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GROUP CLASSIFICATION OF VARIABLE COEFFICIENT $K(m,n)$ EQUATIONS

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Abstract. Lie symmetries of $K(m,n)$ equations with time-dependent coefficients are classified. Group classification is presented up to widest possible equivalence groups, the usual equivalence group of the whole class for the general case and the conditional equivalence groups for special values of the exponents m and n. Examples on reduction of $K(m,n)$ equations (with initial and boundary conditions) to nonlinear ordinary differential equations (with initial conditions) are presented.

1. Introduction

In order to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [17] introduced a generalization of the KdV equation of the form

$$u_t + \varepsilon(u^m)_x + (u^n)_{xxx} = 0$$

where $\varepsilon = \pm 1$. Such equations, that are known as $K(m,n)$ equations, have the property for certain values of m and n their solitary wave solutions are of compact support. In other words, they vanish identically outside a finite core region. Further study followed in the references [13–16].

Here we consider a class of variable coefficient $K(m,n)$ equations of the form

$$u_t + \varepsilon(u^m)_x + f(t)\,(u^n)_{xxx} = 0 \qquad (1)$$

where f is an arbitrary nonvanishing function of the variable t, n and m are arbitrary constants with $n \neq 0$, and $\varepsilon = \pm 1$. Note that the more general class (appeared, e.g., in [24]) of the form

$$u_t + g(t)(u^m)_x + f(t)(u^n)_{xxx} = 0, \qquad fn \neq 0 \tag{2}$$

reduces to class (1) via the transformation $\tilde{t} = \varepsilon \int g(t)\mathrm{d}t$, $\tilde{x} = x$, $\tilde{u} = u$. This transformation maps the class (2) into its subclass (1), where $\tilde{f} = \varepsilon f/g$. This is why without loss of generality it is sufficient to study class (1), since all results on exact solutions, symmetries, conservation laws, etc. for class (2) can be derived from those obtained for (1) using the above transformation.

Lie symmetries have already been classified for many classes of constant coefficient partial differential equations (PDEs) and for many classes of PDEs involving functions with a range of forms. Typically, extra symmetries exist for particular forms of these functions. The classical method of finding Lie symmetries is first to find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations. This method is easy to apply and well-established in the last decades [2,4,5,7,8]. This leads to a continuing interest in finding exact solutions to nonlinear equations using Lie symmetries.

In the present paper, we carry out the Lie group classification for the class (1). All point transformations that link equations from the class are described. Firstly, we find equivalence group of the entire class and then derive three of its subclasses that have nontrivial conditional equivalence groups. The obtained Lie symmetries are employed also to a specific boundary value problem.

2. Equivalence Transformations

If two PDEs are connected by a point transformation then these equations are called *similar* [8] (it is possible also to consider a similarity up to contact transformations). Similar PDEs have similar sets of solutions, symmetries, conservation laws and other related information. Therefore, an important problem is the study of point transformations linking equations from a given class of PDEs. Such transformations are called admissible [11] (or form-preserving [6]) ones. Admissible transformations that preserve the differential structure of the class and transform only its arbitrary elements are called equivalence transformations and form a group. Notions of different kinds of equivalence group can be found, e.g., in [22].

The results on admissible transformations for equations from the class (1) are given in the following theorems. We exclude linear equations, i.e., equations with $(n, m) \in \{(1, 0), (1, 1)\}$, from the present analysis. The proofs of these theorems are omitted. The detailed procedure of how to construct equivalence transformation (or point transformations in general) can be found, for example, in [21,22].

Theorem 1. *The usual equivalence group* $\widetilde{G}$ *of the class* (1) *is formed by the transformations*

$$\tilde{t} = \pm\,\delta_1\delta_3^{1-m}t + \delta_0, \qquad \tilde{x} = \delta_1 x + \delta_2, \qquad \tilde{u} = \delta_3 u$$

$$\tilde{f} = \pm\,\delta_1^2\delta_3^{m-n}f, \qquad \tilde{\varepsilon} = \pm\,\varepsilon, \qquad \tilde{n} = n, \qquad \tilde{m} = m$$

where δ_j*,* $j = 0, 1, 2, 3$*, are arbitrary constants with* $\delta_1\delta_3 \neq 0$*.*

It appears that if $(n, m) \in \{(n, 0), (n, 1), (1, 2)\}$, then there exist nontrivial conditional equivalence groups of the class (1) that are wider than $\widetilde{G}$, namely the following assertions are true.

Theorem 2. *The class* (1) *with* $m = 0$*,*

$$u_t + f(t)\,(u^n)_{xxx} = 0 \tag{3}$$

admits the usual equivalence group $G^{\sim}_{(n,0)}$ *consisting of the transformations*

$$\tilde{t} = T(t), \qquad \tilde{x} = \delta_1 x + \delta_2, \qquad \tilde{u} = \delta_3 u, \qquad \tilde{f} = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\,f, \qquad \tilde{n} = n$$

where δ_j*,* $j = 1, 2, 3$*, are arbitrary constants with* $\delta_1\delta_3 \neq 0$*,* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$*.*

Theorem 3. *The generalized equivalence group* $\widetilde{G}_{(n,1)}$ *of the class* (1) *with* $m = 1$

$$u_t + \varepsilon u_x + f(t)\,(u^n)_{xxx} = 0 \tag{4}$$

comprises the transformations

$$\tilde{t} = T(t), \qquad \tilde{x} = \delta_1(x - \varepsilon t) \pm \varepsilon T(t) + \delta_2, \qquad \tilde{u} = \delta_3 u$$

$$\tilde{f} = \frac{\delta_1^3\delta_3^{1-n}}{T_t}\,f, \qquad \tilde{\varepsilon} = \pm\varepsilon, \qquad \tilde{n} = n$$

where δ_j*,* $j = 1, 2, 3$*, are arbitrary constants with* $\delta_1\delta_3 \neq 0$*,* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$*.*

Theorem 4. *The generalized equivalence group* $\widetilde{G}_{(1,2)}$ *of the class*

$$u_t + \varepsilon(u^2)_x + f(t)u_{xxx} = 0 \tag{5}$$

consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \qquad \tilde{\varepsilon} = \pm\varepsilon$$

$$\tilde{u} = \pm\frac{2\varepsilon\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{2\varepsilon(\alpha\delta - \beta\gamma)}, \qquad \tilde{f} = \frac{\kappa^3}{\alpha\delta - \beta\gamma}\frac{f}{\gamma t + \delta}$$

where $\alpha, \beta, \gamma, \delta, \mu_1, \mu_0$*, and* κ *are constants defined up to a nonzero multiplier,* $\kappa(\alpha\delta - \beta\gamma) \neq 0$*.*

The equations from the class (4) can be reduced to ones (with tilded variables) from the class (3) by the additional equivalence transformation

$$\tilde{t} = t, \qquad \tilde{x} = x - \varepsilon t, \qquad \tilde{u} = u.$$

Therefore, the case $m = 1$ being equivalent to the case $m = 0$ will be excluded from the classification list.

3. Lie Symmetries

We perform the group classification of class (1) within the framework of the classical Lie approach [4, 5, 7, 8]. We search for operators of the form

$$\Gamma = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$$

which generate one-parameter groups of point-symmetry transformations of an equation from class (1). Any such vector field, Γ, satisfies the infinitesimal invariance criterion, i.e., the action of the third prolongation, $\Gamma^{(3)}$, of the operator Γ on equation (1) results in the condition being an identity for all solutions of this equation. That is, we require that

$$\begin{aligned}\Gamma^{(3)}\Big[u_t + \varepsilon m u^{m-1} u_x + n f(t)\big(u^{n-1} u_{xxx} + 3(n-1)u^{n-2} u_x u_{xx} \\ +(n-1)(n-2)u^{n-3} u_x^3\big)\Big] = 0\end{aligned} \tag{6}$$

identically, modulo equation (1).

Equation (6) is an identity in the variables $u_x,\ u_{xx},\ u_{tx},\ u_{xxx}$ and u_{txx}. Coefficients of different powers of these variables, which must be equal to zero, lead to the determining equations on the coefficients τ, ξ and η. Firstly, we use the general results on point transformations between evolution equations [6], which simplify the forms of the coefficient functions. Specifically, we have $\tau = \tau(t)$ and $\xi = \xi(t, x)$. Since the procedure is quite straightforward [4, 5, 7, 8], we omit the detailed analysis. However, we point out that the classification of Lie symmetries is complete.

The coefficient of u_{xxx} gives the equation

$$[f_t \tau + f(\tau_t - 3\xi_x)]\, u + (n-1) f \eta = 0$$

from which we deduce that the analysis needs to be split in two exclusive cases: $n \neq 1$ and $n = 1$.

I. If $n \neq 1$, the coefficient function η takes the form

$$\eta = -\frac{[f_t \tau + f(\tau_t - 3\xi_x)]u}{(n-1)f}.$$

The coefficients of u_{xx} (or u_x^2), u_x and the term independent of the derivatives in (6) produce the following determining equations, respectively

$$n(2n+1)f\xi_{xx}=0$$

$$\varepsilon m\left[(m-n)f\tau_t-(3m-n-2)f\xi_x+(m-1)f_t\tau\right]u^m$$
$$-n(8n+1)f^2\xi_{xxx}u^n+(n-1)f\xi_t u=0$$

$$3\varepsilon m f^2\xi_{xx}u^m+3nf^3\xi_{xxxx}u^n-\left[f^2\tau_{tt}+ff_t\tau_t+ff_{tt}\tau-f_t^2\tau-3f^2\xi_{tx}\right]u=0.$$

Solution of the above determining equations leads to the forms of $\tau(t)$, $\xi(t,x)$ and $f(t)$. The forms of $f(t)$ are determined using *the method of furcate split* suggested in [10]. Lie symmetries according to the forms of $f(t)$ are tabulated in Table 1.

Table 1. Classification of the equations (1) with $n\neq 1$. Here k is an arbitrary nonzero constant. In Cases 6_a and 6_b $\varepsilon=1$ and $\varepsilon=-1$, respectively.

#	n	m	$f(t)$	Basis of $A^{\max}$
1	any	any	any	∂_x
2	any	$\frac{n+2}{3}$	any	$\partial_x,\ x\partial_x+\frac{3}{n-1}u\partial_u$
3	any	0	1	$\partial_t,\ \partial_x,\ x\partial_x+\frac{3}{n-1}u\partial_u,\ 3t\partial_t+x\partial_x$
4	$-\frac{1}{2}$	0	1	$\partial_t,\ \partial_x,\ x\partial_x-2u\partial_u,\ 3t\partial_t+x\partial_x,\ x^2\partial_x-4xu\partial_u$
5	any	any	1	$\partial_t,\ \partial_x,\ (3m-n-2)t\partial_t+(m-n)x\partial_x-2u\partial_u$
6_a	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\partial_t,\ \partial_x,\ 3t\partial_t+2u\partial_u,$ $\sin x\,\partial_x-2\cos x\,u\partial_u,\ \cos x\,\partial_x+2\sin x\,u\partial_u$
6_b	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\partial_t,\ \partial_x,\ 3t\partial_t+2u\partial_u,\ \mathrm{e}^x\partial_x-2\mathrm{e}^x u\partial_u,\ \mathrm{e}^{-x}\partial_x+2\mathrm{e}^{-x}u\partial_u$
7	any	any	t^k	$\partial_x,\ (3m-n-2)t\partial_t+(km-k+m-n)x\partial_x+(k-2)u\partial_u$
8	any	$\frac{n+2}{3}$	t^2	$\partial_x,\ x\partial_x+\frac{3}{n-1}u\partial_u,\ t\partial_t+x\partial_x$
9	any	any	e^t	$\partial_x,\ (3m-n-2)\partial_t+(m-1)x\partial_x+u\partial_u$

Remark 5. *All cases presented in* Table 1 *except* Cases 3 *and* 4 *are classified up to* $\widetilde{G}$*-equivalence. For* Cases 3 *and* 4*, where* $m=0$*, we used the equivalence group* $\widetilde{G}_{(n,0)}$ *that is wider than* $\widetilde{G}$*. Thus, the equation* (3) *with* $n\neq -1/2$ *admits the four-dimensional Lie symmetry algebra with the basis operators*

$$\frac{1}{f(t)}\partial_t,\qquad \partial_x,\qquad x\partial_x+\frac{3}{n-1}u\partial_u,\qquad 3\frac{\int f(t)\mathrm{d}t}{f(t)}\partial_t+x\partial_x$$

irrespectively of the form of the function f. Here and throughout the paper, an integral with respect to t should be interpreted as a fixed antiderivative. If $n=-1/2$, then the Lie symmetry algebra of the equation (3) *is five-dimensional, spanned by the above operators and the additional operator $x^2\partial_x-4xu\partial_u$. Using the equivalence transformation $\tilde t=\int f(t)\mathrm{d}t$, $\tilde x=x$, $\tilde u=u$ from the group $\widetilde G_{(n,0)}$, we reduce these cases to ones with $f=1$* (cf Cases 3 *and* 4 *of* Table 1).

II. If $n=1$, the coefficient of u_xu_{xxx} implies that $\eta_{uu}=0$, and so $\eta=\eta^1(t,x)u+\eta^0(t,x)$. We use the fact that $\tau=\tau(t)$, $\xi=\xi(t,x)$ and the above form for η and from (6) to obtain the following determining equations

$$f_t\tau+f(\tau_t-3\xi_x)=0$$
$$\eta^1_x-\xi_{xx}=0$$
$$\varepsilon m\big(\tau_t-\xi_x+(m-1)\eta^1\big)u^m+\varepsilon m(m-1)\eta^0u^{m-1}+(3f\eta^1_{xx}-\xi_t-f\xi_{xxx})u=0$$
$$\varepsilon m\eta^1_xu^{m+1}+\varepsilon m\eta^0_xu^m+(\eta^1_t+f\eta^1_{xxx})u^2+(\eta^0_t+f\eta^0_{xxx})u=0.$$

We solve the above system and adduce the results in Table 2. It is worthy to note that the group classification problem for the class of equations $u_t+u^{\bar m}u_x+\bar f(t)u_{xxx}=0$ with $\bar m\bar f\neq 0$, that are similar to equations of the form (1) with $n=1$, was carried out in [12,20].

Remark 6. *The group classification of the class* (1) *with $n=1$ and $m\neq 2$ is performed up to $\widetilde G$-equivalence. For the classification of Lie symmetries of the equations* (1) *with $n=1$ and $m=2$ we used the wider conditional equivalence group $\widetilde G_{(1,2)}$. Since transformations from the group $\widetilde G_{(1,2)}$ are quite complicated, we adduce also the additional cases of Lie symmetry extensions of equations* (1) *with $n=1$ and $m=2$ that are inequivalent with respect to the group $\widetilde G$ to* Cases 6–9 *of* Table 2.

1. $f=(t+\beta)^kt^{1-k}$, $k\neq 0,1$, $\beta\neq 0$: $\quad\langle\partial_x,\,2\varepsilon t\partial_x+\partial_u,\,\Gamma_3\rangle$, *where*

$$\Gamma_3=6\varepsilon t(t+\beta)\partial_t+2\varepsilon\,(3t+\beta(2-k))\,x\partial_x+[3x-2\varepsilon(3t+\beta(k+1))u]\partial_u$$

2. $f=t\mathrm{e}^{\frac1t}$: $\quad\langle\partial_x,\,2\varepsilon t\partial_x+\partial_u,\,6\varepsilon t^2\partial_t+2\varepsilon(3t-1)x\partial_x+(3x-2\varepsilon(3t+2)u)\partial_u\rangle$

3. $f=t$: $\quad\langle\partial_x,\,2\varepsilon t\partial_x+\partial_u,\,3t\partial_t+2x\partial_x-u\partial_u,\,2\varepsilon t^2\partial_t+2\varepsilon tx\partial_x+(x-2\varepsilon tu)\partial_u\rangle$.

From the first sight it looks like the counterpart to Case 9 *of* Table 1 *is missed. At the same time it appears that the function $f=\lambda\exp\Big(k\arctan\frac{\alpha t+\beta}{\gamma t+\delta}\Big)$ locally coincides with the function $\check f=\check\lambda\exp(k\arctan(\check\alpha t+\check\beta))$, see* [9] *for details.*

The primary use of Lie symmetries is to obtain a reduction of variables. Similarity variables appear as first integrals of the characteristic system

$$\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}u}{\eta}\cdot$$

Table 2. Classification of the class (1) with $n = 1$. Here k is an arbitrary constant satisfying the following constraints: $k \neq 0$ in Case 3, $k \neq 0, 1$ and $k \geqslant 1/2 \bmod \widetilde{G}_{(1,2)}$ in Case 7, $k \geqslant 0 \bmod \widetilde{G}_{(1,2)}$ in Case 9.

#	$f(t)$	Basis of $A^{\max}$
		$m \neq 2$
1	any	∂_x
2	1	∂_t, ∂_x, $3(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
3	t^k	∂_x, $3(m-1)t\partial_t + (m-1)(k+1)x\partial_x + (k-2)u\partial_u$
4	e^t	∂_x, $3(m-1)\partial_t + (m-1)x\partial_x + u\partial_u$
		$m = 2$
5	any	∂_x, $2\varepsilon t\partial_x + \partial_u$
6	1	∂_t, $2\varepsilon t\partial_x + \partial_u$, ∂_x, $3t\partial_t + x\partial_x - 2u\partial_u$
7	t^k	∂_x, $2\varepsilon t\partial_x + \partial_u$, $3t\partial_t + (k+1)x\partial_x + (k-2)u\partial_u$
8	e^t	∂_x, $2\varepsilon t\partial_x + \partial_u$, $3\partial_t + x\partial_x + u\partial_u$
9	$\mathrm{e}^{k\arctan t}\sqrt{t^2+1}$	∂_x, $2\varepsilon t\partial_x + \partial_u$, $6\varepsilon(t^2+1)\partial_t + 2\varepsilon(3t+k)x\partial_x$ $+(2\varepsilon(k-3t)u + 3x)\partial_u$

Here, we can reduce a PDE in two independent variables into an ordinary differential equation (ODE) using a one-dimensional subalgebra of Lie symmetry algebra. Reductions could be obtained from any symmetry which is an arbitrary linear combination $\sum_{i=1}^{s} a_i\Gamma_i$, where s is the number of basis operators of maximal Lie symmetry algebra of the given PDE. To ensure that a minimal complete set of reductions is obtained from the Lie symmetries of equation (1), we construct the so-called optimal system of one-dimensional subalgebras. Ovsiannikov [8] proved that the optimal system of solutions consists of solutions that are invariant with respect to all proper inequivalent subalgebras of the symmetry algebra. More detail about construction of optimal sets of subalgebras can be found in [7, 8].

As an example for a reduction into an ordinary differential equation, we consider Case 7 of Table 2 which corresponds to the variable coefficient KdV equation

$$u_t + \varepsilon(u^2)_x + t^k u_{xxx} = 0 \tag{7}$$

that admits the three-dimensional Lie symmetry algebra

$$\Gamma_1 = \partial_x,\ \Gamma_2 = 2\varepsilon t\partial_x + \partial_u,\ \Gamma_3 = 3t\partial_t + (k+1)x\partial_x + (k-2)u\partial_u.$$

Depending on the value of k an optimal system of one-dimensional subalgebras of this Lie symmetry algebra consists of the subalgebras

$$\begin{array}{llllll} \langle\Gamma_1\rangle, & \langle\Gamma_2+\sigma\Gamma_1\rangle, & \langle\Gamma_3\rangle & \text{if} & k\neq -1,2 \\ \langle\Gamma_1\rangle, & \langle\Gamma_2+\sigma\Gamma_1\rangle, & \langle\Gamma_3+a\Gamma_1\rangle & \text{if} & k=-1 \\ \langle\Gamma_1\rangle, & \langle\Gamma_2+\sigma\Gamma_1\rangle, & \langle\Gamma_3+a\Gamma_2\rangle & \text{if} & k=2. \end{array}$$

Here, $\sigma\in\{-1,0,1\}$, $a\in\mathbb{R}$.

Reductions associated with the subalgebra $\langle\Gamma_1\rangle$ are not considered since they lead to constant solutions only. The ansatz constructed with the subalgebra $\langle\Gamma_2+\sigma\Gamma_1\rangle$ has the form $u=\dfrac{x}{2\varepsilon t+\sigma}+\phi(\omega)$ with the similarity variable $\omega=t$. This ansatz reduces equation (7) to the ODE $(2\varepsilon\omega+\sigma)\phi_\omega+2\varepsilon\phi=0$, whose general solution is $\phi=\dfrac{c_1}{2\varepsilon\omega+\sigma}$, where c_1 is an arbitrary constant. The corresponding solution of (7) takes the form

$$u=\frac{x+c_1}{2\varepsilon t+\sigma}\cdot$$

It is fair to note that this solution satisfies equations of the form (5) for arbitrary f. Other reductions depend on the value of the exponent k. We adduce the ansatzes together with the corresponding reduced equations

$$k\neq -1,2.\quad \langle\Gamma_3\rangle:\qquad u=t^{\frac{k-2}{3}}\phi(\omega),\qquad \omega=xt^{-\frac{k+1}{3}}$$
$$3\phi_{\omega\omega\omega}+6\varepsilon\phi\phi_\omega-(k+1)\phi_\omega\omega+(k-2)\phi=0$$

$$k=-1.\quad \langle\Gamma_3+a\Gamma_1\rangle:\qquad u=\frac{1}{t}\phi(\omega),\qquad \omega=x-\frac{a}{3}\ln t$$
$$3\phi_{\omega\omega\omega}+6\varepsilon\phi\phi_\omega-a\phi_\omega-3\phi=0$$

$$k=2.\quad \langle\Gamma_3+a\Gamma_2\rangle:\qquad u=\frac{a}{3}\ln t+\phi(\omega),\qquad \omega=\frac{x}{t}-\frac{2a\varepsilon}{3}\ln t$$
$$3\phi_{\omega\omega\omega}+6\varepsilon\phi\phi_\omega-3\phi_\omega\omega-2a\varepsilon\phi_\omega+a=0.$$

We note that the latter two cases are equivalent. Indeed, the equation

$$u_t+\varepsilon(u^2)_x+t^2u_{xxx}=0$$

is mapped to the equation

$$\tilde u_{\tilde t}+\varepsilon(\tilde u)^2{}_{\tilde x}+\tilde t^{-1}\tilde u_{\tilde x\tilde x\tilde x}=0$$

by the following transformation from the group $\widetilde G_{(1,2)}$

$$\tilde t=\frac{1}{t},\qquad \tilde x=-\frac{x}{t},\qquad \tilde u=\frac{2\varepsilon tu-x}{2\varepsilon}\cdot$$

4. Application of Lie Symmetries to a Boundary Value Problem

There exist several approaches exploiting Lie symmetries to reduce of boundary-value problems (BVPs) for PDEs to those for ODEs. The classical technique suggested in [1, 3] is to require that both equation and boundary conditions are left invariant under the one-parameter Lie group of infinitesimal transformations. Of course the infinitesimal approach is usually applied, i.e., a basis of operators of Lie invariance algebras is used instead of finite transformations from the corresponding Lie symmetry group (see, e.g., [2, Section 4.4]). Firstly, the symmetries of a PDE should be derived and then the boundary conditions should be checked to determine whether they are also invariant under the action of the generators of the symmetries found. In the case of a positive answer, the BVP for the PDE was reduced to a BVP for an ODE. Using this technique, a number of boundary-value problems were solved successfully (see, e.g., [18, 19, 23]).

We consider the following initial and boundary value problem

$$u_t + \varepsilon(u^m)_x + t^k(u^n)_{xxx} = 0, \qquad t > 0, \qquad x > 0 \tag{8}$$

$$\begin{gathered} u(x,0) = 0, \qquad x > 0 \\ u(0,t) = q(t), \qquad u_x(0,t) = 0, \qquad u_{xx}(0,t) = 0, \qquad t > 0. \end{gathered} \tag{9}$$

We look for a nonconstant solution using the "direct" approach suggested by Bluman [2, 3].

We have derived the Lie symmetries for the variable coefficient equation (8) and now we examine which of these symmetries leave the initial and boundary conditions of the problem invariant. The procedure starts by assuming a general symmetry of the form

$$\Gamma = \sum_{i=1}^{s} \alpha_i \Gamma_i \tag{10}$$

where s is the number of basis operators of maximal Lie symmetry algebra of the given partial differential equation and $\alpha_i,\ i = 1, \ldots, s$, are constants to be determined.

Equation (8) admits for arbitrary n, m and k a two-dimensional Lie symmetry algebra with the basis operators

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = (3m - n - 2)t\partial_t + (km - k + m - n)x\partial_x + (k - 2)u\partial_u.$$

The general symmetry (10) takes the form

$$\Gamma = \alpha_1\partial_x + \alpha_2\left[(3m - n - 2)t\partial_t + (km - k + m - n)x\partial_x + (k - 2)u\partial_u\right].$$

Application of Γ to the first boundary condition $x = 0,\ u(t,0) = q(t)$ gives $\alpha_1 = 0$ and $q(t) = \gamma t^{\frac{k-2}{3m-n-2}}$, $m \neq \frac{n+2}{3}$. Using the second extension of Γ

$$\Gamma^{(2)} = 3nt\partial_t + (k+1)nx\partial_x + (k-2)u\partial_u + (k-nk-n-2)u_x\partial_{u_x} + (k-2nk-2n-2)u_{xx}\partial_{u_{xx}}$$

where the unused terms have been ignored, it can be shown that it leaves invariant the initial condition and the remaining two boundary conditions of (9). Finally, symmetry Γ produces the transformation

$$u = t^{\frac{k-2}{3m-n-2}}\phi(\omega), \qquad \omega = xt^{-\frac{km-k+m-n}{3m-n-2}} \tag{11}$$

which reduces the problem (8)–(9) into

$$(\phi^n)''' + \varepsilon(\phi^m)' - \frac{km-k+m-n}{3m-n-2}\omega\phi' + \frac{k-2}{3m-n-2}\phi = 0$$

$$\phi(0) = \gamma, \qquad \phi'(0) = 0, \qquad \phi''(0) = 0.$$

The latter Initial Value Problem can be solved numerically and then the solution of Initial BVP (8)–(9) can be recovered using the transformation (11). For details see [20], where a similar problem for generalized KdV equations was solved successfully using finite difference method.

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