IX. Induced Representations and Branching Theorems, 555-614

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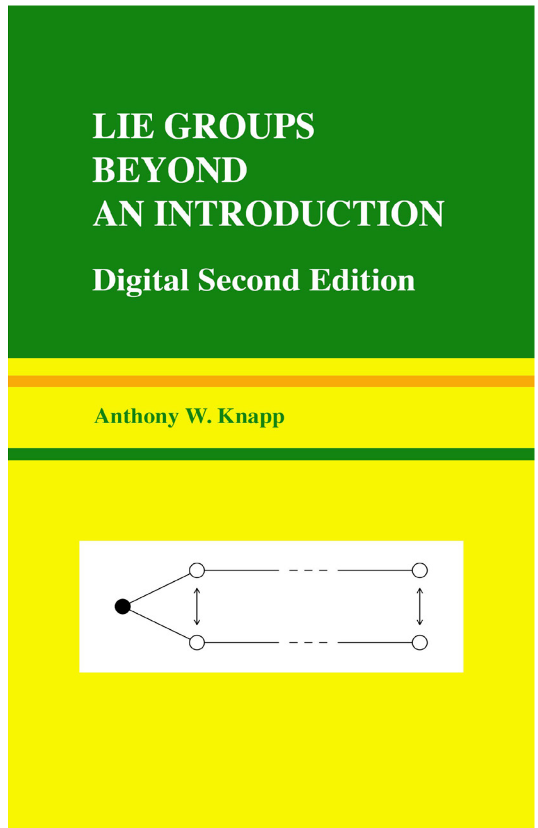

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Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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CHAPTER IX

Induced Representations and Branching Theorems

Abstract. The definition of unitary representation of a compact group extends to the case that the vector space is replaced by an infinite-dimensional Hilbert space, provided care is taken to incorporate a suitable notion of continuity. The theorem is that each unitary representation of a compact group G splits as the orthogonal sum of finite-dimensional irreducible invariant subspaces. These invariant subspaces may be grouped according to the equivalence class of the irreducible representation, and there is an explicit formula for the orthogonal projection on the closure of the sum of all the spaces of a given type. As a result of this formula, one can speak of the multiplicity of each irreducible representation in the given representation.

The left-regular and right-regular representations of G on $L^2(G)$ are examples of unitary representations. So is the left-regular representation of G on $L^2(G/H)$ for any closed subgroup H. More generally, if H is a closed subgroup and σ is a unitary representation of H, the induced representation of σ from H to G is an example. If σ is irreducible, Frobenius reciprocity says that the multiplicity of any irreducible representation τ of G in the induced representation equals the multiplicity of σ in the restriction of τ to H.

Branching theorems give multiplicities of irreducible representations of H in the restriction of irreducible representations of G. Three classical branching theorems deal with passing from $U(n)$ to $U(n-1)$, from $SO(n)$ to $SO(n-1)$, and from $Sp(n)$ to $Sp(n-1)$. These may all be derived from Kostant's Branching Theorem, which gives a formula for multiplicities when passing from a compact connected Lie group to a closed connected subgroup. Under a favorable hypothesis the Kostant formula expresses each multiplicity as an alternating sum of values of a certain partition function.

Some further branching theorems of interest are those for which G/H is a compact symmetric space in the sense that H is the identity component of the group of fixed elements under an involution of G. Helgason's Theorem translates into a theorem in this setting for the case of the trivial representation of H by means of Riemannian duality. An important example of a compact symmetric space is $(G \times G)/\operatorname{diag} G$; a branching theorem for this situation tells how the tensor product of two irreducible representations of G decomposes.

A cancellation-free combinatorial algorithm for decomposing tensor products for the unitary group $U(n)$ is of great utility. It leads to branching theorems for the compact symmetric spaces $U(n)/SO(n)$ and $U(2n)/Sp(n)$. In turn the first of these branching theorems helps in understanding branching for the compact symmetric space $SO(n+m)/(SO(n) \times SO(m))$.

Iteration of branching theorems for compact symmetric spaces permits analysis of some complicated induced representations. Of special note is $L^2(K/(K \cap M_0))$ when G is a reductive Lie group, K is the fixed group under the global Cartan involution, and MAN is the Langlands decomposition of any maximal parabolic subgroup.

1. Infinite-dimensional Representations of Compact Groups

In the discussion of the representation theory of compact groups in Chapter IV, all the representations were finite dimensional. A number of applications of compact groups, however, involve naturally arising infinite-dimensional representations, and a theory of such representations is needed. We address this problem in the first two sections of this chapter.

Throughout this chapter, G will denote a compact group, and dx will denote a two-sided Haar measure on G of total mass 1. To avoid having to discuss some small measure-theoretic complications, we shall state results for general compact groups but assume in proofs that G is separable as a topological group. This matter will not be an issue after §2, when G will always be a Lie group. For commentary about the measure-theoretic complications, see the Historical Notes.

If V is a complex Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$ and norm $\|\cdot\|$, then a **unitary operator** U on V is a linear transformation from V *onto* itself that preserves the norm in the sense that $\|U(v)\| = \|v\|$ for all v in V. Equivalently V is to be a linear operator of V onto itself that preserves the inner product in the sense that $(U(v), U(v')) = (v, v')$ for all v and v' in V. The unitary operators on V form a group. They are characterized by having $U^{-1} = U^*$, where U^* is the adjoint of U.

A **unitary representation** of G on the complex Hilbert space V is a homomorphism of G into the group of unitary operators on V such that a certain continuity property holds. Continuity is a more subtle matter in the present context than it was in §IV.2 because not all possible definitions of continuity are equivalent here. The continuity property we choose is that the group action $G \times V \to V$, given by $g \times v \mapsto \Phi(g)v$, is continuous. When Φ is unitary, this property is equivalent with **strong continuity**, that $g \mapsto \Phi(g)v$ is continuous for every v in V.

Let us see this equivalence. Strong continuity results from fixing the V variable in the definition of continuity of the group action, and therefore continuity of the group action implies strong continuity. In the reverse direction the triangle inequality and the equality $\|\Phi(g)\| = 1$ give

$$\begin{aligned}\|\Phi(g)v - \Phi(g_0)v_0\| &\le \|\Phi(g)(v - v_0)\| + \|\Phi(g)v_0 - \Phi(g_0)v_0\| \\ &= \|v - v_0\| + \|\Phi(g)v_0 - \Phi(g_0)v_0\|,\end{aligned}$$

and it follows that strong continuity implies continuity of the group action.

With this definition of continuity in place, an example of a unitary representation is the **left-regular representation** of G on the complex

Hilbert space $L^2(G)$, given by $(l(g)f)(x) = f(g^{-1}x)$. Strong continuity is satisfied according to Lemma 4.17. The **right-regular representation** of G on $L^2(G)$, given by $(r(g)f)(x) = f(xg)$ also satisfies this continuity property.

In working with a unitary representation Φ of G on V, it is helpful to define $\Phi(f)$ for f in $L^1(G)$ as a smeared-out version of the various $\Phi(x)$ for x in G. Formally $\Phi(f)$ is to be $\int_G f(x)\Phi(x)\,dx$. But to avoid integrating functions whose values are in an infinite-dimensional space, we define $\Phi(f)$ as follows: The function $\int_G f(x)(\Phi(x)v, v')\,dx$ of v and v' is linear in v, conjugate linear in v', and bounded in the sense that $\left|\int_G f(x)(\Phi(x)v, v')\,dx\right| \le \|f\|_1\|v\|\|v'\|$. It follows from the elementary theory of Hilbert spaces that there exists a unique linear operator $\Phi(f)$ such that

$$(\Phi(f)v, v') = \int_G f(x)(\Phi(x)v, v')\,dx \tag{9.1a}$$

for all v and v' in V. This operator satisfies

$$\|\Phi(f)\| \le \|f\|_1 \tag{9.1b}$$

and

$$\Phi(f)^* = \Phi(f^*), \tag{9.1c}$$

where $f^*(x) = \overline{f(x^{-1})}$. From the existence and uniqueness of $\Phi(f)$, it follows that $\Phi(f)$ depends linearly on f.

Another property of the application of Φ to functions is that convolution goes into product. The **convolution** $f * h$ of two L^1 functions f and h is given by $(f * h)(x) = \int_G f(xy^{-1})h(y)\,dy = \int_G f(y)h(y^{-1}x)\,dy$. The result is an L^1 function by Fubini's Theorem. Then we have

$$\Phi(f * h) = \Phi(f)\Phi(h). \tag{9.1d}$$

The formal computation to prove (9.1d) is

$$\begin{aligned}
\Phi(f * h) &= \int_G \int_G f(xy^{-1})h(y)\Phi(x)\,dy\,dx \\
&= \int_G \int_G f(xy^{-1})h(y)\Phi(x)\,dx\,dy \\
&= \int_G \int_G f(x)h(y)\Phi(xy)\,dx\,dy
\end{aligned}$$

$$= \int_G \int_G f(x)h(y)\Phi(x)\Phi(y)\,dx\,dy$$
$$= \Phi(f)\Phi(h).$$

To make this computation rigorous, we put the appropriate inner products in place and use Fubini's Theorem to justify the interchange of order of integration:

$$\begin{aligned}
(\Phi(f*h)v, v') &= \textstyle\int_G\int_G f(xy^{-1})h(y)(\Phi(x)v, v')\,dy\,dx \\
&= \textstyle\int_G\int_G f(xy^{-1})h(y)(\Phi(x)v, v')\,dx\,dy \\
&= \textstyle\int_G\int_G f(x)h(y)(\Phi(xy)v, v')\,dx\,dy \\
&= \textstyle\int_G\int_G f(x)h(y)(\Phi(x)\Phi(y)v, v')\,dx\,dy \\
&= \textstyle\int_G\int_G f(x)h(y)(\Phi(y)v, \Phi(x)^*v')\,dx\,dy \\
&= \textstyle\int_G\int_G f(x)h(y)(\Phi(y)v, \Phi(x)^*v')\,dy\,dx \\
&= \textstyle\int_G f(x)(\Phi(h)v, \Phi(x)^*v')\,dx \\
&= \textstyle\int_G f(x)(\Phi(x)\Phi(h)v, v')\,dx \\
&= (\Phi(f)\Phi(h)v, v').
\end{aligned}$$

This kind of computation translating a formal argument about $\Phi(f)$ into a rigorous argument is one that we shall normally omit from now on.

An important instance of the convolution $f*h$ is the case that f and h are characters of irreducible finite-dimensional representations. The formula in this case is

$$\chi_\tau * \chi_{\tau'} = \begin{cases} d_\tau^{-1}\chi_\tau & \text{if } \tau \cong \tau' \text{ and } d_\tau \text{ is the degree of } \tau \\ 0 & \text{if } \tau \text{ and } \tau' \text{ are inequivalent.} \end{cases} \tag{9.2}$$

To prove (9.2), one expands the characters in terms of matrix coefficients and computes the integrals using Schur orthogonality (Corollary 4.10).

If $f \geq 0$ vanishes outside an open neighborhood N of 1 in G and has $\int_G f(x)\,dx = 1$, then $(\Phi(f)v - v, v') = \int_G f(x)(\Phi(x)v - v, v')\,dx$. When $\|v'\| \leq 1$, the Schwarz inequality therefore gives

$$|(\Phi(f)v - v, v')| \leq \int_N f(x)\|\Phi(x)v - v\|\|v'\|\,dx \leq \sup_{x\in N}\|\Phi(x)v - v\|.$$

Taking the supremum over v' with $\|v'\| \leq 1$ allows us to conclude that

$$\|\Phi(f)v - v\| \leq \sup_{x\in N}\|\Phi(x)v - v\|. \tag{9.3}$$

We shall make use of this inequality shortly.

An **invariant subspace** for a unitary representation Φ on V is a vector subspace U such that $\Phi(g)U \subseteq U$ for all $g \in G$. This notion is useful mainly when U is a closed subspace. In any event if U is invariant, so is the closed orthogonal complement $U^\perp$ since $u^\perp \in U^\perp$ and $u \in U$ imply that

$$(\Phi(g)u^\perp, u) = (u^\perp, \Phi(g)^* u) = (u^\perp, \Phi(g)^{-1} u) = (u^\perp, \Phi(g^{-1})u)$$

is in $(u^\perp, U) = 0$. If $V \neq 0$, the representation is **irreducible** if its only closed invariant subspaces are 0 and V.

Two unitary representations of G, Φ on V and Φ' on V', are said to be **unitarily equivalent** if there is a norm-preserving linear $E : V \to V'$ with a norm-preserving inverse such that $\Phi'(g)E = E\Phi(g)$ for all $g \in G$.

Theorem 9.4. If Φ is a unitary representation of the compact group G on a complex Hilbert space V, then V is the orthogonal sum of finite-dimensional irreducible invariant subspaces.

PROOF. By Zorn's Lemma, choose a maximal orthogonal set of finite-dimensional irreducible invariant subspaces. Let U be the closure of the sum. Arguing by contradiction, suppose that U is not all of V. Then $U^\perp$ is a nonzero closed invariant subspace. Fix $v \neq 0$ in $U^\perp$. For each open neighborhood N of 1 in G, let f_N be the characteristic function of N divided by the measure of N. Then f_N is an integrable function ≥ 0 with integral 1. It is immediate from (9.1a) that $\Phi(f_N)v$ is in $U^\perp$ for every N. Inequality (9.3) and strong continuity show that $\Phi(f_N)v$ tends to v as N shrinks to $\{1\}$. Hence some $\Phi(f_N)v$ is not 0. Fix such an N.

Choose by the Peter–Weyl Theorem (Theorem 4.20) a function h in the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations so that $\|f_N - h\|_2 \leq \frac{1}{2}\|\Phi(f_N)v\|/\|v\|$. Then

$$\begin{aligned}\|\Phi(f_N)v - \Phi(h)v\| &= \|\Phi(f_N - h)v\| \leq \|f_N - h\|_1\|v\| \\ &\leq \|f_N - h\|_2\|v\| \leq \tfrac{1}{2}\|\Phi(f_N)v\|\end{aligned}$$

by (9.1b) and the inequality $\|F\|_1 \leq \|F\|_2$. Hence

$$\|\Phi(h)v\| \geq \|\Phi(f_N)v\| - \|\Phi(f_N)v - \Phi(h)v\| \geq \tfrac{1}{2}\|\Phi(f_N)v\| > 0,$$

and $\Phi(h)v$ is not 0.

The function h lies in some finite-dimensional subspace S of $L^2(G)$ that is invariant under left translation. Let $h_1, \ldots, h_n$ be a basis of S, and write $h_j(g^{-1}x) = \sum_i c_{ij}(g)h_i(x)$. The formal computation

$$\begin{aligned}
\Phi(g)\Phi(h_j)v = \Phi(g)\int_G h_j(x)\Phi(x)v\,dx &= \int_G h_j(x)\Phi(gx)v\,dx \\
&= \int_G h_j(g^{-1}x)\Phi(x)v\,dx = \sum_{i=1}^n c_{ij}(g)\int_G h_i(x)\Phi(x)v\,dx \\
&= \sum_{i=1}^n c_{ij}(g)\Phi(h_i)v
\end{aligned}$$

suggests that the subspace $\sum_j \mathbb{C}\Phi(h_j)v$, which is finite dimensional and lies in $U^\perp$, is an invariant subspace for Φ containing the nonzero vector $\Phi(h)v$. To justify the formal computation, we argue as in the proof of (9.1d), redoing the calculation with an inner product with v' in place throughout. The existence of this subspace of $U^\perp$ contradicts the maximality of U and proves the theorem.

Corollary 9.5. Every irreducible unitary representation of a compact group is finite dimensional.

PROOF. This is immediate from Theorem 9.4.

Corollary 9.6. Let Φ be a unitary representation of the compact group G on a complex Hilbert space V. For each irreducible unitary representation τ of G, let E_τ be the orthogonal projection on the sum of all irreducible invariant subspaces of V that are equivalent with τ. Then E_τ is given by $d_\tau\Phi(\overline{\chi_\tau})$, where d_τ is the degree of τ and χ_τ is the character of τ, and the image of E_τ is the orthogonal sum of irreducible invariant subspaces that are equivalent with τ. Moreover, if τ and τ' are inequivalent, then $E_\tau E_{\tau'} = E_{\tau'}E_\tau = 0$. Finally every v in V satisfies

$$v = \sum_\tau E_\tau v,$$

with the sum taken over a set of representatives τ of all equivalence classes of irreducible unitary representations of G.

PROOF. Let τ be irreducible with degree d_τ, and put $E'_\tau = d_\tau\Phi(\overline{\chi_\tau})$.

Formulas (9.1c), (4.14), (9.1d), and (9.2) give

$$E_\tau'^{*} = d_\tau \Phi(\overline{\chi_\tau}^{\,*}) = d_\tau \Phi(\chi_{\tau^c}) = d_\tau \Phi(\overline{\chi_\tau}) = E_\tau',$$
$$E_\tau' E_{\tau'}' = d_\tau d_{\tau'} \Phi(\overline{\chi_\tau}) \Phi(\overline{\chi_{\tau'}}) = d_\tau d_{\tau'} \Phi(\overline{\chi_\tau} * \overline{\chi_{\tau'}}) = 0 \quad \text{if } \tau \not\cong \tau',$$
$$E_\tau'^{\,2} = d_\tau^2 \Phi(\overline{\chi_\tau} * \overline{\chi_\tau}) = d_\tau \Phi(\overline{\chi_\tau}) = E_\tau'.$$

The first and third of these formulas say that E_τ' is an orthogonal projection, and the second formula says that $E_\tau' E_{\tau'}' = E_{\tau'}' E_\tau' = 0$ if τ and τ' are inequivalent.

Let U be an irreducible finite-dimensional subspace of V on which $\Phi|_U$ is equivalent with τ, and let $u_1, \ldots, u_n$ be an orthonormal basis of U. If we write $\Phi(x)u_j = \sum_{i=1}^n \Phi_{ij}(x)u_i$, then $\Phi_{ij}(x) = (\Phi(x)u_j, u_i)$ and $\chi_\tau(x) = \sum_{i=1}^n \Phi_{ii}(x)$. Thus a formal computation with Schur orthogonality gives

$$E_\tau' u_j = d_\tau \int_G \overline{\chi_\tau(x)} \Phi(x) u_j \, dx = d_\tau \int_G \sum_{i,k} \overline{\Phi_{kk}(x)} \Phi_{ij}(x) u_i \, dx = u_j,$$

and we can justify this computation by using inner products with v' throughout. As a result, we see that E_τ' is the identity on every irreducible subspace of type τ.

Now let us apply E_τ' to a Hilbert space orthogonal sum $V = \sum V_\alpha$ of the kind in Theorem 9.4. We have just seen that E_τ' is the identity on V_α if V_α is of type τ. If V_α is of type τ' with τ' inequivalent with τ, then $E_{\tau'}'$ is the identity on V_α, and we have $E_\tau' u = E_\tau' E_{\tau'}' u = 0$ for all $u \in V_\alpha$. Consequently E_τ' is 0 on V_α, and we conclude that $E_\tau' = E_\tau$. This completes the proof.

It follows from Corollary 9.6 that the number of occurrences of irreducible subspaces of type τ in a decomposition of the kind in Theorem 9.4 is independent of the decomposition. As a result of the corollary, this number may be obtained as the quotient $(\dim \text{image } E_\tau)/d_\tau$. We write $[\Phi : \tau]$ for this quantity and call it the **multiplicity** of τ in Φ. Each multiplicity is a cardinal number, but it may be treated simply as a member of the set $\{0, 1, 2, \ldots, \infty\}$ when the underlying Hilbert space is separable. When Φ is finite dimensional, §IV.2 provides us with a way of computing multiplicities in terms of characters, and the present notion may be regarded as a generalization to the infinite-dimensional case.

For an example, consider the right-regular representation r of G on $L^2(G)$. Let τ be an irreducible unitary representation, let $u_1, \ldots, u_n$ be an orthonormal basis of the space on which τ acts, and form matrices relative to this basis that realize each $\tau(x)$. The formula is $\tau_{ij}(x) = (\tau(x)u_j, u_i)$. The matrix coefficients corresponding to a fixed row, those with i fixed and j varying, form an irreducible invariant subspace for r of type τ, and these spaces are orthogonal to one another by Schur orthogonality. Thus $[r : \tau]$ is at least d_τ. On the other hand, Corollary 4.21 says that such matrix coefficients, as τ varies through representatives of all equivalence classes of irreducible representations, form a complete orthogonal system in $L^2(G)$. The coefficients corresponding to any τ' inequivalent with τ are in the image of $E_{\tau'}$ and are not of type τ. It follows that $[r : \tau]$ equals d_τ and that the spaces of type τ can be taken to be the span of each row of matrix coefficients for τ.

For the left-regular representation l of G on $L^2(G)$, one can reason similarly. The results are that $[l : \tau]$ equals d_τ and that the spaces of type τ can be taken to be the span of the columns of matrix coefficients for the contragredient τ^c.

Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. The multiplicities of each member of $\widehat{G}$ within a unitary representation of G determine the representation up to unitary equivalence. In fact, the various multiplicities are certainly not changed under a unitary equivalence, and if a set of multiplicities is given, any unitary representation of G with those multiplicities is unitarily equivalent to the orthogonal sum of irreducible representations with each irreducible taken as many times as the multiplicity indicates. We shall be interested in techniques for computing these multiplicities.

Proposition 9.7. Let Φ and τ be unitary representations of the compact group G on spaces V^Φ and V^τ, respectively, and suppose τ is irreducible. Then

$$[\Phi : \tau] = \dim_{\mathbb{C}} \operatorname{Hom}_G(V^\Phi, V^\tau) = \dim_{\mathbb{C}} \operatorname{Hom}_G(V^\tau, V^\Phi),$$

where the subscripts G refer to linear maps respecting the indicated actions by G.

PROOF. By Schur's Lemma (Proposition 4.8) and Corollary 9.6, any member of $\operatorname{Hom}_G(V^\Phi, V^\tau)$ annihilates $(E_\tau V^\Phi)^\perp$. Write, by a second application of Corollary 9.6, $E_\tau V^\Phi$ as the orthogonal sum of irreducible subspaces V_α with each V_α equivalent to V^τ. For each V_α, the space of linear maps from V_α to V^τ respecting the action by G is at least 1-dimensional. It is at most 1-dimensional by Schur's Lemma in the form of Corollary 4.9. Then it follows that

$$[\Phi : \tau] = \dim_{\mathbb{C}} \operatorname{Hom}_G(V^\Phi, V^\tau).$$

Taking adjoints, we obtain

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V^\Phi, V^\tau) = \dim_{\mathbb{C}} \operatorname{Hom}_G(V^\tau, V^\Phi).$$

2. Induced Representations and Frobenius Reciprocity

In this section we continue to assume that G is a compact group, and we continue to write out proofs only under the additional assumption that G is separable.

A wider class of examples of infinite-dimensional unitary representations than the regular representations on $L^2(G)$ is obtained as follows: Let H be a closed subgroup of G, and let l be the **left-regular representation** of G on $L^2(G/H)$, given by $(l(g)f)(xH) = f(g^{-1}xH)$.

This is a unitary representation, and it can be realized also as taking place in a certain closed subspace of $L^2(G)$. Namely the identification $f \mapsto F$ given by $F(x) = f(xH)$ carries $L^2(G/H)$ onto the subspace of members of $L^2(G)$ that are right-invariant under H, a closed subspace that we shall denote by $L^2(G, \mathbb{C}, 1_H)$. The result is a unitary equivalence of representations of G.

The realization of $L^2(G/H)$ as $L^2(G, \mathbb{C}, 1_H)$ suggests a generalization in which $\mathbb{C}$ and 1_H are replaced by a Hilbert space V and a unitary representation σ of H on V. The case of most interest is that σ is finite dimensional, but the theory is no more complicated if V is allowed to be infinite dimensional but separable. We shall not have occasion to apply the theory to nonseparable Hilbert spaces, and we defer to the Historical Notes any discussion of the complications in that case.

Let the inner product and norm for V be denoted $(\,\cdot\,,\,\cdot\,)_V$ and $|\,\cdot\,|_V$. A function F from G to V is (weakly) measurable if $x \mapsto (F(x), v)_V$ is Borel measurable for all $v \in V$. In this case let $\{v_n\}$ be an orthonormal basis of V. Then the function $|F(x)|_V^2 = \sum_n |(F(x), v_n)_V|^2$ is measurable and is independent of the choice of orthonormal basis. We say that F is in $L^2(G, V)$ if it is measurable and if $\|F\|_2 = (\int_G |F(x)|_V^2\,dx)^{1/2}$ is finite. Technically the space $L^2(G, V)$ is the Hilbert space of such functions with two such functions identified if they differ on a set of measure 0, but one usually speaks of functions, rather than their equivalence classes, as members of L^2.

We define the **left-regular representation** l of G on $L^2(G, V)$ by $(l(g)F)(x) = F(g^{-1}x)$. To verify the strong continuity, we use the same argument as for Lemma 4.17 once we know that the continuous functions from G to V are dense in $L^2(G, V)$. This density is a consequence of the density in the scalar case, which was proved in §IV.3: if $\{v_n\}$ is an orthonormal basis of V, then the finite linear combinations of functions $f v_n$ with f scalar-valued and continuous are continuous into V and form a dense subset of $L^2(G, V)$.

Let us interject some remarks about Fubini's Theorem. Fubini's Theorem is usually regarded as a statement about the interchange of integrals of nonnegative measurable functions on a product measure space that is totally finite or totally σ-finite, but it says more. For one thing, it says that the result of performing the inner integration is a measurable function of the other variable. For another thing, through its statement in the case of a characteristic function, it gives insight into sets of measure 0; if a measurable set in the product space has the property that almost every slice in one direction has measure 0, then almost every slice in the other direction has measure 0.

Let H be a closed subgroup of G, and let σ be a unitary representation of H on V. Define

$$
\begin{aligned}
L^2(G, V, \sigma) &= \left\{ F \in L^2(G, V) \;\middle|\; \begin{array}{l} F(xh) = \sigma(h)^{-1}F(x) \\ \text{for almost every pair} \\ (x, h) \in G \times H \end{array} \right\} \\
&= \left\{ F \in L^2(G, V) \;\middle|\; \begin{array}{l} \text{For every } h \in H, \\ F(xh) = \sigma(h)^{-1}F(x) \\ \text{for a.e. } x \in G \end{array} \right\}.
\end{aligned}
\tag{9.8}
$$

The equality of the two expressions in braces requires some comment. The equality is meant to convey that an equivalence class of functions in L^2 containing a function having one of the defining properties in (9.8) contains a member that has the other of the defining properties, and vice versa. With this interpretation the second expression is contained in the first by Fubini's Theorem. If F is in the first space, we can adjust F on a subset of G of measure 0 to make it be in the second space. This adjustment is done by integration as follows. Formally we consider $F_1(x) = \int_H \sigma(h)F(xh)\,dh$. By Fubini's Theorem, for almost all $x \in G$, we have $F(xh) = \sigma(h)^{-1}F(x)$ for almost all $h \in H$, and these x's have $F_1(x) = F(x)$. For the remaining

x's, we set $F_1(x) = 0$. Then F_1 is in the second space, and F_1 and F yield the same member of $L^2(G, V)$. This argument is formal in that it used integrals of vector-valued functions. To make it precise, we work throughout with inner products with an arbitrary $v \in V$; we omit these details.

In practice it is a little easier to use the second expression in (9.8), and we shall tend to ignore the first expression. Some authors work instead with the subspace of continuous members of $L^2(G, V, \sigma)$, for which there are no exceptional x's and h's; this approach succeeds because it can be shown that the subspace of continuous members is dense in $L^2(G, V, \sigma)$.

For F in $L^2(G, V, \sigma)$ and g in G, define $(\Phi(g)F)(x) = F(g^{-1}x)$. The system of operators $\Phi(g)$ is nothing more than the restriction to an invariant subspace of the left-regular representation of G on $L^2(G, V)$. Thus Φ is a unitary representation of G on $L^2(G, V, \sigma)$. It is the **induced representation** of σ from H to G and is denoted $\operatorname{ind}_H^G \sigma$.

From the definitions it follows immediately that if σ is the finite or countably infinite orthogonal sum of unitary representations σ_n on separable Hilbert spaces, then $\operatorname{ind}_H^G \sigma$ is unitarily equivalent with the orthogonal sum of the $\operatorname{ind}_H^G \sigma_n$.

Theorem 9.9 (Frobenius reciprocity). Let H be a closed subgroup of the compact group G, let σ be an irreducible unitary representation of H on V^σ, let τ be an irreducible unitary representation of G on V^τ, and let $\Phi = \operatorname{ind}_H^G \sigma$ act on V^Φ. Then there is a canonical vector-space isomorphism

$$\operatorname{Hom}_G(V^\tau, V^\Phi) \cong \operatorname{Hom}_H(V^\tau, V^\sigma),$$

and consequently

$$[\operatorname{ind}_H^G \sigma : \tau] = [\tau|_H : \sigma].$$

REMARKS. Restriction to a subgroup is a way of passing from representations of G to representations of H, and induction is a way of passing in the opposite direction. Frobenius reciprocity gives a sense in which these constructions are adjoint to each other.

PROOF. We shall prove the isomorphism. The equality of multiplicities is then immediate from Proposition 9.7.

The space V^Φ is contained in $L^2(G, V^\sigma)$, and $L^2(G, V^\sigma)$ is simply the direct sum of d_σ copies of $L^2(G)$, d_σ being the degree. Therefore τ occurs exactly $d_\sigma d_\tau$ times in $L^2(G, V^\sigma)$ and at most that many times in V^Φ. By Schur's Lemma we then know that the image of any member of $\mathrm{Hom}_G(V^\tau, V^\Phi)$ lies in the subspace of continuous members of V^Φ. If e denotes evaluation at 1 in G, it therefore makes sense to form the composition eA whenever A is in $\mathrm{Hom}_G(V^\tau, V^\Phi)$. For v in V^τ, we have

$$\begin{aligned}\sigma(h)(eAv) &= \sigma(h)[(Av)(1)] = (Av)(h^{-1})\\ &= (\Phi(h)(Av))(1) = (A\tau(h)v)(1) = eA\tau(h)v.\end{aligned}$$

Thus eA is in $\mathrm{Hom}_H(V^\tau, V^\sigma)$, and the linear map e carries $\mathrm{Hom}_G(V^\tau, V^\Phi)$ into $\mathrm{Hom}_H(V^\tau, V^\sigma)$. To complete the proof, we show that e is an isomorphism.

To see that e is one-one, suppose that $eAv = 0$ for all v in V^τ. Then $(Av)(1) = 0$ for all v. Applying this conclusion to $v = \tau(g)^{-1}v'$ gives

$$0 = (Av)(1) = (A\tau(g)^{-1}v')(1) = (\Phi(g)^{-1}Av')(1) = (Av')(g),$$

and so $Av' = 0$. Since v' is arbitrary, $A = 0$. Thus e is one-one.

To see that e is onto, let a be in $\mathrm{Hom}_H(V^\tau, V^\sigma)$. Define $Av(g) = a(\tau(g)^{-1}v)$ for $v \in V^\tau$ and $g \in G$. Then

$$Av(gh) = a(\tau(h)^{-1}\tau(g)^{-1}v) = \sigma(h)^{-1}(a(\tau(g)^{-1}v)) = \sigma(h)^{-1}(Av(g))$$

shows that Av is in V^Φ. In fact, A is in $\mathrm{Hom}_G(V^\tau, V^\Phi)$ because the equality

$$(\Phi(g_0)Av)(g) = Av(g_0^{-1}g) = a(\tau(g)^{-1}(\tau(g_0)v)) = A(\tau(g_0)v)(g)$$

implies $\Phi(g_0)A = A\tau(g_0)$. Finally e carries A to a because the equality

$$eAv = Av(1) = a(\tau(1)v) = av$$

implies $eA = a$. Thus e is onto, and the proof is complete.

The final topic of this section is "induction in stages," which refers to the legitimacy of forming an induced representation by first inducing to an intermediate group and then inducing from there to the whole group. Induction in stages may be regarded as adjoint to the obvious notion of restriction in stages—that if H and H_1 are closed subgroups of G and $H \subseteq H_1 \subseteq G$, then the effect of restricting from G to H_1 and afterward restricting to H is the same as the effect of restricting from G to H directly. We can quantify this relationship by means of multiplicities as follows. Let τ and σ be irreducible unitary representations of G and H. Decomposing τ under H_1 and the result under H, we see that

$$[\tau : \sigma] = \sum_{\sigma_1 \in \widehat{H}_1} [\tau : \sigma_1][\sigma_1 : \sigma]. \tag{9.10}$$

Induction in stages is more subtle than restriction in stages and requires some justification. When inducing representations in stages, even if we start with an irreducible representation, the intermediate representation is likely to occur in a subspace of some $L^2(G, V)$ with V infinite dimensional. Before stating the result about induction in stages, let us therefore check in the case of interest that all the Hilbert spaces that arise are separable.

Proposition 9.11. Let G be a separable compact group. Then $L^2(G)$ is a separable Hilbert space. In fact, $L^2(G, V)$ is a separable Hilbert space whenever V is a separable Hilbert space.

PROOF. Fix a countable base for the topology of G. For each pair U and V in the countable base such that $\overline{U} \subseteq V$, choose, by Urysohn's Lemma, a continuous real-valued function that is 1 on U and 0 off V. The resulting subset of the space $C(G)$ of continuous complex-valued functions on G is countable and separates points on G. The associative algebra over $\mathbb{Q} + i\mathbb{Q}$ generated by these functions and the constant 1 is countable, is closed under conjugation, and is uniformly dense in the associative algebra over $\mathbb{C}$ generated by these functions and 1. The latter algebra is uniformly dense in $C(G)$ by the Stone–Weierstrass Theorem. Since $C(G)$ is known from §IV.3 to be dense in $L^2(G)$, we conclude that $L^2(G)$ is separable. This proves the first statement.

If V is a separable Hilbert space, let $\{v_n\}$ be a countable orthonormal basis. Choose a countable dense set $\{f_k\}$ in $L^2(G)$. Then the set of finite rational linear combinations of functions $f_k v_n$ is a countable dense set in $L^2(G, V)$.

Proposition 9.12 (induction in stages). Let G be a separable compact group, and let H and H_1 be closed subgroups with $H \subseteq H_1 \subseteq G$. If σ is an irreducible unitary representation of H, then

$$\operatorname{ind}_H^G \sigma \quad \text{is unitarily equivalent with} \quad \operatorname{ind}_{H_1}^G \operatorname{ind}_H^{H_1} \sigma.$$

REMARKS. In fact, the unitary equivalence is canonical, but we shall not need this sharper statement. The functions in the Hilbert space of the doubly induced representation are functions on G whose values are functions on H_1, thus are functions of pairs (g, h_1). Their values are in the space V^σ on which σ acts. The functions in the space of $\operatorname{ind}_H^G \sigma$ are functions from G to V^σ. The unitary equivalence is given in effect by evaluating the functions of pairs (g, h_1) at $h_1 = 1$. Since the functions in question are unaffected by changes on sets of measure 0, some work is needed to make sense of this argument.

PROOF. Let τ and σ be irreducible unitary representations of G and H. Decomposing τ under H_1 and the result under H leads to the multiplicity formula (9.10). Frobenius reciprocity (Theorem 9.9) then gives

$$[\operatorname{ind}_H^G \sigma : \tau] = \sum_{\sigma_1 \in \widehat{H}_1} [\operatorname{ind}_{H_1}^G \sigma_1 : \tau][\operatorname{ind}_H^{H_1} \sigma : \sigma_1]. \tag{9.13}$$

The representation $\operatorname{ind}_H^{H_1} \sigma$ is the orthogonal sum over all σ_1 of $[\operatorname{ind}_H^{H_1} \sigma : \sigma_1]$ copies of σ_1, and hence the induced representation $\operatorname{ind}_{H_1}^G \operatorname{ind}_H^{H_1} \sigma$ is unitarily equivalent with the orthogonal sum over all σ_1 of $[\operatorname{ind}_H^{H_1} \sigma : \sigma_1]$ copies of $\operatorname{ind}_{H_1}^G \sigma_1$. Thus the right side of (9.13) is

$$= [\operatorname{ind}_{H_1}^G \operatorname{ind}_H^{H_1} \sigma : \tau].$$

Therefore the two representations in question have the same respective multiplicities, and they must be unitarily equivalent.

3. Classical Branching Theorems

Let H be a closed subgroup of the compact group G. Frobenius reciprocity deals with the multiplicities of irreducible representations of G in induced representations from H to G, reducing their computation to finding multiplicities of irreducible representations of G when restricted to H. In particular, this approach applies to finding the multiplicities for $L^2(G/H)$. A theorem about computing multiplicities for an irreducible representation upon restriction to a closed subgroup is called a **branching theorem** or **branching rule**. The rest of this chapter will be concerned with results of this type.

We shall concentrate on the case that G is a connected Lie group and that the closed subgroup H is connected. In the next section we shall see that there is a direct formula that handles all examples. However, this formula involves an alternating sum of a great many terms, and it gives a useful answer only in a limited number of situations. It is natural therefore to try to form an arsenal of situations that can be handled recursively, preferably in a small number of steps.

For this purpose a natural first step is to look at the various series of classical compact connected groups and to isolate the effect of restricting an irreducible representation to the next smaller group in the same series.

In this section we list three theorems of this kind, postponing their proofs to §5.

Our groups are as follows. We work with the unitary groups $U(n)$, the rotation groups $SO(N)$ with $N = 2n + 1$ or $N = 2n$, and the quaternion unitary groups $Sp(n)$. The rotation groups are not simply connected, but we omit discussion of their simply connected covers. In each case we use the standard embedding of the subgroup H of next smaller size in the upper left block of the given group G, with the members of H filled out with 1's on the diagonal. A different choice for an embedding of H will yield the same branching if the two subgroups are conjugate via G, as is the case if H is embedded in the lower right block of G, for example.

We parametrize irreducible representations of G and H as usual by highest weights. The maximal tori T are as in §IV.5 for the most part. In the case of $U(n)$, the maximal torus is the diagonal subgroup. For $SO(2n+1)$ it consists of block diagonal matrices with n blocks consisting of 2-by-2 rotation matrices and with 1 block consisting of the entry 1, and for $SO(2n)$ it consists of block diagonal matrices with n blocks consisting of 2-by-2 rotation matrices. To have highest-weight theory apply conveniently to $Sp(n)$, we realize $Sp(n)$ as $Sp(n, \mathbb{C}) \cap U(2n)$; then the maximal torus consists of diagonal matrices whose $(n + j)^{\text{th}}$ entry is the reciprocal of the j^{th} entry for $1 \le j \le n$.

In each case the notation for members of the complexified dual of the Lie algebra of T is to be as in the corresponding example of §II.1. We write $\mathfrak{t}$ for the Lie algebra of T. The positive roots are as in (2.50). The analytically integral members of $(\mathfrak{t}^{\mathbb{C}})^*$ in each case are of the form $a_1e_1 + \cdots + a_ne_n$ with all a_j equal to integers.

We begin with the branching theorem for $U(n)$. For $U(n)$, the condition of dominance is that $a_1 \ge \cdots \ge a_n$.

Theorem 9.14 (Weyl). For $U(n)$, the irreducible representation with highest weight $a_1e_1 + \cdots + a_ne_n$ decomposes with multiplicity 1 under $U(n-1)$, and the representations of $U(n-1)$ that appear are exactly those with highest weights $c_1e_1 + \cdots + c_{n-1}e_{n-1}$ such that

$$a_1 \ge c_1 \ge a_2 \ge \cdots \ge a_{n-1} \ge c_{n-1} \ge a_n. \tag{9.15}$$

EXAMPLE. $L^2(U(n)/U(n-1))$. The space $U(n)/U(n-1)$ may be regarded as the unit sphere in $\mathbb{C}^n$. Frobenius reciprocity says that the multiplicity of an irreducible representation τ of $U(n)$ in $L^2(U(n)/U(n-1))$ equals the multiplicity of the trivial representation of $U(n-1)$ in $\tau|_{U(n-1)}$.

Let τ have highest weight $a_1e_1 + \cdots + a_ne_n$. A brief calculation using Theorem 9.14 shows that

$$[\tau|_{U(n-1)} : 1] = \begin{cases} 1 & \text{if } (a_1, \ldots, a_n) = (q, 0, \ldots, 0, -p) \\ 0 & \text{otherwise.} \end{cases}$$

The representation with highest weight $qe_1 - pe_n$ can be seen to be realized concretely in the subspace $H_{p,q}$ of homogeneous harmonic polynomials in $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ in which p factors of z's and q factors of $\bar{z}$'s are involved; here "harmonic" means that the polynomial is annihilated by the usual Laplacian $\sum_{j=1}^n \big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\big)$. Thus $L^2(U(n)/U(n-1))$ is unitarily equivalent with the sum of all the spaces $H_{p,q}$, each occurring with multiplicity 1. This conclusion, obtained from Theorem 9.14 with just a brief calculation, begs for an analytic interpretation. Here is such an interpretation: Any homogeneous polynomial involving p of the z's and q of the $\bar{z}$'s is uniquely a sum $h_{p,q} + |z|^2 h_{p-1,q-1} + |z|^4 h_{p-2,q-2} + \cdots$ with each of the h's in the indicated space of homogeneous harmonic polynomials. On the unit sphere each of the powers of $|z|$ restricts to the constant 1, and hence every polynomial on the sphere is the sum of harmonic polynomials of the required kind. Compare with Problems 9–17 in Chapter IV.

Now we state the branching theorem for the rotation groups. The condition of dominance for the integral form $a_1e_1 + \cdots + a_ne_n$ for $SO(2n+1)$ and $SO(2n)$ is that

$$\begin{aligned} &a_1 \geq \cdots \geq a_n \geq 0 && \text{for the case of } N = 2n+1, \\ &a_1 \geq \cdots \geq a_{n-1} \geq |a_n| && \text{for the case of } N = 2n. \end{aligned}$$

Theorem 9.16 (Murnaghan).

(a) For $SO(2n+1)$, the irreducible representation with highest weight $a_1e_1 + \cdots + a_ne_n$ decomposes with multiplicity 1 under $SO(2n)$, and the representations of $SO(2n)$ that appear are exactly those with highest weights $(c_1, \ldots, c_n)$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq a_n \geq |c_n|. \tag{9.17a}$$

(b) For $SO(2n)$, the irreducible representation with highest weight $a_1e_1 + \cdots + a_ne_n$ decomposes with multiplicity 1 under $SO(2n-1)$, and the representations of $SO(2n-1)$ that appear are exactly those with highest weights $(c_1, \ldots, c_{n-1})$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq |a_n|. \tag{9.17b}$$

Finally we state the branching theorem for $Sp(n)$. The condition of dominance for the integral form $a_1e_1 + \cdots + a_ne_n$ for $Sp(n)$ is that $a_1 \geq \cdots \geq a_n \geq 0$.

Theorem 9.18 (Zhelobenko). For $Sp(n)$, the irreducible representation with highest weight $a_1e_1 + \cdots + a_ne_n$ decomposes under $Sp(n-1)$ as follows: the number of times the representation of $Sp(n-1)$ with highest weight $(c_1, \ldots, c_{n-1})$ occurs in the given representation of $Sp(n)$ equals the number of integer n-tuples $(b_1, \ldots, b_n)$ such that

$$\begin{gathered} a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_{n-1} \geq a_n \geq b_n \geq 0, \\ b_1 \geq c_1 \geq b_2 \geq \cdots \geq b_{n-1} \geq c_{n-1} \geq b_n. \end{gathered} \tag{9.19}$$

If there are no such n-tuples $(b_1, \ldots, b_n)$, then it is understood that the multiplicity is 0.

Any of the above three theorems can be iterated. For example, the irreducible representation of $U(n)$ with highest weight $a_1e_1 + \cdots + a_ne_n$ decomposes under $U(n-2)$ as follows: the number of times the irreducible representation of $U(n-2)$ with highest weight $c_1e_1 + \cdots + c_{n-2}e_{n-2}$ occurs in the given representation of $U(n)$ equals the number of $(n-1)$-tuples $(b_1, \ldots, b_{n-1})$ such that

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_{n-1} \geq a_n$$

and

$$b_1 \geq c_1 \geq b_2 \geq \cdots \geq b_{n-2} \geq c_{n-2} \geq b_{n-1}.$$

An iterated answer of this kind, however, may be unsatisfactory for some purposes. As the number of iterations increases, this kind of answer becomes more like an algorithm than a theorem. If the result of the algorithm is to be applied by substituting it into some other formula, the answer from the formula may be completely opaque.

4. Overview of Branching

The previous section mentioned that there is a general formula that handles all examples of branching for compact connected Lie groups. This is due to Kostant. The full branching formula of Kostant's involves the same kind of passage to the limit that is involved in §V.6 in deriving the

Weyl Dimension Formula from the Weyl Character Formula. But in this book we shall restrict the treatment of Kostant's formula to the situation where no passage to the limit is needed.

Although the formula can always be used to calculate particular examples, it finds rather few theoretical applications. We shall use it in the next section to derive results implying the classical branching theorems of the previous section, and those will be our only applications of it.

Despite the paucity of theoretical applications, the special hypothesis in the theorem that eliminates any passage to the limit has philosophical implications for us. It will enable us to focus attention on an approach to getting concrete branching formulas in a great many practical situations. We return to this point after stating and proving the theorem.

Let G be a connected compact Lie group, and let H be a connected closed subgroup. The special assumption is that the centralizer in G of a maximal torus S of H is abelian and is therefore a maximal torus T of G. Equivalently the assumption is that some regular element of H is regular in G. We examine the assumption more closely later in this section.

Let us establish some notation for the theorem. Let Δ_G be the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, let Δ_H be the set of roots of $(\mathfrak{h}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}})$, and let W_G be the Weyl group of Δ_G. Introduce compatible positive systems Δ_G^+ and Δ_H^+ by defining positivity relative to an H regular element of $i\mathfrak{s}$, let bar denote restriction from the dual $(\mathfrak{t}^{\mathbb{C}})^*$ to the dual $(\mathfrak{s}^{\mathbb{C}})^*$, and let δ_G be half the sum of the members of Δ_G^+. The restrictions to $\mathfrak{s}^{\mathbb{C}}$ of the members of Δ_G^+, repeated according to their multiplicities, are the nonzero positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$. Deleting from this set the members of Δ_H^+, each with multiplicity 1, we obtain the set Σ of positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}}$, repeated according to multiplicities. The associated **Kostant partition function** is defined as follows: $\mathcal{P}(\nu)$ is the number of ways that a member of $(\mathfrak{s}^{\mathbb{C}})^*$ can be written as a sum of members of Σ, with the multiple versions of a member of Σ being regarded as distinct.

Theorem 9.20 (Kostant's Branching Theorem). Let G be a compact connected Lie group, let H be a closed connected subgroup, suppose that the centralizer in G of a maximal torus S of H is abelian and is therefore a maximal torus T of G, and let other notation be as above. Let $\lambda \in (\mathfrak{t}^{\mathbb{C}})^*$ be the highest weight of an irreducible representation τ of G, and let $\mu \in (\mathfrak{s}^{\mathbb{C}})^*$ be the highest weight of an irreducible representation σ of H. Then the multiplicity of σ in the restriction of τ to H is given by

$$m_\lambda(\mu) = \sum_{w \in W_G} \varepsilon(w)\mathcal{P}(\overline{w(\lambda + \delta_G) - \delta_G} - \mu).$$

PROOF. The theorem generalizes the Kostant Multiplicity Formula for the weights of a representation (Corollary 5.83), and the proof is a variant of the proof of that special case. As in the special case, one needs to make rigorous an argument involving multiplication of formal series; here we define Q^+ to be the set of all nonnegative integer combinations of members of Σ, and matters here are justified by working in a ring $\mathbb{Z}\langle(\mathfrak{s}^{\mathbb{C}})^*\rangle$ defined relative to this Q^+. Namely $\mathbb{Z}\langle(\mathfrak{s}^{\mathbb{C}})^*\rangle$ is the set of all $f \in \mathbb{Z}^{(\mathfrak{s}^{\mathbb{C}})^*}$ whose support is contained in the union of a finite number of sets $\nu_i - Q^+$ with each ν_i in $(\mathfrak{s}^{\mathbb{C}})^*$.

The special assumption about regularity in $\mathfrak{s}^{\mathbb{C}}$ enters as follows. Positivity for both H and G is defined relative to some H regular element $X \in i\mathfrak{s}$; specifically a member α of Δ_G is positive if $\alpha(X) > 0$. Hence the restrictions to $i\mathfrak{s}$ of all members of Σ lie in an open half space of $i\mathfrak{s}^*$, and it follows that $\mathcal{P}(\nu)$ is finite for all $\nu \in (\mathfrak{s}^{\mathbb{C}})^*$. With this finiteness in hand, it follows that

$$\Big(\sum_{\beta\in\Sigma}\big(1-e^{-\beta}\big)^{m_\beta}\Big)\Big(\sum_{\nu\in Q^+}\mathcal{P}(\nu)e^{-\nu}\Big) = 1, \tag{9.21}$$

where m_β is the multiplicity of β in $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}}$. This formula generalizes Lemma 5.72.

Let χ_λ and χ_μ be characters for G and H, respectively. Using bar to indicate restriction, not complex conjugation, we have

$$\overline{\chi_\lambda} = \sum_{\mu\in F} m_\lambda(\mu)\chi_\mu \tag{9.22}$$

as an identity in $\mathbb{Z}[(\mathfrak{s}^{\mathbb{C}})^*]$; here F is a finite set of H dominant weights. The construction of Σ makes

$$\prod_{\alpha\in\Delta_G^+}\big(1-e^{-\bar\alpha}\big) = \Big(\sum_{\beta\in\Sigma}\big(1-e^{-\beta}\big)^{m_\beta}\Big)\Big(\prod_{\gamma\in\Delta_H^+}\big(1-e^{-\gamma}\big)\Big). \tag{9.23}$$

In (9.22) we substitute for χ_μ from the Weyl character for H and obtain

$$\overline{\chi_\lambda}\prod_{\gamma\in\Delta_H^+}\big(1-e^{-\gamma}\big) = \sum_{\substack{p\in W_H,\\ \mu\in F}} m_\lambda(\mu)\varepsilon(p)e^{p(\mu+\delta_H)-\delta_H}, \tag{9.24}$$

where W_H is the Weyl group of H and δ_H is half the sum of the members of Δ_H^+. Substitution from (9.21) and (9.23) into the left side of (9.24) yields

$$\overline{\chi_\lambda}\Big(\prod_{\alpha\in\Delta_G^+}\big(1-e^{-\bar\alpha}\big)\Big)\Big(\sum_{\nu\in Q^+}\mathcal{P}(\nu)e^{-\nu}\Big) = \sum_{\substack{p\in W_H,\\ \mu\in F}} m_\lambda(\mu)\varepsilon(p)e^{p(\mu+\delta_H)-\delta_H}.$$

The Weyl character formula for G implies that

$$\overline{\chi_\lambda} \prod_{\alpha \in \Delta_G^+} \left(1 - e^{-\bar{\alpha}}\right) = \sum_{w \in W_G} \varepsilon(w) e^{\overline{w(\lambda+\delta_G)-\delta_G}}$$

in $\mathbb{Z}[(\mathfrak{s}^{\mathbb{C}})^*]$, and we can substitute and obtain

$$\sum_{\substack{w \in W_G, \\ \nu \in Q^+}} \varepsilon(w) \mathcal{P}(\nu) e^{\overline{w(\lambda+\delta_G)-\delta_G}-\nu} = \sum_{\substack{p \in W_H, \\ \mu \in F}} m_\lambda(\mu) \varepsilon(p) e^{p(\mu+\delta_H)-\delta_H}. \tag{9.25}$$

The theorem will follow by equating the coefficients of e^μ on the two sides of (9.25). On the right side the equation $p(\mu+\delta_H)-\delta_H = \mu$ forces $p = 1$ by Chevalley's Lemma in the form of Corollary 2.73 because μ is H dominant. Thus the coefficient of e^μ on the right side of (9.25) is $m_\lambda(\mu)$. On the left side the coefficient of e^μ is the sum of $\varepsilon(w)\mathcal{P}(\nu)$ over all $w \in W_G$ and $\nu \in Q^+$ such that $\overline{w(\lambda+\delta_G)-\delta_G} - \nu = \mu$. This sum is just $\sum_{w \in W_G} \varepsilon(w)\mathcal{P}(\overline{w(\lambda+\delta_G)-\delta_G} - \mu)$, and the proof is complete.

Let us study in more detail the special assumption in the theorem—that the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$ is abelian. There are two standard situations where this assumption is satisfied. The obvious one of these is when $\mathfrak{s}$ is already maximal abelian in $\mathfrak{g}$. We refer to this as the situation of **equal rank**. This is the case, for example, when $H = T$ and the theorem reduces to the formula for the multiplicity of a weight. The less obvious one is when the subgroup H is the identity component of the set of fixed points of an involution of G. We refer to this situation as that of a **compact symmetric space**.

Let us accept for the moment that the special assumption in Theorem 9.20 is satisfied in the situation of a compact symmetric space, and let us examine the circumstances in the classical branching theorems in the previous section. In the case of branching from $G = SO(n)$ to $H = SO(n-1)$, the subgroup H is the identity component of the set of fixed points of the involution of G given by conjugation by the diagonal matrix $\operatorname{diag}(1, \ldots, 1, -1)$. Thus this is the situation of a compact symmetric space. The case with $G = U(n)$ and $H = U(n-1)$ is not that of a compact symmetric space, nor is it an equal-rank case. Yet this situation does satisfy the special assumption in the theorem, essentially because every root for $U(n)$ is determined by its restriction to $U(n-1)$.

The case with $G = Sp(n)$ and $H = Sp(n-1)$ is more decisive. It does not satisfy the special assumption, and we are led to look for a remedy.

If we think of $G = Sp(n)$ as the unitary group over the quaternions, then the case of $SO(n)$ suggests considering conjugation by $\operatorname{diag}(1, \dots, 1, -1)$. The identity component of the set of fixed points is $H_1 = Sp(n-1) \times Sp(1)$, and thus we have a relevant compact symmetric space. Theorem 9.20 will be applicable with H_1 as subgroup. We can thus handle the branching in two stages, passing from G to H_1 and then from H_1 to H.

For uniformity we can use the same technique with $G = U(n)$, passing from G to $H_1 = U(n-1) \times U(1)$ and then from H_1 to $H = U(n-1)$. In this way all of the classical branching reduces to instances of branching associated with compact symmetric spaces.

What is the scope of compact symmetric spaces? Let U be a compact semisimple Lie group, let Θ be an involution of U, let $\mathfrak{u}_0$ be the Lie algebra of U, and let θ be the corresponding involution of $\mathfrak{u}_0$. Let B be the Killing form for $\mathfrak{u}_0$; this is negative definite by Corollary 4.26 and Cartan's Criterion for Semisimplicity (Theorem 1.45). If K is the identity component of the fixed set of Θ and $\mathfrak{k}_0$ is its Lie algebra, then we can write $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus \mathfrak{q}_0$, where $\mathfrak{q}_0$ is the -1 eigenspace of θ. Corollary 4.22 allows us to regard U as a closed linear group, and then Proposition 7.12 says that U has a complexification $U^{\mathbb{C}}$. We use the Lie algebra of $U^{\mathbb{C}}$ as the complexification $\mathfrak{u}$ of $\mathfrak{u}_0$. Put $\mathfrak{p}_0 = i\mathfrak{q}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. From the definition of $\mathfrak{k}_0$ and $\mathfrak{q}_0$ as eigenspaces for θ, it follows that $[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0$, $[\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0$, and $[\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$. In particular, $\mathfrak{g}_0$ is a real form of $\mathfrak{u}$ and is semisimple. Also the complex extension of B is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$. By the definition in §VI.2, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. If G is the analytic subgroup of $U^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_0$, G/K is called the **noncompact Riemannian dual** of the compact symmetric space U/K.

The proof that the special assumption in Theorem 9.20 is satisfied for the passage from U to K is easy. Proposition 6.60 shows that the centralizer of a maximal abelian subspace $\mathfrak{s}_0$ of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ is abelian, equaling the sum of $\mathfrak{s}_0$ and an abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$. Then the centralizer of $\mathfrak{s}_0$ in $\mathfrak{u}_0$ is the sum of $\mathfrak{s}_0$ and $i\mathfrak{a}_0$ and is abelian. Thus the special assumption is satisfied.

G	K	U/K
$U(n,m)$	$U(n) \times U(m)$	$U(n+m)/(U(n) \times U(m))$
$SO(n,m)_0$	$SO(n) \times SO(m)$	$SO(n+m)/(SO(n) \times SO(m))$
$Sp(n,m)$	$Sp(n) \times Sp(m)$	$Sp(n+m)/(Sp(n) \times Sp(m))$
$GL(n,\mathbb{R})_0$	$SO(n)$	$U(n)/SO(n)$
$GL(n,\mathbb{H})$	$Sp(n)$	$U(2n)/Sp(n)$
$SO^*(2n)$	$U(n)$	$SO(2n)/U(n)$
$Sp(n,\mathbb{R})$	$U(n)$	$Sp(n)/U(n)$

In Chapter VI we took advantage of Cartan decompositions to classify real semisimple Lie algebras. We can refer to that classification now to find, up to isomorphisms and coverings, all the compact semisimple groups and involutions. The ones associated to the classical noncomplex Lie groups are as in the accompanying table, except that special unitary groups have been replaced by unitary groups throughout.

The first three, with $m = 1$, are what govern the classical branching theorems. Later in this chapter we shall observe some things about branching in the context of the other compact symmetric spaces.

One more kind of G of interest along with those in the above table is a group whose Lie algebra $\mathfrak{g}_0$ is complex simple. In this case, $\mathfrak{k}_0$ is a compact form of $\mathfrak{g}_0$. Using Theorem 6.94 to unwind matters, we are led to the compact symmetric space $(K \times K)/\operatorname{diag} K$. The involution in question interchanges the two coordinates.

We can easily make sense of branching from $K \times K$ to $\operatorname{diag} K$. If τ_1 and τ_2 are irreducible representations of K, then the **outer tensor product** $\tau_1 \widehat{\otimes} \tau_2$ given by $(k_1, k_2) \mapsto \tau_1(k_1) \otimes \tau_2(k_2)$ is an irreducible representation of $K \times K$. Application of Corollary 4.21 shows that all irreducible representations of $K \times K$ are of this form. Restricting such a representation to $\operatorname{diag} K$ yields the representation $k \mapsto \tau_1(k) \otimes \tau_2(k)$, which is the ordinary tensor product $\tau_1 \otimes \tau_2$ for K. In other words, branching from $K \times K$ to $\operatorname{diag} K$ is understood as soon as one understands how to decompose representations of K under tensor product.

In practice the list of branching theorems produced from an understanding of branching for compact symmetric spaces is much longer than the above table might suggest. The reason is that many pairs (G, H) arising in practice can be analyzed as a succession of compact symmetric spaces. We give just one example, together with an indication how it can be generalized. The group $Sp(n, 1)$ has real rank one, and it is of interest to know what irreducible representations occur in $L^2(K/M)$, M having been defined in §VI.5. For this example, $K = Sp(n) \times Sp(1)$, and M is isomorphic to $Sp(n-1) \times Sp(1)$. However, the embedding of M in K is subtle. Let $K_1 = (Sp(n-1) \times Sp(1)) \times Sp(1)$ be embedded in K in the expected way. If we regroup K_1 as $Sp(n-1) \times (Sp(1) \times Sp(1))$, then M embeds in K_1 as $Sp(n-1) \times \operatorname{diag} Sp(1)$. Thus K/M is built from two compact symmetric spaces, one that amounts to $Sp(n)/(Sp(n-1) \times Sp(1))$ and another that amounts to $(Sp(1) \times Sp(1))/\operatorname{diag} Sp(1)$.

What is happening in this example is a fairly general phenomenon. Let the restricted-root space decomposition of the Lie algebra be written

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha},$$

with $\mathfrak{a} \oplus \mathfrak{m}$ forming the 0 restricted-root space. The linear transformation φ from $\mathfrak{g}^{\mathbb{C}}$ to itself given as the scalar i^k on $\mathfrak{g}_{k\alpha}$ is an automorphism of $\mathfrak{g}^{\mathbb{C}}$ of order 4. Since $Sp(n,1)^{\mathbb{C}}$ is simply connected, φ lifts to an automorphism Φ of $Sp(n,1)^{\mathbb{C}}$ with $\Phi^4 = 1$. Since φ^2 has real eigenvalues, Φ^2 carries G to itself. Also φ^2 commutes with the Cartan involution, and thus Φ^2 carries K to itself. The map Φ^2 is an involution of K, and K_1 is the identity component of the fixed group under Φ^2. In turn, Φ is an involution of K_1, and M is the identity component of the fixed group under Φ.

5. Proofs of Classical Branching Theorems

In this section we prove Theorems 9.14, 9.16, and 9.18 using Kostant's Branching Theorem (Theorem 9.20). The different cases have a certain similarity to them. Consequently we shall give the proof in full for $U(n)$, but we shall omit parts of the later proofs that consist of easy calculations or repetitive arguments.

1) Branching from $U(n)$ to $U(n-1)$. We use (9.13) with $G = U(n)$, $H_1 = U(n-1) \times U(1)$, and $H = U(n-1)$. The given highest weights are $\lambda = \sum_{j=1}^n a_j e_j$ with $a_1 \geq \cdots \geq a_n$ and $\mu = \sum_{j=1}^{n-1} c_j e_j$ with $c_1 \geq \cdots \geq c_{n-1}$. The only terms σ_1 that can make a contribution to (9.13) are those with highest weight of the form $\mu_1 = \sum_{j=1}^n c_j e_j$ for some c_n. However, τ is scalar on scalar matrices, and it follows for every weight ν of τ that λ and ν have the same inner product with $e_1 + \cdots + e_n$. Since $\nu = \sum_{j=1}^n c_j e_j$ is such a weight, we must have $\sum_{j=1}^n a_j = \sum_{j=1}^n c_j$. In other words, c_n is completely determined.

We may as well therefore assume from the outset that the branching is from $U(n)$ to $U(n-1)\times U(1)$ and that $\mu = \sum_{j=1}^n c_j e_j$ with $c_1 \geq \cdots \geq c_{n-1}$. For the passage from $U(n)$ to $U(n-1) \times U(1)$, we use Theorem 9.20. The multiplicity being computed is

$$m_\lambda(\mu) = \sum_{w \in W_G} \varepsilon(w) \mathcal{P}(w(\lambda + \delta) - (\mu + \delta)). \tag{9.26}$$

Here W_G is the symmetric group on $\{1, \dots, n\}$, the roots in Σ are the $e_i - e_n$ with $1 \leq i \leq n-1$, and $\mathcal{P}$ and δ are given by

$$\mathcal{P}(\nu) = \begin{cases} 1 & \text{if } \langle \nu, e_j \rangle \geq 0 \text{ for all } j < n \text{ and } \langle \nu, e_1 + \cdots + e_n \rangle = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta = \tfrac{1}{2}(n-1)e_1 + \tfrac{1}{2}(n-3)e_2 + \cdots - \tfrac{1}{2}(n-1)e_n.$$

We are to prove that $m_\lambda(\mu)$ is 1 if (9.15) holds and is 0 otherwise.

We begin with two lemmas. The first one gives a necessary condition for $m_\lambda(\mu)$ to be nonzero, and the second one concentrates on the value of the w^{th} term of (9.26). After the two lemmas, we prove two propositions that together prove Theorem 9.14.

Lemma 9.27. Every term of (9.26) is 0 unless $\sum_{j=1}^n a_j = \sum_{j=1}^n c_j$.

PROOF. The formula for $\mathcal{P}$ shows that the w^{th} term of (9.26) is 0 unless

$$\begin{aligned} 0 &= \langle w(\lambda+\delta) - (\mu+\delta), e_1 + \cdots + e_n\rangle \\ &= \langle \lambda+\delta, w^{-1}(e_1+\cdots+e_n)\rangle - \langle \mu+\delta, e_1+\cdots+e_n\rangle \\ &= \langle \lambda-\mu, e_1+\cdots+e_n\rangle \\ &= \sum_{j=1}^n a_j - \sum_{j=1}^n c_j. \end{aligned}$$

Lemma 9.28. Fix i with $i < n$, and suppose that $c_j \geq a_{j+1}$ for $j \leq i$. Then $\mathcal{P}(w(\lambda+\delta)-(\mu+\delta)) = 0$ unless $we_j = e_j$ for $j \leq i$.

PROOF. Fix l with $l \leq i$. Choose $r = r(l)$ with $we_r = e_l$. Then

$$\langle w(\lambda+\delta)-(\mu+\delta), e_l\rangle = \langle \lambda+\delta, e_r\rangle - \langle \mu+\delta, e_l\rangle = (a_r - c_l) - (r-l).$$

For the w^{th} term to be nonzero, this has to be ≥ 0, and thus we must have $a_r \geq c_l + (r-l) \geq a_{l+1} + (r-l)$. The case $l=1$ has $a_r \geq a_2 + (r-1)$. If $r \geq 2$, then $a_2 \geq a_r \geq a_2 + (r-1)$, contradiction. So $l=1$ implies $r=1$, and $we_1 = e_1$. Inductively suppose that $we_j = e_j$ for $j < l$. We have $we_{r(l)} = e_l$. From above,

$$a_{r(l)} \geq a_{l+1} + (r(l)-l).$$

We know that $r(l) \geq l$. If $r(l) > l$, then

$$a_{l+1} \geq a_{r(l)} \geq a_{l+1} + (r(l)-l) > a_{l+1},$$

contradiction. Thus $r(l) = l$, and the induction is complete.

Proposition 9.29. If $c_1 \geq a_2, c_2 \geq a_3, \ldots, c_{n-1} \geq a_n$ hold, then

$$m_\lambda(\mu) = \begin{cases} 1 & \text{if } a_i \geq c_i \text{ for } 1 \leq i \leq n-1 \text{ and } c_n = \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} c_i \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Lemma 9.28 shows that the w^{th} term can contribute to $m_\lambda(\mu)$ only if $we_j = e_j$ for $j \le n-1$. Thus we need consider only $w = 1$. We have

$$\mathcal{P}(1(\lambda+\delta)-(\mu+\delta)) = \mathcal{P}(\lambda-\mu) = \mathcal{P}\big(\textstyle\sum_{j=1}^n (a_j - c_j)\big).$$

The formula for $\mathcal{P}$ shows that $\mathcal{P}$ is 1 if

$$a_j - c_j \ge 0 \ \text{ for } j < n \qquad \text{and} \qquad a_n - c_n = -\sum_{i<n}(a_i - c_i),$$

and it is 0 otherwise. The proposition follows.

Proposition 9.30. If one or more of the inequalities $c_1 \ge a_2$, $c_2 \ge a_3$, $\dots$, $c_{n-1} \ge a_n$ fails, then $m_\lambda(\mu) = 0$.

PROOF. In view of Lemma 9.27, we may assume that $\sum_{i=1}^n c_i = \sum_{i=1}^n a_i$. Choose i as small as possible so that $c_i < a_{i+1}$. Here $1 \le i \le n-1$. Lemma 9.28 shows that the w^{th} term of (9.26) gives 0 unless $we_j = e_j$ for $j < i$. So we may limit consideration to terms in which w has this property. We shall show that the w term cancels with the wp term, where p is the reflection in the root $e_i - e_{i+1}$. Define k and l by $we_i = e_k$ and $we_{i+1} = e_l$. Here $k \ge i$ and $l \ge i$ since $we_j = e_j$ for $j < i$. We have

$$wp(\lambda+\delta)-(\mu+\delta) = w(\lambda+\delta)-(\mu+\delta)-(a_i - a_{i+1}+1)w(e_i - e_{i+1}),$$

and the arguments of $\mathcal{P}$ for w and wp have the same j^{th} component except possibly for $j = k$ and $j = l$. For the k^{th} component,

$$\begin{aligned} \langle wp(\lambda+\delta)-(\mu+\delta), e_k\rangle &= \langle wp(\lambda+\delta), we_i\rangle - \langle \mu+\delta, e_k\rangle \\ &= \langle \lambda+\delta, e_{i+1}\rangle - \langle \mu+\delta, e_k\rangle \\ &= (a_{i+1} - c_k) + (k-i-1) \end{aligned} \tag{9.31}$$

and

$$\begin{aligned} \langle w(\lambda+\delta)-(\mu+\delta), e_k\rangle &= \langle \lambda+\delta, e_i\rangle - \langle \mu+\delta, e_k\rangle \\ &= (a_i - c_k) + (k-i). \end{aligned} \tag{9.32}$$

Assume $k < n$ for the moment. We have

$$c_k - k \le c_i - i < a_{i+1} - i$$

and hence

$$c_k - k \le a_{i+1} - (i+1).$$

So (9.31) is ≥ 0. Since (9.31) is $<$ (9.32), we see that (9.32) is > 0.

Similarly for the l^{th} component,

$$\begin{aligned} \langle wp(\lambda+\delta) - (\mu+\delta), e_l\rangle &= \langle \lambda+\delta, e_i\rangle - \langle \mu+\delta, e_l\rangle \\ &= (a_i - c_l) + (l - i) \end{aligned} \tag{9.33}$$

and

$$\langle w(\lambda+\delta) - (\mu+\delta), e_l\rangle = (a_{i+1} - c_l) + (l - i - 1). \tag{9.34}$$

Under the assumption $l < n$, (9.34) is ≥ 0 and (9.33) is > 0.

Now we want to see that $\mathcal{P}$ has the same value on $w(\lambda+\delta) - (\mu+\delta)$ and $wp(\lambda+\delta) - (\mu+\delta)$. Since we are assuming $\sum_{i=1}^n c_i = \sum_{i=1}^n a_i$, the formula for $\mathcal{P}$ gives

(9.35a)

$$\mathcal{P}(w(\lambda+\delta) - (\mu+\delta)) = 1 \quad \text{if and only if}$$
$$\langle w(\lambda+\delta) - (\mu+\delta), e_j\rangle \ge 0 \ \text{ for } 1 \le j \le n-1$$

(9.35b)

$$\mathcal{P}(wp(\lambda+\delta) - (\mu+\delta)) = 1 \quad \text{if and only if}$$
$$\langle wp(\lambda+\delta) - (\mu+\delta), e_j\rangle \ge 0 \ \text{ for } 1 \le j \le n-1.$$

First suppose that $k < n$ and $l < n$. We have seen that $w(\lambda+\delta) - (\mu+\delta)$ and $wp(\lambda+\delta) - (\mu+\delta)$ match in all components but the k^{th} and l^{th} and that the k^{th} and l^{th} components are ≥ 0 for each. Hence (9.35) gives

$$\mathcal{P}(w(\lambda+\delta) - (\mu+\delta)) = \mathcal{P}(wp(\lambda+\delta) - (\mu+\delta)) \tag{9.36}$$

when $k < n$ and $l < n$.

Next suppose that $k = n$. We have seen that $w(\lambda+\delta) - (\mu+\delta)$ and $wp(\lambda+\delta) - (\mu+\delta)$ match in all components but the n^{th} and l^{th}, hence in all of the first $n-1$ components but the l^{th}. In the l^{th} component, they are ≥ 0. Hence (9.35) gives (9.36) when $k = n$.

Finally if $l = n$, then we argue similarly, and (9.35) gives (9.36) when $l = n$.

2a) Branching from $SO(2n+1)$ to $SO(2n)$. The given highest weights are $\lambda = \sum_{j=1}^{n} a_j e_j$ with $a_1 \geq \cdots \geq a_n \geq 0$ and $\mu = \sum_{j=1}^{n} c_j e_j$ with $c_1 \geq \cdots \geq c_{n-1} \geq |c_n|$.

The multiplicity being computed is again as in (9.26). The members w of the Weyl group W_G are of the form $w = sp$ with s a sign change and p a permutation, the roots in Σ are the e_i with $1 \leq i \leq n$, and the expressions for $\mathcal{P}$ and δ are

$$\mathcal{P}(\nu) = \begin{cases} 1 & \text{if } \langle \nu, e_j \rangle \geq 0 \text{ for all } j \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$\delta = (n + \tfrac{1}{2})e_1 + (n - \tfrac{1}{2})e_2 + \cdots + \tfrac{1}{2}e_n.$$

We are to prove that $m_\lambda(\mu)$ is 1 if (9.17a) holds and is 0 otherwise.

The argument proceeds in the same style as for the unitary groups. There are two lemmas and two propositions.

Lemma 9.37. Write $w = sp$ with s a sign change and p a permutation. Then the w^{th} term can contribute to (9.26) only if s equals 1 or s equals the root reflection s_{e_n}.

PROOF. Consider the expression $\langle w(\lambda + \delta) - (\mu + \delta), e_j \rangle$ for $j < n$. Since $\langle \mu + \delta, e_j \rangle > 0$, we must have $\langle w(\lambda + \delta), e_j \rangle > 0$ for the w^{th} term of (9.26) to be nonzero. Therefore $w^{-1}e_j > 0$ for $j < n$, and hence $p^{-1}s^{-1}e_j > 0$ for $j < n$. This means that $s^{-1}e_j > 0$ for $j < n$, and hence $s = 1$ or $s = s_{e_n}$.

Lemma 9.38. Fix i with $i < n$, and suppose that $c_j \geq a_{j+1}$ for $j \leq i$. Then $\mathcal{P}(w(\lambda + \delta) - (\mu + \delta)) = 0$ unless $we_j = e_j$ for $j \leq i$.

PROOF. The proof is the same as for Lemma 9.28. Lemma 9.37 shows that we need not consider $we_r = -e_l$ since $w^{-1}e_j > 0$ for $j < n$.

Proposition 9.39. If $c_1 \geq a_2, c_2 \geq a_3, \ldots, c_{n-1} \geq a_n$ hold, then

$$m_\lambda(\mu) = \begin{cases} 1 & \text{if } a_i \geq c_i \text{ for } 1 \leq i \leq n-1 \text{ and } a_n \geq |c_n| \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proof is similar to that for Proposition 9.29. The w^{th} term can contribute to $m_\lambda(\mu)$ only if $we_j = e_j$ for $j \leq n-1$. Thus the only possible contributions to $m_\lambda(\mu)$ are from $w = 1$ and $w = s_{e_n}$.

Proposition 9.40. If one or more of the inequalities $c_1 \geq a_2$, $c_2 \geq a_3$, $\dots$, $c_{n-1} \geq a_n$ fails, then $m_\lambda(\mu) = 0$.

PROOF. The proof is along the same lines as the one for Proposition 9.30, and we retain that notation. Again the wp term will cancel with the w term. This time $we_i = \pm e_k$ and $we_{i+1} = \pm e_l$ with $k \geq i$ and $l \geq i$, and the minus signs must be carried along as possibilities if $k = n$ or $l = n$. For the k^{th} component, we readily check that

$$\langle wp(\lambda+\delta) - (\mu+\delta), e_k\rangle \quad \text{and} \quad \langle w(\lambda+\delta) - (\mu+\delta), e_k\rangle \tag{9.41}$$

are both ≥ 0 if $we_i = +e_k$. For $k = n$, if $we_i = -e_n$, then the members of (9.41) are both < 0. Thus the arguments of $\mathcal{P}$ in the wp and w terms have the same sign in the k^{th} component. For the l^{th} component,

$$\langle wp(\lambda+\delta) - (\mu+\delta), e_l\rangle \quad \text{and} \quad \langle w(\lambda+\delta) - (\mu+\delta), e_l\rangle \tag{9.42}$$

are both ≥ 0 if $we_{i+1} = +e_l$. For $l = n$, if $we_{i+1} = -e_l$, then the members of (9.42) are both < 0. Thus the arguments of $\mathcal{P}$ in the wp and w terms have the same sign in the l^{th} component. The proposition follows.

2b) Branching from $SO(2n)$ to $SO(2n-1)$. The given highest weights are $\lambda = \sum_{j=1}^n a_j e_j$ with $a_1 \geq \cdots \geq a_{n-1} \geq |a_n|$ and $\mu = \sum_{j=1}^{n-1} c_j e_j$ with $c_1 \geq \cdots \geq c_{n-1} \geq 0$.

The multiplicity being computed is

$$m_\lambda(\mu) = \sum_{w \in W_G} \varepsilon(w)\mathcal{P}(\overline{w(\lambda+\delta) - \delta} - \mu), \tag{9.43}$$

where the bar indicates restriction to the first $n-1$ components. The members w of the Weyl group W_G are of the form $w = sp$ with s an even sign change and p a permutation, and δ is given by

$$\delta = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}.$$

Let us compute the set of weights Σ. The restrictions of the positive roots of $SO(2n)$ are the $e_i \pm e_j$ with $i < j < n$ and the $e_1, \dots, e_{n-1}$. The $e_i \pm e_j$ have multiplicity 1 as weights in $SO(2n)$ and correspond to roots in $SO(2n-1)$; thus they do not contribute to Σ. The weights $e_1, \dots, e_{n-1}$ have multiplicity 2 in $SO(2n)$ from restriction of $e_j \pm e_n$; one instance of each corresponds to a root of $SO(2n-1)$, and the other instance contributes to

Σ. The $\mathcal{P}$ function is therefore defined relative to the weights $e_1, \dots, e_{n-1}$, each with multiplicity 1. Thus

$$\mathcal{P}(\nu) = \begin{cases} 1 & \text{if } \langle \nu, e_j \rangle \geq 0 \text{ for all } j \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

We are to prove that $m_\lambda(\mu)$ is 1 if (9.17b) holds and is 0 otherwise.

This time we begin with three lemmas, the second and third of which are similar to the lemmas for branching from $SO(2n+1)$ to $SO(2n)$. After the three lemmas, we prove two propositions that together prove Theorem 9.16b.

Lemma 9.44. It is enough to prove the branching formula under the assumption $a_n \geq 0$.

PROOF. The matrix $\operatorname{diag}(1, \dots, 1, -1)$ normalizes $SO(2n)$, and conjugation of $SO(2n)$ by it leaves $SO(2n-1)$ fixed, negates the last variable in the Lie algebra of the maximal torus of $SO(2n)$, and leaves stable the set of positive roots of $SO(2n)$. Thus it carries an irreducible representation of $SO(2n)$ with highest weight $a_1e_1 + \cdots + a_{n-1}e_{n-1} + a_ne_n$ to an irreducible representation with highest weight $a_1e_1 + \cdots + a_{n-1}e_{n-1} - a_ne_n$. Therefore the restrictions to $SO(2n-1)$ of these two irreducible representations of $SO(2n)$ are equivalent.

In both cases restriction to $SO(2n-1)$ is asserted to yield all irreducible representations with highest weights $c_1e_1 + \cdots + c_{n-1}e_{n-1}$ such that $a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq |a_n|$, and the lemma follows.

From now on, we accordingly assume that $a_n \geq 0$.

Lemma 9.45. For w in W_G, the w^{th} term can contribute to $m_\lambda(\mu)$ only if w is a permutation.

PROOF. Consider $\langle w(\lambda+\delta) - (\mu+\delta), e_j \rangle$ for $j < n$. Since $\langle \mu+\delta, e_j \rangle > 0$, we must have $\langle w(\lambda+\delta), e_j \rangle > 0$ for the w^{th} term of $m_\lambda(\mu)$ to be nonzero. Therefore $\langle \lambda + \delta, w^{-1}e_j \rangle > 0$ for $j < n$. Since $\langle \lambda + \delta, e_{j'} \rangle > 0$ if $j' < n$, the only two situations in which we can have $w^{-1}e_j = -e_{j'}$ are $j = n$ and $j' = n$. The number of signs changed by w^{-1} has to be even, and hence this number must be 0 or 2. If it is 0, then w is a permutation. If it is 2, then j and j' cannot both be n. So there is some $j < n$ with $w^{-1}e_j = -e_n$, and we find that $\langle \lambda + \delta, -e_n \rangle > 0$. The left side of this inequality is $-a_n$, and we obtain a contradiction since Lemma 9.44 has allowed us to assume that $a_n \geq 0$.

Lemma 9.46. Fix i with $i < n$, and suppose that $c_j \geq a_{j+1}$ for $j \leq i$. Then $\mathcal{P}(w(\lambda+\delta)-(\mu+\delta)) = 0$ unless $we_j = e_j$ for $j \leq i$.

PROOF. The proof is the same as for Lemma 9.28. Lemma 9.45 shows that w may be assumed to be a permutation.

Proposition 9.47. If $c_1 \geq a_2, c_2 \geq a_3, \ldots, c_{n-1} \geq a_n$ hold, then

$$m_\lambda(\mu) = \begin{cases} 1 & \text{if } a_i \geq c_i \text{ for } 1 \leq i \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proof is similar to that for Proposition 9.29. The w^{th} term can contribute to $m_\lambda(\mu)$ only if w is a permutation and $we_j = e_j$ for $j \leq n-1$. Thus the only possible contribution to $m_\lambda(\mu)$ is from $w = 1$.

Proposition 9.48. If one or more of the inequalities $c_1 \geq a_2, c_2 \geq a_3, \ldots, c_{n-1} \geq a_n$ fails, then $m_\lambda(\mu) = 0$.

PROOF. The proof proceeds along the same lines as the ones for Propositions 9.30 and 9.40, and we retain that earlier notation. Again the wp term will cancel with the w term. This time $we_i = e_k$ and $we_{i+1} = e_l$, and minus signs do not enter. We readily find that

$$\langle wp(\lambda+\delta)-(\mu+\delta), e_k\rangle \quad \text{and} \quad \langle w(\lambda+\delta)-(\mu+\delta), e_k\rangle$$

are both ≥ 0 and that

$$\langle wp(\lambda+\delta)-(\mu+\delta), e_l\rangle \quad \text{and} \quad \langle w(\lambda+\delta)-(\mu+\delta), e_l\rangle$$

are both ≥ 0. Thus the arguments of $\mathcal{P}$ in the wp and w terms have the same sign in the k^{th} component and the same sign in the l^{th} component. The proposition follows.

3) Branching from $Sp(n)$ to $Sp(n-1)$. This case is considerably more complicated than the previous ones and is an indicator of the depth of branching theorems with multiplicities ≥ 1. We use restriction in stages. In (9.13) we take $G = Sp(n)$, $H_1 = Sp(n-1) \times Sp(1)$, and $H = Sp(n-1)$. The given highest weights for G and H are $\lambda = \sum_{j=1}^n a_j e_j$ with $a_1 \geq \cdots \geq a_n \geq 0$ and $\mu = \sum_{j=1}^{n-1} c_j e_j$ with $c_1 \geq \cdots \geq c_{n-1} \geq 0$. Any irreducible representation of H_1 is the outer tensor product of an irreducible representation of $Sp(n-1)$ and an irreducible representation of $Sp(1) \cong SU(2)$. The only terms σ_1 for H_1 that can make a contribution to (9.13) are those for which the representation on the $Sp(n-1)$ factor matches the given σ. Initially we take the representation on the $Sp(1)$ factor to be arbitrary, say with highest weight c_0e_n for an integer $c_0 \geq 0$. Since restriction from $Sp(1)$ to $\{1\}$ yields the trivial representation with multiplicity equal to the dimension, we see that

$$m_\lambda^H\big(\textstyle\sum_{j=1}^{n-1} c_je_j\big) = \displaystyle\sum_{c_0=0}^{\infty}(c_0+1)m_\lambda^{H_1}\big(\textstyle\sum_{j=1}^{n-1} c_je_j + c_0e_n\big), \tag{9.49}$$

where m_λ^H and $m_\lambda^{H_1}$ are the multiplicities of the respective representations of $Sp(n-1)$ and $Sp(n-1)\times Sp(1)$ in the given representation of $Sp(n)$. Thus in principle Theorem 9.18 will follow from an explicit branching theorem for passing from $Sp(n)$ to $Sp(n-1)\times Sp(1)$. We shall state such an explicit branching theorem and sketch its proof, leaving for the Historical Notes a derivation of Theorem 9.18 from it.

Theorem 9.50 (Lepowsky). For $Sp(n)$, the irreducible representation with highest weight $\lambda = a_1e_1+\cdots+a_ne_n$ decomposes under the subgroup $Sp(n-1)\times Sp(1)$ into the sum of representations with highest weights $\mu = c_1e_1+\cdots+c_{n-1}e_{n-1}+c_0e_n$ and multiplicities $m_\lambda(\mu)$ as follows. The multiplicity is 0 unless the integers

$$\begin{aligned}
A_1 &= a_1 - \max(a_2, c_1)\\
A_2 &= \min(a_2, c_1) - \max(a_3, c_2)\\
&\ \vdots\\
A_{n-1} &= \min(a_{n-1}, c_{n-2}) - \max(a_n, c_{n-1})\\
A_n &= \min(a_n, c_{n-1})
\end{aligned}$$

are all ≥ 0 and also c_0 has the same parity as $\sum_{j=1}^n a_j - \sum_{j=1}^{n-1} c_j$. In this case the multiplicity is

$$m_\lambda(\mu) = \mathcal{P}(A_1e_1+\cdots+A_ne_n - c_0e_n) - \mathcal{P}(A_1e_1+\cdots+A_ne_n+(c_0+2)e_n),$$

where $\mathcal{P}$ is the Kostant partition function defined relative to the set $\Sigma = \big\{e_i \pm e_n \,\big|\, 1 \leq i \leq n-1\big\}$.

REMARK. The condition $A_i \geq 0$ for $i \leq n$ is equivalent with the existence of integers b_i as in (9.19) and is equivalent also with the $2n-3$ inequalities $a_i \geq c_i$ for $i \leq n-1$ and $c_i \geq a_{i+2}$ for $i \leq n-2$.

The multiplicity being computed is again as in (9.26). The members w of the Weyl group W_G are of the form $w = sp$ with s a sign change and p a permutation, the roots in Σ are the $e_i \pm e_n$ with $1 \le i \le n-1$, and δ is $ne_1 + (n-1)e_2 + \cdots + 1e_n$. The partition function $\mathcal{P}$ satisfies

$$(9.51) \qquad \mathcal{P}(\nu) = 0 \quad \text{unless} \quad \begin{cases} \langle \nu, e_1 + \cdots + e_n \rangle \text{ is even and} \\ \langle \nu, e_i \rangle \ge 0 \text{ for } 1 \le i \le n-1 \end{cases}$$

because every member of Σ satisfies these properties.

The argument proceeds in the same style as for the unitary and rotation groups except that there are more steps, specifically three lemmas and three propositions. After the first proposition we pause to develop some needed properties of general partition functions. The three propositions, together with the first lemma below, prove Theorem 9.50.

Lemma 9.52. Every term of (9.26) is 0 unless c_0 has the same parity as $\sum_{j=1}^n a_j - \sum_{j=1}^{n-1} c_j$.

PROOF. For any $w \in W_G$, we have the following congruence modulo 2:

$$\begin{aligned} \langle w(\lambda+\delta) - (\mu+\delta), e_1 + \cdots + e_n \rangle &\equiv \langle (\lambda+\delta) - (\mu+\delta), e_1 + \cdots + e_n \rangle \\ &\equiv \textstyle\sum_{j=1}^n a_j - \sum_{j=1}^{n-1} c_j - c_0. \end{aligned}$$

According to the first condition in (9.51), the left side must be even for $\mathcal{P}$ to be nonzero, and hence the right side must be even.

Lemma 9.53. Write $w = sp$ with s a sign change and p a permutation. Then the w^{th} term can contribute to (9.26) only if s equals 1 or s equals the root reflection s_{2e_n}.

PROOF. The proof is the same as for Lemma 9.37.

Lemma 9.53 divides the relevant elements of the Weyl group into two kinds, p and $s_{2e_n}p$ for permutations p. Since $\mathcal{P}(s_{2e_n}\nu) = \mathcal{P}(\nu)$, we have

$$\begin{aligned} \mathcal{P}(s_{2e_n}p(\lambda+\delta) - (\mu+\delta)) &= \mathcal{P}(p(\lambda+\delta) - s_{2e_n}(\mu+\delta)) \\ &= \mathcal{P}(p(\lambda+\delta) - (\mu+\delta) + (2c_0+2)e_n). \end{aligned}$$

In other words the term for $s_{2e_n}p$ behaves like the term for p except that c_0 gets replaced by $-(c_0+2)$. This observation enables us to treat the two kinds of elements separately. In fact, even in the final answer for the multiplicity, the contributions from the two kinds of Weyl groups elements remain separate: the permutations p contribute $\mathcal{P}(A_1e_1 + \cdots + A_ne_n - c_0e_n)$, and the elements $s_{2e_n}p$ contribute $\mathcal{P}(A_1e_1 + \cdots + A_ne_n + (c_0+2)e_n)$ with a minus sign. Thus from now on, we work only with elements w of W_G that are permutations.

Lemma 9.54. Fix a permutation w. If $c_1 \geq a_3, c_2 \geq a_4, \ldots, c_{n-2} \geq a_n$ hold, then $\mathcal{P}(w(\lambda+\delta)-(\mu+\delta)) = 0$ unless every equality $we_i = e_j$ implies $j \geq i-1$.

PROOF. Suppose that the w term is not 0. Fix i, and define j by $we_i = e_j$. We may assume that $j < n$ and $i \geq 3$ since otherwise there is nothing to prove. We have

$$\langle w(\lambda+\delta)-(\mu+\delta), e_j\rangle = \langle \lambda+\delta, e_i\rangle - \langle \mu+\delta, e_j\rangle = (a_i - c_j) + (j-i).$$

By (9.51) the left side is ≥ 0. On the other hand, if $j < i-1$, then the inequalities $a_i \leq c_{i-2}$ and $-c_j \leq -c_{i-2}$ imply

$$(a_i - c_j) + (j-i) < (c_{i-2} - c_{i-2}) - 1 < 0,$$

and we have a contradiction.

Proposition 9.55. If $c_1 \geq a_3, c_2 \geq a_4, \ldots, c_{n-2} \geq a_n$ hold and if $a_i < c_i$ for some $i < n$, then $\mathcal{P}(w(\lambda+\delta)-(\mu+\delta)) = 0$ for every permutation w.

PROOF. Suppose that the w term is nonzero. Define j and k by $e_j = we_i$ and $e_k = w^{-1}e_i$. Lemma 9.54 gives $i \leq j+1$ and $k \leq i+1$. The claim is that $i \leq j-1$ and $k \leq i-1$. For this purpose we may assume that $j < n$. To see that $i \leq j-1$, we write

$$\langle w(\lambda+\delta)-(\mu+\delta), e_j\rangle = (a_i - c_j) + (j-i) < (c_i - c_j) + (j-i).$$

By (9.51) the left side is ≥ 0. If $i > j-1$, then both terms on the right side are ≤ 0, and we have a contradiction. Similarly to see that $k \leq i-1$, we write

$$\langle w(\lambda+\delta)-(\mu+\delta), e_i\rangle = (a_k - c_i) + (i-k) < (a_k - a_i) + (i-k).$$

If $k > i-1$, then both terms on the right side are ≤ 0, and we have a contradiction to the fact that the left side is ≥ 0.

Therefore we have $we_i = e_j$ and $we_k = e_i$ with $k < i < j$. Since $j > i$, $w\{e_{i+1}, \ldots, e_n\}$ does not contain e_j, and thus $w\{e_{i+1}, \ldots, e_n\}$ meets $\{e_1, \ldots, e_i\}$. Since $k < i$, e_i is not in $w\{e_{i+1}, \ldots, e_n\}$. Hence $w\{e_{i+1}, \ldots, e_n\}$ meets $\{e_1, \ldots, e_{i-1}\}$. Consequently there exist indices r and s with $we_s = e_r$, $s \geq i+1$, and $r \leq i-1$. But then $r < s-1$, in contradiction to Lemma 9.54. This completes the proof.

For the proofs of the last two propositions, we shall need three identities concerning partition functions. It will be helpful to derive these in some generality. Let Ω be a finite set lying in an open half space of a Euclidean space. For our purposes each member of Ω will have multiplicity 1, but higher multiplicity can be handled by giving different names to the different versions of the same element. We write $\mathcal{P}^{\Omega}$ for the associated partition function: $\mathcal{P}^{\Omega}(\nu)$ is the number of nonnegative-integer tuples $\{n_\omega \mid \omega \in \Omega\}$ such that $\nu = \sum_{\omega\in\Omega} n_\omega \omega$. If $\alpha_1, \dots, \alpha_k$ are members of Ω, we write $\mathcal{P}^{\Omega}_{\alpha_1,\dots,\alpha_k}$ for $\mathcal{P}^{\Omega'}$ when Ω' is the set Ω with $\alpha_1, \dots, \alpha_k$ removed.

Let us derive the identities. If α is in Ω, then

$$\mathcal{P}^{\Omega}(\nu) = \mathcal{P}^{\Omega}(\nu - \alpha) + \mathcal{P}^{\Omega}_{\alpha}(\nu)$$

for all ν. In fact, the left side counts the number of expansions of ν in terms of Ω, and the right side breaks this count disjointly into two parts—the first part for all expansions containing α at least once and the second part for all expansions not containing α. Iterating this identity $n \geq 0$ times, we obtain

$$\mathcal{P}^{\Omega}(\nu) - \mathcal{P}^{\Omega}(\nu - n\alpha) = \sum_{j=0}^{n-1} \mathcal{P}^{\Omega}_{\alpha}(\nu - j\alpha) \tag{9.56}$$

for all ν. If α and β are both in Ω and if $\gamma = \alpha - \beta$, then we can write a version of (9.56) for β, namely

$$\mathcal{P}^{\Omega}(\nu - n\gamma) - \mathcal{P}^{\Omega}(\nu - n\alpha) = \sum_{j=0}^{n-1} \mathcal{P}^{\Omega}_{\beta}(\nu - n\gamma - j\beta),$$

and the result upon subtraction is

$$\mathcal{P}^{\Omega}(\nu) - \mathcal{P}^{\Omega}(\nu - n\gamma) = \sum_{j=0}^{n-1} \big[\mathcal{P}^{\Omega}_{\alpha}(\nu - j\alpha) - \mathcal{P}^{\Omega}_{\beta}(\nu - n\gamma - j\beta)\big]. \tag{9.57}$$

Now suppose that $\omega \neq 0$ is in the Euclidean space and that ζ is the only member of Ω for which $\langle \zeta, \omega \rangle \neq 0$. Let us normalize ω so that $\langle \zeta, \omega \rangle = 1$. If an expansion of ν in terms of Ω involves $n\zeta$, then $\langle \nu, \omega \rangle = n$. Applying (9.56) for n and then $n + 1$, we obtain

$$\mathcal{P}^{\Omega}(\nu) = \mathcal{P}^{\Omega}(\nu - \langle \nu, \omega \rangle \zeta) = \mathcal{P}^{\Omega}_{\zeta}(\nu - \langle \nu, \omega \rangle \zeta) \tag{9.58}$$

provided $\langle \nu, \omega \rangle$ is an integer ≥ 0.

Proposition 9.59. If $c_1 \geq a_3$, $c_2 \geq a_4$, $\dots$, $c_{n-2} \geq a_n$ hold and if $a_j \geq c_j$ for all $j < n$, then the sum of $\varepsilon(w)\mathcal{P}(w(\lambda+\delta) - (\mu+\delta))$ over all permutations w is $\mathcal{P}(A_1e_1 + \cdots + A_ne_n - c_0e_n)$.

REMARK. By the same proof, an analogous summation formula applies for the elements $s_{2e_n}p$ of the Weyl group and yields the other term $-\mathcal{P}(A_1e_1 + \cdots + A_ne_n + (c_0+2)e_n)$ for the multiplicity in Theorem 9.50.

PROOF. The idea is to reduce matters to the case that

$$c_1 \geq a_2,\ c_2 \geq a_3, \dots,\ c_{n-1} \geq a_n. \tag{9.60}$$

If these inequalities are satisfied, then the proof of Lemma 9.28 shows that $\mathcal{P}(w(\lambda+\delta)-(\mu+\delta)) = 0$ except for $w = 1$. For $w = 1$, these inequalities make $A_j = a_j - c_j$ for $j < n$, and consequently $(\lambda+\delta) - (\mu+\delta) = A_1e_1 + \cdots + A_ne_n - c_0e_n$. Thus the proposition is immediate under the assumption that (9.60) holds.

In the general case suppose that $\lambda' = \sum_{j=1}^n a_j'e_j$ and $\mu' = \sum_{j=1}^{n-1} c_j'e_j + c_0e_n$ are given with $c_1' \geq a_3'$, $c_2' \geq a_4'$, $\dots$, $c_{n-2}' \geq a_n'$, with $a_j' \geq c_j'$ for all $j < n$, and with $c_i' < a_{i+1}'$ for some $i < n$. We may assume that i is as small as possible with this property. Define $c_i = a_{i+1}'$, $a_{i+1} = c_i'$, $c_j = c_j'$ for $j \neq i$, and $a_j = a_j'$ for $j \neq i+1$. Then let $\lambda = \sum_{j=1}^n a_je_j$ and $\mu = \sum_{j=1}^{n-1} c_je_j + c_0e_n$. A quick check shows that λ and μ satisfy the hypotheses of the proposition, that the A_j's are unchanged, and that the first index j, if any, with $c_j < a_{j+1}$ has $j > i$. Writing $(i\ \ i{+}1)$ for the transposition of i and $i+1$, we shall show that

$$\begin{aligned} &\mathcal{P}(w(\lambda+\delta) - (\mu+\delta)) - \mathcal{P}(w(i\ \ i{+}1)(\lambda+\delta) - (\mu+\delta)) \\ &\quad \stackrel{?}{=} \mathcal{P}(w(\lambda'+\delta) - (\mu'+\delta)) - \mathcal{P}(w(i\ \ i{+}1)(\lambda'+\delta) - (\mu'+\delta)) \end{aligned} \tag{9.61}$$

for all permutations w. When this identity is multiplied by $\varepsilon(w)$ and summed on w, it shows that twice the sum of $\varepsilon(w)\mathcal{P}(w(\lambda+\delta) - (\mu+\delta))$ equals twice the sum of $\varepsilon(w)\mathcal{P}(w(\lambda'+\delta) - (\mu'+\delta))$. Consequently an induction on the index i reduces the proposition to the case where (9.60) holds, and we have seen that it holds there.

Thus the proposition will follow once (9.61) is proved. Possibly replacing w by $w(i\ \ i{+}1)$ in this identity, we may assume that $w(e_i - e_{i+1}) > 0$. Define r and s by $e_r = we_i$ and $e_s = we_{i+1}$. Our normalization of w makes $r < s$. The argument of Lemma 9.28, applied with $i-1$ in place of i, shows that all four terms in (9.61) are 0 unless $we_j = e_j$ for $j \leq i-1$. Thus we may assume that $r \geq i$. Let us prove that we may take $r = i$.

If $r > i$, then the j with $we_j = e_i$ cannot be i or $i+1$ and thus has to satisfy $j \geq i+2$. Consequently Lemma 9.54 shows that the first term on each side of (9.61) is 0. Similarly the j' with $w(i\ \ i+1)e_{j'} = e_i$ cannot be i or $i+1$ and thus has to satisfy $j' \geq i+2$. Hence Lemma 9.54 shows that the second term on each side of (9.61) is 0. Therefore we may assume that $r = i$.

We now compute the respective sides of (9.61) using (9.56), (9.57), and (9.58). There will be two cases, $s < n$ and $s = n$. The first case will be the harder, and we handle that first. At the end we indicate what happens when $s = n$. To simplify some of the notation, we abbreviate $e_a - e_b$ as e_{ab}.

We begin with the left side of (9.61). The difference of the arguments of $\mathcal{P}$ in the two terms on the left side is $\langle \lambda + \delta, e_{i,i+1} \rangle e_{is}$. We are going to apply (9.57) with $\gamma = e_{is}$. Here $\gamma = \alpha - \beta$ with $\alpha = e_{in}$ and $\beta = e_{sn}$. Application of (9.57) shows that the left side of (9.61) is

$$(9.62) \quad = \sum_{j=0}^{a_i - a_{i+1}} \Big[\mathcal{P}_{e_{in}}(w(\lambda+\delta) - (\mu+\delta) - je_{in}) \\ - \mathcal{P}_{e_{sn}}(w(\lambda+\delta) - (\mu+\delta) - \langle \lambda+\delta, e_{i,i+1} \rangle e_{is} - je_{sn}) \Big].$$

In the first term of (9.62), the i^{th} component of the argument of $\mathcal{P}$ is

$$\langle w(\lambda+\delta) - (\mu+\delta) - je_{in}, e_i \rangle = a_i - c_i - j.$$

For $j > a_i - c_i$, the term drops out by (9.51). Thus we need not sum the first term beyond $j = a_i - c_i$. Since we have arranged that $c_i \geq a_{i+1}$, we can change the upper limit of the sum for the first term from $a_i - a_{i+1}$ to $a_i - c_i$. In the second term of (9.62), the i^{th} component of the argument of $\mathcal{P}$ is $a_{i+1} - c_i - 1$, and this is < 0 for every j. Thus every member of the second sum in (9.62) is 0.

We apply (9.58) to the first term of (9.62), taking $\Omega = \Sigma - \{e_{in}\}$, $\zeta = e_i + e_n$, and $\omega = e_i$. In the second term of (9.62), we subtract from the argument a multiple of $e_s + e_n$ to make the s^{th} component 0; this does not affect anything since every member of the sum remains equal to 0. After these steps we interchange the i^{th} and s^{th} arguments in the second term, taking advantage of symmetry. The resulting expression for (9.62) simplifies to

$$\sum_{j=0}^{a_i - c_i} \Big[\mathcal{P}(w(\lambda+\delta) - (\mu+\delta) + (c_i - a_i)e_i + (c_i - a_i + 2j)e_n) \\ - \mathcal{P}(w(\lambda+\delta) - (\mu+\delta) + (c_i - a_i)e_i - ((c_i - c_s) + (s-i))e_s \\ + ((c_s - a_i) + (i-s) + 2j)e_n) \Big].$$

The difference in the arguments of the two terms works out to be $((c_i - c_s) + (s - i))(e_s + e_n)$. Thus (9.56) with $\alpha = e_s + e_n$ shows that the above expression is

$$= \sum_{j=0}^{a_i - c_i} \sum_{k=0}^{(c_i - c_s)+(s-i-1)} \mathcal{P}_{e_s+e_n}(w(\lambda+\delta) - (\mu+\delta) + (c_i - a_i)e_i + (c_i - a_i + 2j)e_n - k(e_s + e_n)).$$

The coefficient of e_s in the argument is

$$\langle \lambda + \delta, e_{i+1} \rangle - \langle \mu + \delta, e_s \rangle - k = (a_{i+1} - c_s) + (s - i - 1) - k,$$

and so the term drops out if $k > (a_{i+1} - c_s) + (s - i - 1)$. Since $c_i \geq a_{i+1}$, we can replace the upper limit in the sum by $(a_{i+1} - c_s) + (s - i - 1)$. For the terms that have not dropped out, we apply (9.58) with $\zeta = e_{sn}$, and the result is that the left side of (9.61) is

$$= \sum_{j=0}^{a_i - c_i} \sum_{k=0}^{(a_{i+1} - c_s)+(s-i-1)} \mathcal{P}(w(\lambda + \delta) - (\mu + \delta) - ((a_{i+1} - c_s) + (s - i - 1))e_s + (c_i - a_i)e_i + (a_{i+1} - a_i + c_i - c_s + (s - i - 1) + 2j - 2k)e_n). \tag{9.63}$$

Now we compute the right side of (9.61). The formulas that relate λ' to λ and μ' to μ are

$$\lambda' = \lambda + (c_i - a_{i+1})e_{i+1} \quad \text{and} \quad \mu' = \mu - (c_i - a_{i+1})e_i. \tag{9.64}$$

The difference of the arguments of $\mathcal{P}$ in the two terms on the right side of (9.61) is $(a_i - c_i + 1)e_{is}$. Thus (9.57) shows that the right side of (9.61) is

$$= \sum_{j=0}^{a_i - c_i} \big[\mathcal{P}_{e_{in}}(w(\lambda' + \delta) - (\mu' + \delta) - je_{in}) - \mathcal{P}_{e_{sn}}(w(\lambda' + \delta) - (\mu' + \delta) - \langle \lambda' + \delta, e_{i,i+1} \rangle e_{is} - je_{sn})\big]. \tag{9.65}$$

In the first term of (9.65), the i^{th} component of the argument of $\mathcal{P}$ is $a_i - a_{i+1} - j \geq c_i - a_{i+1} \geq 0$. In the second term the s^{th} component of the argument is $(a_i - c_s) + (s - i) - j \geq (c_i - c_s) + (s - i) \geq 0$. We apply (9.58) to both terms, using $\zeta = e_i + e_n$ in the first and $\zeta = e_s + e_n$

in the second, and then we interchange the i^{th} and s^{th} components in the second term. The result is that (9.65) simplifies to

$$\sum_{j=0}^{a_i-c_i}\Big[\mathcal{P}(w(\lambda'+\delta)-(\mu'+\delta)-(a_i-a_{i+1})e_i-(a_i-a_{i+1}-2j)e_n) \\ -\mathcal{P}(w(\lambda'+\delta)-(\mu'+\delta)-(a_i-a_{i+1})e_i-(a_{i+1}-c_s+s-i)e_s \\ -(a_i-c_s+s-i-2j)e_n)\Big].$$

The difference in the arguments for the two terms is now equal to $(a_{i+1}-c_s+s-i)(e_s+e_n)$. Thus (9.56) with $\alpha=e_s+e_n$ shows that (9.65) simplifies further to

$$=\sum_{j=0}^{a_i-c_i}\sum_{k=0}^{(a_{i+1}-c_s)+(s-i-1)} \\ \mathcal{P}_{e_s+e_n}(w(\lambda'+\delta)-(\mu'+\delta)+(a_{i+1}-a_i)e_i-ke_s+(a_{i+1}-a_i+2j-k)e_n).$$

The coefficient of e_s in the argument is

$$\langle\lambda'+\delta,e_{i+1}\rangle-\langle\mu'+\delta,e_s\rangle-k,$$

and the smallest that this gets to be is $c_i-a_{i+1}\geq 0$. Thus we can apply (9.58) with $\zeta=e_{sn}$, and we find that (9.65) simplifies finally to (9.63). Thus the left side in (9.61) agrees with the right side, and (9.61) is proved in the case that $s<n$.

When $s=n$, we proceed similarly with each side of (9.61), but the simpler formula (9.56) may be used in place of (9.57). Once (9.58) has been used once with each side, no further steps are necessary, and we find that the left and right sides of (9.61) have been simplified to the same expression.

Proposition 9.66. If one or more of the inequalities $c_1\geq a_3$, $c_2\geq a_4$, $\dots$, $c_{n-2}\geq a_n$ fails, then $m_\lambda(\mu)=0$.

PROOF. Fix an $i\leq n-2$ with $c_i<a_{i+2}$. The idea is to show that the sum of $\varepsilon(w)\mathcal{P}(w(\lambda+\delta)-(\mu+\delta))$ over all permutations w cancels in sets of six. To describe the sets of six, we need some facts about the symmetric group $\mathcal{S}_{u,v}$ on the integers $\{u,u+1,\dots,v\}$. Let us write c_{kl} for the cyclic permutation with $k\leq l$ that sends k into $k+1$, $k+1$ into $k+2$, $\dots$, $l-1$ into l, and l into k. If $k\neq k'$ are integers $\geq u$, then $c_{kv}^{-1}c_{k'v}$ cannot be in $\mathcal{S}_{u,v-1}$, and it follows that $\mathcal{S}_{u,v}=\bigcup_{k=u}^{v}c_{kv}\mathcal{S}_{u,v-1}$. Similarly we have

$\mathcal{S}_{u,v} = \mathcal{S}_{u+1,v} \bigcup_{l=u}^{v} c_{ul}$. Iterating the first kind of decomposition and then the second, we find that each member w of $\mathcal{S}_{1,n}$ has a unique decomposition as $w = pzq$ with

$$p = c_{k_n,n} c_{k_{n-1},n-1} \cdots c_{k_{i+3},i+3} \quad \text{and} \quad q = c_{i-1,l_{i-1}} c_{i-2,l_{i-2}} \cdots c_{1,l_1}$$

and with all $k_j \geq i$, all $l_j \leq n$, and $z \in \mathcal{S}_{i,i+2}$. A set of six consists of all w with a common p and a common q. The properties of p and q that we need are

$$\begin{aligned} i \leq p(i) < p(i+1) < p(i+2), \\ q^{-1}(i) < q^{-1}(i+1) < q^{-1}(i+2) \leq i+2. \end{aligned} \tag{9.67}$$

Define $i' = i + 1$ and $i'' = i + 2$, and abbreviate $e_a - e_b$ as e_{ab} during the remainder of the proof. Fix p and q as above, and define $r = p(i)$, $s = p(i')$, and $t = p(i'')$, so that $i \leq r < s < t$ by (9.67). The proof divides into two cases, $t < n$ and $t = n$. The case that $t = n$ is the simpler, and its proof can be obtained from the proof when $t < n$ by replacing t by n and by dropping some of the terms. Thus we shall assume that $t < n$ from now on.

For z equal to 1 or $c_{ii'}$ or $c_{ii''}$, an application of (9.57) with $\alpha = e_{sn}$, $\beta = e_{tn}$, and $\gamma = e_{st}$ gives

$$\begin{aligned} &\mathcal{P}(pzq(\lambda+\delta) - (\mu+\delta)) - \mathcal{P}(pc_{i'i''}zq(\lambda+\delta) - (\mu+\delta)) \\ &\quad = \sum_{j=0}^{\langle\lambda+\delta, q^{-1}z^{-1}e_{i'i''}\rangle - 1} \Big[\mathcal{P}_{e_{sn}}(pzq(\lambda+\delta) - (\mu+\delta) - je_{sn})) \\ &\qquad - \mathcal{P}_{e_{tn}}(pzq(\lambda+\delta) - (\mu+\delta) - \langle\lambda+\delta, q^{-1}z^{-1}e_{i'i''}\rangle e_{st} - je_{tn}))\Big]. \end{aligned}$$

We multiply this equation by $\varepsilon(z)$ and add for the three values of z. On the left side we have our desired sum of six terms of (9.26), apart from a factor of $\varepsilon(pq)$, and on the right side we have six sums, three involving $\mathcal{P}_{e_{sn}}$ and three involving $\mathcal{P}_{e_{tn}}$. The limits of summation for the two sets of three sums are the same; with their coefficient signs in place, they are

$$\sum_{j=0}^{\langle\lambda+\delta, q^{-1}e_{i'i''}\rangle - 1}, \quad -\sum_{j=0}^{\langle\lambda+\delta, q^{-1}e_{ii''}\rangle - 1}, \quad \sum_{j=0}^{\langle\lambda+\delta, q^{-1}e_{ii'}\rangle - 1}. \tag{9.68}$$

The middle one we break into two parts as

$$(9.69)\qquad -\sum_{j=0}^{\langle\lambda+\delta,q^{-1}e_{ii''}\rangle-1} = -\sum_{j=\langle\lambda+\delta,q^{-1}e_{ii'}\rangle}^{\langle\lambda+\delta,q^{-1}e_{ii''}\rangle-1} - \sum_{j=0}^{\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-1}.$$

With the first sum on the right side of (9.69), we change variables using $j' = j - \langle\lambda+\delta, q^{-1}e_{ii'}\rangle$, and then we change j' back to j. The new limits of summation are from 0 to $\langle\lambda+\delta, q^{-1}e_{i'i''}\rangle - 1$. This adjusted sum gets lumped with the first sum in (9.68), and the second sum on the right side of (9.69) gets lumped with the third sum in (9.68). The expression we get is

$$\begin{aligned}
&= \sum_{j=0}^{\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle-1} \big[\mathcal{P}_{e_{sn}}(pq(\lambda+\delta)-(\mu+\delta)-je_{sn}) \\
&\qquad - \mathcal{P}_{e_{sn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-(j+\langle\lambda+\delta,q^{-1}e_{ii'}\rangle)e_{sn})\big] \\
&- \sum_{j=0}^{\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle-1} \big[\mathcal{P}_{e_{tn}}(pq(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{st}-je_{tn}) \\
&\qquad - \mathcal{P}_{e_{tn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{ii''}\rangle e_{st} \\
&\qquad\qquad\qquad - (j+\langle\lambda+\delta,q^{-1}e_{ii'}\rangle)e_{tn})\big] \\
&- \sum_{k=0}^{\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-1} \big[\mathcal{P}_{e_{sn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-ke_{sn}) \\
&\qquad - \mathcal{P}_{e_{sn}}(pc_{ii''}q(\lambda+\delta)-(\mu+\delta)-ke_{sn})\big] \\
&+ \sum_{k=0}^{\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-1} \big[\mathcal{P}_{e_{tn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{ii''}\rangle e_{st}-ke_{tn}) \\
&\qquad - \mathcal{P}_{e_{tn}}(pc_{ii''}q(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{ii'}\rangle e_{st}-ke_{tn})\big].
\end{aligned}$$

In this expression we have four sums of differences, and we find that the respective differences of the arguments of $\mathcal{P}$ are

$$\begin{gathered}
\langle\lambda+\delta,q^{-1}e_{ii'}\rangle e_{rn}, \quad \langle\lambda+\delta,q^{-1}e_{ii'}\rangle e_{rn}, \\
\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{rt}, \ \text{ and } \ \langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{rs}.
\end{gathered}$$

To handle the first and second sums of differences, we use (9.56) with $\alpha = e_{rn}$. For the third sum of differences, we use (9.57) with $\alpha = e_{rn}$ and $\beta = e_{tn}$. For the fourth sum of differences, we use (9.57) with $\alpha = e_{rn}$ and

$\beta = e_{sn}$. The expression is then

$$= \sum_{j=0}^{\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle-1} \sum_{k=0}^{\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-1}$$

$$\begin{aligned}
&\big[\mathcal{P}_{e_{sn},e_{rn}}(pq(\lambda+\delta)-(\mu+\delta)-je_{sn}-ke_{rn})\\
&-\mathcal{P}_{e_{tn},e_{rn}}(pq(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{st}-je_{tn}-ke_{rn})\\
&-\mathcal{P}_{e_{sn},e_{rn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-ke_{sn}-je_{rn})\\
&+\mathcal{P}_{e_{sn},e_{tn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-ke_{sn}-je_{tn}-\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{rt})\\
&+\mathcal{P}_{e_{tn},e_{rn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{ii''}\rangle e_{st}-ke_{tn}-je_{rn})\\
&-\mathcal{P}_{e_{tn},e_{sn}}(pc_{ii'}q(\lambda+\delta)-(\mu+\delta)-\langle\lambda+\delta,q^{-1}e_{ii''}\rangle e_{st}-ke_{tn}-je_{sn}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\langle\lambda+\delta,q^{-1}e_{i'i''}\rangle e_{rs})\big].
\end{aligned}$$

Let us call the terms within brackets A, B, C, D, E, F. The proof is completed by showing for each j and k that A cancels with C, B cancels with E, and D cancels with F. We compute the differences of the arguments of $\mathcal{P}$ for the three pairs, seeing that they are $(\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-k+j)$ times e_{rs}, e_{rt}, and e_{st} in the three cases. The proofs of cancellation are similar in the three cases, and we give only the one for canceling A and C.

The idea is to apply (9.58) twice to each of A and C, once with $\zeta = e_r+e_n$ and once with $\zeta = e_s+e_n$. The arguments of A and C differ only in the r^{th} and s^{th} components, and the inner products of the arguments with e_r+e_s are equal. Hence simplification of A and C by means of (9.58) will make the arguments equal, and the terms will cancel.

To be able to apply (9.58) in this way, we have to know that the r^{th} and s^{th} components of the arguments of A and C are ≥ 0 for every j and k. This verification will be the only place where we use the hypothesis $c_i < a_{i+2}$. To begin with, we know that $k \leq \langle\lambda+\delta,q^{-1}e_{ii'}\rangle$, and thus $(\langle\lambda+\delta,q^{-1}e_{ii'}\rangle-k+j)$ is ≥ 0. Then for each (j,k), we have

$$\langle\text{argument}(A),e_r\rangle-\langle\text{argument}(C),e_r\rangle=\langle(\geq 0)e_r,e_r\rangle\geq 0,$$

from which it follows that both arguments have r^{th} component ≥ 0 if C does. Similarly both arguments have s^{th} component ≥ 0 if A does. We have

$$\begin{aligned}
\langle\text{argument}(C),e_r\rangle &= \langle pc_{ii'}q(\lambda+\delta),e_r\rangle-\langle\mu+\delta,e_r\rangle-j\\
&= \langle\lambda+\delta,q^{-1}e_{i'}\rangle-\langle\mu+\delta,e_r\rangle-j
\end{aligned}$$

$$\begin{aligned}
&\geq \langle \lambda + \delta, q^{-1} e_{i''} \rangle - \langle \mu + \delta, e_r \rangle + 1 \\
&\geq \langle \lambda + \delta, e_{i''} \rangle - \langle \mu + \delta, e_i \rangle + 1 \\
&= a_{i''} - c_i + \langle \delta, e_{i''} - e_i \rangle + 1 \\
&= a_{i+2} - c_i - 1 \\
&\geq 0.
\end{aligned}$$

The three inequalities above respectively use the upper bound on j, the inequalities (9.67), and the hypothesis $c_i < a_{i+2}$. Also

$$\begin{aligned}
\langle \text{argument}(A), e_s \rangle &= \langle pq(\lambda + \delta), e_s \rangle - \langle \mu + \delta, e_s \rangle - j \\
&= \langle \lambda + \delta, q^{-1} e_{i'} \rangle - \langle \mu + \delta, e_s \rangle - j \\
&\geq \langle \lambda + \delta, q^{-1} e_{i'} \rangle - \langle \mu + \delta, e_r \rangle - j \quad \text{since } r < s \\
&\geq 0,
\end{aligned}$$

the last step following from the preceding computation. This completes the proof.

6. Tensor Products and Littlewood–Richardson Coefficients

Let us return to the framework of §4 of finding the multiplicities of the irreducible representations of G in $L^2(G/H)$ when G/H can be constructed from a succession of compact symmetric spaces. The starting point is branching theorems in the context of compact symmetric spaces U/K. In this section we begin a discussion of some further results of this kind beyond those proved in §5. Some of them have the property of handling only some representations of U or K, but they are still applicable to the problem of analyzing $L^2(G/H)$.

The first such result, given below as Theorem 9.70, handles the trivial representation of K. When U is semisimple, Theorem 9.70 is a direct translation, via Riemannian duality, of part of Helgason's Theorem (Theorem 8.49) because Lemma 8.48 shows that M fixes a nonzero highest weight vector if and only if M acts by the trivial representation in the highest restricted-weight space. For general U, Theorem 9.70 follows from the result in the semisimple case because Theorem 4.29 shows that the semisimple part of U is closed and because the additional contribution to M comes from the identity component of the subgroup of the center fixed by Φ.

Theorem 9.70. Let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$, let K be the identity component of the set of fixed elements under an involution Φ, let φ be the differential of Φ, and let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ be the eigenspace decomposition of $\mathfrak{u}$ under φ. Choose a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{q}$, let $\mathfrak{s}$ be a maximal abelian subspace of the centralizer of $\mathfrak{b}$ in $\mathfrak{k}$, and put $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{s}$. Let M be the centralizer of $\mathfrak{b}$ in K. Impose an ordering on $(i\mathfrak{t})^*$ that takes $i\mathfrak{b}$ before $i\mathfrak{s}$. Then an irreducible finite-dimensional representation π of U has a nonzero K fixed vector if and only if M fixes a nonzero highest-weight vector of π.

A particularly simple yet illuminating example is the case of tensor products for a compact connected Lie group G. As we saw in §4, this case arises from the compact symmetric space U/K with $U = G \times G$ and $K = \operatorname{diag} G$. Let us examine this case in detail.

First let us consider the example directly, writing τ_λ for an irreducible representation of G with highest weight λ and writing χ_λ for its character. By (4.13), (4.15), and Corollary 4.16, the multiplicity of τ_μ in $\tau_{\lambda_1} \otimes \tau_{\lambda_2}$ is just

$$[\tau_{\lambda_1} \otimes \tau_{\lambda_2} : \tau_\mu] = \int_G \chi_{\lambda_1} \chi_{\lambda_2} \overline{\chi_\mu}\, dx. \tag{9.71}$$

If $\mu = 1$, then the integral is nonzero if and only if $\chi_{\lambda_2} = \overline{\chi_{\lambda_1}}$, thus if and only if τ_{λ_2} is equivalent with $\tau_{\lambda_1}^c$. In this case the multiplicity is 1.

Now let us consider this example from the point of view of Theorem 9.70. If $\mathfrak{c}$ is a Cartan subalgebra of the Lie algebra of G, then we can take $\mathfrak{b} = \{(X, -X) \mid X \in \mathfrak{c}\}$. We are forced to let $\mathfrak{s} = \operatorname{diag} \mathfrak{c}$, and we have $\mathfrak{t} = \mathfrak{c} \oplus \mathfrak{c}$. A member (λ_1, λ_2) of $(i\mathfrak{t})^*$ decomposes as $\frac{1}{2}(\lambda_1 - \lambda_2,\ \lambda_2 - \lambda_1) + \frac{1}{2}(\lambda_1 + \lambda_2,\ \lambda_1 + \lambda_2)$ with the first term carried on $i\mathfrak{b}$ and the second term carried on $i\mathfrak{s}$. Roots are of the form $(\alpha, 0)$ and $(0, \alpha)$ with $\alpha \in \Delta_G$, and their corresponding decompositions are $\frac{1}{2}(\alpha, -\alpha) + \frac{1}{2}(\alpha, \alpha)$ and $\frac{1}{2}(-\alpha, \alpha) + \frac{1}{2}(\alpha, \alpha)$. Since $i\mathfrak{b}$ comes before $i\mathfrak{s}$, according to the hypotheses of Theorem 9.70, the sign of $(\alpha, 0)$ is determined by $\frac{1}{2}(\alpha, -\alpha)$. Thus $(\alpha, 0) > 0$ implies $(0, -\alpha) > 0$. Consequently Δ_U^+ is determined by a choice of Δ_G^+ and is given by

$$\Delta_U^+ = \{(\alpha, 0) \mid \alpha \in \Delta_G^+\} \cup \{(0, -\alpha) \mid \alpha \in \Delta_G^+\}.$$

Dominance for (λ_1, λ_2) therefore means that $\langle \lambda_1, \alpha \rangle \geq 0$ and $\langle \lambda_2, \alpha \rangle \leq 0$ for all $\alpha \in \Delta_G^+$. That is, λ_1 and $-\lambda_2$ are to be dominant for Δ_G^+. We know from §4 that every irreducible representation of $G \times G$ is an outer tensor product; suppose that the irreducible representation of U with highest weight (λ_1, λ_2) is the outer tensor product $\tau \widehat{\otimes} \tau'$. Then τ is just τ_{λ_1} up to equivalence, but τ' has *lowest* weight λ_2. So τ' is an irreducible representation whose contragredient has highest weight $-\lambda_2$. In other words, $\tau'^c = \tau_{-\lambda_2}$ and $\tau' = \tau^c_{-\lambda_2}$, up to equivalence. Thus the irreducible representation of U with highest weight (λ_1, λ_2) is equivalent with $\tau_{\lambda_1} \widehat{\otimes} \tau^c_{-\lambda_2}$. To understand the content of Theorem 9.70 for this example, we need to identify M. The group M is the subgroup of elements (x, x) in $G \times G$ with $\mathrm{Ad}(x, x)(X, -X) = (X, -X)$ for all X in $\mathfrak{c}$. By Corollary 4.52 an element x of G with $\mathrm{Ad}(x)X = X$ for all X in $\mathfrak{c}$ must itself be in $\exp \mathfrak{c}$, and hence $M = \exp \mathfrak{s}$. The condition of Theorem 9.70 is that (λ_1, λ_2) vanish on $\mathfrak{s}$, hence that $\lambda_1 + \lambda_2 = 0$. Then $-\lambda_2 = \lambda_1$ and $\tau_{\lambda_1} \widehat{\otimes} \tau^c_{-\lambda_2}$ is equivalent with $\tau_{\lambda_1} \widehat{\otimes} \tau^c_{\lambda_1}$.

Theorem 9.70 detects only what tensor products contain the trivial representation. With any of our tools so far—namely the multiplicity formula (9.71), Kostant's Branching Theorem (Theorem 9.20), or even Problem 17 at the end of this chapter—we are left with a great deal of computation to decompose any particular tensor product. For example, if N is the order of the Weyl group of G, then the Kostant formula for checking a multiplicity within a tensor product has N^2 terms.

For particular groups G, there are better methods for decomposing tensor products. Of particular interest is the unitary group $G = U(n)$. Before giving results in that case, we need one general fact.

Proposition 9.72. In a compact connected Lie group G, let λ'' be any highest weight in $\tau_\lambda \otimes \tau_{\lambda'}$, i.e., the highest weight of some irreducible constituent. Then λ'' is of the form $\lambda'' = \lambda + \mu'$ for some weight μ' of $\tau_{\lambda'}$.

PROOF. Write a λ'' highest weight vector in terms of weight vectors of τ_λ and $\tau_{\lambda'}$ as $v = \sum_{\mu+\mu'=\lambda''}(v_\mu \otimes v_{\mu'})$, allowing more than one term per choice of μ, if necessary, and taking the $v_{\mu'}$'s to be linearly independent. Choose $\mu = \mu_0$ as large as possible so that there is a nonzero term $v_\mu \otimes v_{\mu'}$. If E_α is a root vector for a positive root α, then

$$0 = E_\alpha v = \sum_{\mu+\mu'=\lambda''} (E_\alpha v_\mu \otimes v_{\mu'}) + \sum_{\mu+\mu'=\lambda''} (v_\mu \otimes E_\alpha v_{\mu'}).$$

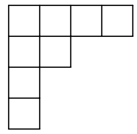
The only way a vector of weight $\mu_0 + \alpha$ can occur in the first member of the tensor products on the right side is from terms $E_\alpha v_{\mu_0} \otimes v_{\mu'}$ with $\mu' = \lambda'' - \mu_0$. Since the corresponding vectors $v_{\mu'}$ are linearly independent, $E_\alpha v_{\mu_0}$ is 0 for each v_{μ_0} that occurs. Therefore any such v_{μ_0} is a highest weight vector for τ_λ. We conclude that $\mu_0 = \lambda$ and that λ'' is of the required form.

Now we examine tensor products when G is the unitary group $U(n)$. It is traditional to study representations of $U(n)$ in a normalized form that can be obtained by multiplying by a suitable power of the 1-dimensional determinant representation: A representation τ of $U(n)$ is a **polynomial representation** if all of its matrix coefficients $x \mapsto (\tau(x)\psi', \psi)$ are polynomial functions of the entries x_{ij}. Equivalently all of the matrix coefficients of the holomorphic extension of τ to $GL(n, \mathbb{C})$ are to be holomorphic polynomials of the entries of the matrix in $GL(n, \mathbb{C})$. This notion is preserved under passage from a representation to an equivalent representation and under direct sums, tensor products, and subrepresentations. Consequently any irreducible constituent of the tensor product of two polynomial representations is again a polynomial representation.

An integral form $\nu = \sum_{j=1}^{n} \nu_j e_j$ for $U(n)$ is **nonnegative** if $\nu_j \geq 0$ for all j. Restricting a polynomial representation to the diagonal matrices, we see that every weight of a polynomial representation is nonnegative. Conversely we can see that any irreducible representation whose highest weight is nonnegative is a polynomial representation. In fact, the standard representation, with highest weight e_1, is a polynomial representation. The usual representation in alternating tensors of rank k lies in the k-fold tensor product of the standard representation with itself and is therefore polynomial; its highest weight is $\sum_{j=1}^{k} e_j$. Finally, if we adopt the convention that $\lambda_{n+1} = 0$, a general highest weight $\lambda = \sum_{j=1}^{n} \lambda_j e_j$ can be rewritten as the sum $\lambda = \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) \sum_{i=1}^{k} e_i$. An irreducible representation with highest weight λ thus lies in a suitable tensor product of alternating-tensor representations and is polynomial.

The classical representation theory for the unitary group deals with irreducible polynomial representations, which we now know are the irreducible representations with nonnegative highest weight or, equivalently, with all weights nonnegative. The restriction that an irreducible representation have nonnegative highest weight is not a serious one, since any irreducible τ is of the form $\tau' \otimes (\det)^{-N}$ with τ' polynomial if the integer N is large enough.

Let τ_λ be an irreducible polynomial representation with highest weight $\lambda = \sum_{j=1}^{n} \lambda_j e_j$. We define the **depth** of τ_λ or λ to be the largest $j \geq 0$ such that $\lambda_j \neq 0$. If λ has depth d, the **parts** of λ are the d positive integers λ_j. To τ_λ or λ, we associate a **diagram**, sometimes called a "Ferrers diagram." This consists of a collection of left-justified rows of boxes: λ_1 in the first row, λ_2 in the second row, $\ldots$, λ_d in the d^{th} row. The integer n is suppressed. For example, the highest weight $4e_1 + 2e_2 + e_3 + e_4$ is associated to the diagram



We shall allow ourselves to replace the boxes in a diagram by various integers, retaining the pattern. Thus if we use 0's in place of boxes above, we obtain

$$\begin{array}{llll} 0 & 0 & 0 & 0 \\ 0 & 0 & & \\ 0 & & & \\ 0 & & & \end{array}$$

as the diagram.

If ν is a nonnegative integral form, we write $\|\nu\|$ for $\langle \nu, e_1 + \cdots + e_n \rangle$. This number is the same for all weights of an irreducible representation. In the example above of a diagram with boxes, the depth is the number of rows, namely 4, and the common value of $\|\nu\|$ is the total number of boxes, namely 8.

Let us suppose that the tensor product of two irreducible polynomial representations τ_μ and τ_ν of $U(n)$ decomposes into irreducible representations as

$$\tau_\mu \otimes \tau_\nu \cong \sum_{\mathrm{depth}(\lambda) \leq n} c^\lambda_{\mu\nu} \tau_\lambda. \tag{9.73}$$

The integers $c^\lambda_{\mu\nu}$, which are ≥ 0, are called **Littlewood-Richardson coefficients**. We shall give without proof a recipe for computing these coefficients that is rapid and involves no cancellation of terms.

Fix μ and ν and suppose that τ_λ actually occurs in $\tau_\mu \otimes \tau_\nu$ in the sense that $c^\lambda_{\mu\nu} \neq 0$. Then λ is nonnegative and $\|\lambda\| = \|\mu\| + \|\nu\|$ because every weight of the tensor product has these properties. A more subtle property of λ is that λ is the sum of μ and a nonnegative integral form (and also the sum of ν and a nonnegative integral form); this follows immediately from Proposition 9.72. In terms of diagrams, this relationship means that the diagram of μ is a subset of the diagram of λ, and we consequently write $\mu \subseteq \lambda$ for this relationship. To find all possible λ's, we may think of enlarging the diagram of μ with $\|\nu\|$ additional boxes or 0's and hoping to determine which enlarged diagrams correspond to λ's that actually occur. Of course, the enlarged diagram needs to correspond to a dominant form, and thus the lengths of its rows are decreasing. But that condition is not enough. The additional data that are needed to describe which λ's actually occur are what we shall call the "symbols" of ν: if $\nu = \sum \nu_j e_j$ has depth d, the **symbols** of ν are ν_1 occurrences of the integer 1, ν_2 occurrences of the integer 2, ..., and ν_d occurrences of the integer d. The diagram of μ is written with 0's in place, and the enlargement is formed by putting the symbols of ν into place in such a way that the diagram of a dominant form results. For example, let $\mu = 4e_1 + 2e_2 + e_3 + e_4$ and $\nu = 3e_1 + e_2 + e_3 + e_4$. The symbols of ν are $\{1, 1, 1, 2, 3, 4\}$. One conceivable enlargement of the diagram of μ is

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & & \\ 0 & 3 & 4 & & & \\ 0 & & & & & \end{matrix}$$

In fact, this particular enlargement will not be an allowable one in the theorem below because it does not satisfy condition (c).

Theorem 9.74 (Littlewood–Richardson). Let τ_μ and τ_ν be irreducible polynomial representations of $U(n)$, and let τ_λ be a polynomial representation of $U(n)$ with $\|\lambda\| = \|\mu\| + \|\nu\|$ and $\mu \subseteq \lambda$. Represent μ by a diagram of 0's, and consider enlargements of that diagram, using the symbols of ν, to diagrams of λ. Then the number $c^\lambda_{\mu\nu}$ of times that τ_λ occurs in $\tau_\mu \otimes \tau_\nu$ equals the number of enlarged diagrams such that

(a) the integers along each row of the enlarged diagram are increasing but not necessarily strictly increasing,
(b) the nonzero integers down each column are strictly increasing, and
(c) the nonzero integers in the enlarged diagram, when read from right to left and row by row starting from the top row, are such that each initial segment never has more of an integer i than an integer j with $1 \le j < i$.

In the enlarged diagram before the statement of the theorem, the sequence of integers addressed by (c) is 112143. This does not satisfy (c) because the initial segment 11214 has more 4's than 3's.

In the theorem if $c^\lambda_{\mu\nu} \neq 0$, then $\nu \subseteq \lambda$ is forced.

EXAMPLE. Tensor product $\tau_\mu \otimes \tau_\nu$ in $U(3)$, where $\mu = \nu = 2e_1 + e_2$. The diagram for μ is $\begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix}$, and the symbols of ν are $\{1, 1, 2\}$. The first symbol of ν that we encounter in (c) has to be a 1, and then no symbol 2 can be placed in the first row, by (a). An enlarged diagram can have at most 3 rows, in order to correspond to a highest weight for $U(3)$. We find 6 enlarged diagrams as follows:

$$\begin{array}{lll} 0\ 0\ 1\ 1 & 0\ 0\ 1\ 1 & 0\ 0\ 1 \\ 0\ 2 & 0 & 0\ 1\ 2 \\ & 2 & \end{array}$$

$$\begin{array}{lll} 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0 \\ 0\ 2 & 0\ 1 & 0\ 1 \\ 1 & 2 & 1\ 2 \end{array}$$

The highest weights of the corresponding irreducible constituents of the tensor product are the dominant forms corresponding to the above 6 diagrams: $4e_1+2e_2$, $4e_1+e_2+e_3$, $3e_1+3e_2$, $3e_1+2e_2+e_3$, $3e_1+2e_2+e_3$, and $2e_1+2e_2+2e_3$. The respective multiplicities equal the number of times that the forms appear in this list. Thus the constituent with highest weight $3e_1+2e_2+e_3$ appears with multiplicity 2, and the four others appear with multiplicity 1. It would be easy to err by omitting one of the diagrams in the above computation, but a check of dimensions will detect an error of this kind if there are no other errors. The given τ_μ has dimension 8, and thus the tensor product has dimension 64. The dimension of each constituent is 27, 10, 10, and 1 in the case of the representations of multiplicity 1, and 8 in the case of the representation of multiplicity 2. We have $27+10+10+1+2(8)=64$, and thus the dimensions check. One final remark is in order. Our computation retained enlarged diagrams only when they had at most 3 rows. For $U(n)$ with $n\geq 4$, we would encounter two additional diagrams, namely

$$\begin{array}{l} 0\ 0 \\ 0\ 1 \\ 1 \\ 2 \end{array} \qquad \text{and} \qquad \begin{array}{l} 0\ 0\ 1 \\ 0 \\ 1 \\ 2 \end{array}$$

These correspond to $2e_1+2e_2+e_3+e_4$ and $3e_1+e_2+e_3+e_4$.

7. Littlewood's Theorems and an Application

We continue our discussion of branching theorems in the context of compact symmetric spaces U/K. The first two theorems are due to D. E.

Littlewood and handle branching for the compact symmetric spaces $U(n)/SO(n)$ and $U(2n)/Sp(n)$, but only under a hypothesis limiting the depth of the given representation of the unitary group. We state these theorems without proof, giving examples for each.

The statements of the theorems involve the Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$ defined in (9.73). In computing these coefficients, we are given λ, μ, and several possibilities for ν; we seek the ν's and the coefficients. These may be computed by changing the emphasis in the method of the previous section. Here is an example: Let $\lambda = 3e_1 + 3e_2$ and $\mu = 2e_1 + e_2$. The formula for μ tells us the diagram of 0's in the earlier method of computation, and the formula for λ tells us the total shape of the diagram. Let us insert the symbol x for the unknown values in the diagram of λ. Then we are to start from

$$\begin{matrix} 0 & 0 & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} \end{matrix}$$

Each possibility for ν gives us a set of symbols. For example, $\nu = 2e_1 + e_2$ gives us the set $\{1, 1, 2\}$, and we can complete the diagram in just one way that is allowed by Theorem 9.74, namely to

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}$$

Thus $c^{\lambda}_{\mu\nu} = 1$ for this ν.

The hypothesis on the depth can be dropped at the expense of introducing something called "Newell's Modification Rules," but we shall not pursue this topic.

Theorem 9.75 (Littlewood). Let τ_λ be an irreducible polynomial representation of $U(n)$ with highest weight λ, and suppose that λ has depth $\leq [n/2]$. Let σ_ν be an irreducible representation of $SO(n)$ with highest weight ν.

(a) If n is odd,

$$\tau_\lambda\big|_{SO(n)} \cong \sum_{\substack{\mu \text{ nonnegative}, \\ \mu \subseteq \lambda, \\ \mu \text{ has even parts}}} \ \sum_{\substack{\nu \text{ nonnegative}, \\ \nu \subseteq \lambda, \\ \|\mu\| + \|\nu\| = \|\lambda\|}} c^{\lambda}_{\mu\nu} \sigma_\nu.$$

(b) If n is even and s denotes the Weyl-group element that changes the last sign, then

$$\tau_\lambda\big|_{SO(n)} \cong \sum_{\substack{\mu \text{ nonnegative}, \\ \mu \subseteq \lambda, \\ \mu \text{ has even parts}}} \Bigg(\sum_{\substack{\nu \text{ nonnegative}, \\ s\nu = \nu \\ \nu \subseteq \lambda, \\ \|\mu\| + \|\nu\| = \|\lambda\|}} c^{\lambda}_{\mu\nu} \sigma_\nu + \sum_{\substack{\nu \text{ nonnegative}, \\ s\nu \neq \nu \\ \nu \subseteq \lambda, \\ \|\mu\| + \|\nu\| = \|\lambda\|}} c^{\lambda}_{\mu\nu} (\sigma_\nu + \sigma_{s\nu}) \Bigg).$$

EXAMPLES.

1) With $n = 4$, let $\lambda = 5e_1 + 2e_2$. We seek the restriction of τ_λ from $U(4)$ to $SO(4)$. We form a list of the nonnegative μ's with even parts such that $\mu \subseteq \lambda$, namely

$$0, \quad 2e_1, \quad 4e_1, \quad 2e_1 + 2e_2, \quad 4e_1 + 2e_2.$$

Each of these tells us a value for $\|\nu\|$, and we list the ν's that must be examined for each μ:

$$\begin{array}{lll}
\mu = 0, & \|\nu\| = 7, & \nu = 5e_1 + 2e_2 \\
\mu = 2e_1, & \|\nu\| = 5, & \nu = 5e_1 \ \text{ or } \ 4e_1 + e_2 \ \text{ or } \ 3e_1 + 2e_2 \\
\mu = 4e_1, & \|\nu\| = 3, & \nu = 3e_1 \ \text{ or } \ 2e_1 + e_2 \\
\mu = 2e_1 + 2e_2, & \|\nu\| = 3, & \nu = 3e_1 \ \text{ or } \ 2e_1 + e_2 \\
\mu = 4e_1 + 2e_2, & \|\nu\| = 1, & \nu = e_1.
\end{array}$$

Then we do the computation with the 0's and x's, seeing how many ways Theorem 9.74 allows for placing the symbols of ν. For a sample let us do $\mu = 4e_1$ and then $\mu = 2e_1 + 2e_2$. First consider $\mu = 4e_1$. The ν's to examine are $3e_1$ and $2e_1 + e_2$, and the diagram to complete is

$$\begin{array}{lllll} 0 & 0 & 0 & 0 & \text{x} \\ \text{x} & \text{x} \end{array}$$

The respective sets of symbols are $\{1, 1, 1\}$ and $\{1, 1, 2\}$. With the first set we can complete the diagram with each x $= 1$, and with the second set we can put the 2 in the second position on the second line. Thus $\mu = 4e_1$ gives us a contribution of one occurrence of each σ_ν. Next consider $\mu = 2e_1 + 2e_2$. We are interested in the same ν's, and the diagram to complete is

$$\begin{array}{lllll} 0 & 0 & \text{x} & \text{x} & \text{x} \\ 0 & 0 \end{array}$$

We can complete the diagram with the symbols $\{1, 1, 1\}$ but not with $\{1, 1, 2\}$. Thus this time we get a contribution from $\nu = 3e_1$ but not from $2e_1 + 2e_2$. A similar computation shows for each of the other three μ's that the diagram can be completed in one allowable way for each ν. We now add the contributions from each ν. The theorem tells us also to include $s\nu$ when the coefficient of e_2 in ν is not 0. Abbreviating $ae_1 + be_2$ as (a, b), we find that the restriction of τ_λ from $U(4)$ to $SO(4)$ is

$$\sigma_{5,0} + 2\sigma_{3,0} + \sigma_{1,0} + \sigma_{5,2} + \sigma_{5,-2} + \sigma_{4,1} + \sigma_{4,-1} + \sigma_{3,2} + \sigma_{3,-2} + \sigma_{2,1} + \sigma_{2,-1}.$$

For a check we can compute the dimension in two ways, verifying that it comes to 224 both times.

2) With $n = 3$, let $\lambda = ae_1$ for some $a \geq 0$. We seek to restrict τ_λ from $U(3)$ to $SO(3)$. The values of μ to consider are $0, 2e_1, 4e_1, \ldots, 2[a/2]e_1$. For each μ, we are to consider just one ν, namely $\lambda - \mu$. The symbols for ν are $\{1, \ldots, 1\}$, and the relevant diagram of 0's and x's can be completed in exactly one allowable way. Thus the restriction of τ_λ to $SO(3)$ is

$$\sigma_{ae_1} + \sigma_{(a-2)e_1} + \cdots + (\sigma_{e_1} \text{ or } \sigma_0).$$

This decomposition has the following interpretation: One realization of τ_λ for $U(3)$ is in the space of homogeneous polynomials of degree a in variables $\bar{z}_1, \bar{z}_2, \bar{z}_3$. The restriction to $SO(3)$ breaks into irreducible representations in a manner described by Problems 9–14 of Chapter IV and Problem 2 of Chapter V.

Theorem 9.76 (Littlewood). Let τ_λ be an irreducible polynomial representation of $U(2n)$ with highest weight λ, and suppose that λ has depth $\leq n$. Let σ_ν be an irreducible representation of $Sp(n)$ with highest weight ν. Then

$$\tau_\lambda\big|_{Sp(n)} \cong \sum_{\substack{\mu \text{ nonnegative,} \\ \mu \subseteq \lambda, \\ \mu \text{ has an even number of} \\ \text{parts of each magnitude}}} \; \sum_{\substack{\nu \text{ nonnegative,} \\ \nu \subseteq \lambda, \\ \|\mu\| + \|\nu\| = \|\lambda\|}} c^\lambda_{\mu\nu} \sigma_\nu.$$

EXAMPLE. For $\lambda = 5e_1 + 2e_2$, we seek the restriction of τ_λ from $U(4)$ to $Sp(2)$. The list of μ's in question is

$$0, \quad e_1 + e_2, \quad 2e_1 + 2e_2;$$

the list includes $e_1 + e_2$, for instance, because $e_1 + e_2$ has 2 parts of magnitude 1 and 0 parts of all other magnitudes. For $\mu = 0$, we are led to $\nu = 5e_1 + 2e_2$ and one way of completing the diagram. For $\mu = e_1 + e_2$, we have $\|\nu\| = 5$, and the ν's to consider are $5e_1$, $4e_1 + e_2$, and $3e_1 + 2e_2$. These have the respective sets of symbols $\{1, 1, 1, 1, 1\}$, $\{1, 1, 1, 1, 2\}$, and $\{1, 1, 1, 2, 2\}$, and the diagram to complete is

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0 x x x x
0 x
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The diagram can be completed in one allowable way in the second case and in no allowable way in the other two cases. Thus we get a contribution to the restriction from $4e_1 + e_2$. For $\mu = 2e_1 + 2e_2$, we have $\|\nu\| = 3$, and the ν's to consider are $3e_1$ and $2e_1 + e_2$. These have the respective sets of symbols {1, 1, 1} and {1, 1, 2}, and the diagram to complete is

$$\begin{matrix} 0 & 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ 0 & 0 & & & \end{matrix}$$

The diagram can be completed in one allowable way in the first case and in no allowable way in the second case. Thus we get a contribution to the restriction from $3e_1$. The conclusion is that the restriction of τ_λ to $Sp(2)$ is

$$\sigma_{5e_1+2e_2} + \sigma_{4e_1+e_2} + \sigma_{3e_1}.$$

The dimensions of these constituents are 140, 64, and 20, and they add to 224, as they must.

Now let us pull together some of the threads of this chapter. We have concentrated on branching theorems for compact symmetric spaces because so many compact homogeneous spaces can be built from symmetric spaces. The example suggested at the end of §4 is $L^2(K/(K \cap M_0))$ whenever G is semisimple, K is the fixed group of a Cartan involution, and MAN is the Langlands decomposition of a maximal parabolic subgroup. For example, consider $G = SO(p,q)_0$ with $p \geq q$, K being $SO(p) \times SO(q)$. One parabolic subgroup has $K \cap M_0 = SO(p-q) \times \operatorname{diag} SO(q)$. If we introduce $K_1 = SO(p-q) \times SO(q) \times SO(q)$, then K/K_1 and $K_1/(K \cap M_0)$ are compact symmetric spaces. To analyze $L^2(K/(K \cap M_0))$, we can use induction in stages, starting from the trivial representation of $K \cap M_0$. We pass to K_1, and the result is the sum of $1 \widehat{\otimes} \sigma \widehat{\otimes} \sigma^c$ over all irreducible representations σ of $SO(q)$. The passage from K_1 to K requires understanding those representations of $SO(p)$ that contain $1 \widehat{\otimes} \sigma$ when restricted to $SO(p-q) \times SO(q)$. These are addressed in the following theorem, which reduces matters to the situation studied in Theorem 9.75 if $p \geq 2q$. Certain maximal parabolic subgroups in other semisimple groups lead to a similar analysis with groups $U(n)$ and $Sp(n)$, and the theorem below has analogs for these groups reducing matters to the situation in Theorem 9.74 or 9.76.

Theorem 9.77. Let $1 \leq n \leq m$, and regard $SO(n)$ and $SO(m)$ as embedded as block diagonal subgroups of $SO(n+m)$ in the standard way with $SO(n)$ in the upper left diagonal block and with $SO(m)$ in the lower right diagonal block.

(a) If $a_1e_1+\cdots+a_{[\frac{1}{2}(n+m)]}e_{[\frac{1}{2}(n+m)]}$ is the highest weight of an irreducible representation (σ, V) of $SO(n+m)$, then a necessary and sufficient condition for the subspace $V^{SO(m)}$ of vectors fixed by $SO(m)$ to be nonzero is that $a_{n+1}=\cdots=a_{[\frac{1}{2}(n+m)]}=0$.

(b) Let $\lambda = a_1e_1+\cdots+a_ne_n$ be the highest weight of an irreducible representation (σ_λ, V) of $SO(n+m)$ with a nonzero subspace $V^{SO(m)}$ of vectors fixed by $SO(m)$, and let $(\tau_{\lambda'}, V')$ be an irreducible representation of $U(n)$ with highest weight $\lambda' = a_1e_1+\cdots+a_{n-1}e_{n-1}+|a_n|e_n$. Then the representation $(\sigma_\lambda\big|_{SO(n)}, V^{SO(m)})$ is equivalent with the restriction to $SO(n)$ of the representation $(\tau_{\lambda'}, V')$ of $U(n)$.

EXAMPLE. Consider branching from $SO(10)$ to $SO(4)\times SO(6)$. If σ is an irreducible representation of $SO(10)$ with highest weight written as $a_1e_1+\cdots+a_5e_5$, then (a) says that the restriction of σ to $SO(4)\times SO(6)$ contains some $\sigma'\widehat{\otimes}1$ if and only if $a_5=0$. In this case, (b) says that the representations σ', with their multiplicities, are determined by restricting from $U(4)$ to $SO(4)$ the irreducible representation of $U(4)$ with highest weight $a_1e_1+\cdots+a_4e_4$. Theorem 9.75 identifies this restriction if $a_3=a_4=0$. For example, if $\lambda = 5e_1+2e_2$ is the given highest weight for $SO(10)$, then Example 1 following that theorem identifies the representations σ' of $SO(4)$, together with their multiplicities, that occur in the restriction of τ_λ from $U(4)$ to $SO(4)$. Then the representations $\sigma'\widehat{\otimes}1$ of $SO(4)\times SO(6)$, with the same multiplicities, are the ones in the restriction of σ from $SO(10)$ to $SO(4)\times SO(6)$ for which the representation on the $SO(6)$ factor is trivial.

SKETCH OF PROOF OF THEOREM. Conclusion (a) is an easy exercise starting from Theorem 9.16. Let us consider (b) under the assumption $a_n\geq 0$. Write $K_1=SO(n)$, $K_2=SO(m)$, and $K=K_1\times K_2$. We introduce the noncompact Riemannian dual of $SO(n+m)/K$, which is isomorphic to $SO(n,m)_0/K$. The isomorphism $\pi_1(SO(n+m),1)\cong\pi_1(SO(n+m)^{\mathbb{C}},1)$ and the unitary trick allow us to extend σ_λ holomorphically to $SO(n+m)^{\mathbb{C}}$ and then to restrict to a representation, which we still call σ_λ, of $SO(n,m)_0$. Form the usual maximally noncompact Cartan subalgebra of the Lie algebra $\mathfrak{so}(n,m)$ of $SO(n,m)_0$ and the usual positive system of roots relative to it that takes the noncompact part $\mathfrak{a}$ before the compact part. The restricted-root system is of type $(BC)_n$ or B_n or D_n, depending on the size of $m-n$.

In all cases the restricted roots of the form e_i-e_j form a subsystem of type A_{n-1} in which each restricted root has multiplicity 1. The associated Lie subalgebra of $\mathfrak{so}(n,m)$, with all of $\mathfrak{a}$ included, is isomorphic to $\mathfrak{gl}(n,\mathbb{R})$. Let $L\cong GL(n,\mathbb{R})_0$ be the corresponding analytic subgroup of $SO(n,m)_0$.

Let $K_L = K \cap L$ be the standard copy of $SO(n)$ inside L. The subgroup K_L is embedded block diagonally as $K_L = \{\mathrm{diag}(k, 1, \pi(k) \mid k \in K_1\}$, where π is some mapping. Projection of K_L to the first factor gives an isomorphism $\iota : K_L \to K_1$.

Let v_0 be a nonzero highest weight vector of σ_λ in the new ordering. The cyclic span of v_0 under L is denoted V', and the restriction of $\sigma_\lambda\big|_L$ to the subspace V' is denoted τ_λ. The representation (τ_λ, V') of L is irreducible. Let E be the projection of V onto V^{K_2} given by $E(v) = \int_{K_2} \sigma_\lambda(k)v\,dk$. If we take the isomorphism $\iota : K_L \to K_1$ into account, then the linear map E is equivariant with respect to K_1. An argument that uses the formula $K = K_2K_L$ and the Iwasawa decomposition in G^d shows that E carries the subspace V' *onto* V^{K_2}.

The group L and the representation (τ_λ, V') are transferred from $SO(n, m)_0$ back to $SO(n + m)$, and the result is a strangely embedded subgroup G' of $SO(n+m)$ isomorphic to $U(n)$, together with an irreducible representation of G' that we still write as (τ_λ, V'). The group K_L, being contained in K, does not move in the passage from $SO(n, m)_0$ back to $SO(n + m)$ and may be regarded as a subgroup of G', embedded in the standard way that $SO(n)$ is embedded in $U(n)$.

Unwinding the highest weights in question and taking care of any possible ambiguities in the above construction that might lead to outer automorphisms of $G' \cong U(n)$, we find that the highest weights match those in the statement of the theorem.

To complete the proof, it suffices to show that the map E of V' onto V^{K_2} is one-one. This is done by proving that $\dim V' = \dim V^{K_2}$. We limit ourselves to proving this equality for one example that will illustrate how the proof goes in general. We take $n = 2$ and $m = 4$, and we write highest weights as tuples. Say the given highest weight of $SO(6)$ is $(2, 1, 0)$. We do $n = 2$ steps of branching via Theorem 9.16 to determine the irreducible constituents under $SO(m) = SO(4)$, and we are interested only in the constituents where $SO(4)$ acts trivially. Branching from $SO(6)$ to $SO(5)$ leads from $(2, 1, 0)$ for $SO(6)$ to $(2, 1) + (2, 0) + (1, 1) + (1, 0)$ for $SO(5)$. The pieces $(2, 1)$ and $(1, 1)$, not ending in 0, do not contain the trivial representation of $SO(4)$, according to conclusion (a) above. For the other two, branching from $SO(5)$ to $SO(4)$ gives

$$(2, 0) \mapsto (2, 0) + (1, 0) + (0, 0)$$
$$(1, 0) \mapsto (1, 0) + (0, 0).$$

Thus we obtain one constituent each time as much as possible of the highest weight becomes 0 at each step, namely twice. So $\dim V^{K_2} = 2$. To compute

$\dim V'$, we start with $(2, 1, 0)$ truncated so as to be a highest weight for $U(n) = U(2)$. That is, we start with $(2, 1)$. We do branching via Theorem 9.14 a step at a time to $U(1)$ and then one more time to arrive at empty tuples. Specifically we pass from $(2, 1)$ to $(2) + (1)$ and then to $() + ()$. The $U(1)$ representations are all 1-dimensional, and hence the number of empty tuples equals the dimension of the representation with highest weight $(2, 1)$. That is, it equals $\dim V'$. The point is that there is a correspondence between the steps with SO leading to $(0, 0)$ and the steps with U leading to $()$. It is given by padding out the tuples for U with a suitable number of 0's. Thus $\dim V' = \dim V^{K_2}$.

8. Problems

1. For $U(n)$, let $\lambda = \sum a_j e_j$ be a dominant integral form, define $\delta' = ne_1 + (n-1)e_2 + \cdots + 1e_n$, and let $t = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Write ξ_ν for the multiplicative character corresponding to an integral linear form ν.
   (a) Show from the Weyl character formula that the character χ_λ of an irreducible representation with highest weight λ is given by
   $$\chi_\lambda(t) = \xi_{-\delta'}(t) \sum_{w \in W} \varepsilon(w) \xi_{w(\lambda+\delta')}(t) \Big/ \prod_{k<l} \big(1 - e^{-i\theta_k + i\theta_l}\big).$$
   at every point t where $\xi_\alpha(t) = 1$ for no root α.
   (b) Show that the formula in (a) can be rewritten as
   $$\chi_\lambda(t) = \xi_{-\delta'}(t) \det\big\{e^{i(a_k+n+1-k)\theta_l}\big\} \Big/ \prod_{k<l} \big(1 - e^{-i\theta_k + i\theta_l}\big).$$
   (c) Derive Theorem 9.14 by carrying out the following manipulations with the determinant in (b): Put $\theta_n = 0$. Replace the first row by the difference of the first and second rows, the second row by the difference of the second and third rows, and so on until the last column is 1 in the n^{th} entry and 0 elsewhere. Reduce the size of the determinant to $n-1$. Divide the factor $\big(1 - e^{-i\theta_l}\big)$ of the product in the denominator into the l^{th} column of the determinant, $1 \le l \le n-1$. Recognize the first row of the determinant as the sum of $a_1 - a_2 + 1$ natural row vectors of exponentials and expand the determinant by linearity. Repeat for the second row of each resulting determinant, using a sum of $a_2 - a_3 + 1$ row vectors. Continue through the $(n-1)^{\text{st}}$ row, and match the answer with the sum of the characters of $U(n-1)$ indicated by Theorem 9.14.
2. In Theorem 9.18, the branching theorem for passing from $Sp(n)$ to $Sp(n-1)$, prove that the number of integer n-tuples $(b_1, \dots, b_n)$ satisfying (9.19) is equal to $\prod_{i=1}^n (A_i + 1)$, where A_i is as in the statement of Theorem 9.50 and A_i is assumed to be ≥ 0 for all i.

3. In §4 identify the set Σ that arises in Kostant's Branching Theorem when passing from $U(2n)$ to $SO(2n)$.

4. Suppose that a permutation w satisfies the condition of Lemma 9.54 that every equality $we_i = e_j$ implies $j \geq i - 1$. Prove that w is a product of certain transpositions of consecutive integers, with the pairs decreasing from left to right. For example, with $n = 3$, show that w is of the form $\big((1) \text{ or } (2\ 3)\big) \times \big((1) \text{ or } (1\ 2)\big)$.

5. Theorem 9.75 shows how certain irreducible representations of $U(n)$ reduce when restricted to $SO(n)$. Starting from the irreducibility of the action of $U(n)$ on each $\bigwedge^l \mathbb{C}^n$, use Theorem 9.75 to derive the conclusions of Problems 8–10 of Chapter V concerning irreducibility and reducibility of the alternating-tensor representations of $SO(n)$.

6. View $Sp(n)$ embedded in $U(2n)$ in the standard way so that its Lie algebra is $\mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(2n)$. Root vectors are given in Example 3 of §II.1.
   (a) Theorem 9.76 shows that the irreducible alternating-tensor representation of $U(6)$ on $\bigwedge^3 \mathbb{C}^6$ decomposes under $Sp(3)$ into exactly two irreducible pieces, with highest weights $e_1 + e_2 + e_3$ and e_1. Show that $e_1 \wedge e_2 \wedge e_3$ and $e_1 \wedge (e_2 \wedge e_5 + e_3 \wedge e_6)$ are respective highest weight vectors.
   (b) For $k \leq n$, use Theorem 9.76 to find the highest weights of the irreducible constituents of $\bigwedge^k \mathbb{C}^{2n}$ under the action of $Sp(n)$. Find a nonzero highest weight vector for each constituent.

Problems 7–10 deal with the construction of many elements in the space of an induced representation. Let H be a closed subgroup of a compact group G, and let σ be a unitary representation of H on a separable Hilbert space V.

7. For each continuous $f : G \to \mathbb{C}$ and v in V, define $I_{f,v} : G \to V$ by
$$(I_{f,v}(x), v')_V = \int_H f(xh)(\sigma(h)v, v')_V \, dh \qquad \text{for } v' \in V.$$
Prove that $I_{f,v}$ is continuous and is a member of the space for $\operatorname{ind}_H^G \sigma$.

8. Prove that the linear span of all the functions $I_{f,v}$ in Problem 7 is dense in the space for $\operatorname{ind}_H^G \sigma$ by showing that the 0 function is the only member of the space for $\operatorname{ind}_H^G \sigma$ that is orthogonal to all the $I_{f,v}$.

9. Assuming that the given Hilbert space V is not 0, prove that the Hilbert space for $\operatorname{ind}_H^G \sigma$ is not 0.

10. Prove that if σ is irreducible, then σ lies in the restriction from G to H of some irreducible representation of G.

Problems 11–14 address in two ways the analysis of L^2 of the sphere S^{4n-1} under the action of $Sp(n)$. In the first way, $Sp(n)$ acts transitively on the unit sphere in the space $\mathbb{H}^n$ of n-dimensional column vectors of quaternions, with isotropy subgroup $Sp(n-1)$ at $(0, \ldots, 0, 1)$. In the second way, the unit sphere is realized as K/M for $Sp(n,1)$. The connection between the two ways results from an action of the group $Sp(1)$ on column vectors by *right* multiplication entry-by-entry by the group of unit quaternions.

11. Using Frobenius reciprocity and Theorem 9.18, prove that $L^2(S^{4n-1})$ decomposes under $Sp(n)$ as a Hilbert-space sum $\sum_{a\geq 0, b\geq 0}(b+1)\tau_{(a+b)e_1+ae_2}$, where τ_λ is an irreducible representation of $Sp(n)$ with highest weight λ.

12. Introduce notation for $Sp(n,1)$ as in the next-to-last paragraph of §4, so that $K \supset K_1 \supset M$. The proof of Theorem 7.66 shows that K/M is the sphere S^{4n-1}. Using Frobenius reciprocity, induction in stages, and Theorem 9.50, prove that $L^2(S^{4n-1})$ decomposes under $K = Sp(n) \times Sp(1)$ as a Hilbert-space sum $\sum_{a\geq 0, b\geq 0}\tau_{(a+b)e_1+ae_2}\widehat{\otimes}\sigma_{be_{n+1}}$, where τ_λ is an irreducible representation of $Sp(n)$ with highest weight λ and σ_μ is an irreducible representation of $Sp(1)$ with highest weight μ.

13. The subspace of $L^2(S^{4n-1})$ in Problem 12 of functions invariant under the unit-quaternion subgroup $Sp(1)$ of K may be regarded as the L^2 functions on quaternionic projective space. What is the decomposition of this subspace under the action of $Sp(n)$?

14. Similarly regard S^{2n-1} both as $U(n)/U(n-1)$ and as K/M for a group we could call $U(n,1)$. What are the decompositions of $L^2(S^{2n-1})$ that are analogous to those in Problems 11 and 12? In analogy with Problem 13, what is the decomposition of L^2 of complex projective space under the action of $U(n)$?

Problems 15–18 deal with decomposing tensor products into irreducible representations. Let G be a compact connected Lie group, fix a maximal abelian subspace of its Lie algebra, and let W be the Weyl group. If λ is a dominant integral form relative to some system of positive roots, let τ_λ be an irreducible representation of G with highest weight λ and let χ_λ be the character of this representation. Denote the multiplicative character corresponding to a linear form ν by ξ_ν.

15. Prove that if all weights of τ_λ have multiplicity one, then each irreducible constituent of $\tau_\lambda \otimes \tau_{\lambda'}$ has multiplicity one.

16. If λ is an integral form and if there exists $w_0 \neq 1$ in W fixing λ, prove that $\sum_{w\in W}\varepsilon(w)\xi_{w\lambda} = 0$.

17. (**Steinberg's Formula**) Let $m_\lambda(\mu)$ be the multiplicity of the weight μ in τ_λ, and define $\operatorname{sgn}\mu$ by

$$\operatorname{sgn}\mu = \begin{cases} 0 & \text{if some } w \neq 1 \text{ in } W \text{ fixes } \mu \\ \varepsilon(w) & \text{otherwise, where } w \text{ is chosen in } W \text{ to make} \\ & w\mu \text{ dominant.} \end{cases}$$

Write the character of τ_λ as $\chi_\lambda = \sum m_\lambda(\lambda'')\xi_{\lambda''}$, write $\chi_{\lambda'}$ as in the Weyl Character Formula, and multiply. With $\mu^\vee$ denoting the result of applying an element of W to μ to obtain something dominant, obtain the formula

$$\chi_\lambda\chi_{\lambda'} = \sum_{\lambda''=\text{weight of }\tau_\lambda} m_\lambda(\lambda'')\operatorname{sgn}(\lambda''+\lambda'+\delta)\chi_{(\lambda''+\lambda'+\delta)^\vee-\delta}.$$

18. Let $-\mu$ be the lowest weight of τ_λ. Deduce from Problem 17 that if $\lambda'-\mu$ is dominant, then $\tau_{\lambda'-\mu}$ occurs in $\tau_\lambda \otimes \tau_{\lambda'}$ with multiplicity one.

Problems 19–21 use Problem 17 to identify a particular constituent of a tensor product of irreducible representations, beyond the one in Problem 18. Let λ and λ' be dominant integral. Let w be in W, and suppose that $\lambda'+w\lambda$ is dominant. The goal is to prove that $\tau_{\lambda'+w\lambda}$ occurs in $\tau_\lambda \otimes \tau_{\lambda'}$ with multiplicity one.

19. Prove that $\lambda'' = w\lambda$ contributes $\chi_{\lambda'+w\lambda}$ to the right side of the formula in Problem 17.

20. To see that there is no other contribution of $\chi_{\lambda'+w\lambda}$, suppose that λ'' contributes. Then $(\lambda'+\delta+\lambda'')^\vee - \delta = \lambda' + w\lambda$. Solve for λ'', compute its length squared, and use the assumed dominance to obtain $|\lambda''|^2 \ge |w\lambda|^2$. Show how to conclude that $\lambda'' = w\lambda$.

21. Complete the proof that $\tau_{\lambda'+w\lambda}$ occurs in $\tau_\lambda \otimes \tau_{\lambda'}$ with multiplicity one.

Problems 22–24 deal with the reduction of tensor products into irreducible representations, comparing Steinberg's Formula in Problem 17 with the appropriate special case of Kostant's Branching Theorem (Theorem 9.20). Let G be a compact connected Lie group, fix a maximal abelian subspace of its Lie algebra, let W_G be the Weyl group of G, fix a positive system Δ_G^+ for the roots, let δ be half the sum of the positive roots, and let τ_ν be an irreducible representation of G with highest weight ν. Let $\mathcal{P}^{\mathrm{wt}}$ be the Kostant partition function defined relative to $\Sigma = \Delta_G^+$.

22. Combining Steinberg's Formula with the formula in Corollary 5.83 for the multiplicity of a weight, show that the multiplicity of τ_μ in $\tau_\lambda \otimes \tau_{\lambda'}$ is
$$\sum_{w\in W_G}\sum_{w'\in W_G} \varepsilon(w)\varepsilon(w')\mathcal{P}^{\mathrm{wt}}(w(\lambda+\delta) - w'(\mu+\delta) + \lambda').$$

23. Using Kostant's Branching Theorem for restriction from $G\times G$ to G, show that the multiplicity of τ_μ in $\tau_\lambda \otimes \tau_{\lambda'}$ is
$$\sum_{w\in W_G}\sum_{w'\in W_G} \varepsilon(w)\varepsilon(w')\mathcal{P}^{\mathrm{wt}}(w(\lambda+\delta) + w'(\lambda'+\delta) - 2\delta - \mu).$$

24. Reconcile the formulas obtained in the previous two problems by using the fact that multiplicities of weights are invariant under the Weyl group.

Problems 25–30 give a combinatorial description, involving no cancellation, for the multiplicity of a weight in an irreducible representation of $U(n)$. For this set of problems, the diagram of a nonnegative dominant integral form will consist of boxes, and each such box will get an integer from 1 to n put into it. The result is a **Young tableau** if (a) the integers in each row are increasing but not necessarily strictly increasing and (b) the integers in each column are strictly increasing. If m_j denotes the number of integers j in a Young tableau, the tuple $(m_1, \dots, m_n)$ will be called the **pattern** of the tableau. Let $\mu = \sum_{j=1}^{n} a_j e_j$ and $\mu' = \sum_{j=1}^{n-1} c_j e_j$ be dominant integral forms. We say μ' **interleaves** μ if (9.15) holds. For $0 \le r \le n-1$, a **branching system** for $U(n)$ of **level** r coming from a dominant integral λ is a set $\{\lambda^{(k)} \mid 0 \le k \le r\}$ such that $\lambda^{(0)} = \lambda$, $\lambda^{(k)}$ is a dominant integral form for $U(n-k)$, and $\lambda^{(k)}$ interleaves $\lambda^{(k-1)}$ for all $k \ge 1$; the **end** of the system is $\lambda^{(r)}$.

25. Let τ_λ and $\tau_{\lambda^{(r)}}$ be irreducible representations of $U(n)$ and $U(n-r)$, respectively, with highest weights λ and $\lambda^{(r)}$. For $0 \le r \le n-1$, prove that the number of branching systems for $U(n)$ of level r coming from λ and having end $\lambda^{(r)}$ equals the multiplicity of $\tau_{\lambda^{(r)}}$ in $\tau_\lambda|_{U(n-r)}$. Conclude that the number of branching systems of level $n-1$ coming from λ equals the degree of τ_λ.

26. Let (τ_λ, V) be an irreducible representation of $U(n)$ whose highest weight λ is nonnegative, and let $\{\lambda^{(k)}\}$ be a branching system of level r coming from λ and ending with $\lambda^{(r)}$. For $0 \le r \le n-1$, prove that there exists a unique decreasing chain of subspaces V_j of V, $0 \le j \le r$, such that V_j is invariant and irreducible under the rank n subgroup $U(n-j) \times U(1) \times \cdots \times U(1)$ with highest weight $\lambda^{(j)} + \sum_{l=n-j+1}^{n} \big(\|\lambda^{(n-l)}\| - \|\lambda^{(n-l+1)}\|\big) e_l$.

27. In Problem 26, prove for $0 \le r \le n-1$ that distinct branching systems $\{\lambda^{(k)}\}$ of level r coming from λ and ending with $\lambda^{(r)}$ yield orthogonal subspaces V_r.

28. Taking $r = n-1$ in Problem 27, show that the result is a spanning orthogonal system of 1-dimensional invariant subspaces under the diagonal subgroup.

29. Let λ be nonnegative dominant integral, let $\{\lambda^{(k)}\}$ be a branching system of level $n-1$ for it, and define $\lambda^{(n)} = \emptyset$. Associate to the system a placement of integers in the diagram of λ as follows: put the integer l in a box if that box is part of the diagram of $\lambda^{(n-l)}$ but not part of the diagram of $\lambda^{(n-l+1)}$, $1 \le l \le n$. Prove that the result is a Young tableau and that the pattern of the tableau is
$$\big(\|\lambda^{(n-1)}\| - \|\lambda^{(n)}\|,\ \|\lambda^{(n-2)}\| - \|\lambda^{(n-1)}\|,\ \dots,\ \|\lambda^{(0)}\| - \|\lambda^{(1)}\|\big).$$

30. Let λ be nonnegative integral dominant for $U(n)$, and let τ_λ be an irreducible representation with highest weight λ. Prove that if $\mu = \sum_{j=1}^{n} m_j e_j$ is an integral form, then the multiplicity of the weight μ in τ_λ equals the number of Young tableaux for the diagram of λ whose pattern is $(m_1, \dots, m_n)$.