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NEW PARAMETERIZATIONS OF THE CASSINIAN OVALS

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Abstract. Here we present a general scheme which leads effectively to the reconstruction of any plane curve whose curvature is specified by a function of the radial coordinate. As a concrete example we have derived two new parametrizations of the Cassinian ovals.

1. Introduction

Surprising or not it turns out that the curvature of a lot of the famous plane curves such as conic sections, Bernoulli's lemniscate [8, 15], Cassinian ovals [2, 12, 16], Delaunay surfaces [13, 17] and their generalization [7], Euler's elastica [5, 11], Sturmian spirals [4, 17], and many others, depends solely on the distance from a certain point or a line in the Euclidean plane. Let us remind also that the most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [3, p. 296] or [19, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that if the curvature is given by a function of its position the situation is even more complicated. Viewing the Frenet-Serret equations as a ficticious dynamical system in [22] it was proven that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [21] has already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [22] or mechanical [21] ones proposed in the above cited papers. Actually, the method developed below is applicable (modulo evaluating in explicit form the arising integrals) to the whole class of curves whose curvature depend solely on the distance from the origin.

2. Cassinian Ovals

These curves (sometimes called Cassinian ellipses) were introduced by Giovanni Domenico Cassini in 1680 as an alternative trajectories of planets to the Newtonian ones, and despite of the fact that this turns out to be wrong idea they have found a lot of interesting applications in mathematics, physics and other sciences.

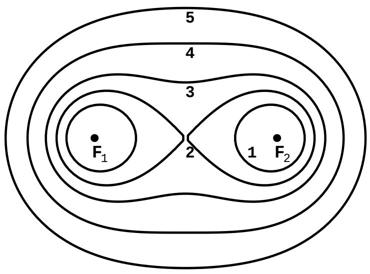

Figure 1. Various Cassinian ovals.

E.g., the elliptic integrals of the second kind [1, 18] appear in the attempt to rectify the arcs of ellipses and Legendre rise the question which are the curves whose arcs are in one to one correspondence with the elliptic integrals of the first kind. The answer has been given by the French geometer J.-A. Serret who proved that these are exactly Cassinian ovals termed by him as ***hyperbolic***, ***parabolic*** and ***elliptic lemniscates*** [20]. For the reader's curiosity we will mention also that the equipotential curves of two currents flowing along two infinitely long parallel straight wires coincide with them in any perpendicular plane. These planes are punctured by the wires at two points called *foci* (see Fig. 1 and below). Elliptic like ovals have been used also to model the sections of human red blood cells [2, 9, 10].

Actually, by their very definition **Cassinian ovals** obey to the following simple geometrical condition. The product of the distances of its points to two given points $\mathbf{F}_1$ and $\mathbf{F}_2$ in the XOZ plane called focuses which are $2a$ apart is a constant denoted by c^2, i.e.,

$$\sqrt{(x-a)^2+z^2}\sqrt{(x+a)^2+z^2}=c^2 \tag{1}$$

or what is the same

$$(x^2+z^2)^2-2a^2(x^2-z^2)+a^4-c^4=0. \tag{2}$$

By making use of the general formula for the curvature of implicitly defined curves $F(x,z) = 0$

$$\kappa(x,z) = \frac{|F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2|}{(F_x^2 + F_z^2)^{3/2}}{}_{|F=0}. \tag{3}$$

one can easily find that the curvature of the Cassinian curves (2) is given by the formula

$$\kappa = \frac{a^4 - c^4}{2c^2r^3} + \frac{3r}{2c^2} \tag{4}$$

which turns out to belong exactly to the class in which we are interested.

3. Integration

Here we will present a scheme by which one can reconstruct (in principle) any plane curves whose curvature depends solely on the distance from the origin, i.e., $\kappa \equiv \kappa(r)$, where r is the polar radius in the plane. For this purpose let us remind the general formula for the **curvature** in polar coordinates

$$\kappa = \frac{r^2 + 2\dot{r}^2 - r\ddot{r}}{(r^2 + \dot{r}^2)^{3/2}} \tag{5}$$

where the dots denote the derivatives with respect to the polar angle ϕ. The introduction of the new variable

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi} = \tau \tag{6}$$

has as a result the useful formula

$$\phi = \int \frac{\mathrm{d}r}{\tau}. \tag{7}$$

Further on we have also

$$\ddot{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}\phi^2} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}\phi} = \frac{\mathrm{d}\tau}{\mathrm{d}\phi} = \frac{\mathrm{d}\tau}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\phi} = \tau\frac{\mathrm{d}\tau}{\mathrm{d}r} \tag{8}$$

and therefore

$$\kappa = \frac{r^2 + 2\tau^2 - r\tau\frac{\mathrm{d}\tau}{\mathrm{d}r}}{(r^2 + \tau^2)^{3/2}} = \frac{2}{(r^2 + \tau^2)^{1/2}} - \frac{r^2 + r\tau\frac{\mathrm{d}\tau}{\mathrm{d}r}}{(r^2 + \tau^2)^{3/2}}\,. \tag{9}$$

The above formulas suggest to introduce also the notation

$$r^2 = \xi, \qquad r^2 + \tau^2 = \zeta \tag{10}$$

and making use of them to write

$$r^2 + r\tau\frac{\mathrm{d}\tau}{\mathrm{d}r} = \xi\frac{\mathrm{d}\zeta}{\mathrm{d}\xi} \tag{11}$$

which finally leads to the formula

$$\kappa(\xi) = \frac{2}{\sqrt{\zeta}} - \frac{\xi}{\sqrt{\zeta^3}}\frac{\mathrm{d}\zeta}{\mathrm{d}\xi} = 2\frac{\mathrm{d}}{\mathrm{d}\xi}(\frac{\xi}{\sqrt{\zeta}})\cdot \tag{12}$$

The integration of the last equation gives

$$\frac{\xi}{\sqrt{\zeta}} = \frac{1}{2}\int \kappa(\xi)\mathrm{d}\xi \tag{13}$$

and going back to the original coordinates one ends with the equation

$$\frac{r^2}{\sqrt{r^2+\tau^2}} = \int r\kappa(r)\mathrm{d}r. \tag{14}$$

Performing the integration on the right hand side produces

$$\int r\kappa(r)\mathrm{d}r = m(r) - \omega \tag{15}$$

where ω denotes the integration constant. Solving equations (14) and (15) for τ one gets

$$\tau = \frac{r\sqrt{r^2-(m(r)-\omega)^2}}{m(r)-\omega} \tag{16}$$

and respectively

$$\phi = \int \frac{m(r)-\omega}{r\sqrt{r^2-(m(r)-\omega)^2}}\,\mathrm{d}r \tag{17}$$

which is the result that will be used extensively in what follows. Here, the integration constant is omitted as it is responsible for the choice of the polar axis which can be done arbitrarily.

In order to obtain concrete results one has to specify the curvature in explicit form and this is what we will do below.

4. Parameterizations

The result for the Cassinian oval (4) suggests considering the curves whose curvature is given by the formula

$$\kappa = \frac{\lambda}{r^3} + \mu r = \frac{\lambda}{r^3} + 3\nu r, \qquad \lambda \in \mathbb{R}, \quad \nu \in \mathbb{R}^+. \tag{18}$$

In what follows we will present the respective parametric equations of the curves whose curvature is specified in (18) by using the approach described in the previous section. It is easy to see that strongly positive values of the parameters λ and μ reproduce exactly the Cassinian oval due to the relations

$$a^4 = \frac{4\lambda\mu + 1}{4\mu^2}, \qquad c^2 = \frac{1}{2\mu}\cdot \tag{19}$$

The above range of parameters could be easily extended by adding negative values of λ which fulfill together with μ the inequality

$$4\lambda\mu + 1 > 0. \tag{20}$$

Further on it will be taken as granted but let us mention that it includes also some negative values of λ and μ which means that the corresponding curve should be considered actually as a deformation of the parent curve [16].
Taking (18) as input, formulas (15) and (17) produce respectively

$$m(r) = -\frac{\lambda}{r} + \nu r^3 \tag{21}$$

and

$$\phi = \nu\int\frac{r^3\mathrm{d}r}{\sqrt{-\nu^2 r^8 + (2\lambda\nu+1)r^4 - \lambda^2}} - \lambda\int\frac{\mathrm{d}r}{r\sqrt{-\nu^2 r^8 + (2\lambda\nu+1)r^4 - \lambda^2}}\cdot \tag{22}$$

The above integrals can be uniformized by the following chain of substitutions

$$r^2 = \chi = n\mathrm{dn}(u,k), \qquad n = \frac{\sqrt{2\,\lambda\,\nu + 1 + \sqrt{4\,\lambda\,\nu + 1}}}{\sqrt{2}\,\nu} \tag{23}$$

where $\mathrm{dn}(u,k)$ is one of the Jacobian elliptic functions, u is the uniformizing parameter and

$$k = \sqrt{\frac{2\sqrt{4\,\lambda\,\nu + 1}}{2\,\lambda\,\nu + 1 + \sqrt{4\,\lambda\,\nu + 1}}}$$

is the so called elliptic modulus (more details about elliptic functions and integrals can be found in [1] and [18]). As a result one gets

$$\phi = \frac{1}{2}\left[\frac{\lambda}{\nu n^2\tilde{k}}\int\frac{\mathrm{d}u}{\mathrm{dn}(u,k)} - \int\mathrm{dn}(u,k)\mathrm{d}u\right], \qquad \tilde{k} = \sqrt{\frac{2\,\lambda\,\nu + 1 - \sqrt{4\,\lambda\,\nu + 1}}{2\,\lambda\,\nu + 1 + \sqrt{4\,\lambda\,\nu + 1}}}$$

and finally

$$\phi(u) = \frac{1}{2}\left[\frac{\lambda}{\nu n^2\tilde{k}}\arccos\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)} - \mathrm{am}(u,k)\right] \tag{24}$$

where $\mathrm{am}(u,k)$ is the Jacobian amplitude function and $\mathrm{cn}(u,k) = \cos\mathrm{am}(u,k)$. Alternatively, the integrals in (22) can be evaluated via the substitution $\omega = r^4$

which gives

$$\phi(r) = \frac{1}{4}\left[\arcsin\frac{2\,\nu^2\,r^4\; - 2\,\lambda\,\nu - 1}{\sqrt{4\,\lambda\,\nu + 1}} + \arcsin\frac{2\,\lambda^2 - (2\,\lambda\,\nu + 1)\,r^4}{\sqrt{4\,\lambda\,\nu + 1}\;r^4}\right]\cdot \tag{25}$$

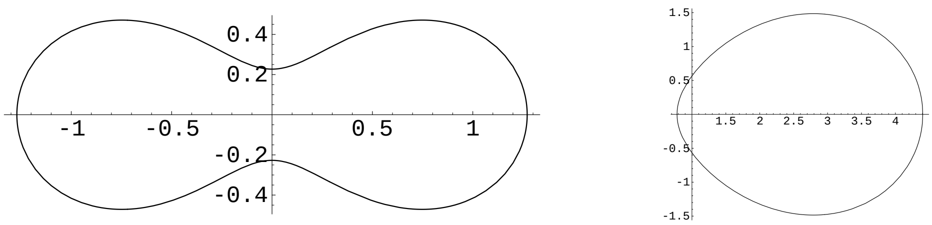

Figure 2. The biconcave curve on the left hand side is produced via (24) with parameters $\lambda = -0.05$ and $\nu = 0.6$. The (internal) oval on the right hand side is generated via (25) with $\lambda = -0.59$ and $\nu = 0.05$.

5. Concluding Remarks

Let us point out that the above parameterizations are entirely different from those reported in [2, 12, 17]. Besides, one should note a subtle difference between the formulas (24) and (25) - the first one is capable to produce both external and internal Cassinian ovals, but not the Bernoullian lemniscate while the second one can be used to draw the lemniscate and the internal ovals but omits the biconcave curves.

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