III. Universal Enveloping Algebra, 213-232

DOI: 10.3792/euclid/9798989504206-4

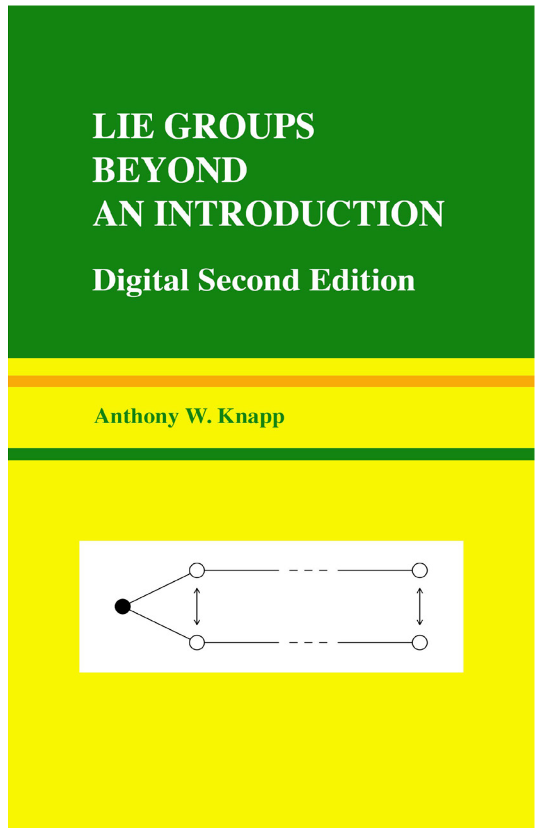
from

Lie Groups Beyond an Introduction

Digital Second Edition, 2023

Anthony W. Knapp

Full Book DOI: 10.3792/euclid/9798989504206
ISBN: 979-8-9895042-0-6



Distributed by Project Euclid.
For copyright information, see the following page.

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729
U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/~aknapp

Lie Groups Beyond an Introduction, Digital Second Edition

Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

AMS Subject Classifications: 17-01, 22-01

©1996 Anthony W. Knapp, First Edition, ISBN 0-8176-3926-8
©2002 Anthony W. Knapp, Printed Second Edition, ISBN 0-8176-4259-5
©2023 Anthony W. Knapp, Digital Second Edition, ISBN 979-8-9895042-0-6

All rights reserved. The Author (Anthony W. Knapp) has granted a license to the original Publisher (Birkhäuser Boston, c/o Springer Science+Business Media, 233 Spring Street, New York, NY 10013, USA) for the full and exclusive rights covered under the copyright of this book and all revisions of it insofar as they concern print media. These rights include the right to publish and distribute printed copies of the book throughout the world in English and in all other languages. Accordingly this work may not be transcribed or translated in whole or in part without the written permission of the Publisher, except for brief excerpts in connection with reviews or scholarly analysis and except for other fair use as understood in the copyright law.

The Author has reserved to himself all electronic rights to this book. The electronic file of the book is made available throughout the world for limited noncommercial use for purposes of education, scholarship, and research, and for these purposes only, or for fair use as understood in the United States copyright law. Users may freely download this file for their own use and may store it, post full pages of it online, and transmit full pages of it digitally for purposes of education, scholarship, and research. They may not convert it from PDF to any other format (e.g., EPUB), they may not edit it, and they may not do reverse engineering with it. In transmitting full pages of the file to others or posting pages online, users must include the copyright page, they must charge no fee, and they may not include the file in any collection of files for which a fee is charged. Any exception to these rules requires written permission from the author.

CHAPTER III

Universal Enveloping Algebra

Abstract. For a complex Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ is an explicit complex associative algebra with identity having the property that any Lie algebra homomorphism of $\mathfrak{g}$ into an associative algebra A with identity "extends" to an associative algebra homomorphism of $U(\mathfrak{g})$ into A and carrying 1 to 1. The algebra $U(\mathfrak{g})$ is a quotient of the tensor algebra $T(\mathfrak{g})$ and is a filtered algebra as a consequence of this property. The Poincaré–Birkhoff–Witt Theorem gives a vector-space basis of $U(\mathfrak{g})$ in terms of an ordered basis of $\mathfrak{g}$.

One consequence of this theorem is to identify the associated graded algebra for $U(\mathfrak{g})$ as canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. This identification allows the construction of a vector-space isomorphism called "symmetrization" from $S(\mathfrak{g})$ onto $U(\mathfrak{g})$. When $\mathfrak{g}$ is a direct sum of subspaces, the symmetrization mapping exhibits $U(\mathfrak{g})$ canonically as a tensor product.

Another consequence of the Poincaré–Birkhoff–Witt Theorem is the existence of a free Lie algebra on any set X. This is a Lie algebra $\mathfrak{F}$ with the property that any function from X into a Lie algebra extends uniquely to a Lie algebra homomorphism of $\mathfrak{F}$ into the Lie algebra.

1. Universal Mapping Property

Throughout this chapter we suppose that $\mathfrak{g}$ is a complex Lie algebra. We shall be interested only in Lie algebras whose dimension is at most countable, but our discussion will apply in general. Usually, but not always, $\mathfrak{g}$ will be finite dimensional. When we are studying a Lie group G with Lie algebra $\mathfrak{g}_0$, $\mathfrak{g}$ will be the complexification of $\mathfrak{g}_0$.

If we have a (complex-linear) representation π of $\mathfrak{g}$ on a complex vector space V, then the investigation of invariant subspaces in principle involves writing down all iterates $\pi(X_1)\pi(X_2)\cdots\pi(X_n)$ for members of $\mathfrak{g}$, applying them to members of V, and seeing what elements of V result. In the course of computing the resulting elements of V, one might be able to simplify an expression by using the identity $\pi(X)\pi(Y) = \pi(Y)\pi(X) + \pi[X, Y]$. This identity really has little to do with π, and our objective in this section will be to introduce a setting in which we can make such calculations without reference to π; to obtain an identity for the representation π, one simply applies π to both sides of a universal identity.

For a first approximation of what we want, we can use the tensor algebra $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k(\mathfrak{g})$. (Appendix A gives the definition and elementary properties of $T(\mathfrak{g})$.) The representation π is a linear map of $\mathfrak{g}$ into the associative algebra $\operatorname{End}_{\mathbb{C}} V$ and extends to an algebra homomorphism $\widetilde{\pi} : T(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}} V$ with $\widetilde{\pi}(1) = 1$. Then $\pi(X_1)\pi(X_2)\cdots\pi(X_n)$ can be replaced by $\widetilde{\pi}(X_1 \otimes X_2 \otimes \cdots \otimes X_n)$. The difficulty with using $T(\mathfrak{g})$ is that it does not take advantage of the Lie algebra structure of $\mathfrak{g}$ and does not force the identity $\pi(X)\pi(Y) = \pi(Y)\pi(X) + \pi[X, Y]$ for all X and Y in $\mathfrak{g}$ and all π. Thus instead of the tensor algebra, we use the following quotient of $T(\mathfrak{g})$:

(3.1a) $$U(\mathfrak{g}) = T(\mathfrak{g})/J,$$

where

(3.1b) $$J = \left(\begin{array}{l} \text{two-sided ideal generated by all} \\ X \otimes Y - Y \otimes X - [X, Y] \text{ with } X \\ \text{and } Y \text{ in } T^1(\mathfrak{g}) \end{array} \right).$$

The quotient $U(\mathfrak{g})$ is an associative algebra with identity and is known as the **universal enveloping algebra** of $\mathfrak{g}$. Products in $U(\mathfrak{g})$ are written without multiplication signs.

The canonical map $\mathfrak{g} \to U(\mathfrak{g})$ given by embedding $\mathfrak{g}$ into $T^1(\mathfrak{g})$ and then passing to $U(\mathfrak{g})$ is denoted ι. Because of (3.1), ι satisfies

(3.2) $$\iota[X, Y] = \iota(X)\iota(Y) - \iota(Y)\iota(X) \qquad \text{for } X \text{ and } Y \text{ in } \mathfrak{g}.$$

The algebra $U(\mathfrak{g})$ is harder to work with than the exterior algebra $\bigwedge(\mathfrak{g})$ or the symmetric algebra $S(\mathfrak{g})$, which are both quotients of $T(\mathfrak{g})$ and are discussed in Appendix A. The reason is that the ideal in (3.1b) is not generated by homogeneous elements. Thus, for example, it is not evident that the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is one-one. However, when $\mathfrak{g}$ is *abelian*, $U(\mathfrak{g})$ reduces to $S(\mathfrak{g})$, and we have a clear notion of what to expect of $U(\mathfrak{g})$. Even when $\mathfrak{g}$ is nonabelian, $U(\mathfrak{g})$ and $S(\mathfrak{g})$ are still related, and we shall make the relationship precise later in this chapter.

Let $U_n(\mathfrak{g})$ be the image of $T_n(\mathfrak{g}) = \bigoplus_{k=0}^{n} T^k(\mathfrak{g})$ under the passage to the quotient in (3.1). Then $U(\mathfrak{g}) = \bigcup_{n=0}^{\infty} U_n(\mathfrak{g})$. Since $\{T_n(\mathfrak{g})\}$ exhibits

$T(\mathfrak{g})$ as a filtered algebra, $\{U_n(\mathfrak{g})\}$ exhibits $U(\mathfrak{g})$ as a filtered algebra. If $\mathfrak{g}$ is finite dimensional, each $U_n(\mathfrak{g})$ is finite dimensional.

Proposition 3.3. The algebra $U(\mathfrak{g})$ and the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ have the following universal mapping property: Whenever A is a complex associative algebra with identity and $\pi : \mathfrak{g} \to A$ is a linear mapping such that

$$\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi[X, Y] \qquad \text{for all } X \text{ and } Y \text{ in } \mathfrak{g}, \tag{3.4}$$

then there exists a unique algebra homomorphism $\widetilde{\pi} : U(\mathfrak{g}) \to A$ such that $\widetilde{\pi}(1) = 1$ and the diagram

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[dr, dashed, "\widetilde{\pi}"] & \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[rr, "\pi"'] & & A \end{tikzcd} \tag{3.5}$$

commutes.

REMARKS.

1) We regard $\widetilde{\pi}$ as an "extension" of π. This notion will be more appropriate after we prove that ι is one-one.

2) This proposition allows us to make an alternative definition of **universal enveloping algebra** for $\mathfrak{g}$. It is a pair $(U(\mathfrak{g}), \iota)$ such that $U(\mathfrak{g})$ is an associative algebra with identity, $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a linear mapping satisfying (3.2), and whenever $\pi : \mathfrak{g} \to A$ is a linear mapping satisfying (3.4), then there exists a unique algebra homomorphism $\widetilde{\pi} : U(\mathfrak{g}) \to A$ such that $\widetilde{\pi}(1) = 1$ and the diagram (3.5) commutes. The proposition says that the constructed $U(\mathfrak{g})$ has this property, and we can use this property to see that any other candidate, say $(U'(\mathfrak{g}), \iota')$, has $U'(\mathfrak{g})$ canonically isomorphic with the constructed $U(\mathfrak{g})$. In fact, if we use (3.5) with $A = U'(\mathfrak{g})$ and $\pi = \iota'$, we obtain an algebra map $\widetilde{\iota'} : U(\mathfrak{g}) \to U'(\mathfrak{g})$. Reversing the roles of $U(\mathfrak{g})$ and $U'(\mathfrak{g})$ yields $\widetilde{\iota} : U'(\mathfrak{g}) \to U(\mathfrak{g})$. To see that $\widetilde{\iota} \circ \widetilde{\iota'} = 1_{U(\mathfrak{g})}$, we use the uniqueness of the extension $\widetilde{\pi}$ in (3.5) when $A = U(\mathfrak{g})$ and $\pi = 1$. Similarly $\widetilde{\iota'} \circ \widetilde{\iota} = 1_{U'(\mathfrak{g})}$.

PROOF. Uniqueness follows from the fact that 1 and $\iota(\mathfrak{g})$ generate $U(\mathfrak{g})$. For existence let $\pi_1 : T(\mathfrak{g}) \to A$ be the extension given by the universal mapping property of $T(\mathfrak{g})$ in Proposition A.14. To obtain $\widetilde{\pi}$, we are to show that π_1 annihilates the ideal J in (3.1b). It is enough to consider π_1

on a typical generator of J, where we have

$$\begin{aligned}
&\pi_1(\iota X \otimes \iota Y - \iota Y \otimes \iota X - \iota[X,Y]) \\
&\qquad\qquad = \pi_1(\iota X)\pi_1(\iota Y) - \pi_1(\iota Y)\pi_1(\iota X) - \pi_1(\iota[X,Y]) \\
&\qquad\qquad = \pi(X)\pi(Y) - \pi(Y)\pi(X) - \pi[X,Y] \\
&\qquad\qquad = 0.
\end{aligned}$$

Corollary 3.6. Representations of $\mathfrak{g}$ on complex vector spaces stand in one-one correspondence with unital left $U(\mathfrak{g})$ modules (under the correspondence $\pi \to \widetilde{\pi}$ of Proposition 3.3).

REMARK. **Unital** means that 1 operates as 1.

PROOF. If π is a representation of $\mathfrak{g}$ on V, we apply Proposition 3.3 to $\pi : \mathfrak{g} \to \operatorname{End}_{\mathbb{C}} V$, and then V becomes a unital left $U(\mathfrak{g})$ module under $uv = \widetilde{\pi}(u)v$ for $u \in U(\mathfrak{g})$ and $v \in V$. Conversely if V is a unital left $U(\mathfrak{g})$ module, then V is already a complex vector space with scalar multiplication given by the action of the scalar multiples of 1 in $U(\mathfrak{g})$. If we define $\pi(X)v = (\iota X)v$, then (3.2) implies that π is a representation of $\mathfrak{g}$. The two constructions are inverse to each other since $\widetilde{\pi} \circ \iota = \pi$ in Proposition 3.3.

Proposition 3.7. There exists a unique antiautomorphism $u \mapsto u^t$ of $U(\mathfrak{g})$ such that $\iota(X)^t = -\iota(X)$ for all $X \in \mathfrak{g}$.

REMARK. The map $(\,\cdot\,)^t$ is called **transpose**.

PROOF. It is unique since $\iota(\mathfrak{g})$ and 1 generate $U(\mathfrak{g})$. Let us prove existence. For each $n \geq 1$, the map

$$(X_1, \ldots, X_n) \mapsto (-1)^n X_n \otimes \cdots \otimes X_1$$

is n-multilinear from $\mathfrak{g} \times \cdots \times \mathfrak{g}$ into $T^n(\mathfrak{g})$ and hence extends to a linear map of $T^n(\mathfrak{g})$ into itself with

$$X_1 \otimes \cdots \otimes X_n \mapsto (-1)^n X_n \otimes \cdots \otimes X_1.$$

Taking the direct sum of these maps as n varies, we obtain a linear map $x \mapsto x^t$ of $T(\mathfrak{g})$ into itself sending 1 into 1. It is clear that this map is an antiautomorphism and extends $X \mapsto -X$ in $T^1(\mathfrak{g})$. Composing with passage to the quotient by J, we obtain an antihomomorphism of $T(\mathfrak{g})$ into

$U(\mathfrak{g})$. Its kernel is an ideal. To show that the map descends to $U(\mathfrak{g})$, it is enough to show that each generator

$$X \otimes Y - Y \otimes X - [X, Y]$$

maps to 0. But this element maps in $T(\mathfrak{g})$ to itself and then maps to 0 in $U(\mathfrak{g})$. Hence the transpose map descends to $U(\mathfrak{g})$. It is clearly of order two and thus is one-one onto.

The transpose map $u \mapsto u^t$ allows us to regard left $U(\mathfrak{g})$ modules V also as right $U(\mathfrak{g})$ modules, and vice versa: To convert a left $U(\mathfrak{g})$ module into a right $U(\mathfrak{g})$ module, we just define $vu = u^t v$ for $u \in U(\mathfrak{g})$ and $v \in V$. Conversion in the opposite direction is accomplished by $uv = vu^t$.

2. Poincaré–Birkhoff–Witt Theorem

The main theorem about $U(\mathfrak{g})$ gives a basis for $U(\mathfrak{g})$ as a vector space. Let $\{X_i\}_{i \in A}$ be a basis of $\mathfrak{g}$. A set such as A always admits a **simple ordering**, i.e., a partial ordering in which every pair of elements is comparable. In cases of interest, the dimension of $\mathfrak{g}$ is at most countable, and we can think of this ordering as quite elementary. For example, it might be the ordering of the positive integers, or it might be something quite different but still reasonable.

Theorem 3.8 (Poincaré–Birkhoff–Witt). Let $\{X_i\}_{i \in A}$ be a basis of $\mathfrak{g}$, and suppose a simple ordering has been imposed on the index set A. Then the set of all monomials

$$(\iota X_{i_1})^{j_1} \cdots (\iota X_{i_n})^{j_n}$$

with $i_1 < \cdots < i_n$ and with all $j_k \geq 0$, is a basis of $U(\mathfrak{g})$. In particular the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is one-one.

REMARKS.

1) If A is finite, say $A = \{1, \dots, N\}$, the basis consists of all monomials $(\iota X_1)^{j_1} \cdots (\iota X_N)^{j_N}$ with all $j_k \geq 0$.

2) The proof will be preceded by two lemmas, which will essentially establish the spanning. The main step will be to prove the linear independence. For this we have to prove that $U(\mathfrak{g})$ is suitably large. The motivation for carrying out this step comes from assuming the theorem to be true. Then

we might as well drop ι from the notation, and monomials $X_{i_1}^{j_1} \cdots X_{i_n}^{j_n}$ with $i_1 < \cdots < i_n$ will form a basis. These same monomials, differently interpreted, are a basis of $S(\mathfrak{g})$. Thus the theorem is asserting a particular vector-space isomorphism $U(\mathfrak{g}) \to S(\mathfrak{g})$. Since $U(\mathfrak{g})$ is naturally a unital left $U(\mathfrak{g})$ module, this isomorphism suggests that $S(\mathfrak{g})$ should be a left unital $U(\mathfrak{g})$ module. By Corollary 3.6 we should look for a natural representation of $\mathfrak{g}$ on $S(\mathfrak{g})$ consistent with left multiplication of $\mathfrak{g}$ on $U(\mathfrak{g})$ and consistent with the particular isomorphism $U(\mathfrak{g}) \to S(\mathfrak{g})$. The proof consists of constructing this representation, and then the linear independence follows easily. Actually the proof will make use of a polynomial algebra, but the polynomial algebra is canonically isomorphic to $S(\mathfrak{g})$ once a basis of $\mathfrak{g}$ has been specified.

Lemma 3.9. Let $Z_1, \ldots, Z_p$ be in $\mathfrak{g}$, and let σ be a permutation of $\{1, \ldots, p\}$. Then

$$(\iota Z_1) \cdots (\iota Z_p) - (\iota Z_{\sigma(1)}) \cdots (\iota Z_{\sigma(p)})$$

is in $U_{p-1}(\mathfrak{g})$.

PROOF. Without loss of generality, let σ be the transposition of j with $j+1$. Then the lemma follows from

$$(\iota Z_j)(\iota Z_{j+1}) - (\iota Z_{j+1})(\iota Z_j) = \iota[Z_j, Z_{j+1}]$$

by multiplying through on the left by $(\iota Z_1) \cdots (\iota Z_{j-1})$ and on the right by $(\iota Z_{j+2}) \cdots (\iota Z_p)$.

For the remainder of the proof of Theorem 3.8, we shall use the following notation: For $i \in A$, let $Y_i = \iota X_i$. For any tuple $I = (i_1, \ldots, i_p)$ of members of A, we say that I is **increasing** if $i_1 \leq \cdots \leq i_p$. Whether or not I is increasing, we write $Y_I = Y_{i_1} \cdots Y_{i_p}$. Also $i \leq I$ means $i \leq \min\{i_1, \ldots, i_p\}$.

Lemma 3.10. The Y_I, for all increasing tuples from A of length $\leq p$, span $U_p(\mathfrak{g})$.

PROOF. If we use *all* tuples of length $\leq p$, we certainly have a spanning set, since the obvious preimages in $T(\mathfrak{g})$ span $T_p(\mathfrak{g})$. Lemma 3.9 then implies inductively that the increasing tuples suffice.

PROOF OF THEOREM 3.8. Let P be the polynomial algebra over $\mathbb{C}$ in the variables z_i, $i \in A$, and let P_p be the subspace of members of total degree $\leq p$. For a tuple $I = (i_1, \ldots, i_p)$, define $z_I = z_{i_1} \cdots z_{i_p}$ as a member of P_p. We shall construct a representation π of $\mathfrak{g}$ on P such that

$$\pi(X_i)z_I = z_i z_I \qquad \text{if } i \leq I. \tag{3.11}$$

Let us see that the theorem follows from the existence of such a representation. In fact, let us use Corollary 3.6 to regard P as a unital left $U(\mathfrak{g})$ module. Then (3.11) and the identity $\pi(X)v = (\iota X)v$ imply that

$$Y_i z_I = z_i z_I \qquad \text{if } i \leq I.$$

If $i_1 \leq \cdots \leq i_p$, then as a consequence we obtain

$$\begin{aligned}(Y_{i_1} \cdots Y_{i_p})1 &= (Y_{i_1} \cdots Y_{i_{p-1}})Y_{i_p}1 \\ &= (Y_{i_1} \cdots Y_{i_{p-1}})z_{i_p} \\ &= (Y_{i_1} \cdots Y_{i_{p-2}})Y_{i_{p-1}}z_{i_p} \\ &= (Y_{i_1} \cdots Y_{i_{p-2}})(z_{i_{p-1}}z_{i_p}) \\ &= \cdots = z_{i_1} \cdots z_{i_p}.\end{aligned}$$

Thus the set $\{Y_I 1 \mid I \text{ increasing}\}$ is linearly independent within P, and $\{Y_I \mid I \text{ increasing}\}$ must be independent in $U(\mathfrak{g})$. The independence in Theorem 3.8 follows, and the spanning is given in Lemma 3.10.

Thus we have to construct π satisfying (3.11). We shall define linear maps $\pi(X) : P_p \to P_{p+1}$ for X in $\mathfrak{g}$, by induction on p so that they are compatible and satisfy

- (A_p) $\pi(X_i)z_I = z_i z_I$ for $i \leq I$ and z_I in P_p,
- (B_p) $\pi(X_i)z_I - z_i z_I$ is in P_p for all I with z_I in P_p,
- (C_p) $\pi(X_i)(\pi(X_j)z_J) = \pi(X_j)(\pi(X_i)z_J) + \pi[X_i, X_j]z_J$ for all J with z_J in P_{p-1}.

With $\pi(X)$ defined on P as the union of its definitions on the P_p's, π will be a representation by (C_p) and will satisfy (3.11) by (A_p). Hence we will be done.

For $p = 0$, we define $\pi(X_i)1 = z_i$. Then (A_0) holds, (B_0) is valid, and (C_0) is vacuous.

Inductively assume that $\pi(X)$ has been defined on P_{p-1} for all $X \in \mathfrak{g}$ in such a way that (A_{p-1}), (B_{p-1}), and (C_{p-1}) hold. We are to define $\pi(X_i)z_I$ for each increasing sequence I of p indices in such a way that

(A_p), (B_p), and (C_p) hold. If $i \le I$, we make the definition according to (A_p). Otherwise we can write in obvious notation $I = (j, J)$ with $j < i$, $j \le J$, $|J| = p-1$. We are forced to define

$$\begin{aligned}
\pi(X_i)z_I &= \pi(X_i)(z_j z_J) \\
&= \pi(X_i)\pi(X_j)z_J && \text{since } \pi(X_j)z_J \text{ is already defined by } (A_{p-1}) \\
&= \pi(X_j)\pi(X_i)z_J + \pi[X_i, X_j]z_J && \text{by } (C_p) \\
&= \pi(X_j)(z_i z_J + w) + \pi[X_i, X_j]z_J && \text{with } w \text{ in } P_{p-1} \text{ by } (B_{p-1}) \\
&= z_j z_i z_J + \pi(X_j)w + \pi[X_i, X_j]z_J && \text{by } (A_p) \\
&= z_i z_I + \pi(X_j)w + \pi[X_i, X_j]z_J.
\end{aligned}$$

We make this definition, and then (B_p) holds. Therefore $\pi(X_i)z_I$ has now been defined in all cases on P_p, and we have to show that (C_p) holds.

Our construction above was made so that (C_p) holds if $j < i$, $j \le J$, $|J| = p-1$. Since $[X_j, X_i] = -[X_i, X_j]$, it holds also if $i < j$, $i \le J$, $|J| = p-1$. Also (C_p) is trivial if $i = j$. Thus it holds whenever $i \le J$ or $j \le J$. So we may assume that $J = (k, K)$, where $k \le K$, $k < i$, $k < j$, $|K| = p-2$. We know that

$$\begin{aligned}
\pi(X_j)z_J &= \pi(X_j)z_k z_K \\
&= \pi(X_j)\pi(X_k)z_K \\
&= \pi(X_k)\pi(X_j)z_K + \pi[X_j, X_k]z_K && \text{by } (C_{p-1}) \\
&= \pi(X_k)(z_j z_K + w) + \pi[X_j, X_k]z_K
\end{aligned} \tag{3.12}$$

for a certain element w in P_{p-2} given by (B_{p-2}), which is assumed valid since $(B_{p-2}) \subseteq (B_{p-1})$. We apply $\pi(X_i)$ to both sides of this equation, calling the three terms on the right T_1, T_2, and T_3. We can use what we already know for (C_p) to handle $\pi(X_i)$ of T_1 because $k \le (j, K)$, and we can use (C_{p-1}) with $\pi(X_i)$ of T_2 and T_3. Reassembling T_1 and T_2 as in line (3.12), we conclude that we can use known cases of (C_p) with the sum $\pi(X_i)\pi(X_k)\pi(X_j)z_K$, and we can use (C_{p-1}) with $\pi(X_i)$ of T_3. Thus we have

$$\begin{aligned}
\pi(X_i)\pi(X_j)z_J &= \pi(X_i)\pi(X_k)\pi(X_j)z_K + \pi(X_i)\pi[X_j, X_k]z_K \quad \text{from (3.12)} \\
&= \pi(X_k)\pi(X_i)\pi(X_j)z_K + \pi[X_i, X_k]\pi(X_j)z_K \\
&\quad + \pi[X_j, X_k]\pi(X_i)z_K + \pi[X_i, [X_j, X_k]]z_K \\
&\qquad\qquad \text{by known cases of } (C_p)
\end{aligned}$$

$$= T_1' + T_2' + T_3' + T_4'.$$

Interchanging i and j and subtracting, we see that the terms of type T_2' and T_3' cancel and that we get

$$\begin{aligned}
\pi(X_i)\pi(X_j)z_J - \pi(X_j)\pi(X_i)z_J \qquad & \\
= \pi(X_k)\{\pi(X_i)\pi(X_j)z_K - \pi(X_j)\pi(X_i)z_K\} & \\
+ \{\pi[X_i, [X_j, X_k]] - \pi[X_j, [X_i, X_k]]\}z_K & \\
= \pi(X_k)\pi[X_i, X_j]z_K + \pi[[X_i, X_j], X_k]z_K & \quad \text{by } (C_{p-1}) \text{ and Jacobi} \\
= \pi[X_i, X_j]\pi(X_k)z_K & \quad \text{by } (C_{p-1}) \\
= \pi[X_i, X_j]z_k z_K & \\
= \pi[X_i, X_j]z_J. &
\end{aligned}$$

We have obtained (C_p) in the remaining case, and the proof of Theorem 3.8 is complete.

Now that ι is known to be one-one, there is no danger in dropping it from the notation. We shall freely use Corollary 3.6, identifying representations of $\mathfrak{g}$ with unital left $U(\mathfrak{g})$ modules. Moreover we shall feel free either to drop the name of a representation from the notation (to emphasize the module structure) or to include it even when the argument is in $U(\mathfrak{g})$ (to emphasize the representation structure).

The Poincaré–Birkhoff–Witt Theorem appears in a number of guises. Here is one such.

Corollary 3.13. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the associative subalgebra of $U(\mathfrak{g})$ generated by 1 and $\mathfrak{h}$ is canonically isomorphic to $U(\mathfrak{h})$.

PROOF. If $\rho : \mathfrak{h} \to \mathfrak{g}$ denotes inclusion, then ρ yields an inclusion (also denoted ρ) of $\mathfrak{h}$ into $U(\mathfrak{g})$ such that $\rho(X)\rho(Y) - \rho(Y)\rho(X) = \rho[X, Y]$ for X and Y in $\mathfrak{h}$. By the universal mapping property of $U(\mathfrak{h})$, we obtain a corresponding algebra map $\widetilde{\rho} : U(\mathfrak{h}) \to U(\mathfrak{g})$ with $\widetilde{\rho}(1) = 1$. The image of $\widetilde{\rho}$ is certainly the subalgebra of $U(\mathfrak{g})$ generated by 1 and $\rho(\mathfrak{h})$. Theorem 3.8 says that monomials in an ordered basis of $\mathfrak{h}$ span $U(\mathfrak{h})$, and a second application of the theorem says that these monomials in $U(\mathfrak{g})$ are linearly independent. Thus $\widetilde{\rho}$ is one-one and the corollary follows.

If $\mathfrak{g}$ happens to be the vector-space direct sum of two Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$, then it follows that we have a vector-space isomorphism

$$U(\mathfrak{g}) \cong U(\mathfrak{a}) \otimes_{\mathbb{C}} U(\mathfrak{b}). \tag{3.14}$$

Namely we obtain a linear map from right to left from the inclusions in Corollary 3.13. To see that the map is an isomorphism, we apply Theorem 3.8 to a basis of $\mathfrak{a}$ followed by a basis of $\mathfrak{b}$. The monomials in the separate bases are identified within $U(\mathfrak{g})$ as bases for $U(\mathfrak{a})$ and $U(\mathfrak{b})$, respectively, by Corollary 3.13, while the joined-together bases give both a basis of the tensor product and a basis of $U(\mathfrak{g})$, again by Theorem 3.8. Thus our map sends basis to basis and is an isomorphism.

3. Associated Graded Algebra

If A is a complex associative algebra with identity and if A is filtered in the sense of Appendix A, say as $A = \cup_{n=0}^{\infty} A_n$, then Appendix A shows how to define the associated graded algebra $\operatorname{gr} A = \bigoplus_{n=0}^{\infty}(A_n/A_{n-1})$, where $A_{-1} = 0$. In this section we shall compute $\operatorname{gr} U(\mathfrak{g})$, showing that it is canonically isomorphic with the symmetric algebra $S(\mathfrak{g})$. Then we shall derive some consequences of this isomorphism.

The idea is to use the Poincaré–Birkhoff–Witt Theorem. The theorem implies that a basis of $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is all monomial cosets

$$X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(\mathfrak{g})$$

for which the indices have $i_1 < \cdots < i_k$ and the sum of the exponents is exactly n. The monomials $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, interpreted as in $S(\mathfrak{g})$, are a basis of $S^n(\mathfrak{g})$, and the linear map that carries basis to basis ought to be the desired isomorphism. In fact, this statement is true, but this approach does not conveniently show that the isomorphism is independent of basis. We shall therefore proceed somewhat differently.

We shall construct the map in the opposite direction without using the Poincaré–Birkhoff–Witt Theorem, appeal to the theorem to show that we have an isomorphism, and then compute what the map is in terms of a basis. Let $T_n(\mathfrak{g}) = \bigoplus_{k=0}^{n} T^k(\mathfrak{g})$ be the n^{th} member of the usual filtration of $T(\mathfrak{g})$. We have defined $U_n(\mathfrak{g})$ to be the image in $U(\mathfrak{g})$ of $T_n(\mathfrak{g})$ under the passage $T(\mathfrak{g}) \to T(\mathfrak{g})/J$. Thus we can form the composition

$$T_n(\mathfrak{g}) \to (T_n(\mathfrak{g}) + J)/J = U_n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}).$$

This composition is onto and carries $T_{n-1}(\mathfrak{g})$ to 0. Since $T^n(\mathfrak{g})$ is a vector-space complement to $T_{n-1}(\mathfrak{g})$ in $T_n(\mathfrak{g})$, we obtain an onto linear map

$$T^n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}).$$

Taking the direct sum over n gives an onto linear map

$$\widetilde{\psi} : T(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$$

that respects the grading.

Appendix A uses the notation I for the two-sided ideal in $T(\mathfrak{g})$ such that $S(\mathfrak{g}) = T(\mathfrak{g})/I$:

$$I = \begin{pmatrix} \text{two-sided ideal generated by all} \\ X \otimes Y - Y \otimes X \text{ with } X \text{ and } Y \text{ in} \\ T^1(\mathfrak{g}) \end{pmatrix}. \tag{3.15}$$

Proposition 3.16. The linear map $\widetilde{\psi} : T(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$ respects multiplication and annihilates the defining ideal I for $S(\mathfrak{g})$. Therefore ψ descends to an algebra homomorphism

$$\psi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g}) \tag{3.17}$$

that respects the grading. This homomorphism is an isomorphism.

PROOF. Let x be in $T^r(\mathfrak{g})$ and let y be in $T^s(\mathfrak{g})$. Then $x + J$ is in $U_r(\mathfrak{g})$, and we may regard $\widetilde{\psi}(x)$ as the coset $x + T_{r-1}(\mathfrak{g}) + J$ in $U_r(\mathfrak{g})/U_{r-1}(\mathfrak{g})$, with 0 in all other coordinates of $\operatorname{gr} U(\mathfrak{g})$ since x is homogeneous. Arguing in a similar fashion with y and xy, we obtain

$$\widetilde{\psi}(x) = x + T_{r-1}(\mathfrak{g}) + J, \qquad \widetilde{\psi}(y) = y + T_{s-1}(\mathfrak{g}) + J,$$
$$\text{and} \qquad \widetilde{\psi}(xy) = xy + T_{r+s-1}(\mathfrak{g}) + J.$$

Since J is an ideal, $\widetilde{\psi}(x)\widetilde{\psi}(y) = \widetilde{\psi}(xy)$. General members x and y of $T(\mathfrak{g})$ are sums of homogeneous elements, and hence $\widetilde{\psi}$ respects multiplication.

Consequently $\ker \widetilde{\psi}$ is a two-sided ideal. To show that $\ker \widetilde{\psi} \supseteq I$, it is enough to show that $\ker \widetilde{\psi}$ contains all generators $X \otimes Y - Y \otimes X$. We have

$$\begin{aligned} \widetilde{\psi}(X \otimes Y - Y \otimes X) &= X \otimes Y - Y \otimes X + T_1(\mathfrak{g}) + J \\ &= [X, Y] + T_1(\mathfrak{g}) + J \\ &= T_1(\mathfrak{g}) + J, \end{aligned}$$

and thus $\widetilde{\psi}$ maps the generator to 0. Hence $\widetilde{\psi}$ descends to a homomorphism ψ as in (3.17).

Now let $\{X_i\}$ be an ordered basis of $\mathfrak{g}$. The monomials $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$ in $S(\mathfrak{g})$ with $i_1 < \cdots < i_k$ and with $\sum_m j_m = n$ form a basis of $S^n(\mathfrak{g})$. Let us follow the effect of (3.17) on such a monomial. A preimage of this monomial in $T^n(\mathfrak{g})$ is the element

$$X_{i_1} \otimes \cdots \otimes X_{i_1} \otimes \cdots \otimes X_{i_k} \otimes \cdots \otimes X_{i_k},$$

in which there are j_m factors of X_{i_m} for $1 \leq m \leq k$. This element maps to the monomial in $U_n(\mathfrak{g})$ that we have denoted $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, and then we pass to the quotient $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. Theorem 3.8 shows that such monomials modulo $U_{n-1}(\mathfrak{g})$ form a basis of $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. Consequently (3.17) is an isomorphism.

Inspecting the proof of Proposition 3.16, we see that if $i_1 < \cdots < i_k$ and $\sum_m j_m = n$, then

$$\psi(X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}) = X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(\mathfrak{g}). \tag{3.18a}$$

Hence

$$\psi^{-1}(X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(\mathfrak{g})) = X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}, \tag{3.18b}$$

as asserted in the second paragraph of this section. Note that the restriction $i_1 < \cdots < i_k$ can be dropped in (3.18) as a consequence of Lemma 3.9.

Corollary 3.19. Let W be a subspace of $T^n(\mathfrak{g})$, and suppose that the quotient map $T^n(\mathfrak{g}) \to S^n(\mathfrak{g})$ sends W isomorphically onto $S^n(\mathfrak{g})$. Then the image of W in $U_n(\mathfrak{g})$ is a vector-space complement to $U_{n-1}(\mathfrak{g})$.

PROOF. Consider the diagram

$$\begin{tikzcd} T^n(\mathfrak{g}) \arrow[r] \arrow[d] & U_n(\mathfrak{g}) \arrow[d] \\ S^n(\mathfrak{g}) \arrow[r, "\psi"] & U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}) \end{tikzcd}$$

The fact that this diagram is commutative is equivalent with the conclusion in Proposition 3.16 that $\widetilde{\psi} : T^n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ descends to a map $\psi : S^n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. The proposition says that ψ on the bottom of the diagram is an isomorphism, and the hypothesis is that the map on the left, when restricted to W, is an isomorphism onto $S^n(\mathfrak{g})$. Therefore the composition of the map on the top followed by the map on the right is an isomorphism when restricted to W, and the corollary follows.

We apply Corollary 3.19 to the space $\widetilde{S}^n(\mathfrak{g})$ of symmetrized tensors within $T^n(\mathfrak{g})$. As in §A.2, $\widetilde{S}^n(\mathfrak{g})$ is the linear span, for all n-tuples $X_1, \ldots, X_n$ from $\mathfrak{g}$, of the elements

$$\frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} X_{\tau(1)}\cdots X_{\tau(n)},$$

where $\mathfrak{S}_n$ is the symmetric group on n letters. According to Proposition A.25, we have a direct sum decomposition

$$T^n(\mathfrak{g}) = \widetilde{S}^n(\mathfrak{g}) \oplus (T^n(\mathfrak{g}) \cap I). \tag{3.20}$$

We shall use this decomposition to investigate a map known as "symmetrization."

For $n \geq 1$, define a symmetric n-multilinear map

$$\sigma_n : \mathfrak{g} \times \cdots \times \mathfrak{g} \to U(\mathfrak{g})$$

by
$$\sigma_n(X_1, \ldots, X_n) = \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} X_{\tau(1)}\cdots X_{\tau(n)}.$$

By Proposition A.20a we obtain a corresponding linear map, also denoted σ_n, from $S^n(\mathfrak{g})$ into $U(\mathfrak{g})$. The image of $S^n(\mathfrak{g})$ in $U(\mathfrak{g})$ is clearly the same as the image of the subspace $\widetilde{S}^n(\mathfrak{g})$ of $T^n(\mathfrak{g})$ in $U_n(\mathfrak{g})$. By (3.20) and Corollary 3.19, σ_n is one-one from $S^n(\mathfrak{g})$ onto a vector-space complement to $U_{n-1}(\mathfrak{g})$ in $U_n(\mathfrak{g})$, i.e.,

$$U_n(\mathfrak{g}) = \sigma_n(S^n(\mathfrak{g})) \oplus U_{n-1}(\mathfrak{g}). \tag{3.21}$$

The direct sum of the maps σ_n for $n \geq 0$ (with $\sigma_0(1) = 1$) is a linear map $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ such that

$$\sigma(X_1\cdots X_n) = \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} X_{\tau(1)}\cdots X_{\tau(n)}.$$

The map σ is called **symmetrization**.

Lemma 3.22. The symmetrization map $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ has associated graded map $\psi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$, with ψ as in (3.17).

REMARK. The "associated graded map" is defined in §A.4.

PROOF. Let $\{X_i\}$ be a basis of $\mathfrak{g}$, and let $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, with $\sum_m j_m = n$, be a basis vector of $S^n(\mathfrak{g})$. Under σ, this vector is sent to a symmetrized sum, but each term of the sum is congruent mod $U_{n-1}(\mathfrak{g})$ to $(n!)^{-1} X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, by Lemma 3.9. Hence the image of $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$ under the associated graded map is

$$= X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(\mathfrak{g}) = \psi(X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}),$$

as asserted.

Proposition 3.23. Symmetrization σ is a vector-space isomorphism of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$ satisfying

$$U_n(\mathfrak{g}) = \sigma(S^n(\mathfrak{g})) \oplus U_{n-1}(\mathfrak{g}). \tag{3.24}$$

PROOF. Formula (3.24) is a restatement of (3.21), and the other conclusion follows by combining Lemma 3.22 and Proposition A.39.

The canonical decomposition of $U(\mathfrak{g})$ from $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ when $\mathfrak{a}$ and $\mathfrak{b}$ are merely vector spaces is given in the following proposition.

Proposition 3.25. Suppose $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ and suppose $\mathfrak{a}$ and $\mathfrak{b}$ are subspaces of $\mathfrak{g}$. Then the mapping $a \otimes b \mapsto \sigma(a)\sigma(b)$ of $S(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{b})$ into $U(\mathfrak{g})$ is a vector-space isomorphism onto.

PROOF. The vector space $S(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{b})$ is graded consistently for the given mapping, the n^{th} space of the grading being $\bigoplus_{p=0}^{n} S^p(\mathfrak{a}) \otimes_{\mathbb{C}} S^{n-p}(\mathfrak{b})$. The given mapping operates on an element of this space by

$$\sum_{p=0}^{n} a_p \otimes b_{n-p} \mapsto \sum_{p=0}^{n} \sigma(a_p)\sigma(b_{n-p}),$$

and the image of this under the associated graded map is

$$= \sum_{p=0}^{n} \sigma(a_p)\sigma(b_{n-p}) + U_{n-1}(\mathfrak{g}).$$

In turn this is

$$= \sigma\Big(\sum_{p=0}^{n} a_p \otimes b_{n-p}\Big) + U_{n-1}(\mathfrak{g})$$

by Lemma 3.9. In other words the associated graded map is just the same as for σ. Hence the result follows by combining Propositions 3.23 and A.39.

Corollary 3.26. Suppose that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Then the mapping $(u, p) \mapsto u\sigma(p)$ of $U(\mathfrak{k}) \otimes_{\mathbb{C}} S(\mathfrak{p})$ into $U(\mathfrak{g})$ is a vector-space isomorphism onto.

PROOF. The composition

$$(k, p) \mapsto (\sigma(k), p) \mapsto \sigma(k)\sigma(p),$$

sending

$$S(\mathfrak{k}) \otimes_{\mathbb{C}} S(\mathfrak{p}) \to U(\mathfrak{k}) \otimes_{\mathbb{C}} S(\mathfrak{p}) \to U(\mathfrak{g}),$$

is an isomorphism by Proposition 3.25, and the first map is an isomorphism by Proposition 3.23. Therefore the second map is an isomorphism, and the notation corresponds to the statement of the corollary when we write $u = \sigma(k)$.

Proposition 3.27. If $\mathfrak{g}$ is finite dimensional, then the ring $U(\mathfrak{g})$ is left Noetherian.

PROOF. The associated graded algebra for $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$, according to Proposition 3.16, and $S(\mathfrak{g})$ is a commutative Noetherian ring by the Hilbert Basis Theorem (Theorem A.45) and the examples that follow it. By Proposition A.47, $U(\mathfrak{g})$ is left Noetherian.

Corollary 3.28. If $\mathfrak{g}$ is finite dimensional and $I_1, \ldots, I_m$ are left ideals of finite codimension in $U(\mathfrak{g})$, then the product ideal $I_1 \cdots I_m$ is of finite codimension in $U(\mathfrak{g})$.

REMARK. The product ideal by definition consists of all finite sums of products $x_1 \cdots x_m$ with each x_j in I_j.

PROOF. By induction it is enough to handle $m = 2$. The vector space $U(\mathfrak{g})/I_1$ is finite dimensional by assumption, and we let $x_1 + I_1, \ldots, x_r + I_1$ be a vector-space basis. Since $U(\mathfrak{g})$ is left Noetherian by Proposition 3.27, Proposition A.44 shows that the left ideal I_2 is finitely generated, say with $y_1, \ldots, y_s$ as generators.

The claim is that $\{x_i y_j + I_1 I_2\}$ is a spanning set for the vector space $I_2/I_1 I_2$. In fact, any x in I_2 is of the form $x = \sum_{j=1}^{s} u_j y_j$ with u_j in $U(\mathfrak{g})$. For each j, write $u_j + I_1 = \sum_{i=1}^{r} c_{ij} x_i + I_1$ with $c_{ij} \in \mathbb{C}$. Then $u_j y_j + I_1 I_2 = \sum_{i=1}^{r} c_{ij} x_i y_j + I_1 I_2$, and the claim follows when we sum on j.

Thus $I_2/I_1 I_2$ is finite dimensional. Since $\dim U(\mathfrak{g})/I_1 I_2$ is equal to $\dim U(\mathfrak{g})/I_2 + \dim I_2/I_1 I_2$, we conclude that $U(\mathfrak{g})/I_1 I_2$ is finite dimensional.

4. Free Lie Algebras

Using the Poincaré–Birkhoff–Witt Theorem, we can establish the existence of "free Lie algebras." A **free Lie algebra** on a set X is a pair $(\mathfrak{F}, \iota)$ consisting of a Lie algebra $\mathfrak{F}$ and a function $\iota : X \to \mathfrak{F}$ with the following universal mapping property: Whenever $\mathfrak{l}$ is a complex Lie algebra and $\varphi : X \to \mathfrak{l}$ is a function, there exists a unique Lie algebra homomorphism $\widetilde{\varphi}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{F} \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ X \arrow[ur, "\iota"] \arrow[rr, "\varphi"'] & & \mathfrak{l} \end{tikzcd} \tag{3.29}$$

commutes. We regard $\widetilde{\varphi}$ as an extension of φ.

Let us construct such a Lie algebra. Let V consist of all formal complex linear combinations of the members of X, so that V can be regarded as a complex vector space with X as basis. We embed V in its tensor algebra via $\iota_V : V \to T(V)$, obtaining $T^1(V) = \iota_V(V)$ as usual. Since $T(V)$ is an associative algebra, we can regard it as a Lie algebra in the manner of Example 2 in §I.1. Let $\mathfrak{F}$ be the Lie subalgebra of $T(V)$ generated by $T^1(V)$.

In the setting of (3.29), we are to construct a Lie algebra homomorphism $\widetilde{\varphi}$ so that (3.29) commutes, and we are to show that $\widetilde{\varphi}$ is unique. Extend $\varphi : X \to \mathfrak{l}$ to a linear map $\varphi : V \to \mathfrak{l}$, and let $\iota_{\mathfrak{l}} : \mathfrak{l} \to U(\mathfrak{l})$ be the canonical map. The universal mapping property of $T(V)$ allows us in the diagram

$$\begin{tikzcd} & T(V) \arrow[dr, dashed, "a"] & \\ V \arrow[ur, "\iota_V"] \arrow[rr, "\iota_{\mathfrak{l}} \circ \varphi"'] & & U(\mathfrak{l}) \end{tikzcd}$$

to extend $\iota_{\mathfrak{l}} \circ \varphi$ to an associative algebra homomorphism a with $a(1) = 1$. For $x \in X$, the commutativity of this diagram implies that

$$a(\iota_V(x)) = \iota_{\mathfrak{l}}(\varphi(x)). \tag{3.30}$$

Let us think of a as a Lie algebra homomorphism in (3.30). The right side of (3.30) is in image $\iota_{\mathfrak{l}}$, and it follows that $a(\mathfrak{F}) \subseteq$ image $\iota_{\mathfrak{l}}$.

Now we use the Poincaré–Birkhoff–Witt Theorem, which implies that $\iota_{\mathfrak{l}} : \mathfrak{l} \to$ image $\iota_{\mathfrak{l}}$ is one-one. We write $\iota_{\mathfrak{l}}^{-1}$ for the inverse of this Lie algebra isomorphism, and we put $\widetilde{\varphi} = \iota_{\mathfrak{l}}^{-1} \circ a$. Then $\widetilde{\varphi}$ is the required Lie algebra homomorphism making (3.29) commute.

To see that $\widetilde{\varphi}$ is unique when $\mathfrak{F}$ is defined this way, we observe that (3.29) forces $\widetilde{\varphi}(\iota_V(x)) = \varphi(x)$ for all $x \in X$. Since the elements $\iota_V(x)$ generate $\mathfrak{F}$ and since $\widetilde{\varphi}$ is a Lie algebra homomorphism, $\widetilde{\varphi}$ is completely determined on all of $\mathfrak{F}$. This proves the first statement in the following proposition.

Proposition 3.31. If X is a nonempty set, then there exists a free Lie algebra $\mathfrak{F}$ on X, and the image of X in $\mathfrak{F}$ generates $\mathfrak{F}$. Any two free Lie algebras on X are canonically isomorphic.

REMARK. This result was stated in Chapter II as Proposition 2.96, and the proof was deferred until now.

PROOF. Existence of $\mathfrak{F}$ was proved before the statement of the proposition. We still have to prove that $\mathfrak{F}$ is unique up to canonical isomorphism. Let $(\mathfrak{F}, \iota)$ and $(\mathfrak{F}', \iota')$ be two free Lie algebras on X. We set up the diagram (3.29) with $\mathfrak{l} = \mathfrak{F}'$ and $\varphi = \iota'$ and invoke existence to obtain a Lie algebra homomorphism $\widetilde{\iota'} : \mathfrak{F} \to \mathfrak{F}'$. Reversing the roles of $\mathfrak{F}$ and $\mathfrak{F}'$, we obtain a Lie algebra homomorphism $\widetilde{\iota} : \mathfrak{F}' \to \mathfrak{F}$. To see that $\widetilde{\iota} \circ \widetilde{\iota'} = 1_{\mathfrak{F}}$, we set up the diagram (3.29) with $\mathfrak{l} = \mathfrak{F}$ and $\varphi = \iota_X$ to see that $\widetilde{\iota} \circ \widetilde{\iota'}$ is an extension of ι. By uniqueness of the extension, $\widetilde{\iota} \circ \widetilde{\iota'} = 1_{\mathfrak{F}}$. Similarly $\widetilde{\iota'} \circ \widetilde{\iota} = 1_{\mathfrak{F}'}$.

5. Problems

1. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, let Ω be the member of $U(\mathfrak{g})$ given by $\Omega = \frac{1}{2}h^2 + ef + fe$, where h, e, and f are as in (1.5).
   (a) Prove that Ω is in the center of $U(\mathfrak{g})$.
   (b) Let π be a representation of $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space V, and regard V as a $U(\mathfrak{g})$ module. Show that Ω acts in V by the operator Z of Lemma 1.65.
2. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and define $\operatorname{ad} X$ on $U(\mathfrak{g})$ for $X \in \mathfrak{g}$ by $(\operatorname{ad} X)u = Xu - uX$. Prove that ad is a representation of $\mathfrak{g}$ and that each element of $U(\mathfrak{g})$ lies in a finite-dimensional space invariant under $\operatorname{ad} \mathfrak{g}$.
3. Let $U(\mathfrak{g})$ be the universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$. Prove that $U(\mathfrak{g})$ has no zero divisors.
4. (a) Identify a free Lie algebra on a set consisting of one element.
   (b) Prove that a free Lie algebra on a set consisting of two elements is infinite dimensional.

5. Let $\mathfrak{F}$ be a free Lie algebra on the set $\{X_1, X_2, X_3\}$, and let $\mathfrak{g}$ be the quotient obtained by setting to 0 all brackets involving three or more members of $\mathfrak{F}$.
   (a) Prove that $\dim \mathfrak{g} = 6$ and that $\mathfrak{g}$ is nilpotent but not abelian.
   (b) Define $B(X_i, X_j) = 0$, $B([X_i, X_j], [X_{i'}, X_{j'}] = 0$, and

$$B(X_3, [X_1, X_2]) = B(X_2, [X_3, X_1]) = B(X_1, [X_2, X_3]) = 1.$$

   Prove that B extends to a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$.

6. Say that a complex Lie algebra $\mathfrak{h}$ is two-step nilpotent if $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = 0$. Prove for each integer $n \geq 1$ that that there is a finite-dimensional two-step nilpotent Lie algebra $\mathfrak{g}$ such that every two-step nilpotent Lie algebra of dimension $\leq n$ is isomorphic to a homomorphic image of $\mathfrak{g}$.

7. The construction of a free Lie algebra $\mathfrak{F}$ on X in §4 first built a complex vector space V with X as basis. Then $\mathfrak{F}$ was obtained as the Lie algebra generated by V within $T(V)$. Prove that $U(\mathfrak{F})$ can be identified with $T(V)$.

Problems 8–10 concern the diagonal mapping for a universal enveloping algebra. Fix a complex Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$.

8. Use the 4-multilinear map $(u_1, u_2, u_3, u_4) \mapsto u_1u_2 \otimes u_3u_4$ of $U(\mathfrak{g}) \times U(\mathfrak{g}) \times U(\mathfrak{g}) \times U(\mathfrak{g})$ into $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ to define a multiplication in $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$. Prove that $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ becomes an associative algebra with identity.

9. Prove that there exists a unique associative algebra homomorphism Δ from $U(\mathfrak{g})$ into $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ such that $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in \mathfrak{g}$ and such that $\Delta(1) = 1$.

10. If φ_1 and φ_2 are in the dual space $U(\mathfrak{g})^*$, then $\varphi_1 \otimes \varphi_2$ is well defined as a linear functional on $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ since $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ canonically. Define a product $\varphi_1\varphi_2$ in $U(\mathfrak{g})^*$ by

$$(\varphi_1\varphi_2)(u) = (\varphi_1 \otimes \varphi_2)(\Delta(u)),$$

   where Δ is as in Problem 9. Prove that this product makes $U(\mathfrak{g})^*$ into a commutative associative algebra (without necessarily an identity).

Problems 11–13 identify $U(\mathfrak{g})$ with an algebra of differential operators. Let G be a Lie group, let $\mathfrak{g}_0$ be the Lie algebra, and let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. For $X \in \mathfrak{g}_0$, let $\widetilde{X}$ be the left-invariant vector field on G corresponding to X, regarded as acting in the space $C^\infty(G)$ of all *complex*-valued functions on G. The vector field $\widetilde{X}$ is a **left-invariant differential operator** in the sense that it is a member D of $\operatorname{End}_{\mathbb{C}}(C^\infty(G))$ commuting with left translations such that, for each $g \in G$, there is a chart (φ, V) about g, say $\varphi = (x_1, \ldots, x_n)$, and there are functions $a_{k_1 \cdots k_n}$

in $C^\infty(V)$ with the property that

$$Df(x) = \sum_{\text{bounded}} a_{k_1\cdots k_n}(x) \frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(x)$$

for all $x \in V$ and $f \in C^\infty(G)$. Such operators form a complex subalgebra $D(G)$ of $\operatorname{End}_{\mathbb{C}}(C^\infty(G))$ containing the identity. Moreover, any D of this kind has such an expansion in any chart about x.

11. Prove that the map $X \mapsto \widetilde{X}$ extends to an algebra homomorphism of $U(\mathfrak{g})$ into $D(G)$ sending 1 to 1.
12. Prove that the map in Problem 11 is onto.
13. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}_0$.
    (a) For each tuple $(i_1, \ldots, i_n)$ of integers ≥ 0, prove that there is a function $f \in C^\infty(G)$ with the property that $(\widetilde{X}_1)^{j_1} \cdots (\widetilde{X}_n)^{j_n} f(1)$ equals 1 if $j_1 = i_1, \ldots, j_n = i_n$, and equals 0 if not.
    (b) Deduce that the map in Problem 11 is one-one.

Problems 14–22 concern the Weyl algebra and a higher-dimensional version of the Heisenberg Lie algebra discussed in Problems 25–27 in Chapter I. Let V be a real finite-dimensional vector space, and let $\langle \cdot, \cdot \rangle$ be a nondegenerate skew-symmetric bilinear form on $V \times V$. The **Heisenberg Lie algebra** $H(V)$ on V is the Lie algebra $V \oplus \mathbb{R}X_0$ in which X_0 is central and V brackets with itself by $[u, v] = \langle u, v \rangle X_0$. The complex **Weyl algebra** $W(V^{\mathbb{C}})$ on V is the quotient of $T(V^{\mathbb{C}})$ by the two-sided ideal generated by all $u \otimes v - v \otimes u - \langle u, v \rangle 1$ with u and v in V.

14. Using Problem 45b of Chapter II, prove that the Heisenberg algebra and the Weyl algebra on V are determined up to isomorphism by the dimension of V, which must be even, say $2n$.
15. Verify that an example of a $2n$-dimensional V with its form $\langle \cdot, \cdot \rangle$ is $V = \mathbb{C}^n$ with $\langle u, v \rangle = \operatorname{Im}(u, v)$, where $(\cdot, \cdot)$ is the usual Hermitian inner product on $\mathbb{C}^n$. For this V, exhibit an isomorphism of $H(V)$ with the Lie algebra of all complex $(n+1)$-by-$(n+1)$ matrices of the form $\begin{pmatrix} 0 & \bar{z}^t & ir \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$ with $z \in \mathbb{C}^n$ and $r \in \mathbb{R}$.
16. Show that the linear map $\iota(v + cX_0) = v + c1$ is a Lie algebra homomorphism of $H(V)$ into $W(V^{\mathbb{C}})$ and that its extension to an associative algebra homomorphism $\widetilde{\iota} : U(H(V)^{\mathbb{C}}) \to W(V^{\mathbb{C}})$ is onto and has kernel equal to the two-sided ideal generated by $X_0 - 1$.
17. Prove that $W(V^{\mathbb{C}})$ has the following universal mapping property: For any Lie algebra homomorphism π of $H(V)$ into a complex associative algebra A with identity such that X_0 maps to 1, there exists a unique associative algebra homomorphism $\widetilde{\pi}$ of $W(V^{\mathbb{C}})$ into A such that $\pi = \widetilde{\pi} \circ \iota$.

18. Let $v_1, \dots, v_{2n}$ be any vector space basis of V. Prove that the elements $v_1^{k_1} \cdots v_{2n}^{k_{2n}}$ with integer exponents ≥ 0 span $W(V^{\mathbb{C}})$.

19. If $\dim_{\mathbb{R}} V = 2n$, prove that V is the vector-space direct sum $V = V^+ \oplus V^-$ of two n-dimensional subspaces on which $\langle \cdot\,, \cdot \rangle$ is identically 0. Show that it is possible to choose bases $p_1, \dots, p_n$ of V^+ and $q_1, \dots, q_n$ of V^- such that $\langle p_i, q_j \rangle = \delta_{ij}$.

20. Let $\mathcal{S}$ be the space of all complex-valued functions $P(x)e^{-\pi|x|^2}$, where $P(x) = P(x_1, \dots, x_n)$ is a polynomial in n variables. Show that $\mathcal{S}$ is mapped into itself by the linear operators $\partial/\partial x_i$ and $m_j =$ (multiplication-by-x_j).

21. In the notation of Problems 19 and 20, let φ be the linear map of V into $\mathrm{End}_{\mathbb{C}}\mathcal{S}$ given by $\varphi(p_i) = \partial/\partial x_i$ and $\varphi(q_j) = m_j$. Use Problem 17 to extend φ to an algebra homomorphism $\widetilde{\varphi}$ of $W(V^{\mathbb{C}})$ into $\mathrm{End}_{\mathbb{C}}\mathcal{S}$ with $\widetilde{\varphi}(1) = 1$, and use Problem 16 to obtain a representation of $H(V)$ of $\mathcal{S}$. Prove that this representation is irreducible.

22. In Problem 21 prove that the algebra homomorphism $\widetilde{\varphi} : W(V^{\mathbb{C}}) \to \mathrm{End}_{\mathbb{C}}\,\mathcal{S}$ is one-one. Conclude that the elements $v_1^{k_1} \cdots v_{2n}^{k_{2n}}$ of Problem 18 form a vector space basis of $W(V^{\mathbb{C}})$.