \tag{53}$$

Note that $1 - \boldsymbol{c}.\eta\boldsymbol{c} = \left(\dfrac{4-x}{4+x}\right)^2$ and thus $1 - \boldsymbol{c}.\eta\boldsymbol{c} \geq 0$ with $1 - \boldsymbol{c}.\eta\boldsymbol{c} = 0$ if and only if $x = 4$. Thus, equation (53) has two real roots, namely

$$\begin{aligned} \mathbf{m}_+.(\eta\mathbf{m}_+) &= 4\frac{2 - \boldsymbol{c}.\eta\boldsymbol{c} + 2\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}} = 4\frac{1+\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{1-\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}} \\ \mathbf{m}_-.(\eta\mathbf{m}_-) &= 4\frac{2 - \boldsymbol{c}.\eta\boldsymbol{c} - 2\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}} = 4\frac{1-\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{1+\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}. \end{aligned} \tag{54}$$

Direct calculation shows that

$$1 + \frac{\mathbf{m}_\pm}{2}\cdot\eta\frac{\mathbf{m}_\pm}{2} = 2\frac{1\pm\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}$$

and thus we find the two sections (47) of (46). Note that the first two properties in (49) follow directly from (47) and (54). To prove the formulas from the second row in (49) let us calculate for example

$$\mathbf{m}_+(\boldsymbol{c}) \;= -2\frac{1+\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}\eta\boldsymbol{c} = -\frac{1+1\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{1-1\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}2\frac{1-\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}\eta\boldsymbol{c}$$

which in accordance with the results in (54) simplifies exactly to $-\dfrac{4}{\mathbf{m}_-.\eta\mathbf{m}_-}\mathbf{m}_-$. For $\mathbf{m}_-$ the calculations are analogical. ■

Note that if we assume the axis-rapidity representation of SO(2,1) elements via hyperbolic vector-parameters then for $\mathbf{m}_+(\boldsymbol{c})$ and $\mathbf{m}_-(\boldsymbol{c})$ we obtain

$$\mathbf{m}_-(\boldsymbol{c}) = -2\mathrm{th}\frac{\theta}{4}\eta\mathbf{n}, \qquad \mathbf{m}_+(\boldsymbol{c}) = -2\mathrm{cth}\frac{\theta}{4}\eta\mathbf{n}.$$

If we assume the representation of SO(2,1) elements via *elliptic* vector-parameters then for $\mathbf{m}_+(\boldsymbol{c})$ and $\mathbf{m}_-(\boldsymbol{c})$ we would have

$$\mathbf{m}_-(\boldsymbol{c}) = 2\mathrm{tg}\frac{\theta}{4}\eta\mathbf{n}, \qquad \mathbf{m}_+(\boldsymbol{c}) = -2\mathrm{ctg}\frac{\theta}{4}\eta\mathbf{n}.$$

Observe that in the vector-parameterization of $\mathrm{SO}(2,1)$ the elements $\mathcal{O}_h$ obeying the condition $\eta\mathcal{O}_h^t = \mathcal{O}_h\eta$ are not parametrized. In $\mathrm{SU}(1,1)$ vector-parameter form they are however well defined and are represented by vectors $\mathbf{m} \in \mathbb{R}^3$ which fulfill the equality $\mathbf{m}.\eta\mathbf{m} = -4$, i.e., $\Delta_{\mathbf{m}} = 2$. According to (33) these are elliptic elements. Substituting $\Delta_{\mathbf{m}} = 2$ in (45) we obtain the vector-parameterized matrix form of the pseudo half-turns, i.e.,

$$\mathcal{O}_h(\mathbf{m}) = \frac{1}{2}\begin{pmatrix} -m_1^2 & m_1 m_2 & m_1 m_3 \\ m_1 m_2 & -m_2^2 & -m_2 m_3 \\ -m_1 m_3 & m_2 m_3 & m_3^2 \end{pmatrix} - \mathcal{I}, \qquad m_1^2 + m_2^2 - m_3^2 = -4. \tag{55}$$

Taken together equations (17) and (55) imply

$$\mathbf{m} = 2(\beta_1, \beta_2, \alpha_2), \qquad \mathcal{O}_h = \mathcal{O}_h(\mathbf{m}) = \mathcal{O}_h(0, \alpha_2, \beta_1, \beta_2). \tag{56}$$

For practical applications, we may need to extract the vector-parameter of an arbitrary $\mathrm{SO}(2,1)$ element H. If H is a regular element we extract its vector-parameter from (21). If this is not the case, i.e., $\eta H^t = H\eta$, $H = \mathcal{O}_h$ we can take any of the nonzero columns of $H_1 = (H + \mathcal{I})\eta$, lets say $\boldsymbol{h}_1$, *normalize* it to $\hat{\boldsymbol{h}}_1 = \dfrac{\boldsymbol{h}_1}{\sqrt{-\boldsymbol{h}_1.\eta\boldsymbol{h}_1}}$ and then multiply the result by two. Note that $\sqrt{-\boldsymbol{h}_1.\eta\boldsymbol{h}_1}$ is well-defined real number since $\boldsymbol{h}_1.\eta\boldsymbol{h}_1$ equals (see equation (17)) to one of these: $-\beta_1^2, -\beta_2^2$ or $-\alpha_2^2$. Notice also that the matrix $\eta(\mathcal{O}_h^t + \mathcal{I})$ is invariant under left multiplication by $\mathcal{O}_h$

$$\mathcal{O}_h\eta(\mathcal{O}_h^t + \mathcal{I}) = \mathcal{O}_h\eta\mathcal{O}_h^t + \mathcal{O}_h\eta = \eta + \eta\mathcal{O}_h^t = \eta(\mathcal{O}_h^t + \mathcal{I}).$$

Thus, the columns of $\eta(\mathcal{O}_h^t + \mathcal{I})$ (and thus $\mathbf{m}$) are eigenvectors of $\mathcal{O}_h$.
In $\mathrm{SU}(1,1)$, the matrix form of these elements is (see equation (32))

$$\mathcal{W}_h(\mathbf{m}) = \mathbf{m}\cdot\mathbf{E} = \begin{pmatrix} \mathrm{i}\dfrac{m_3}{2} & \dfrac{m_1}{2} + \mathrm{i}\dfrac{m_2}{2} \\ \dfrac{m_1}{2} - \mathrm{i}\dfrac{m_2}{2} & -\mathrm{i}\dfrac{m_3}{2} \end{pmatrix}, \qquad \mathbf{m}.\eta\mathbf{m} = -4. \tag{57}$$

Actually, formula (30) says that the vectors in the fixed point set of the *Cayley* map are only those $\mathbf{m} \in \mathbb{R}^3$ for which $\mathbf{m}.\eta\mathbf{m} = -4$.

Taking into account the sections (47) of the vector-parameter form of φ (cf. (3) and Theorem 9) we can obtain the matrix form of a $\mathrm{SU}(1,1)$ element in terms of $\mathrm{SO}(2,1)$ vector-parameter $\boldsymbol{c}$. After calculating $\Delta_{\mathbf{m}_\pm}$ in terms of $\boldsymbol{c}$ we substitute in formula (30) and obtain

$$\mathcal{M}_h(\boldsymbol{c}) = \frac{\pm 1}{\sqrt{1 - \boldsymbol{c}.\eta\boldsymbol{c}}}\begin{pmatrix} 1 + \mathrm{i}c_3 & c_1 + \mathrm{i}c_2 \\ c_1 - \mathrm{i}c_2 & 1 - \mathrm{i}c_3 \end{pmatrix}. \tag{58}$$

Because φ is group-homomorphism, its vector-parameter form also preserves the composition law. It turns out that its two sections $\mathbf{m}_\pm$ also preserve it in the following sense

Theorem 10. *Let* $\boldsymbol{c}$ *and* $\boldsymbol{a}$ *be such that* $\boldsymbol{c}.\eta\boldsymbol{c} \neq 1, \boldsymbol{a}.\eta\boldsymbol{a} \neq 1$ *and* $1 + \boldsymbol{a}.\eta\boldsymbol{c} \neq 0$. *Let* $H(\boldsymbol{c}), H(\boldsymbol{c})$ *are two* SO(2,1) *elements represented by* $\boldsymbol{c}$ *and* $\boldsymbol{a}$. *Let* $\mathcal{U}_{h1}$ *and* $\mathcal{U}_{h2}$ *be the* SU(1,1) *elements in matrix form, obtained by the "+" or "-" section of* (47), *i.e.,*

$$\mathcal{U}_{h1} = \frac{\varepsilon}{\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}\begin{pmatrix} 1+\mathrm{i}c_3 & c_1+\mathrm{i}c_2 \\ c_1-\mathrm{i}c_2 & 1-\mathrm{i}c_3 \end{pmatrix}, \qquad \mathcal{U}_{h2} = \frac{\varepsilon}{\sqrt{1-\boldsymbol{a}.\eta\boldsymbol{a}}}\begin{pmatrix} 1+\mathrm{i}a_3 & a_1+\mathrm{i}a_2 \\ a_1-\mathrm{i}a_2 & 1-\mathrm{i}a_3 \end{pmatrix}$$

where $\varepsilon = \pm 1$ depending on the chosen section. Let $\mathcal{M}_h = \mathcal{U}_{h2}\mathcal{U}_{h1}$ be their composition in SU(1,1)*. Then*

$$\mathcal{M}_h = \frac{\pm 1}{\sqrt{1-\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}}\begin{pmatrix} 1+\mathrm{i}\widetilde{c}_3 & \widetilde{c}_1+\mathrm{i}\widetilde{c}_2 \\ \widetilde{c}_1-\mathrm{i}\widetilde{c}_2 & 1-\mathrm{i}\widetilde{c}_3 \end{pmatrix} \tag{59}$$

where $\widetilde{\boldsymbol{c}} = \langle \boldsymbol{a}, \boldsymbol{c}\rangle_{\mathrm{SO}(2,1)}$, i.e., each of the sections in (30) *preserves the composition law up to a sign.*

Proof: Direct calculation of $\mathcal{M}_h = \mathcal{U}_2\mathcal{U}_1$ yields

$$\mathcal{M}_h = \frac{1}{\sqrt{1-\boldsymbol{a}.\eta\boldsymbol{a}}\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}\begin{pmatrix} \alpha+\mathrm{i}\beta & \gamma+\mathrm{i}\delta \\ \gamma-\mathrm{i}\delta & \alpha-\mathrm{i}\beta \end{pmatrix}$$

where

$$\begin{aligned} \alpha+\mathrm{i}\beta &= 1+\boldsymbol{a}.\eta\boldsymbol{c} + \mathrm{i}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{a}\curlywedge\boldsymbol{c})_3 \\ \gamma+\mathrm{i}\delta &= (\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{a}\curlywedge\boldsymbol{c})_1 + \mathrm{i}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{a}\curlywedge\boldsymbol{c})_2. \end{aligned} \tag{60}$$

We can also calculate $1-\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}} = \dfrac{(1-\boldsymbol{a}.\eta\boldsymbol{a})(1-\boldsymbol{c}.\eta\boldsymbol{c})}{(1+\boldsymbol{a}.\eta\boldsymbol{c})^2}$ and thus

$$\frac{\pm 1}{\sqrt{1-\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}} = \frac{|1+\boldsymbol{a}.\eta\boldsymbol{c}|}{\sqrt{(1-\boldsymbol{a}.\eta\boldsymbol{a})(1-\boldsymbol{c}.\eta\boldsymbol{c})}} = \frac{\varepsilon_1(1+\boldsymbol{a}.\eta\boldsymbol{c})}{\sqrt{(1-\boldsymbol{a}.\eta\boldsymbol{a})(1-\boldsymbol{c}.\eta\boldsymbol{c})}} \tag{61}$$

where $\varepsilon_1 = \mathrm{sgn}(1+\boldsymbol{a}.\eta\boldsymbol{c})$. Now (59) readily follows from (60) and (61). ■

4. Composition of Pseudo Rotations

The vector-parameter form (22) of the composition law in SO(2,1) is well defined in the case when neither of the two elements is a pseudo-half-turn and the result is not a pseudo half-turn as well. All other cases, i.e., when at least one of the two rotations is a pseudo half-turn and/or when their composition is a pseudo half-turn has to be treated separately. The exhaustive list of all possible scenarios is:

- composition of two proper pseudo rotations
- composition of a pseudo half-turn and a pseudo proper rotation
- composition of a pseudo proper rotation and a pseudo half-turn
- composition of two pseudo half-turns.

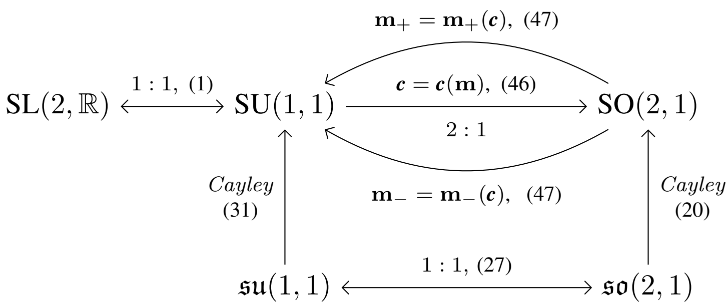

Figure 1. Relations between the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$ and respective Lie groups $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(2,1)$ in terms of the vector-parameterizations generated by the *Cayley* map.

Of course, if the identity matrix is involved in the composition, the problem is trivial. For the most of these scenarios we will not compose the given rotations in SO(2,1) but their images in the covering group SU(1,1) making use of the vector-parameter form of the latter, see Figure 1.

In this way we will find and describe all possibilities listed above and describe the result either as a SO(2,1) vector-parameter $\boldsymbol{c}$ representing a regular SO(2,1) element $\mathcal{R}_h(\boldsymbol{c})$ or as SU(1,1) vector-parameter $\mathbf{m}$ representing a pseudo half-turn $\mathcal{O}_h(\mathbf{m})$. In this way, an extension of the vector-parameter form of the composition law in SO(2,1) will be made which leads to more informative and intuitive group description of SO(2,1).

4.1. Composition of Two Regular Pseudo Rotations

Proposition 11. *Let* $\mathcal{R}_{h1} = \mathcal{R}_h(\boldsymbol{c}_1)$, $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 < 1$ *and* $\mathcal{R}_{h2} = \mathcal{R}_h(\boldsymbol{c}_2)$, $\boldsymbol{c}_2.\eta\boldsymbol{c}_2 < 1$ *be two regular* SO(2,1) *elements. Then for their composition*

$$H = \mathcal{R}_{h2}(\boldsymbol{c}_2)\mathcal{R}_{h1}(\boldsymbol{c}_1)$$

we have that it is represented by the SO(2,1) *vector-parameter*

$$\boldsymbol{c} = \frac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \curlywedge \boldsymbol{c}_1}{1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}) \tag{62}$$

provided that $\boldsymbol{c}_2.\eta\boldsymbol{c}_1 \neq -1$. *If* $\boldsymbol{c}_2.\eta\boldsymbol{c}_1 = -1$ *then* H *is a pseudo half-turn, represented by the* SU(1,1) *vector-parameter*

$$\mathbf{m} = -2\frac{\eta\boldsymbol{c}_2 + \eta\boldsymbol{c}_1 - (\eta\boldsymbol{c}_2) \curlywedge (\eta\boldsymbol{c}_1)}{\sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}}, \qquad H \equiv \mathcal{O}_h(\mathbf{m}). \tag{63}$$

Proof: The case $\boldsymbol{c}_2.\eta\boldsymbol{c}_1 \neq -1$ is already covered by Theorem 3. Let $\boldsymbol{c}_2.\eta\boldsymbol{c}_1 = -1$ and lets suppose that $\boldsymbol{c}_i.\eta\boldsymbol{c}_i \neq 0$, $i = 1, 2$. Let $\mathbf{m}_1 = \mathbf{m}_+(\boldsymbol{c}_1)$, $\mathbf{m}_2 = \mathbf{m}_+(\boldsymbol{c}_2)$ be the SU(1,1) vector-parameters obtained by the "+" section of (47). We will firstly calculate the denominator of (37) which is equal to $e_1 := \frac{1}{4}E(\mathbf{m}_2, \mathbf{m}_1)$ (see (36)) for arbitrary value of $\boldsymbol{c}_2.\eta\boldsymbol{c}_1$ because we will use this more general formula later on. For simplicity of the calculations we will use the additional notation

$$\Upsilon_{\boldsymbol{c}} := \frac{1 + \sqrt{1 - \boldsymbol{c}.\eta\boldsymbol{c}}}{\boldsymbol{c}.\eta\boldsymbol{c}}.$$

We have

$$\begin{aligned} e_1 =& 1 - 2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\boldsymbol{c}_2.\eta\boldsymbol{c}_1 + (2\Upsilon_{\boldsymbol{c}_2} - 1)(2\Upsilon_{\boldsymbol{c}_1} - 1) \\ =& \frac{2}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1\boldsymbol{c}_2.\eta\boldsymbol{c}_2}\big(\boldsymbol{c}_1.\eta\boldsymbol{c}_1\boldsymbol{c}_2.\eta\boldsymbol{c}_2 + (1 + \sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2})(1 + \sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}) \\ & - \boldsymbol{c}_1.\eta\boldsymbol{c}_1(1 + \sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}) - \boldsymbol{c}_2.\eta\boldsymbol{c}_2(1 + \sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1})\big) \\ =& 2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\left(1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1 + \sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}\right). \end{aligned} \tag{64}$$

Substituting $\boldsymbol{c}_2.\eta\boldsymbol{c}_1 = -1$ in (64) implies $e_1 = 2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}$ is always non-zero. Taking into account equation (54), for the composition vector-parameter $\mathbf{m} = \langle \mathbf{m}_2, \mathbf{m}_1 \rangle_{\mathrm{SU}(1,1)}$ we obtain

$$\mathbf{m} = \frac{-2\Upsilon_{\boldsymbol{c}_1}2\Upsilon_{\boldsymbol{c}_2}\eta\boldsymbol{c}_2 - 2\Upsilon_{\boldsymbol{c}_2}2\Upsilon_{\boldsymbol{c}_1}\eta\boldsymbol{c}_1 + 2\Upsilon_{\boldsymbol{c}_2}2\Upsilon_{\boldsymbol{c}_1}(\eta\boldsymbol{c}_2) \curlywedge (\eta\boldsymbol{c}_1)}{2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}}$$

which becomes (63) after cancellation of the non-zero term $2\Upsilon_{\boldsymbol{c}_1}\Upsilon_{\boldsymbol{c}_2}$. We just proved the Proposition in the case $\boldsymbol{c}_i.\eta\boldsymbol{c}_i \neq 0$, $i = 1, 2$. If at least one of the terms $\boldsymbol{c}_i.\eta\boldsymbol{c}_i, i = 1, 2$ equals zero in the proof we take the preimage $\mathbf{m}_i = -\eta\boldsymbol{c}_i$ of the corresponding vector-parameter $\boldsymbol{c}_i$ in accordance with equation (48). In these three cases the calculations follow the same idea, are easier to perform and will be omitted here. ∎

Corollary 12. *To determine the type of the composition element of two regular* $\mathrm{SO}(2,1)$ *elements we substitute* (64) *in* (40) *after making analogical algebraic simplification with the denominator. We obtain*

$$\frac{\mathbf{m}}{2} \cdot \eta\frac{\mathbf{m}}{2} = \frac{2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\left(1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1 - \sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}\right)}{2\Upsilon_{\boldsymbol{c}_2}\Upsilon_{\boldsymbol{c}_1}\left(1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1 + \sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}\right)} = \frac{r-1}{r+1}$$

where $r = \dfrac{1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1}{\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}\cdot$ *Thus from criterion* (40) *we obtain that*

$$H \text{ is } \begin{cases} hyperbolic & \text{if} \quad r \in (-\infty, -1) \cup (1, +\infty) \\ parabolic & \text{if} \quad r = \pm 1 \\ elliptic & \text{if} \quad r \in (-1, 1). \end{cases} \tag{65}$$

4.2. Composition of a Proper Pseudo Rotation and a Pseudo Half-Turn

Proposition 13. *Let* $\mathcal{O}_{h2} = \mathcal{O}_h(\mathbf{m}_2)$, $\mathbf{m}_2.\eta\mathbf{m}_2 = -4$ *be a pseudo half-turn and* $\mathcal{R}_{h1} = \mathcal{R}_h(\boldsymbol{c}_1)$, $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 < 1$ *be a regular* SO(2,1) *element. Then for their composition*

$$H = \mathcal{O}_{h2}(\mathbf{m}_2)\mathcal{R}_{h1}(\boldsymbol{c}_1)$$

we have that it is represented by the SO(2,1) *vector-parameter*

$$\boldsymbol{c} = \eta\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{\mathbf{m}_2.\boldsymbol{c}_1}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}) \tag{66}$$

provided that $\mathbf{m}_2.\boldsymbol{c}_1 \neq 0$ *and* $\mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} \neq 0$*. If* $\mathbf{m}_2.\boldsymbol{c}_1 = 0$ *then* H *is a pseudo half-turn, represented by the* SU(1,1) *vector-parameter*

$$\mathbf{m} = -\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}, \qquad H \equiv \mathcal{O}_h(\mathbf{m}). \tag{67}$$

Finally, if $\mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} = 0$ *then* H *is represented by the parabolic* SO(2,1) *vector-parameter*

$$\boldsymbol{c} = -\eta\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge \eta.\boldsymbol{c}_1}{2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}). \tag{68}$$

Proof: Let $\mathbf{a}_1 = \mathbf{a}_1(\boldsymbol{c}_1)$ be the "+" section of (47) in the case $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 \neq 0$. Let $e_2 := E(\mathbf{a}_2, \mathbf{m}_1)$. Before making a composition of $\mathbf{a}_1$ and $\mathbf{m}_2$ in SU(1,1) we need to check the condition $e_2 \neq 0$ for existence of the composition vector-parameter. We calculate consequently

$$\begin{aligned} e_2 &= 1 + 2\frac{\mathbf{m}_2}{2}\cdot(\eta\frac{\mathbf{a}_1}{2}) + \big(\frac{\mathbf{m}_2}{2}\cdot(\eta\frac{\mathbf{m}_2}{2})\big)\big(\frac{\mathbf{a}_1}{2}\cdot(\eta\frac{\mathbf{a}_1}{2})\big) \\ &= \frac{1}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1}\Big(2\boldsymbol{c}_1.\eta\boldsymbol{c}_1 - (1+\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1})\mathbf{m}_2.\boldsymbol{c}_1 - 2 - 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}\Big) \\ &= -\frac{1+\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1}(\mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}). \end{aligned}$$

It is clear that $e_2 = 0 \Leftrightarrow \mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} = 0$. Let $e_2 = 0$. From Remark 6 we know that H is a parabolic element in SU(1,1) generated by the vector-parameter

$$\widehat{\mathbf{m}} = -\frac{2\frac{1+\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1}\mathbf{m}_2 - 2\frac{1+\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1}\mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{-\frac{4}{\boldsymbol{c}_1.\eta\boldsymbol{c}_1}\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}(1+\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1})} = \frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}.$$

The corresponding SO(2,1) vector-parameter is $\boldsymbol{c} = -\eta\widehat{\mathbf{m}}$ and this leads exactly to the formula (68).

Now let $e_2 \neq 0$ and let $\mathbf{m} = \langle \mathbf{m}_2, \mathbf{a}_1 \rangle_{\mathrm{SU}(1,1)}$ by (37). Using some of the already done calculations and (41) we obtain

$$\mathbf{m} = -2\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} + \mathbf{m}_2.\boldsymbol{c}_1}, \qquad 1 + \frac{\mathbf{m}}{2}\cdot\eta\frac{\mathbf{m}}{2} = \frac{2\mathbf{m}_2.\boldsymbol{c}_1}{2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} + \mathbf{m}_2.\boldsymbol{c}_1} \tag{69}$$

and thus the condition for $\mathbf{m}$ to represent a pseudo half-turn is $\mathbf{m}_2.\boldsymbol{c}_1 = 0$. If this is the case then formula (67) follows directly from the first equality in (69). Finally, if $\mathbf{m}_2.\boldsymbol{c}_1 \neq 0$ substituting the results of (69) in (46) we obtain (66) which ends the proof in the case $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 \neq 0$. If the last holds true then $\mathbf{a}_1 = -\eta\boldsymbol{c}_1$ is also a parabolic element. If one repeats the same scheme of the proof in the case $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 \neq 0$ will find out the three formulas and the corresponding conditions are consistent with the case $\boldsymbol{c}_1.\eta\boldsymbol{c}_1 = 0$. The calculations in this case are much easier and we skip them here. ■

Corollary 14. *After performing algebraic simplification of the expression* $\mathbf{m}.\eta\mathbf{m}$ *in the case* $\mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} \neq 0$ *we end up with*

$$\mathbf{m}.\eta\mathbf{m} = 4\frac{\mathbf{m}_2\boldsymbol{c}_1 - 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}{\mathbf{m}_2\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}} = 4\frac{q-1}{q+1}, \qquad q = \frac{\mathbf{m}_2.\boldsymbol{c}_1}{2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1}}.$$

The case $q = -1$ *is covered* (*see equation* (68)) *and in this case* H *is of parabolic type. Now by performing simple calculus, from* (42) (*or, equivalently from* (40)) *we obtain the following condition for the type of the composition, i.e.,*

$$H \text{ is } \begin{cases} \textit{hyperbolic} & \text{if} \quad q \in (-\infty, -1) \cup (1, +\infty) \\ \textit{parabolic} & \text{if} \quad q = \pm 1 \\ \textit{elliptic} & \text{if} \quad q \in (-1, 1). \end{cases} \tag{70}$$

Remark 15. *Note that equations* (66) *and* (68) *are actually coherent and thus the case* $\mathbf{m}_2.\boldsymbol{c}_1 + 2\sqrt{1-\boldsymbol{c}_1.\eta\boldsymbol{c}_1} = 0$ *does not really have a nature of singularity. Rather, it signifies turning point for the type as it can be seen from* Corollary 14. *The same observation can be made for the next* Proposition *and its* Corollary.

4.3. Composition of a Pseudo Half-Turn and a Proper Pseudo Rotation

Proposition 16. *Let* $\mathcal{O}_{h1} = \mathcal{O}_h(\mathbf{m}_1)$, $\mathbf{m}_1.\eta\mathbf{m}_1 = -4$ *be a pseudo half-turn and* $\mathcal{R}_{h2} = \mathcal{R}_h(\boldsymbol{c}_2)$, $\boldsymbol{c}_2.\eta\boldsymbol{c}_2 < 1$ *be a regular* SO(2,1) *element. Their composition*

$$H = \mathcal{R}_{h2}(\boldsymbol{c}_2)\mathcal{O}_{h1}(\mathbf{m}_1)$$

will be represented by the SO(2,1) *vector-parameter*

$$\boldsymbol{c} = \eta\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}_2) \curlywedge \mathbf{m}_1}{\boldsymbol{c}_2.\mathbf{m}_1}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}) \tag{71}$$

provided that $\boldsymbol{c}_2.\mathbf{m}_1 \neq 0$ *and* $\boldsymbol{c}_2.\mathbf{m}_1 + 2\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2} \neq 0$. *If* $\boldsymbol{c}_2.\mathbf{m}_1 = 0$ *then* H *will be a pseudo half-turn, represented by the* SU(1,1) *vector-parameter*

$$\mathbf{m} = -\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}_2) \curlywedge \mathbf{m}_1}{\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2}}, \qquad H \equiv \mathcal{O}_h(\mathbf{m}). \tag{72}$$

Finally, if $\boldsymbol{c}_2.\mathbf{m}_1 + 2\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2} = 0$ *then* $\mathcal{R}_h$ *is represented by the parabolic* SO(2,1) *vector-parameter*

$$\boldsymbol{c} = -\eta\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}_2) \curlywedge \mathbf{m}_2}{2\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2}}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}). \tag{73}$$

Proof: The proof is analogical to that one of Proposition 16 and will be omitted. ■

Corollary 17. *Introducing the parameter* $p = \dfrac{\boldsymbol{c}_2.\mathbf{m}_1}{2\sqrt{1-\boldsymbol{c}_2.\eta\boldsymbol{c}_2}}$, *in the same manner as in* Corollary 14 *we can obtain the criterion*

$$H \text{ is } \begin{cases} hyperbolic & \text{if} \quad p \in (-\infty,-1) \cup (1,+\infty) \\ parabolic & \text{if} \quad p = \pm 1 \\ elliptic & \text{if} \quad p \in (-1,1). \end{cases} \tag{74}$$

4.4. Composition of Two Pseudo Half-Turns

Before we start with the composition of two pseudo half-turns we will need the following

Lemma 18. *Let* $\mathbf{m} = (m_1, m_2, m_3)$, $\mathbf{a} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) \in \mathbb{R}^3$. *The system of equations*

$$\begin{array}{rcccccccc} \mathbf{m}.\eta\mathbf{m} & = & m_1^2 & + & m_2^2 & - & m_3^2 & = & -4 \\ \mathbf{a}.\eta\mathbf{a} & = & \mathrm{a}_1^2 & + & \mathrm{a}_2^2 & - & \mathrm{a}_3^2 & = & -4 \\ \mathbf{m}.\eta\mathbf{a} & = & m_1\mathrm{a}_1 & + & m_2\mathrm{a}_2 & - & m_3\mathrm{a}_3 & = & -\lambda^2 \end{array} \tag{75}$$

where $|\lambda| \leq 2$ *has a solution* if and only if $|\lambda| = 2$ *and in this case* $\mathbf{m} \equiv \mathbf{a}$.

Proof: Let the system of equations has a solution. From the third equation we have that $(m_3\mathrm{a}_3)^2 = (m_1\mathrm{a}_1 + m_2\mathrm{a}_2 + \lambda^2)^2$ and substituting m_3^2 and a_3^2 from the first and the second equation we obtain

$$(m_1\mathrm{a}_1 + m_2\mathrm{a}_2 + \lambda^2)^2 = (m_1^2 + m_2^2 + 4)(\mathrm{a}_1^2 + \mathrm{a}_2^2 + 4). \tag{76}$$

Let us define the vectors $\mathbf{x} = (m_1, m_2, \lambda)$ and $\mathbf{y} = (\mathrm{a}_1, \mathrm{a}_2, \lambda)$. Then the *Cauchy-Bunyakovsky-Schwarz* inequality for $\mathbf{x}$ and $\mathbf{y}$ reads as

$$(m_1\mathrm{a}_1 + m_2\mathrm{a}_2 + \lambda^2)^2 \le (m_1^2 + m_2^2 + \lambda^2)(\mathrm{a}_1^2 + \mathrm{a}_2^2 + \lambda^2) \tag{77}$$

with equality if and only if $\mathbf{x} = \mu\mathbf{y}$ for some $\mu \in \mathbb{R}^*$. From (76) and (77) we have

$$(m_1^2 + m_2^2 + \lambda^2)(\mathrm{a}_1^2 + \mathrm{a}_2^2 + \lambda^2) \ge (m_1^2 + m_2^2 + 4)(\mathrm{a}_1^2 + \mathrm{a}_2^2 + 4)$$

which is impossible for $|\lambda| < 2$. Thus, $|\lambda| = 2$ and therefore $\mathbf{x}$ and $\mathbf{y}$ are proportional. But the third components of the vectors $\mathbf{x}$ and $\mathbf{y}$ are equal to λ and thus $\mathbf{x} \equiv \mathbf{y}$. Therefore, also $m_1 = \mathrm{a}_1$ and $m_2 = \mathrm{a}_2$. Now from the equations of the system (75) we obtain $m_3^2 = \mathrm{a}_3^2 = m_3\mathrm{a}_3$ and therefore $m_3 = \mathrm{a}_3$. In this way we have proved that $\mathbf{m} = \mathbf{a}$. Note that in this case the three equations become one and the same equation the solutions for both vectors lying on a double sheeted hyperboloid. ∎

Proposition 19. *Let* $\mathcal{O}_{h1} = \mathcal{O}_h(\mathbf{m}_1)$, $\mathbf{m}_1.\eta\mathbf{m}_1 = -4$ *and* $\mathcal{O}_{h2} = \mathcal{O}_h(\mathbf{m}_2)$, $\mathbf{m}_2.\eta\mathbf{m}_2 = -4$ *be two pseudo half-turns in* $\mathrm{SO}(2,1)$ *and* $\mathbf{m}_1 \neq \mathbf{m}_2$. *Then for their composition*

$$H = \mathcal{O}_{h2}(\mathbf{m}_2)\mathcal{O}_{h1}(\mathbf{m}_1)$$

we have that it is represented by the SO(2,1) *vector-parameter*

$$\boldsymbol{c} = -\frac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{m}_2.\eta\mathbf{m}_1}, \qquad H \equiv \mathcal{R}_h(\boldsymbol{c}). \tag{78}$$

Proof: Before composing the SU(1,1) the vector-parameters $\mathbf{m}_1$ and $\mathbf{m}_2$ we need to check the condition $E(\mathbf{m}_2, \mathbf{m}_1) \neq 0$. In the present setting it is equivalent to $\mathbf{m}_2.\eta\mathbf{m}_1 \neq -4$. Supposing $\mathbf{m}_2.\eta\mathbf{m}_1 = -4$, the conditions of Lemma 18 are fulfilled (for $\lambda = 2$) and thus it follows that $\mathbf{m}_1 = \mathbf{m}_2$, a contradiction. Now, according to (37), the composition of the vector-parameters $\mathbf{m}_1$ and $\mathbf{m}_2$ in SU(1,1) is $\mathbf{m}_3 = \dfrac{\mathbf{m}_2 \curlywedge \mathbf{m}_1}{2 + 2\frac{\mathbf{m}_2}{2} \cdot \eta\frac{\mathbf{m}_1}{2}}$. From (41) we obtain $1 + \frac{\mathbf{m}_3}{2} \cdot \eta\frac{\mathbf{m}_3}{2} = \dfrac{\mathbf{m}_2.\eta\mathbf{m}_1}{2 + 2\frac{\mathbf{m}_2}{2} \cdot \eta\frac{\mathbf{m}_1}{2}}$ and thus $\mathbf{m}_3$ represents a pseudo half-turn if and only if $\mathbf{m}_2.\eta\mathbf{m}_1 = 0$. From Lemma 18, applied for $\lambda = 0$ we conclude that this is not possible. Finally, using

Table 1. Systematized results from Propositions 11, 13, 16, 19 and their Corollaries.

Product of pseudo rotations	Compound rotations	Conditions	Results	Type
$\mathcal{R}_h(\boldsymbol{c}_2)\mathcal{R}_h(\boldsymbol{c}_1)$	$\mathcal{R}_h(\boldsymbol{c})$	$\boldsymbol{c}_2.\eta\boldsymbol{c}_1 \neq -1$	$\boldsymbol{c} = \frac{\boldsymbol{c}_2 + \boldsymbol{c}_1 + \boldsymbol{c}_2 \curlywedge \boldsymbol{c}_1}{1 + \boldsymbol{c}_2.\eta\boldsymbol{c}_1}$	eq. (65)
	$\mathcal{O}_h(\mathbf{m})$	$\boldsymbol{c}_2.\eta\boldsymbol{c}_1 = -1$	$\mathbf{m} = -2\frac{\eta\boldsymbol{c}_2 + \eta\boldsymbol{c}_1 - (\eta\boldsymbol{c}_2) \curlywedge (\eta\boldsymbol{c}_1)}{\sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}}$	
$\mathcal{O}_h(\mathbf{m}_2)\mathcal{R}_h(\boldsymbol{c}_1)$	$\mathcal{R}_h(\boldsymbol{c})$	$\mathbf{m}_2.\boldsymbol{c}_1 \neq 0$	$\boldsymbol{c} = \eta\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{\mathbf{m}_2.\boldsymbol{c}_1}$	eq. (70)
	$\mathcal{O}_h(\mathbf{m})$	$\mathbf{m}_2.\boldsymbol{c}_1 = 0$	$\mathbf{m} = -\frac{\mathbf{m}_2 - \mathbf{m}_2 \curlywedge (\eta\boldsymbol{c}_1)}{\sqrt{1 - \boldsymbol{c}_1.\eta\boldsymbol{c}_1}}$	
$\mathcal{R}_h(\boldsymbol{c}_2)\mathcal{O}_h(\mathbf{m}_1)$	$\mathcal{R}_h(\boldsymbol{c})$	$\boldsymbol{c}_2.\mathbf{m}_1 \neq 0$	$\boldsymbol{c} = \eta\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}_2) \curlywedge \mathbf{m}_1}{\boldsymbol{c}_2.\mathbf{m}_1}$	eq. (74)
	$\mathcal{O}_h(\mathbf{m})$	$\boldsymbol{c}_2.\mathbf{m}_1 = 0$	$\mathbf{m} = -\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}_2) \curlywedge \mathbf{m}_1}{\sqrt{1 - \boldsymbol{c}_2.\eta\boldsymbol{c}_2}}$	
$\mathcal{O}_h(\mathbf{m}_2)\mathcal{O}_h(\mathbf{m}_1)$	$\mathcal{R}_h(\boldsymbol{c})$	$\mathbf{m}_1 \neq \mathbf{m}_2$	$\boldsymbol{c} = -\frac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{m}_2.\eta\mathbf{m}_1}$	eq. (80)
	$\mathcal{I}$	$\mathbf{m}_1 = \mathbf{m}_2$	$\boldsymbol{c} = \mathbf{0}$	

formula (46) we calculate directly

$$\boldsymbol{c} = -\frac{\eta \mathbf{m}_3}{1 + \frac{\mathbf{m}_3}{2} \cdot \eta \frac{\mathbf{m}_3}{2}} = \frac{-\eta(\mathbf{m}_2 \curlywedge \mathbf{m}_1)}{\mathbf{m}_2.\eta \mathbf{m}_1} = -\frac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{m}_2.\eta \mathbf{m}_1}.$$

■

Remark 20. *Let us finally discuss the case* $\mathbf{m}_2 = \mathbf{m}_1 = \mathbf{m}$ *which is not covered by the proposition. From* Remark 7 *we can conclude that for an arbitrary pseudo half-turn, represented by the* SU(1,1) *vector* $\mathbf{m}$ *we have that*

$$\mathcal{O}_h(\mathbf{m})\mathcal{O}_h(\mathbf{m}) = \mathcal{I}. \tag{79}$$

Corollary 21. *To determine the type of the composition* H *from* Propositon 19 *we can use* Corollary 8*. In this case it simplifies to*

$$H \text{ is } \begin{cases} hyperbolic & \text{if} \quad t \in (-\infty, -4) \cup (4, +\infty) \\ parabolic & \text{if} \quad t = 4 \end{cases} \tag{80}$$

for $t = \mathbf{m}_2.\eta \mathbf{m}_1$*. Note that the case* $t \in (-4, 4)$ *is not possible due to* Lemma 18 *applied for* $t = \lambda^2$*.*

A systematics of the results from Propositions 11, 13, 16, 19 and their Corollaries is presented in Table 1.

5. Some Applications

5.1. Finding a Square Root in SO(2,1)

In Lemma 1 we have characterized the pseudo half-turns. We can clarify also which are the pseudo rotations equivalent to quarter-turns. They should be SO(2,1) elements $\mathcal{Q}_h$ such that $\mathcal{Q}_h^2$ are pseudo half-turns. Let $\widetilde{\mathbf{m}}$ be a SU(1,1) vector-parameter associated with the pseudo half-turn $\mathcal{O}_h(\widetilde{\mathbf{m}})$. We seek such vector-parameter $\mathbf{m}$ for which $\langle \mathbf{m}, \mathbf{m} \rangle_{\mathrm{SU}(1,1)} = \widetilde{\mathbf{m}}$. For that purpose we can use the results from Remark 7 and write down the equation

$$\widetilde{\mathbf{m}} = \frac{2\mathbf{m}}{1 + \frac{\mathbf{m}}{2} \cdot \eta \frac{\mathbf{m}}{2}}, \qquad -4 = \widetilde{\mathbf{m}}.\eta \widetilde{\mathbf{m}} = \frac{4}{\left(1 + \frac{x}{4}\right)^2} x$$

for the respective unknown $x = \mathbf{m}.\eta \mathbf{m}$. In other words, we are led to the quadratic equation

$$y^2 + 4y - 4 = 0, \qquad y = 1 + \frac{x}{4}$$

which solutions obviously are $y_\pm = -2(1 \pm \sqrt{2})$. Taking all this into account we have that $\mathbf{m} = \frac{y}{2}\widetilde{\mathbf{m}} = -(1 \pm \sqrt{2})\widetilde{\mathbf{m}}$. However, we are interested in finding those $\mathrm{SO}(2,1)$ vector-parameters $\boldsymbol{c}$ corresponding to $\widetilde{\mathbf{m}}$ which fulfill the defining relation, i.e., $\mathcal{Q}_h = \mathcal{R}_h(\boldsymbol{c})$, $\mathcal{Q}_h^2 = \mathcal{O}_h(\widetilde{\mathbf{m}})$. Fortunately, such solution can be obtained immediately, namely $\boldsymbol{c}_- \overset{(46)}{=} \boldsymbol{c}(\widetilde{\mathbf{m}}) = -\eta\frac{\widetilde{\mathbf{m}}}{2}$. Note that both $\widetilde{\mathbf{m}}$ and $-\widetilde{\mathbf{m}}$ represent the same pseudo half-turn $\mathcal{O}_h$. Thus, by repetition of the calculations above we obtain that $\boldsymbol{c}_+ \overset{(46)}{=} \boldsymbol{c}(-\widetilde{\mathbf{m}}) = \eta\frac{\widetilde{\mathbf{m}}}{2}$ will do also the job. Finally we obtain that the equation $\mathcal{Q}_h^2 = \mathcal{O}_h$ has two roots of elliptic type, i.e.,

$$\mathcal{Q}_h^2(\boldsymbol{c}_\pm) = \mathcal{O}_h(\widetilde{\mathbf{m}}), \qquad \boldsymbol{c}_\pm = \pm\eta\frac{\widetilde{\mathbf{m}}}{2}, \qquad \boldsymbol{c}_\pm.\eta\boldsymbol{c}_\pm = -1.$$

In a similar way one can find the pseudo "octa-turns" and so on, only this time using (22) because it is already well defined. Not only that, given an arbitrary $\widetilde{\boldsymbol{c}} \in \mathbb{R}^{2,1}$, $\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}} < 1$, we can solve the equation

$$\mathcal{R}_h^2(\boldsymbol{c}) = \mathcal{R}_h(\widetilde{\boldsymbol{c}}), \qquad \boldsymbol{c}.\eta\boldsymbol{c} < 1$$

explicitly for $\boldsymbol{c}$ in general. From equation (22) we have $\widetilde{\boldsymbol{c}} = \frac{2\boldsymbol{c}}{1 + \boldsymbol{c}.\eta\boldsymbol{c}}$ and thus $\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}} = \frac{4\boldsymbol{c}.\eta\boldsymbol{c}}{(1 + \boldsymbol{c}.\eta\boldsymbol{c})^2}\cdot$ If $\widetilde{\boldsymbol{c}}$ is parabolic then it is obvious that $\boldsymbol{c} = \frac{1}{2}\widetilde{\boldsymbol{c}}$. In the other cases, i.e., when $\widetilde{\boldsymbol{c}}$ is of hyperbolic or elliptic type, we obtain the quadratic equation

$$(\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}})z^2 - 4z + 4 = 0, \qquad z = 1 + \boldsymbol{c}.\eta\boldsymbol{c}$$

from which we obtain $z_\pm = \frac{2}{\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}(1 \pm \sqrt{1 - \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}})$. After respective algebraic simplification we obtain

$$\boldsymbol{c}_\pm = \frac{1 \mp \sqrt{1 - \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}}{\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}\widetilde{\boldsymbol{c}}, \qquad \boldsymbol{c}_\pm.\eta\boldsymbol{c}_\pm = \frac{(1 \mp \sqrt{1 - \widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}})^2}{\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}}\cdot$$

Another simple computations show that that $\boldsymbol{c}_+.\eta\boldsymbol{c}_+ > 1$ for $\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}} > 0$ (see Fig. 2) and thus $\boldsymbol{c}_+ \notin \mathrm{SO}(2,1)$ in this case. So, it turns out that the equation $\mathcal{R}_h^2(\boldsymbol{c}) = \mathcal{R}_h(\widetilde{\boldsymbol{c}})$ has two roots only for the *elliptic* elements and just one root for hyperbolic and parabolic elements.

5.2. On the Analogue of Cartan's Theorem in SO(2,1)

Cartan's theorem states, that every $\mathrm{O}(n, \mathbb{R})$ rotation can be decomposed into a product of at most n symmetric $\mathrm{SO}(n, \mathbb{R})$ matrices, i.e., reflections. In the case

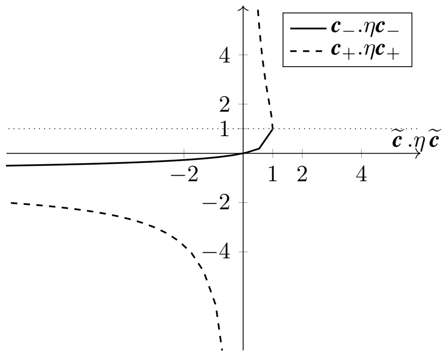

Figure 2. Graphs of $\boldsymbol{c}_+.\eta\boldsymbol{c}_+$ and $\boldsymbol{c}_-.\eta\boldsymbol{c}_-$ as functions of $\widetilde{\boldsymbol{c}}.\eta\widetilde{\boldsymbol{c}}$.

$n \equiv 3$ any rotation can be decomposed into a product of two half-turns. The vector-parameterization of SO(3, ℝ) has been used in [5] to find such explicit solution. Below we discuss an analogue of *Cartan*'s theorem for the set of hyperbolic SO(2,1) elements.

Proposition 22. *Let $\mathcal{R}_h(\boldsymbol{c}), \boldsymbol{c} = (c_1, c_2, c_3)$ be a proper pseudo rotation which is of hyperbolic type, i.e., $1 > \boldsymbol{c}.\eta\boldsymbol{c} > 0$. Then*

$$\mathcal{R}_h(\boldsymbol{c}) = \mathcal{O}_h(\mathbf{m}_2)\mathcal{O}_h(\mathbf{m}_1) \tag{81}$$

where

$$\mathbf{m}_1 = (\sqrt{s^2-4}\cos\varphi, \sqrt{s^2-4}\sin\varphi, s), \qquad \mathbf{m}_2 = -\frac{\mathbf{m}_1 - (\eta\boldsymbol{c}) \curlywedge \mathbf{m}_1}{2\sqrt{1-\boldsymbol{c}.\eta\boldsymbol{c}}}$$

for arbitrary $s^2 \geq 4\dfrac{c_1^2+c_2^2}{\boldsymbol{c}.\eta\boldsymbol{c}}$, $s^2 \neq 4\dfrac{c_1^2}{c_1^2-c_3^2}$ where $\operatorname{tg}\dfrac{\varphi}{2} = \frac{B\pm\sqrt{A^2+B^2-C^2}}{A+C}$, $A = c_1\sqrt{s-4}$, $B = c_2\sqrt{s-4}$ and $C = -sc_3$.

Proof: Consider the plane perpendicular to $\boldsymbol{c}$ in $\mathbb{R}^3$

$$\alpha_{\boldsymbol{c}} : c_1x + c_2y + c_3z = 0.$$

From the assumptions of the theorem it follows that at least one of the coordinates of $\boldsymbol{c}$ is nonzero. We will choose a vector $\mathbf{m}_1 = (m_1, m_2, m_3) \in \alpha_{\boldsymbol{c}}$ such that $\mathbf{m}.\eta\mathbf{m} - 4$. Thus we need to find a solution of the system of equations

$$\begin{array}{rcrcrcl} m_1c_1 & + & m_2c_2 & + & m_3c_3 & = & 0 \\ m_1^2 & + & m_2^2 & - & m_3^2 & = & -4 \end{array}$$

under the condition $\boldsymbol{c}.\eta\boldsymbol{c} = c_1^2 + c_2^2 - c_3^2 \in (0,1)$. Let $m_3 = s$ be an arbitrary real number but such that $s^2 \geq 4\dfrac{c_1^2 + c_2^2}{\boldsymbol{c}.\eta\boldsymbol{c}} \geq 4$. This choice reduces the above system to

$$\begin{array}{rcrcl} c_1 m_1 & + & c_2 m_2 & = & -c_3 s \\ m_1^2 & + & m_2^2 & = & s^2 - 4. \end{array} \tag{82}$$

Since $\boldsymbol{c}$ is such that $\boldsymbol{c}.\eta\boldsymbol{c} \in (0,1)$ it is not possible to have $c_1 = c_2 = 0$ because this will imply $c_3^2 < 0$, i.e., a contradiction. Thus, $c_1 m_1 + c_2 m_2 = -c_3 s$ is a line in the plane $\mathbb{R}^2$. The system (82) can be viewed geometrically as a problem about the intersection of a circle and a line. If we introduce polar coordinates, $m_1 = r\cos\varphi$, $m_2 = r\sin\varphi$ we are led to

$$A\cos\varphi + B\sin\varphi = C, \qquad r = \sqrt{s^2 - 4} \tag{83}$$

where $A = c_1 r$, $B = c_2 r$ and $C = -c_3 s$. Equation (83) can be solved by the substitution $t = \operatorname{tg}\dfrac{\varphi}{2}$ which leads to the quadratic equation for t in the form

$$(A + C)t^2 - 2Bt - (A - C) = 0.$$

We have that $A + C \neq 0$ exactly when $s^2 \neq 4\dfrac{c_1^2}{c_1^2 - c_3^2}\cdot$ The discriminant of the so obtained quadratic equation is

$$D = A^2 + B^2 - C^2 = (s^2 - 4)(c_1^2 + c_2^2) - s_3^2 c_3^2 = s^2(\boldsymbol{c}.\eta\boldsymbol{c}) - 4(c_1^2 + c_2^2).$$

Our choice for s implies that $D \geq 0$ and thus the solutions of the system (82) are

$$m_1 = \sqrt{s^2 - 4}\cos\varphi_\pm, \qquad m_2 = \sqrt{s^2 - 4}\sin\varphi_\pm \tag{84}$$

where $\operatorname{tg}\dfrac{\varphi_\pm}{2} = \frac{B \pm \sqrt{A^2 + B^2 - C^2}}{A + C}\cdot$ In this way we have described all vectors $\mathbf{m}_1$ that are perpendicular to $\boldsymbol{c}$ and for which $\mathbf{m}_1.\eta\mathbf{m}_1 = -4$. Let $\mathbf{m}_1$ be such vector. Equation (81) is equivalent to the identity

$$\mathcal{O}_h(\mathbf{m}_2) = \mathcal{R}_h(\boldsymbol{c})\mathcal{O}_h(\mathbf{m}_1) \tag{85}$$

and from Proposition 16 and $\boldsymbol{c}.\mathbf{m}_1 = 0$ it follows that $\mathcal{O}_h(\mathbf{m}_2)$ is represented by the $\mathrm{SU}(1,1)$ vector-parameter $\mathbf{m}_2 = -\dfrac{\mathbf{m}_1 - (\eta\boldsymbol{c}) \curlywedge \mathbf{m}_1}{2\sqrt{1 - \boldsymbol{c}.\eta\boldsymbol{c}}}\cdot$ Indeed, having in mind that $\mathbf{m}_1.\eta\mathbf{m}_1 = -4$ and $\boldsymbol{c}.\mathbf{m}_1 = 0$ we get

$$-\frac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{m}_2.\eta\mathbf{m}_1} = -\frac{-\dfrac{\mathbf{m}_1 - (\eta\boldsymbol{c}) \curlywedge \mathbf{m}_1}{2\sqrt{1 - \boldsymbol{c}.\eta\boldsymbol{c}}} \times \mathbf{m}_1}{-\dfrac{\mathbf{m}_1 - (\eta\boldsymbol{c}) \curlywedge \mathbf{m}_1}{2\sqrt{1 - \boldsymbol{c}.\eta\boldsymbol{c}}} \cdot \eta\mathbf{m}_1} = \frac{\eta(\boldsymbol{c} \times \mathbf{m}_1) \times \mathbf{m}_1}{\mathbf{m}_1.\eta\mathbf{m}_1} = \boldsymbol{c}$$

that ends the proof. ■

Remark 23. *From* Remark 7 *if follows that the identity matrix $\mathcal{I} \in \mathrm{SO}(2,1)$ can be decomposed into $\mathcal{O}_h(\mathbf{m})\mathcal{O}_h(\mathbf{m})$ for each pseudo half-turn $\mathcal{O}_h(\mathbf{m})$.*

6. Concluding Remarks

The Lie groups discussed in this paper are of great importance in modern mechanics, crystallography, quantum physics and mathematics. The parameterizations developed here along corresponding composition laws and the relationships among these groups offer some alternative viewpoints. The vector-parameteterization of the regular pseudo-rotations inside $\mathrm{SO}(2,1)$ is extended to cover the whole group, including the pseudo half-turns which had been characterized in detail. This is done by going to the covering group $\mathrm{SU}(1,1)$ and its parameterization via the *Cayley* map. Although the set of pseudo half-turns $\mathcal{O}_H$ is of measure zero within the group $\mathrm{SO}(2,1)$, they appear to be important since it can be shown that according to the analogue of *Cartan*'s theorem they are generators for the set of all hyperbolic elements in $\mathrm{SO}(2,1)$.

Furthermore it is interesting to note also that the projections of the pair of the matrices $\mathcal{L}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{L}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ in $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathcal{I}\}$ form a (2,3)-generating set of the latter [16]. The group $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup of SL(2, ℝ) and thus can be parameterized by the *Cayley* map. The $\mathrm{SU}(1,1)$ vector-parameters that correspond to $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\mathbf{m}_1 = (0,0,-2)$, $\mathbf{m}_1.\eta\mathbf{m}_1 = -4$ and $\mathbf{m}_2 = (0, -\frac{2}{3}, -\frac{4}{3})$, $\mathbf{m}_2.\eta\mathbf{m}_2 = -\frac{4}{3}\cdot$ The vector-parameter $\mathbf{m}_1$ is of elliptic type and it corresponds to a pseudo half-turn in SO(2,1).

Finally, using the developed formalism the problem of finding the square root in the group $\mathrm{SO}(2,1)$ is solved explicitly.

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