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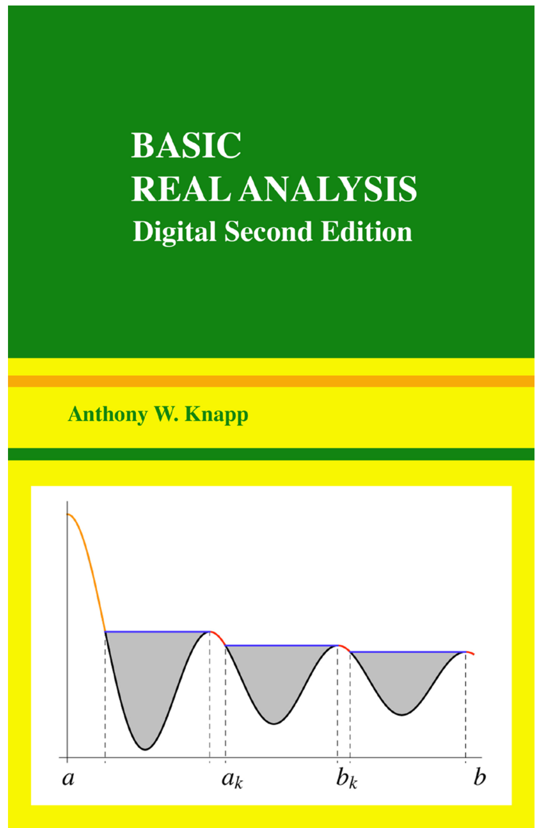

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CHAPTER III

Theory of Calculus in Several Real Variables

Abstract. This chapter gives a rigorous treatment of parts of the calculus of several variables.

Sections 1–3 handle the more elementary parts of the differential calculus. Section 1 introduces an operator norm that makes the space of linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ or from $\mathbb{C}^n$ to $\mathbb{C}^m$ into a metric space. Section 2 goes through the definitions and elementary facts about differentiation in several variables in terms of linear transformations and matrices. The chain rule and Taylor's Theorem with integral remainder are two of the results of the section. Section 3 supplements Section 2 in order to allow vector-valued and complex-valued extensions of all the results.

Sections 4–5 are digressions. The material in these sections uses the techniques of the present chapter but is not needed until later. Section 4 develops the exponential function on complex square matrices and establishes its properties; it will be applied in Chapter IV. Section 5 establishes the existence of partitions of unity in Euclidean space; this result will be applied at the end of Section 10.

Section 6 returns to the development in Section 2 and proves two important theorems about differential calculus. The Inverse Function Theorem gives sufficient conditions under which a differentiable function from an open set in $\mathbb{R}^n$ into $\mathbb{R}^n$ has a locally defined differentiable inverse, and the Implicit Function Theorem gives sufficient conditions for the local solvability of m nonlinear equations in $n+m$ variables for m of the variables in terms of the other n. The Inverse Function Theorem is proved on its own, and the Implicit Function Theorem is derived from it.

Sections 7–10 treat Riemann integration in several variables. Elementary properties analogous to those in the one-variable case are in Section 7, a useful necessary and sufficient condition for Riemann integrability is established in Section 8, Fubini's Theorem for interchanging the order of integration is in Section 9, and a preliminary change-of-variables theorem for multiple integrals is in Section 10.

Sections 11–13 give a careful treatment of integrals of scalar-valued and vector-valued functions on simple arcs and other curves in $\mathbb{R}^n$. The main theorem, proved in Section 13, is Green's Theorem for the plane, which for a suitably nice region of $\mathbb{R}^2$ relates a line integral over the boundary to a double integral over the region. Section 13 concludes with some remarks about higher-dimensional generalizations.

1. Operator Norm

This section works with linear functions from n-dimensional column-vector space to m-dimensional column-vector space. It will have applications within this chapter both when the scalars are real and when the scalars are complex. To be neutral let us therefore write $\mathbb{F}$ for $\mathbb{R}$ or $\mathbb{C}$. Material on the correspondence between linear functions and matrices may be found in Section A7 of Appendix A.

Specifically for $m > 0$ and $n > 0$, let $L(\mathbb{F}^n, \mathbb{F}^m)$ be the vector space of all linear functions from $\mathbb{F}^n$ into $\mathbb{F}^m$. This space corresponds to the vector space of m-by-n matrices with entries in $\mathbb{F}$, as follows: In the notation in Section A7 of Appendix A, we let $(e_1, \dots, e_n)$ be the standard ordered basis of $\mathbb{F}^n$, and $(u_1, \dots, u_m)$ the standard ordered basis of $\mathbb{F}^m$. We define a dot product in $\mathbb{F}^m$ by

$$(a_1, \dots, a_m) \cdot (b_1, \dots, b_m) = a_1 b_1 + \cdots + a_m b_m$$

with no complex conjugations involved. The correspondence of a linear function T in $L(\mathbb{F}^n, \mathbb{F}^m)$ to a matrix A with entries in $\mathbb{F}$ is then given by $A_{ij} = T(e_j) \cdot u_i$.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{F}^n$ or $\mathbb{F}^m$, given as in Section II.1 by the square root of the sum of the absolute values squared of the entries. The Euclidean norm makes $\mathbb{F}^n$ and $\mathbb{F}^m$ into metric spaces, the distance between two points being the Euclidean norm of the difference.

Proposition 3.1. If T is a member of the space $L(\mathbb{F}^n, \mathbb{F}^m)$ of linear functions from $\mathbb{F}^n$ to $\mathbb{F}^m$, then there exists a finite M such that $|T(x)| \le M|x|$ for all x in $\mathbb{F}^n$. Consequently T is uniformly continuous on $\mathbb{F}^n$.

PROOF. Each x in $\mathbb{F}^n$ has $x = \sum_{j=1}^n (x \cdot e_j) e_j$, and linearity gives $T(x) = \sum_{j=1}^n (x \cdot e_j) T(e_j)$. Thus

$$|T(x)| = \Big| \sum_{j=1}^n (x \cdot e_j) T(e_j) \Big| \le \sum_{j=1}^n |T(e_j)| |x \cdot e_j|.$$

The expression $x \cdot e_j$ is just the j^{th} entry of x, and hence $|x \cdot e_j| \le |x|$. Therefore $|T(x)| \le \big(\sum_{j=1}^n |T(e_j)| \big) |x|$, and the first conclusion has been proved with $M = \sum_{j=1}^n |T(e_j)|$. Replacing x by $x - y$ gives

$$|T(x) - T(y)| = |T(x-y)| \le M|x - y|,$$

and uniform continuity of T follows with $\delta = \epsilon / M$. $\square$

Let T be in $L(\mathbb{F}^n, \mathbb{F}^m)$. Using Proposition 3.1, we define the **operator norm** $\|T\|$ of T to be the nonnegative number

$$\|T\| = \inf_{M \ge 0} \big\{ M \bigm| |T(x)| \le M|x| \text{ for all } x \in \mathbb{F}^n \big\}.$$

Then $$|T(x)| \le \|T\|\,|x| \qquad \text{for all } x \in \mathbb{F}^n.$$

Since $|T(cx)| = |c||T(x)|$ for any scalar c, the inequality $|T(x)| \le M|x|$ holds for all $x \ne 0$ if and only if it holds for all x with $0 < |x| \le 1$, if and only if it holds for all x with $|x| = 1$. Also, we have $T(0) = 0$. It follows that two other expressions for $\|T\|$ are

$$\|T\| = \sup_{|x|\le 1} |T(x)| = \sup_{|x|=1} |T(x)|.$$

Proposition 3.2. The operator norm on $L(\mathbb{F}^n, \mathbb{F}^m)$ satisfies

(a) $\|T\| \ge 0$ with equality if and only if $T = 0$,
(b) $\|cT\| = |c|\,\|T\|$ for c in $\mathbb{F}$,
(c) $\|T + S\| \le \|T\| + \|S\|$,
(d) $\|TS\| \le \|T\|\|S\|$ if S is in $L(\mathbb{F}^n, \mathbb{F}^m)$ and T is in $L(\mathbb{F}^m, \mathbb{F}^k)$,
(e) $\|1\| = 1$ if $n = m$ and 1 denotes the identity function on $\mathbb{F}^n$.

PROOF. All the properties but (d) are immediate. For (d), we have

$$|(TS)(x)| = |T(S(x))| \le \|T\|\,|S(x)| \le \|T\|\|S\|\,|x|.$$

Taking the supremum for $|x| \le 1$ yields $\|TS\| \le \|T\|\|S\|$. $\square$

Corollary 3.3. The space $L(\mathbb{F}^n, \mathbb{F}^m)$ becomes a metric space when a metric d is defined by $d(T, S) = \|T - S\|$.

PROOF. Conclusion (a) of Proposition 3.2 shows that $d(T, S) \ge 0$ with equality if and only if $T = S$, conclusion (b) shows that $d(T, S) = d(S, T)$, and conclusion (c) yields the triangle inequality because substitution of $T = T' - V'$ and $S = V' - U'$ into (c) yields $d(T', U') \le d(T', V') + d(V', U')$. $\square$

Suppose that $\mathbb{F} = \mathbb{C}$. If the matrix A that corresponds to some T in $L(\mathbb{C}^n, \mathbb{C}^m)$ has real entries, we can regard T as a member of $L(\mathbb{R}^n, \mathbb{R}^m)$, as well as a member of $L(\mathbb{C}^n, \mathbb{C}^m)$. Two different definitions of $\|T\|$ are in force. Let us check that they yield the same value for $\|T\|$.

Proposition 3.4. Let T be in $L(\mathbb{C}^n, \mathbb{C}^m)$, and suppose that the vector $T(e_j)$ lies in $\mathbb{R}^m$ for $1 \le j \le n$. Then T carries $\mathbb{R}^n$ into $\mathbb{R}^m$, and $\|T\|$ is consistently defined in the sense that

$$\|T\| = \sup_{x\in\mathbb{R}^n,\ |x|\le 1} |T(x)| = \sup_{z\in\mathbb{C}^n,\ |z|\le 1} |T(z)|.$$

PROOF. The first conclusion follows since T is $\mathbb{R}$ linear. For the second conclusion, let $\|T\|_{\mathbb{R}}$ and $\|T\|_{\mathbb{C}}$ be the middle and right expressions, respectively, in the displayed equation above. Certainly we have $\|T\|_{\mathbb{R}} \le \|T\|_{\mathbb{C}}$. If z is in $\mathbb{C}^n$, write $z = x + iy$ with x and y in $\mathbb{R}^n$. Since $T(x)$ and $T(y)$ are in $\mathbb{R}^n$ and T is $\mathbb{C}$ linear,

$$\begin{aligned}|T(z)|^2 &= |T(x) + iT(y)|^2 = |T(x)|^2 + |T(y)|^2 \\ &\le (\|T\|_{\mathbb{R}}|x|)^2 + (\|T\|_{\mathbb{R}}|y|^2) = \|T\|_{\mathbb{R}}^2(|x|^2 + |y|^2) = \|T\|_{\mathbb{R}}^2|z|^2.\end{aligned}$$

Hence $|T(z)| \le \|T\|_{\mathbb{R}}|z|$, and it follows that $\|T\|_{\mathbb{C}} \le \|T\|_{\mathbb{R}}$. The second conclusion follows. $\square$

We shall encounter limits of linear functions in the metric d given in Corollary 3.3, and it is worth knowing just what these limits mean. For this purpose, let T be in $L(\mathbb{F}^n, \mathbb{F}^m)$, and define the **Hilbert–Schmidt norm** of T to be

$$|T| = \Big(\sum_{j=1}^{n} |T(e_j)|^2\Big)^{1/2}.$$

This quantity has an interpretation in terms of the m-by-n matrix A that is associated to the linear function T by the above formula $A_{ij} = T(e_j) \cdot u_i$. Namely, $|T|$ equals $\big(\sum_{i,j} |A_{ij}|^2\big)^{1/2}$, which is just the Euclidean norm of the matrix A if we think of A as lying in $\mathbb{F}^{nm}$. This correspondence provides the license for using the notation of a Euclidean norm for the Hilbert–Schmidt norm of T. The Hilbert–Schmidt norm has the same three properties as the operator norm that allow us to use it to define a metric:

(i) $|T| \ge 0$ with equality if and only if $T = 0$,
(ii) $|cT| = |c|\,|T|$ for c in $\mathbb{F}$,
(iii) $|T + S| \le |T| + |S|$.

Let us write $d_2(T, S) = |T - S|$ for the associated metric. Parenthetically we might mention that the analogs of (d) and (e) for the Hilbert–Schmidt norm are

(iv) $|TS| \le |T|\,|S|$ if S is in $L(\mathbb{F}^n, \mathbb{F}^m)$ and T is in $L(\mathbb{F}^m, \mathbb{F}^k)$,
(v) $|1| = \sqrt{n}$ if $n = m$ and 1 denotes the identity function on $\mathbb{F}^n$.

We shall have no need for these last two properties, and their proofs are left to be done in Problem 1 at the end of the chapter.

Proposition 3.5. The operator norm and Hilbert–Schmidt norm on $L(\mathbb{F}^n, \mathbb{F}^m)$ are related by

$$\|T\| \le |T| \le \sqrt{n}\,\|T\|.$$

Consequently the associated metrics are related by

$$d \le d_2 \le \sqrt{n}\,d.$$

PROOF. If $|x| \leq 1$, then the triangle inequality and the classical Schwarz inequality of Section A5 give

$$|T(x)| = \Big|\sum_{j=1}^{n} (x \cdot e_j) T(e_j)\Big| \leq \sum_{j=1}^{n} |x \cdot e_j|\, |T(e_j)|$$
$$\leq \Big(\sum_{j=1}^{n} |x \cdot e_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{n} |T(e_j)|^2\Big)^{1/2} = |x| \Big(\sum_{j=1}^{n} |T(e_j)|^2\Big)^{1/2} \leq |T|.$$

Taking the supremum over x yields $\|T\| \leq |T|$. In addition,

$$|T|^2 = \sum_{j=1}^{n} |T(e_j)|^2 \leq \sum_{j=1}^{n} \|T\|^2 |e_j|^2 = n\|T\|^2,$$

and the second asserted inequality follows. □

Proposition 3.5 implies that the identity map between the two metric spaces $(L(\mathbb{F}^n, \mathbb{F}^m), d)$ and $(L(\mathbb{F}^n, \mathbb{F}^m), d_2)$ is uniformly continuous and has a uniformly continuous inverse. Therefore open sets, convergent sequences, and even Cauchy sequences are the same in the two metrics. Briefly said, convergence in the operator norm means entry-by-entry convergence of the associated matrices, and similarly for Cauchy sequences.

2. Nonlinear Functions and Differentiation

We begin a discussion of more general functions between Euclidean spaces by defining the multivariable derivative for such a function and giving conditions for its existence. Let E be an open set in $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function. We can write $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$, where $f_i(x) = f(x) \cdot u_i$. Then $f(x) = \sum_{i=1}^{m} f_i(x) u_i$. The functions $f_i : E \to \mathbb{R}$ are called the **components** of f. The associated **partial derivatives** are given by

$$\frac{\partial f_i}{\partial x_j}(x) = \frac{d}{dt} f_i(x + te_j)\Big|_{t=0}.$$

We say that f is **differentiable** at x in E if there is some T in $L(\mathbb{R}^n, \mathbb{R}^m)$ with

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - T(h)|}{|h|} = 0.$$

The linear function T is unique if it exists. In fact, if T_1 and T_2 both serve as T in this limit relation, then we write

$$T_2(h) - T_1(h) = \big(f(x+h) - f(x) - T_1(h)\big) - \big(f(x+h) - f(x) - T_2(h)\big)$$

and find that

$$\begin{aligned}\frac{|T_1(h) - T_2(h)|}{|h|} &\le \frac{|f(x+h) - f(x) - T_1(h)|}{|h|} + \frac{|f(x+h) - f(x) - T_2(h)|}{|h|} \\ &\longrightarrow 0.\end{aligned}$$

If $T_1 \ne T_2$, choose some $v \in \mathbb{R}^n$ with $|v| = 1$ and $T_1(v) \ne T_2(v)$. As a nonzero real parameter t tends to 0, we must have

$$\begin{aligned}&|T_1(v) - T_2(v)| \\ &\quad = |tv|^{-1}\big|\big(f(x+tv) - f(x) - T_1(tv)\big) - \big(f(x+tv) - f(x) - T_2(tv)\big)\big| \\ &\quad \longrightarrow 0.\end{aligned}$$

Since t does not appear on the left side but the right side tends to 0, the result is a contradiction. Thus $T_1 = T_2$, and T is unique in the definition of "differentiable."

If T exists, we write $f'(x)$ for it and call $f'(x)$ the **derivative** of f at x. If f is differentiable at every point x in E, then $x \mapsto f'(x)$ defines a function $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$. We deal with the differentiability of this function presently.

A differentiable function is necessarily continuous. In fact, differentiability at x implies that $|f(x+h) - f(x) - T(h)| \to 0$ as $h \to 0$. Since T is continuous, $T(h) \to 0$ also. Thus $f(x+h) \to f(x)$, and f is continuous at x.

Proposition 3.6. Let E be an open set of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function. If $f'(x)$ exists, then $\dfrac{\partial f_i}{\partial x_j}(x)$ exists for all i and j, and

$$\frac{\partial f_i}{\partial x_j}(x) = f'(x)(e_j) \cdot u_i.$$

REMARKS. In other words, if $f'(x)$ exists at some point x, then it has to be the linear function whose matrix is $\Big[\dfrac{\partial f_i}{\partial x_j}(x)\Big]$. This matrix is called the **Jacobian matrix** of f at x. We shall denote it by $[f'(x)]$.

PROOF. We are given that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - f'(x)(h)|}{|h|} = 0.$$

Dot product with a particular vector is continuous by Proposition 3.1. Take $h = te_j$ with t real in the displayed equation, and form the dot product with u_i. Then we obtain

$$\lim_{t\to 0} \frac{|f_i(x+te_j) - f_i(x) - tf'(x)(e_j)\cdot u_i|}{|t|} = 0.$$

The result follows. □

The natural converse to Proposition 3.6 is false: the first partial derivatives of a function may all exist at a point, and it can still happen that f is discontinuous.

If $f'(x)$ exists at all points of the open set E in $\mathbb{R}^n$, then we obtain a function $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$, and we have seen that we can regard $L(\mathbb{R}^n, \mathbb{R}^m)$ as a Euclidean space by means of the Hilbert–Schmidt norm. Let us examine what continuity of f' means and then what differentiability of f' means.

Theorem 3.7. Let E be an open set of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function. If $f'(x)$ exists for all x in E and $x \mapsto f'(x)$ is continuous at some x_0, then $x \mapsto \dfrac{\partial f_i}{\partial x_j}(x)$ is continuous at x_0 for all i and j. Conversely if each $\dfrac{\partial f_i}{\partial x_j}(x)$ exists at every point of E and is continuous at a point x_0, then $f'(x_0)$ exists. If all $\dfrac{\partial f_i}{\partial x_j}$ are continuous on E, then $x \mapsto f'(x)$ is continuous on E.

PROOF OF DIRECT PART. The partial derivative $\dfrac{\partial f_i}{\partial x_j}(x)$ is one of the entries of $f'(x)$, regarded as a matrix, and has to be continuous if $f'(x)$ is continuous. □

PROOF OF CONVERSE PART. For the moment, let x be fixed. Regard h as $(h_1, \dots, h_n)$, and for $1 \le j \le n$, put $h^{(j)} = (h_1, \dots, h_j, 0, \dots, 0)$. Define T to be the member of $L(\mathbb{R}^n, \mathbb{R}^m)$ with matrix $\Big[\dfrac{\partial f_i}{\partial x_j}(x)\Big]$. Use of the Mean Value Theorem gives

$$\begin{aligned}
[f(x+h) - f(x)]_i &= \sum_{j=1}^n [f(x+h^{(j)}) - f(x+h^{(j-1)})]_i \\
&= \sum_{j=1}^n h_j \frac{d}{dt} f_i(x + h^{(j-1)} + th_je_j)\Big|_{t=t_{ij}} \qquad \text{with } 0 < t_{ij} < 1 \\
&= \sum_{j=1}^n h_j \frac{\partial f_i}{\partial x_j}(x + h^{(j-1)} + t_{ij}h_je_j) \\
&= \sum_{j=1}^n h_j \frac{\partial f_i}{\partial x_j}(x) + \sum_{j=1}^n h_j \Big[\frac{\partial f_i}{\partial x_j}(x + h^{(j-1)} + t_{ij}h_je_j) - \frac{\partial f_i}{\partial x_j}(x)\Big]
\end{aligned}$$

and hence

$$\frac{[f(x+h)-f(x)-T(h)]_i}{|h|} = \sum_{j=1}^{n} \frac{h_j}{|h|}\Big[\frac{\partial f_i}{\partial x_j}(x+h^{(j-1)}+t_{ij}h_je_j)-\frac{\partial f_i}{\partial x_j}(x)\Big].$$

Consequently

$$\frac{|f(x+h)-f(x)-T(h)|}{|h|} \le \sum_{i=1}^{m}\sum_{j=1}^{n}\Big|\frac{\partial f_i}{\partial x_j}(x+h^{(j-1)}+t_{ij}h_je_j)-\frac{\partial f_i}{\partial x_j}(x)\Big|.$$

Let $\epsilon > 0$ be given, and recall that the partial derivatives are assumed to be continuous at x_0. If $\delta > 0$ is chosen such that $|h| < \delta$ implies

$$\Big|\frac{\partial f_i}{\partial x_j}(x_0+h)-\frac{\partial f_i}{\partial x_j}(x_0)\Big| < \frac{\epsilon}{mn},$$

then we see that $|h| < \delta$ implies

$$\frac{|f(x_0+h)-f(x_0)-T(h)|}{|h|} < \epsilon.$$

Thus $f'(x_0)$ exists.

Now assume that all the partial derivatives are continuous on E. Since $L(\mathbb{R}^n, \mathbb{R}^m)$ is identified with $\mathbb{R}^{nm}$, the continuity of the entries $\dfrac{\partial f_i}{\partial x_j}(x)$ of the matrix of $[f'(x)]$ of $f'(x)$ implies the continuity of $f'(x)$ itself. This completes the proof. $\square$

If $x \mapsto f'(x)$ is continuous on E, we say that f is of **class** C^1 on E or is a C^1 **function** on E. Let us iterate the above construction: Suppose that E is open in $\mathbb{R}^n$ and that $f : E \to \mathbb{R}^m$ is of class C^1, so that $x \mapsto f'(x)$ is continuous from E into $L(\mathbb{R}^n, \mathbb{R}^m)$. We introduce second partial derivatives of f and the derivative of f'. Namely, define

$$\frac{\partial^2 f_i}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_k}\Big(\frac{\partial f_i}{\partial x_j}\Big).$$

Since the entries of the matrix of $f'(x)$ are $\dfrac{\partial f_i}{\partial x_j}(x) = f'(x)e_j \cdot u_i$, the expression $\dfrac{\partial^2 f_i}{\partial x_k \partial x_j}$ is the partial derivative with respect to x_k of an entry of the matrix of $f'(x)$. Thus we can say that f is of **class** C^2 from E into $\mathbb{R}^m$ if $f'(x)$ is of class C^1, and so on. We say that f is of **class** C^∞ or is a C^∞ **function** if it is of class C^k for all k. A C^∞ function is also said to be **smooth**.[1] We write $C^k(E)$ and $C^\infty(E)$ for the sets of C^k functions and C^∞ functions on E.

[1] *Warning:* Many authors use the word "smooth" in the context of curves to mean something less than C^∞, but we shall be careful to avoid this practice. The curves in question arise in Sections 11–13 and also in Appendix B.

Corollary 3.8. Let E be an open set of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function. The function f is of class C^k on E if and only if all l^{th}-order partial derivatives of each f_i exist and are continuous on E for $l \leq k$.

This is immediate from Theorem 3.7 and the intervening definitions. The definition of a second partial derivative was given in a careful way that stresses the order in which the partial derivatives are to be computed. Reversing the order of two partial derivatives is a problem involving an interchange of limits. In addressing sufficient conditions for this interchange to be valid, it is enough to consider a function of two variables, since $n-2$ variables will remain fixed when we consider a mixed second partial derivative. The different components of the function do not interfere with each other for these purposes, and thus we may assume that the range is $\mathbb{R}^1$.

Proposition 3.9. Let E be an open set in $\mathbb{R}^2$. Suppose that $f : E \to \mathbb{R}^1$ is a function such that $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist in E and $\dfrac{\partial^2 f}{\partial y \partial x}$ is continuous at $(x, y) = (a, b)$. Then $\dfrac{\partial^2 f}{\partial x \partial y}(a, b)$ exists and equals $\dfrac{\partial^2 f}{\partial y \partial x}(a, b)$.

PROOF. Put

$$\Delta(h, k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk},$$

and let $u(t) = f(t, b+k) - f(t, b)$. The function u is a function of one variable t whose derivative is $\frac{\partial f}{\partial x}(t, b+k) - \frac{\partial f}{\partial x}(t, b)$. Use of the Mean Value Theorem produces ξ between a and $a+h$, as well as η between b and $b+k$, such that

$$\begin{aligned} \Delta(h, k) &= \frac{u(a+h) - u(a)}{hk} = \frac{u'(\xi)}{k} \\ &= \frac{\frac{\partial f}{\partial x}(\xi, b+k) - \frac{\partial f}{\partial x}(\xi, b)}{k} = \frac{\partial^2 f}{\partial y \partial x}(\xi, \eta). \end{aligned} \qquad (*)$$

Let $\epsilon > 0$ be given. By the assumed continuity of $\partial^2 f \big/ \partial y \partial x$ at (a, b), choose $\delta > 0$ such that $|(h, k)| < \delta$ implies

$$\left| \frac{\partial^2 f}{\partial y \partial x}(a+h, b+k) - \frac{\partial^2 f}{\partial y \partial x}(a, b) \right| < \epsilon.$$

Then $(*)$ shows that $|(h, k)| < \delta$ implies

$$\left| \Delta(h, k) - \frac{\partial^2 f}{\partial y \partial x}(a, b) \right| < \epsilon.$$

Letting k tend to 0 shows, for $|h| < \delta/2$, that

$$\Big|\frac{\frac{\partial f}{\partial y}(a+h,b) - \frac{\partial f}{\partial y}(a,b)}{h} - \frac{\partial^2 f}{\partial y \partial x}(a,b)\Big| \le \epsilon.$$

Since ϵ is arbitrary, $\dfrac{\partial^2 f}{\partial x \partial y}(a,b)$ exists and equals $\dfrac{\partial^2 f}{\partial y \partial x}(a,b)$. □

Now that the order of partial derivatives up through order k can be interchanged arbitrarily in the case of a scalar-valued C^k function, we can introduce the usual notation $\dfrac{\partial^k f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$ to indicate the result of differentiating f a total of k times, namely k_1 times with respect to x_1, etc., through k_n times with respect to x_n. Simpler notation will be introduced later to indicate such iterated partial derivatives.

Theorem 3.10 (chain rule). Let E be an open set in $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function differentiable at a point x in E. Suppose that g is a function with range $\mathbb{R}^k$ whose domain contains $f(E)$ and is a neighborhood of $f(x)$. Suppose further that g is differentiable at $f(x)$. Then the composition $g \circ f : E \to \mathbb{R}^k$ is differentiable at x, and $(g \circ f)'(x) = g'(f(x))f'(x)$.

PROOF. With x fixed, define $y = f(x)$, $T = f'(x)$, $S = g'(y)$, and also

$$u(h) = f(x+h) - f(x) - T(h) \quad \text{and} \quad v(k) = g(y+k) - g(y) - S(k).$$

Continuity of f at x and of g at y implies that

$$|u(h)| = \varepsilon(h)|h| \quad \text{and} \quad |v(k)| = \eta(k)|k|$$

with $\varepsilon(h)$ tending to 0 as h tends to 0 and with $\eta(k)$ tending to 0 as k tends to 0.

Given $h \neq 0$, put $k = f(x+h) - f(x)$. Then

$$|k| = |T(h) + u(h)| \le [\|T\| + \varepsilon(h)]|h| \tag{$*$}$$

and

$$\begin{aligned} g(f(x+h)) - g(f(x)) - (ST)(h) &= g(y+k) - g(y) - S(T(h)) \\ &= v(k) + S(k) - S(T(h)) \\ &= S(k - T(h)) + v(k) \\ &= S(u(h)) + v(k). \end{aligned}$$

Therefore

$$\begin{aligned} |h|^{-1}|g(f(x+h)) - g(f(x)) - (ST)(h)| &\le \|S\|\,|u(h)|/|h| + |v(k)|/|h| \\ &\le \|S\|\,\varepsilon(h) + \eta(k)|k|/|h| \\ &\le \|S\|\,\varepsilon(h) + \eta(k)[\|T\| + \varepsilon(h)], \end{aligned}$$

the last inequality following from the upper bound obtained in ($*$) for $|k|$. As h tends to 0, k tends to 0, by that same bound. Thus $\varepsilon(h)$ and $\eta(k)$ tend to 0. The theorem follows. □

Let us clarify in the context of a simple example how the notation in Theorem 3.10 corresponds to the traditional notation for the chain rule. Let f and g be given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad \text{and} \qquad z = g\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2.$$

In traditional notation one of the partial derivatives of the composite function is computed by starting from

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2x\cos\theta - 2y\sin\theta$$

and then substituting for x and y in terms of r and θ. In notation closer to that of the theorem, we replace derivatives by Jacobian matrices and obtain

$$\begin{aligned}\begin{pmatrix} \dfrac{\partial(g\circ f)}{\partial r} & \dfrac{\partial(g\circ f)}{\partial\theta} \end{pmatrix} &= \begin{pmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}\bigg|_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = f\left(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\right)} \begin{pmatrix} \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial\theta} \\ \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial\theta} \end{pmatrix} \\ &= \begin{pmatrix} 2x & -2y \end{pmatrix}\bigg|_{\substack{x = r\cos\theta, \\ y = r\sin\theta}} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.\end{aligned}$$

The formula above for $\partial z/\partial r$ is just the first entry of this matrix equation.

The chain rule in several variables is a much more powerful result than its one-variable prototype, permitting one to handle differentiations when a particular variable occurs in several different ways within a function. For example, consider the rule for differentiating a product in one-variable calculus. The function $x \mapsto f(x)g(x)$ can be regarded as a composition if we recognize that one of the ingredients is the multiplication function from $\mathbb{R}^2$ to $\mathbb{R}^1$. Thus let $u = f(x)$ and $v = g(x)$. If we define $F(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ and $G\begin{pmatrix} u \\ v \end{pmatrix} = uv$, then $(G\circ F)(x) = f(x)g(x)$. Theorem 3.10 therefore gives

$$\begin{aligned}\frac{d}{dx}(G\circ F)(x) &= \begin{pmatrix} \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v} \end{pmatrix} \begin{pmatrix} f'(x) \\ g'(x) \end{pmatrix} = \begin{pmatrix} v & u \end{pmatrix}\bigg|_{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = F(x)} \begin{pmatrix} f'(x) \\ g'(x) \end{pmatrix} \\ &= \begin{pmatrix} g(x) & f(x) \end{pmatrix} \begin{pmatrix} f'(x) \\ g'(x) \end{pmatrix} = g(x)f'(x) + f(x)g'(x).\end{aligned}$$

Theorem 3.11 (Taylor's Theorem). Let N be an integer ≥ 0, and let E be an open set in $\mathbb{R}^n$. Suppose that $F : E \to \mathbb{R}^1$ is a function of class C^{N+1} on E and that the line segment from $x = (x_1, \dots, x_n)$ to $x + h$, where $h = (h_1, \dots, h_n)$, lies in E. Then

$$F(x+h) = F(x) + \sum_{K=1}^{N} \sum_{\substack{k_1+\cdots+k_n=K,\\ \text{all } k_j \geq 0}} (k_1! \cdots k_n!)^{-1} \frac{\partial^K F(x)}{\partial x_1^{k_1} \cdots x_n^{k_n}}\, h_1^{k_1} \cdots h_n^{k_n}$$
$$+ \sum_{\substack{l_1+\cdots+l_n=N+1,\\ \text{all } l_j \geq 0}} \frac{N+1}{l_1! \cdots l_n!}\, h_1^{l_1} \cdots h_n^{l_n} \int_0^1 (1-s)^N \frac{\partial^{N+1} F(x+sh)}{\partial x_1^{l_1} \cdots x_n^{l_n}}\, ds.$$

PROOF. Define a function f of one variable by $f(t) = F(x + th)$. Taylor's Theorem in one variable (Theorem 1.36) gives

$$f(t) = f(0) + \sum_{K=1}^{N} (K!)^{-1} f^{(K)}(0)\, t^K + \frac{1}{N!} \int_0^t (t-s)^N f^{(N+1)}(s)\, ds,$$

and we put $t = 1$ in this formula. If $g(t) = G(x + th)$, the function g is the composition of $t \mapsto x + th$ followed by G, and the chain rule (Theorem 3.10) allows us to compute its derivative as

$$g'(t) = \begin{pmatrix} \dfrac{\partial G}{\partial x_1} & \cdots & \dfrac{\partial G}{\partial x_n} \end{pmatrix}\Bigg|_{x+th} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \sum_{j=1}^{n} h_j \frac{\partial G}{\partial x_j}(x+th).$$

Taking G equal to any of various iterated partial derivatives of F and doing an easy induction, we obtain

$$f^{(K)}(s) = \sum_{\substack{k_1+\cdots+k_n=K,\\ \text{all } k_j \geq 0}} \binom{K}{k_1, \dots, k_n} h_1^{k_1} \cdots h_n^{k_n} \frac{\partial^K F(x+sh)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

where $\binom{K}{k_1,\dots,k_n}$ is the multinomial coefficient $\frac{K!}{(k_1)! \cdots (k_n)!}$. Substitution of this expression into the one-variable expansion with $t = 1$ yields the theorem. $\square$

3. Vector-Valued Partial Derivatives and Riemann Integrals

It is useful to extend the results of Section 2 so that they become valid for functions $f : E \to \mathbb{C}^m$, where E is an open set in $\mathbb{R}^n$. Up to the chain rule in Theorem

3.10, these extensions are consequences of what has been proved in Section 2 if we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$. Achieving the extensions by this identification is preferable to trying to modify the original proofs because of the use of the Mean Value Theorem in the proofs of Theorem 3.7 and Proposition 3.9.

The chain rule extends in the same fashion, once we specify what kinds of functions are to be involved in the composition. We always want the domain to be a subset of some $\mathbb{R}^l$, and thus in a composition $g \circ f$, we can allow g to have values in some $\mathbb{C}^k$, but we insist as in Theorem 3.10 that f have values in $\mathbb{R}^m$.

Now let us turn our attention to Taylor's Theorem as in Theorem 3.11. The statement of Theorem 3.11 allows $\mathbb{R}^1$ as range but not a general $\mathbb{R}^m$. Thus the above extension procedure is not immediately applicable. However, if we allow the given F to take values in $\mathbb{R}^m$, a vector-valued version of Taylor's Theorem will be valid if we adapt our definitions so that the formula remains true component by component. For this purpose we need to enlarge two definitions—that of partial derivatives of any order and that of 1-dimensional Riemann integration—so that both can operate on vector-valued functions. There is no difficulty in doing so, and we may take it that our definitions have been extended in this way.

In the case of vector-valued partial derivatives, let $f : E \to \mathbb{R}^m$ be given. Then $\dfrac{\partial f}{\partial x_j}$ is now defined without passing to components. The entries of this vector-valued partial derivative are exactly the entries of the j^{th} column of the Jacobian matrix of f. Thus the Jacobian matrix consists of the various vector-valued partial derivatives of f, lined up as the columns of the matrix.

Riemann integration is being extended so that the integrand can have values in $\mathbb{R}^m$ or $\mathbb{C}^m$, rather than just $\mathbb{R}^1$. Among the expected properties of the extended version of the Riemann integral, one inequality needs proof because it involves interactions among the various components of the function, namely

$$\Big| \int_a^b F(t)\,dt \Big| \le \int_a^b |F(t)|\,dt.$$

The Riemann integral on the left side is that of a vector-valued function, while the one on the right side is that of a real-valued function. To prove this inequality, let $(\,\cdot\,,\,\cdot\,)$ be the usual inner product for the range space—the dot product if the range is Euclidean space $\mathbb{R}^m$ or the usual Hermitian inner product as in Section II.1 if the range is complex Euclidean space $\mathbb{C}^m$. If u is any vector in the range space with $|u| = 1$, then linearity gives

$$\Big(\int_a^b F(t)\,dt,\ u \Big) = \int_a^b (F(t), u)\,dt.$$

Hence

$$\Big| \Big(\int_a^b F(t)\,dt,\ u \Big) \Big| = \Big| \int_a^b (F(t), u)\,dt \Big| \le \int_a^b |(F(t), u)|\,dt \le \int_a^b |F(t)|\,dt,$$

the two inequalities following from the known scalar-valued version of our inequality and from the Schwarz inequality. If $\int_a^b F(t)\,dt$ is the 0 vector, then our desired inequality is trivial. Otherwise, we specialize the above computation to $u = \left|\int_a^b F(t)\,dt\right|^{-1} \int_a^b F(t)\,dt$, and we obtain our desired inequality.

4. Exponential of a Matrix

In Chapter IV, we shall make use of the **exponential of a matrix** in connection with ordinary differential equations. If A is an n-by-n complex matrix, then we define

$$\exp A = e^A = \sum_{N=0}^{\infty} \frac{1}{N!} A^N.$$

This definition makes sense, according to the following proposition.

Proposition 3.12. For any n-by-n complex matrix A, e^A is given by a convergent series entry by entry. Moreover, the series $X \mapsto e^X$ and every partial derivative of an entry of it is uniformly convergent on any bounded subset of matrix space ($= \mathbb{R}^{2n^2}$), and therefore $X \mapsto e^X$ is a C^∞ function.

REMARK. The proof will be tidier if we use derivatives of n-by-n matrix-valued functions. If F and G are two such functions, the same argument as for the usual product rule shows that $\frac{d}{dt}(F(t)G(t)) = F'(t)G(t) + F(t)G'(t)$.

PROOF. Let us define $\|A\|$ for an n-by-n matrix A to be the operator norm of the member of $L(\mathbb{C}^n, \mathbb{C}^n)$ with matrix A. Fix $M \geq 1$. On the set where $\|A\| \leq M$, we have

$$\Big\| \sum_{N=N_1}^{N_2} \frac{1}{N!} A^N \Big\| \leq \sum_{N=N_1}^{N_2} \frac{1}{N!} \|A^N\| \leq \sum_{N=N_1}^{N_2} \frac{1}{N!} \|A\|^N \leq \sum_{N=N_1}^{N_2} \frac{1}{N!} M^N,$$

and the right side tends to 0 as N_1 and N_2 tend to infinity. Hence for $\|A\| \leq M$, the series for e^A is uniformly Cauchy in the metric built from the operator norm and therefore, by Proposition 3.5, uniformly Cauchy in the metric built from the Hilbert–Schmidt norm. Uniformly Cauchy in the latter metric means that the series is uniformly Cauchy entry by entry, and hence it is uniformly convergent.

The matrices that are 1 or i in one entry and 0 in all other entries form a $2n^2$ member basis over $\mathbb{R}$ of the n-by-n complex matrices. Call these matrices by the names E_j, $1 \leq j \leq 2n^2$. To compute the k^{th} partial derivative of A^N in a succession of not necessarily distinct directions $E_1, \ldots, E_k$, we form $\frac{\partial^k}{\partial t_1 \ldots \partial t_k}(A + \sum t_j E_j) \cdots (A + \sum t_j E_j)$ with N factors, evaluated with all $t_j = 0$.

We apply each derivative in turn, using the product rule in the remark. Each differentiation replaces a product of N factors with a sum of N products of N factors. The new factors are each the full expression $A + \sum t_j E_j$ or else a single E_j or else 0, the 0 occurring when the factor to differentiate is some $E_{j'}$. When all k differentiations have been computed, we evaluate the resulting expression at $t_1 = \cdots = t_k = 0$. The result is a sum of N^k terms, and each nonzero term is the product of N factors equal to A or some E_j.

For a factor E_j, Proposition 3.5 gives $\|E_j\| \leq |E_j| = 1 \leq M$. For a factor of A, we have $\|A\| \leq M$. Thus the operator norm of one such product is $\leq M^N$. The operator norm of the sum of all N^k terms for a k^{th}-order partial derivative is therefore $\leq N^k M^N$. Taking into account the coefficient $1/(N!)$ for the original A^N, we see that the operator norm of terms N_1 through N_2 of the term-by-term k-times differentiated series is

$$\leq \sum_{N=N_1}^{N_2} \frac{N^k M^N}{N!}.$$

We see as a consequence that the term-by-term k-times differentiated series obtained from $\sum (N!)^{-1} A^N$ is uniformly convergent entry by entry. By the complex-valued version of Theorem 1.23, applied recursively to handle k^{th} order partial derivatives, we conclude that $\exp A$ is of class C^k and that the partial derivatives can be computed term by term. Since k is arbitrary, the proof is complete. □

Proposition 3.13. The exponential function for matrices satisfies

(a) $e^X e^Y = e^{X+Y}$ if X and Y commute,
(b) e^X is nonsingular,
(c) $\frac{d}{dt}(e^{tX}) = Xe^{tX}$,
(d) $e^{W^{-1}XW} = W^{-1}e^X W$ if W is nonsingular,
(e) $\det e^X = e^{\operatorname{Tr} X}$, where the trace $\operatorname{Tr} X$ is the sum of the diagonal entries of X.

REMARKS. The conclusion of (a) fails for general X and Y, as one sees by taking $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Relevant properties of the determinant function det that appears in the statement of (e) are summarized in Section A7 of Appendix A.

PROOF. The rate of convergence determined in Proposition 3.12 is good enough to justify the manipulations that follow. For (a), we have

$$\begin{aligned} e^X e^Y &= \Big(\sum_{r=0}^{\infty} \frac{1}{r!}\, X^r\Big)\Big(\sum_{s=0}^{\infty} \frac{1}{s!}\, Y^s\Big) = \sum_{r,s\geq 0} \frac{1}{r!s!}\, X^r Y^s \\ &= \sum_{N=0}^{\infty}\sum_{k=0}^{N} \frac{X^k Y^{N-k}}{k!(N-k)!} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{k=0}^{N} \binom{N}{k} X^k Y^{N-k} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!}(X+Y)^N = e^{X+Y}. \end{aligned}$$

Conclusion (b) follows by taking $Y = -X$ in (a) and using $e^0 = 1$. For (c), we have

$$\begin{aligned} \frac{d}{dt}\big(e^{tX}\big) &= \frac{d}{dt} \sum_{N=0}^{\infty} \frac{1}{N!}\,(tX)^N = \sum_{N=0}^{\infty} \frac{d}{dt}\Big[\frac{1}{N!}\,(tX)^N\Big] \\ &= \sum_{N=0}^{\infty} \frac{N}{N!}\, t^{N-1} X^N = X \sum_{N=1}^{\infty} \frac{1}{(N-1)!}\,(tX)^{N-1} = Xe^{tX}. \end{aligned}$$

Conclusion (d) follows from the computation

$$e^{W^{-1}XW} = \sum_{N=0}^{\infty} \frac{1}{N!}(W^{-1}XW)^N = \sum_{N=0}^{\infty} \frac{1}{N!} W^{-1} X^N W = W^{-1} e^X W.$$

For conclusion (e), define a complex-valued function f of one variable by $f(t) = \det e^{tX}$. By (a), we have

$$\begin{aligned} f'(t) &= \frac{d}{ds} \det e^{(t+s)X}\big|_{s=0} = \frac{d}{ds} \det(e^{tX} e^{sX})\big|_{s=0} = \frac{d}{ds}\,(\det e^{tX})(\det e^{sX})\big|_{s=0} \\ &= (\det e^{tX})\frac{d}{ds}\,(\det e^{sX})\big|_{s=0} = f(t)\frac{d}{ds}\,(\det e^{sX})\big|_{s=0}. \end{aligned}$$

Now $e^{sX} = 1 + sX + \frac{1}{2}s^2X^2 + \cdots = 1 + sX + s^2F(s)$ for some smooth matrix-valued function F with entries F_{ij}. If X has entries X_{ij}, then

$$\begin{aligned} \det e^{sX} &= \det \begin{pmatrix} 1 + sX_{11} + s^2F_{11}(s) & sX_{12} + s^2F_{12}(s) & \cdots \\ sX_{21} + s^2F_{21}(s) & 1 + sX_{22} + s^2F_{22}(s) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &= 1 + s\operatorname{Tr} X + s^2 G(s) \end{aligned}$$

for some smooth function G. Thus $\frac{d}{ds}\,(\det e^{sX})\big|_{s=0} = \operatorname{Tr} X$, and we obtain $f'(t) = (\operatorname{Tr} X) f(t)$ for all t. Consequently

$$\frac{d}{dt}\big(e^{-(\operatorname{Tr} X)t} f(t)\big) = e^{-(\operatorname{Tr} X)t} f'(t) - (\operatorname{Tr} X) e^{-(\operatorname{Tr} X)t} f(t) = 0$$

for all t, and $e^{-(\operatorname{Tr} X)t} f(t)$ is a constant. The constant is seen to be 1 by putting $t = 0$. Therefore $f(t) = e^{(\operatorname{Tr} X)t}$. Conclusion (e) follows by taking $t = 1$. $\square$

5. Partitions of Unity

In Section 10 we shall use a "partition of unity" in proving a change-of-variables formula for multiple integrals. As a general matter in analysis, a partition of unity serves as a tool for localizing analysis problems to a neighborhood of each point. The result we shall use in Section 10 is as follows.

Proposition 3.14. Let K be a compact subset of $\mathbb{R}^n$, and let $\{U_1, \ldots, U_k\}$ be a finite open cover of K. Then there exist continuous functions $\varphi_1, \ldots, \varphi_k$ on $\mathbb{R}^n$ with values in [0, 1] such that

(a) each φ_i is 0 outside of some compact set contained in U_i,
(b) $\sum_{i=1}^k \varphi_i$ is identically 1 on K.

REMARKS. The system $\{\varphi_1, \ldots \varphi_k\}$ is an instance of a "partition of unity." For a general metric space X, a **partition of unity** is a family Φ of continuous functions from X into [0, 1] with sum identically 1 such that for each point x in X, there is a neighborhood of x where only finitely many of the functions are not identically 0. The side condition about neighborhoods ensures that the sum $\sum_{\varphi\in\Phi} \varphi(x)$ has only finitely many nonzero terms at each point and that arbitrary partial sums are well-defined continuous functions on X. If $\mathcal{U}$ is an open cover of X, the partition of unity is said to be **subordinate to the cover** $\mathcal{U}$ if each member of Φ vanishes outside some member of $\mathcal{U}$. Further discussion of partitions of unity beyond the present setting appears in the problems at the end of Chapter X. The use of partitions of unity involving continuous functions tends to be good enough for applications to integration problems, but applications to partial differential equations and smooth manifolds are often aided by partitions of unity involving smooth functions, rather than just continuous functions.[2]

We require a lemma.

Lemma 3.15. In $\mathbb{R}^N$,

(a) if L is a compact set and U is an open set with $L \subseteq U$, then there exists an open set V with V^{cl} compact and $L \subseteq V \subseteq V^{\mathrm{cl}} \subseteq U$,
(b) if K is a compact set and $\{U_1, \ldots, U_n\}$ is a finite open cover of K, then there exists an open cover $\{V_1, \ldots, V_n\}$ of K such that V_i^{cl} is a compact subset of U_i for each i.

[2]Partitions of unity involving smooth functions play no role in the present volume, but they occur in several places in the companion volume, *Advanced Real Analysis*, and their existence is addressed there.

PROOF. In (a), if $L = \varnothing$, we can take $V = \varnothing$. If $L \neq \varnothing$, then the continuous function $x \mapsto D(x, U^c)$ on $\mathbb{R}^N$ is everywhere positive on L since $L \subseteq U$. Corollary 2.39 and the compactness of L show that this function attains a positive minimum c on L. If R is chosen large enough so that $L \subseteq B(R; 0)$ and if we take $V = \left\{x \in U \mid D(x, U^c) > \frac{1}{2}c\right\} \cap B(R; 0)$, then $L \subseteq V$, V^{cl} is compact (being closed and bounded), and $V^{\mathrm{cl}} \subseteq \left\{x \in \mathbb{R}^N \mid D(x, U^c) \geq \frac{1}{2}c\right\} \subseteq U$.

For (b), since $\{U_1, \dots, U_n\}$ is a cover of K, we have $K - (U_2 \cup \cdots \cup U_n) \subseteq U_1$. Part (a) produces an open set V_1 with V_1^{cl} compact such that

$$K - (U_2 \cup \cdots \cup U_n) \subseteq V_1 \subseteq V_1^{\mathrm{cl}} \subseteq U_1.$$

The first inclusion shows that $\{V_1, U_2, \dots, U_n\}$ is an open cover of K. Proceeding inductively, let V_i be an open set with

$$K - (V_1 \cup \cdots \cup V_{i-1} \cup U_{i+1} \cup \cdots \cup U_n) \subseteq V_i \subseteq V_i^{\mathrm{cl}} \subseteq U_i.$$

At each stage, $\{V_1, \dots, V_i, U_{i+1}, \dots, U_n\}$ is an open cover of K, and $V_i^{\mathrm{cl}} \subseteq U_i$. Thus $\{V_1, \dots, V_n\}$ is an open cover of K, and $V_i^{\mathrm{cl}} \subseteq U_i$ for all i. $\square$

PROOF OF PROPOSITION 3.14. Apply Lemma 3.15b to produce an open cover $\{W_1, \dots, W_k\}$ of K such that W_i^{cl} is compact and $W_i^{\mathrm{cl}} \subseteq U_i$ for each i. Then apply it a second time to produce an open cover $\{V_1, \dots, V_k\}$ of K such that V_i^{cl} is compact and $V_i^{\mathrm{cl}} \subseteq W_i$ for each i. Proposition 2.30e produces a continuous function $g_i \geq 0$ that is 1 on V_i^{cl} and is 0 off W_i. Then $g = \sum_{i=1}^n g_i$ is continuous and ≥ 0 on $\mathbb{R}^n$ and is > 0 everywhere on K. A second application of Proposition 2.30e produces a continuous function $h \geq 0$ that is 1 on the set where g is 0 and is 0 on K. Then $g + h$ is everywhere positive on $\mathbb{R}^n$, and the functions $\varphi_i = g_i/(g + h)$ have the required properties. $\square$

6. Inverse and Implicit Function Theorems

The Inverse Function Theorem and the Implicit Function Theorem are results for working with coordinate systems and for defining functions by means of solving equations. Let us use the latter application as a device for getting at the statements of both the theorems.

In the one-variable situation we are given some equation, such as $x^2 + y^2 = a^2$, and we are to think of solving for y in terms of x, choosing one of the possible y's for each x. For example, one solution is $y = -\sqrt{a^2 - x^2}$, $-a < x < a$; unless some requirement like continuity is imposed, there are infinitely many such solutions. In one-variable calculus the terminology is that this solution is "defined implicitly" by the given equation. In terms of functions, the functions $F(x, y) = x^2 + y^2 - a^2$ and $y = f(x) = -\sqrt{a^2 - x^2}$ are such that $F(x, f(x))$ is identically 0. It is then possible to compute dy/dx for this solution in two ways. Only one of these methods remains within the subject of one-variable calculus, namely to compute the "total differential" of $x^2 + y^2 - a^2$, however that is defined, and to set the result equal to 0. One obtains $2x\,dx + 2y\,dy = 0$ with x and y playing symmetric roles. The declaration that x is to be an independent variable and y is to be dependent means that we solve for dy/dx, obtaining $dy/dx = -x/y$. The other way is more transparent conceptually but makes use of multivariable calculus: it uses the chain rule in two-variable calculus to compute d/dx of $F(x, f(x))$ as the derivative of a composition, the result being set equal to 0 because $(d/dx)F(x, f(x))$ is the derivative of the 0 function. This second method gives $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} f'(x) = 0$, with the partial derivatives evaluated where $(x, y) = (x, f(x))$. Then we can solve for $f'(x)$ provided $\partial F\big/\partial y$ is not zero at a point of interest, again obtaining $f'(x) = -x/y$. It is an essential feature of both methods that the answer involves both x and y; the reason is that there is more than one choice of y for some x's, and thus specifying x alone does not determine all possibilities for $f'(x)$.

In the general situation we have m equations in $n + m$ variables. Some n of the variables are regarded as independent, and we think in terms of solving for the other m. An example is

$$\begin{aligned} z^3x + w^2y^3 + 2xy &= 0, \\ xyzw - 1 &= 0, \end{aligned}$$

with x and y regarded as the independent variables.

The classical method of implicit differentiation, which is a version of the first method above, is again to form "total differentials"

$$\begin{aligned} 2wy^3\,dw + 3z^2x\,dz + (z^3 + 2y)\,dx + (3w^2y^2 + 2x)\,dy &= 0, \\ xyz\,dw + xyw\,dz + yzw\,dx + xzw\,dy &= 0, \end{aligned}$$

and then to solve the resulting system of equations for dw and dz in terms of dx and dy. The system is

$$\begin{pmatrix} 2wy^3 & 3z^2x \\ xyz & xyw \end{pmatrix} \begin{pmatrix} dw \\ dz \end{pmatrix} = \begin{pmatrix} -(z^3 + 2y)\,dx - (3w^2y^2 + 2x)\,dy \\ -(yzw)\,dx - (xzw)\,dy \end{pmatrix},$$

and the solution is of the form

$$\begin{aligned} dw &= \text{coefficient}\,dx + \text{coefficient}\,dy, \\ dz &= \text{coefficient}\,dx + \text{coefficient}\,dy. \end{aligned}$$

Here the coefficients are the various partial derivatives of interest. Specifically

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy,$$
$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

The analog of the second method above is to set up matters as a computation of the derivative of a composition. Namely, we write

$$F\begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} z^3x + w^2y^3 + 2xy \\ xyzw - 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} w \\ z \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}.$$

We view the given equations as saying that a composition of

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}$$

followed by F is the 0 function, i.e.,

$$F\begin{pmatrix} x \\ y \\ f\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = 0.$$

We apply the chain rule and compute Jacobian matrices throughout, keeping the variables in the same order x, y, w, z. The Jacobian matrix of the 0 function is a 0 matrix of the appropriate size, and the other side of the differentiated equation is the product of two matrices. Thus

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} z^3 + 2y & 3w^2y^2 + 2x & 2wy^3 & 3z^2x \\ yzw & xzw & xyz & xyw \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} z^3 + 2y & 3w^2y^2 + 2x \\ yzw & xzw \end{pmatrix} + \begin{pmatrix} 2wy^3 & 3z^2x \\ xyz & xyw \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}.\end{aligned}$$

In other words,

$$\begin{pmatrix} 2wy^3 & 3z^2x \\ xyz & xyw \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = -\begin{pmatrix} z^3 + 2y & 3w^2y^2 + 2x \\ yzw & xzw \end{pmatrix},$$

and we have the same system of linear equations as before. Comparing the two methods, we see that we have computed the same things in both methods, merely giving them different names; thus the two methods will lead to the same result in general, not merely in this one example.

The theoretical question is whether the given system of equations, which was $F(x, y, w, z) = 0$ above, can in principle be solved to give a differentiable function; the latter was $\begin{pmatrix} w \\ x \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}$ above. The two computational methods show what the partial derivatives are if the equations can be solved, but these methods by themselves give no information about the theoretical question. The theoretical question is answered by the Implicit Function Theorem, which says that there is no problem if the coefficient matrix of our system of linear equations, namely $\begin{pmatrix} 2wy^3 & 3z^2x \\ xyz & xyw \end{pmatrix}$ in the above example, is invertible at a point of interest.

Theorem 3.16 (Implicit Function Theorem). Suppose that F is a C^1 function from an open set E in $\mathbb{R}^{n+m}$ into $\mathbb{R}^m$ and that $F(a, b) = 0$ for some (a, b) in E, with a understood to be in $\mathbb{R}^n$ and b understood to be in $\mathbb{R}^m$. If the matrix $\left[\dfrac{\partial F_i}{\partial y_j}\right]\Big|_{x=a,\, y=b}$ is invertible, then there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^n$ with (a, b) in U and a in W with this property: to each x in W corresponds a unique y in $\mathbb{R}^m$ such that (x, y) is in U and $F(x, y) = 0$. If this y is defined as $f(x)$, then f is a C^1 function from W into $\mathbb{R}^m$ such that $f(a) = b$, the expression $F(x, f(x))$ is identically 0 for x in W, and the Jacobian matrix of f at x is

$$[f'(x)] = -\left[\frac{\partial F_i}{\partial y_j}\right]^{-1}\left[\frac{\partial F_i}{\partial x_j}\right] \qquad \text{at } (x, y) = (x, f(x)).$$

We shall come to the proof shortly. In the example above, $[f'(x)]$ is the matrix $\begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}$, $\left[\frac{\partial F_i}{\partial y_j}\right]$ is $\begin{pmatrix} 2wy^3 & 3z^2x \\ xyz & xyw \end{pmatrix}$, and $\left[\frac{\partial F_i}{\partial x_j}\right]$ is $\begin{pmatrix} z^3+2y & 3w^2y^2+2x \\ yzw & xzw \end{pmatrix}$.

Let us use the same approach to the question of introducing a new coordinate system in place of an old one. For example, we start with ordinary Euclidean coordinates (u, v) for $\mathbb{R}^2$, and we want to know whether polar coordinates (r, θ) define a legitimate coordinate system in their place. The formula for passing from one system to the other is $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$, but this formula does not really define r and θ. Defining r and θ entails solving for r and θ in terms of u and v. Thus let us set up the system

$$\begin{aligned} r\cos\theta - u &= 0, \\ r\sin\theta - v &= 0. \end{aligned}$$

This is a system of the kind in the Implicit Function Theorem, and the considerations in that theorem apply. The independent vector variable is to be

$x = \binom{u}{v}$, and the dependent vector variable is to be $y = \binom{r}{\theta}$. The system itself is $F(u, v, r, \theta) = 0$, where

$$F(u, v, r, \theta) = \begin{pmatrix} F_1(u, v, r, \theta) \\ F_2(u, v, r, \theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta - u \\ r\sin\theta - v \end{pmatrix}.$$

The sufficient condition for solving the equations locally is that the matrix $\left[\frac{\partial F_i}{\partial y_j}\right]$ be invertible at a point of interest. This is just the matrix

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

The determinant is r, and hence the matrix is invertible except where $r = 0$. The Implicit Function Theorem is therefore telling us in this special case that r and θ give us good *local* coordinates for $\mathbb{R}^2$ except possibly where $r = 0$. The Implicit Function Theorem gives no information about what happens when $r = 0$.

The general result about introducing a new coordinate system in place of an old one is as follows.

Theorem 3.17 (Inverse Function Theorem). Suppose that φ is a C^1 function from an open set E of $\mathbb{R}^n$ into $\mathbb{R}^n$, and suppose that $\varphi'(a)$ is invertible for some a in E. Put $b = \varphi(a)$. Then

(a) there exist open sets $U \subseteq E \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ such that a is in U, b is in V, φ is one-one from U onto V, and
(b) the inverse $f : V \to U$ is of class C^1.

Consequently, $f'(\varphi(x)) = \varphi'(x)^{-1}$ for x in U.

REMARKS. Theorems 3.16 and 3.17 are closely related. We saw in the context of polar coordinates that the Implicit Function Theorem implies the Inverse Function Theorem, and Problem 6 at the end of the chapter points out that this implication is valid in complete generality. Actually, the implication goes both ways, and within this section we shall follow the more standard approach of deriving the Implicit Function Theorem from the Inverse Function Theorem and subsequently proving the Inverse Function Theorem on its own.

PROOF OF THEOREM 3.16 IF THEOREM 3.17 IS KNOWN. Let n, m, E, F, and (a, b) be given as in the statement of Theorem 3.16. We define a function $\varphi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ to which we shall apply Theorem 3.17 in dimension $n + m$. The function is

$$\varphi(x, y) = (x, F(x, y)) \quad \text{for } (x, y) \text{ in } E.$$

This satisfies $\varphi(a, b) = (a, F(a, b)) = (a, 0)$, and its Jacobian matrix at (a, b) is

$$[\varphi'(a, b)] = \begin{pmatrix} \begin{matrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{matrix} & 0 \\ \left[\dfrac{\partial F_i}{\partial x_j}\right]\Big|_{\substack{x=a,\\ y=b}} & \left[\dfrac{\partial F_i}{\partial y_j}\right]\Big|_{\substack{x=a,\\ y=b}} \end{pmatrix}.$$

The upper left block of $[\varphi'(a, b)]$ is the n-by-n identity matrix, and the lower right block is of size m-by-m. Since Theorem 3.16 has assumed that $\left[\dfrac{\partial F_i}{\partial y_j}\right]\Big|_{x=a,\, y=b}$ is invertible, $[\varphi'(a, b)]$ is invertible. Theorem 3.17 therefore applies to φ and produces an open neighborhood W' of $\varphi(a, b) = (a, 0)$ such that φ^{-1} exists on W' and carries W' to an open set. Let $U = \varphi^{-1}(W')$. Define W to be the open neighborhood $W' \cap (\mathbb{R}^n \times \{0\})$ of a in $\mathbb{R}^n$, and define $f(x)$ for x in W by $(x, f(x)) = \varphi^{-1}(x, 0)$. Then f is of class C^1 on W, and $f(a) = b$ because $(a, f(a)) = \varphi^{-1}(a, 0) = (a, b)$. The identity

$$(x, 0) = \varphi(\varphi^{-1}(x, 0)) = \varphi(x, f(x)) = (x, F(x, f(x)))$$

shows that $F(x, f(x)) = 0$ for x in W. The latter equation and the chain rule (Theorem 3.10) give the formula for $[f'(x)]$.

Finally we are to see that $y = f(x)$ is the unique y in $\mathbb{R}^m$ for which (x, y) is in U and $F(x, y) = 0$. Thus suppose that x is in W and that y_1 and y_2 are in $\mathbb{R}^m$ with (x, y_1) and (x, y_2) in U and $F(x, y_1) = F(x, y_2) = 0$. Then we have $\varphi(x, y_1) = (x, F(x, y_1)) = (x, 0) = (x, F(x, y_2)) = \varphi(x, y_2)$. Since (x, y_1) and (x, y_2) are in U, we can apply φ^{-1} to this equation and obtain $(x, y_1) = (x, y_2)$. Therefore $y_1 = y_2$. This completes the proof of Theorem 3.16 if Theorem 3.17 is known. □

Let us turn our attention to a direct proof of the Inverse Function Theorem (Theorem 3.17). When the dimension n is 1, a nonzero derivative at a point yields monotonicity, and the theorem is greatly simplified; this special case is the subject of Section A3 of Appendix A.

For general dimension n, it may be helpful to begin with an outline of the proof. The first step is to show that φ is one-one near the point a in question; this is relatively easy. The hard step is to prove that φ is locally onto some open set; this uses either the compactness of closed balls or else their completeness, and we return to a discussion of this step in a moment. The argument for differentiability of the inverse function depends on the continuity of the inverse function; this dependence was already seen in the 1-dimensional case in Section A3 of Appendix A. Continuity of the inverse function amounts to the fact that small open neighborhoods of a get carried to open sets, and this is part of the proof that φ is locally onto some open set. Finally the chain rule gives $(\varphi^{-1})'(x) = \big(\varphi'(\varphi^{-1}(x))\big)^{-1}$, and the continuity of $(\varphi^{-1})'$ follows. Thus φ^{-1} is of class C^1.

In carrying out the hard step, one has a choice of using either the compactness of closed balls or else their completeness. The argument using completeness lends itself to certain infinite-dimensional generalizations that are well beyond the scope of this book. Since the argument using compactness is the easier one, we shall use that.

The first step and the hard step mentioned above will be carried out in three lemmas below. After them we address the continuity and differentiability of the inverse function, and the proof of the Inverse Function Theorem will be complete.

Lemma 3.18. If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function that is invertible, then there exists a real number $m > 0$ such that $|L(y)| \geq m|y|$ for all y in $\mathbb{R}^n$.

REMARK. We shall apply this lemma in Lemma 3.19 with $L = \varphi'(a)$.

PROOF. The linear inverse function $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and onto. Thus if y is given, there exists x with $y = L^{-1}(x)$, and we have $|y| = |L^{-1}(x)| \leq \|L^{-1}\||x| \leq \|L^{-1}\||L(y)|$. The lemma follows with $m = \|L^{-1}\|^{-1}$. $\square$

Lemma 3.19. In the notation of Theorem 3.17 and Lemma 3.18, choose $m > 0$ such that $|\varphi'(a)(y)| \geq m|y|$ for all $y \in \mathbb{R}^n$, and choose, by continuity of φ', any $\delta > 0$ for which $x \in B(\delta; a)$ implies $\|\varphi'(x) - \varphi'(a)\| \leq \frac{m}{2\sqrt{n}}$. Then $|\varphi(x') - \varphi(x)| \geq \frac{m}{2\sqrt{n}}\,|x' - x|$ whenever x' and x are both in $B(\delta; a)$.

REMARKS. This proves immediately that φ is one-one on $B(\delta; a)$, and it gives an estimate that will establish that φ^{-1} is continuous, once φ^{-1} is known to exist. It proves also that the linear function $\varphi'(x)$ is invertible for $x \in B(\delta; a)$ because

$$\begin{aligned} m|y| &\leq |\varphi'(a)(y)| \\ &\leq |\varphi'(x)(y)| + |\varphi'(x)(y) - \varphi'(a)(y)| \\ &\leq |\varphi'(x)(y)| + \|\varphi'(x) - \varphi'(a)\|\,|y| \\ &\leq |\varphi'(x)(y)| + \frac{m|y|}{2\sqrt{n}}; \end{aligned}$$

if $\varphi'(x)$ were not invertible, then any nonzero y in the kernel of $\varphi'(x)$ would contradict this chain of inequalities.

PROOF. The line segment from x to x' lies within $B(\delta; a)$. Put $z = x' - x$, write this line segment as $t \mapsto x + tz$ for $0 \leq t \leq 1$, and apply the Mean Value

Theorem to each component φ_k of φ to obtain

$$\begin{aligned}\varphi_k(x') - \varphi_k(x) &= \varphi_k(x+tz)\big|_{t=1} - \varphi_k(x+tz)\big|_{t=0} \\ &= \varphi'(x+t_kz)(z)\cdot e_k \qquad \text{with } 0 < t_k < 1 \\ &= \varphi'(a)(z)\cdot e_k + (\varphi'(x+t_kz) - \varphi'(a))(z)\cdot e_k.\end{aligned}$$

Taking the absolute value of both sides allows us to write

$$\begin{aligned}|\varphi(x') - \varphi(x)| &\geq |\varphi_k(x') - \varphi_k(x)| \\ &\geq |\varphi'(a)(z)\cdot e_k| - |(\varphi'(x+t_kz) - \varphi'(a))(z)| \\ &\geq |\varphi'(a)(z)\cdot e_k| - \frac{m}{2\sqrt{n}}\,|x'-x|.\end{aligned}$$

Therefore

$$\begin{aligned}|\varphi(x') - \varphi(x)| &\geq \frac{1}{\sqrt{n}}\,|\varphi'(a)(z)| - \frac{m}{2\sqrt{n}}\,|x'-x| \\ &\geq \frac{m}{\sqrt{n}}\,|x'-x| - \frac{m}{2\sqrt{n}}\,|x'-x| \\ &= \frac{m}{2\sqrt{n}}\,|x'-x|. \qquad \square\end{aligned}$$

Lemma 3.20. With notation as in Lemma 3.19, $\varphi(B(\delta; a))$ is open in $\mathbb{R}^n$.

PROOF. Let $c = m\big/(2\sqrt{n}\,)$ be the constant in the statement of Lemma 3.19. Fix x_0 in $B(\delta; a)$ and let $y_0 = \varphi(x_0)$, so that y_0 is the most general element of $\varphi(B(\delta; a))$. Find $\delta_1 > 0$ such that $B(\delta_1; x_0)^{\mathrm{cl}} \subseteq B(\delta; a)$. It is enough to prove that $B(c\delta_1/2; y_0) \subseteq \varphi(B(\delta; a))$. Even better, we prove that $B(c\delta_1/2; y_0) \subseteq \varphi(B(\delta_1; x_0)^{\mathrm{cl}})$.

Thus let y_1 have $|y_1 - y_0| < c\delta_1/2$. Choose, by compactness of $B(\delta_1; x_0)^{\mathrm{cl}}$, a member $x = x_1$ of $B(\delta_1; x_0)^{\mathrm{cl}}$ for which $|\varphi(x) - y_1|^2$ is minimized. Let us show that x_1 is not on the edge of $B(\delta_1; x_0)^{\mathrm{cl}}$, i.e., that $|x_1 - x_0| < \delta_1$. In fact, if $|x_1 - x_0| = \delta_1$, then Lemma 3.19 gives

$$\begin{aligned}|\varphi(x_1) - y_1| &\geq |\varphi(x_1) - y_0| - |y_1 - y_0| \\ &> |\varphi(x_1) - \varphi(x_0)| - c\delta_1/2 \\ &\geq c|x_1 - x_0| - c\delta_1/2 \\ &= c\delta_1/2 \\ &> |y_1 - y_0| \\ &= |\varphi(x_0) - y_1|,\end{aligned}$$

in contradiction to the fact that $|\varphi(x)-y_1|^2$ is minimized on $B(\delta_1; x_0)^{\mathrm{cl}}$ at $x = x_1$. Thus $|x_1-x_0| < \delta_1$. In this case the scalar-valued function $(\varphi(x)-y_1)\cdot(\varphi(x)-y_1)$ is minimized at an interior point of $B(\delta_1; x_0)^{\mathrm{cl}}$, and all its partial derivatives must be 0. Therefore $\varphi'(x_1)(z)\cdot(\varphi(x_1)-y_1) = 0$ for all z in $\mathbb{R}^n$. Since the linear function $\varphi'(x_1)$ is onto $\mathbb{R}^n$, we conclude that $\varphi(x_1) - y_1 = 0$, and the lemma follows. □

COMPLETION OF PROOF OF THEOREM 3.17. Lemma 3.19 showed that the restriction of φ to $B(\delta; a)$ is one-one, and Lemma 3.20 showed that the image is an open set in $\mathbb{R}^n$. Let $f : \varphi(B(\delta; a)) \to B(\delta; a)$ be the inverse function. To complete the proof of Theorem 3.17, we need to see that f is differentiable on $\varphi(B(\delta; a))$. Fix x in $B(\delta; a)$, and suppose that $x + h$ is in $B(\delta; a)$ with $h \neq 0$. Define y and k by $y = \varphi(x)$ and $y + k = \varphi(x + h)$. Since φ is one-one on $B(\delta; a)$, k is not 0. in fact, Lemma 3.19 gives

$$|k| \geq c|h|, \tag{$*$}$$

where $c = m\big/(2\sqrt{n}\,)$. The definitions give

$$\begin{aligned} f(y+k) - f(y) - \varphi'(x)^{-1}(k) &= (x+h) - x - \varphi'(x)^{-1}(k) \\ &= h - \varphi'(x)^{-1}\big(\varphi(x+h) - \varphi(x)\big) \\ &= -\varphi'(x)^{-1}\big(\varphi(x+h) - \varphi(x) - \varphi'(x)(h)\big). \end{aligned}$$

Combining this identity with $(*)$ gives

$$\frac{|f(y+k) - f(y) - \varphi'(x)^{-1}(k)|}{|k|} \leq \frac{\|\varphi'(x)^{-1}\|}{c}\,\frac{|\varphi(x+h) - \varphi(x) - \varphi'(x)(h)|}{|h|}.$$

If $\epsilon > 0$ is given, choose $\eta > 0$ small enough so that

$$\frac{\|\varphi'(x)^{-1}\|}{c}\,\frac{|\varphi(x+h) - \varphi(x) - \varphi'(x)h|}{|h|} < \epsilon$$

as long as $|h| < \eta$. If $|k| < c\eta$, then $|h| < \eta$ by $(*)$ and hence

$$\frac{|f(y+k) - f(y) - \varphi'(x)^{-1}(k)|}{|k|} < \epsilon.$$

In other words, f is differentiable at y, and $f'(y) = \varphi'(x)^{-1}$. □

Suppose that the given function φ in the Inverse Function Theorem is better than a C^1 function. What can be said about the inverse function f? The answer is carried by the formula $f'(\varphi(x)) = \varphi'(x)^{-1}$ for the derivative of the inverse function f. This formula implies that the partial derivatives of f are quotients of polynomials in partial derivatives of φ by a nonvanishing polynomial (the determinant) in partial derivatives of φ. Thus the iterated partial derivatives of f can be computed harmlessly in terms of the iterated partial derivatives of φ and this same determinant polynomial. Consequently if φ is of class C^k with $k \geq 1$, then so is f. If φ is smooth, so is f. In the case that φ and f are both smooth, we say that φ is a **diffeomorphism**. Let us summarize these facts in a corollary.

Corollary 3.21. Suppose, for some $k \geq 1$, that φ is a C^k function from an open set E of $\mathbb{R}^n$ into $\mathbb{R}^n$, and suppose that $\varphi'(a)$ is invertible for some a in E. Put $b = \varphi(a)$. Let U and V be open subsets of $\mathbb{R}^n$ as in the Inverse Function Theorem such that a is in U, b is in V, and φ is one-one from U onto V. Then the inverse function $f : V \to U$ is of class C^k. If φ is smooth, then φ is a diffeomorphism of U onto V.

7. Definition and Properties of Riemann Integral

Section I.4 contained a careful but limited development of the Riemann integral in one variable. The present section extends that development to several variables. A certain amount of the theory parallels what happened in one variable, and proofs for that part of the theory can be obtained by adjusting the notation and words of Section I.4 in simple ways. Results of that kind are much of the subject matter of this section.

In later sections we shall take up results having no close analog in Section I.4. The main results of this kind are

(i) a necessary and sufficient condition for a function to be Riemann integrable,
(ii) Fubini's Theorem, concerning the relationship between multiple integrals and iterated integrals in the various possible orders,
(iii) a change-of-variables formula for multiple integrals.

We begin a discussion of these in the next section.

The one-variable theory worked with a bounded function $f : [a, b] \to \mathbb{R}$, with domain a closed bounded interval, and we now work with a bounded function $f : A \to \mathbb{R}$ with domain A a "closed rectangle" in $\mathbb{R}^n$. For this purpose a **closed rectangle** (or "closed geometric rectangle") in $\mathbb{R}^n$ is a bounded set of the form

$$A = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

with $a_j \le b_j$ for all j. Let us abbreviate $[a_j, b_j]$ as A_j. In geometric terms the sides or faces are assumed parallel to the axes or coordinate hyperplanes. We shall use the notion of **open rectangle** in later sections and chapters, an open rectangle being a similar product of bounded open intervals (a_j, b_j) for $1 \le j \le n$. However, in this section the term "rectangle" will always mean closed rectangle.

If P_j is a one-variable partition of A_j, then we can form an n-variable **partition** $P = (P_1, \dots, P_n)$ of the given rectangle A into **component rectangles** $[c_1, d_1] \times \cdots \times [c_n, d_n]$, where c_j and d_j are consecutive subdivision points of P_j. A typical component rectangle is denoted by R, and its n-dimensional volume $\prod_{j=1}^n (d_j - c_j)$ is denoted by $|R|$. The **mesh** $\mu(P)$ of the partition P is the maximum of the meshes of the one-dimensional partitions P_j, hence the largest length of a side of all component rectangles of P.

Relative to our given function f and a given partition P, define $M_R(f) = \sup_{x\in R} f(x)$ and $m_R(f) = \inf_{x\in R} f(x)$ for each component rectangle R of P. Put

$$U(P, f) = \sum_R M_R(f)\,|R| = \text{upper Riemann sum for } P,$$

$$L(P, f) = \sum_R m_R(f)\,|R| = \text{lower Riemann sum for } P,$$

$$\overline{\int_A} f\,dx = \inf_P U(P, f) = \text{upper Riemann integral of } f,$$

$$\underline{\int_A} f\,dx = \sup_P L(P, f) = \text{lower Riemann integral of } f.$$

We say that f is **Riemann integrable** on A if $\overline{\int_A} f\,dx = \underline{\int_A} f\,dx$, and in this case we write $\int_A f\,dx$ for the common value of these two numbers. We write $\mathcal{R}(A)$ for the set of Riemann integrable functions on A. The following lemma is proved in the same way as Lemma 1.24.

Lemma 3.22. Suppose that $f : A \to \mathbb{R}$ has $m \le f(x) \le M$ for all x in A. Then for any partition P of A,

$$m|A| \le L(P, f) \le U(P, f) \le M|A|,$$

$$m|A| \le \underline{\int_A} f\,dx \le M|A|,$$

$$m|A| \le \overline{\int_A} f\,dx \le M|A|.$$

A **refinement** of a partition P of A is a partition P^* such that every component rectangle for P^* is a subset of a component rectangle for P. If $P = (P_1, \dots, P_n)$

and $P' = (P'_1, \ldots, P'_n)$ are two partitions of A, then P and P' have at least one common refinement $P^* = (P^*_1, \ldots, P^*_n)$; specifically, for each j, we can take P^*_j to be a common refinement of P_j and P'_j. Arguing as in Lemma 1.25 and Theorem 1.26, we obtain the following two results. The key to the second one of these is the uniform continuity of any continuous function $f : A \to \mathbb{R}$; for the uniform continuity we appeal to the Heine–Borel Theorem (Corollary 2.37) and Proposition 2.41 in several variables, the corresponding one-variable result being Theorem 1.10.

Lemma 3.23. Let $f : A \to \mathbb{R}$ satisfy $m \le f(x) \le M$ for all x in A. Then

(a) $L(P, f) \le L(P^*, f)$ and $U(P^*, f) \le U(P, f)$ whenever P is a partition of A and P^* is a refinement,
(b) $L(P_1, f) \le U(P_2, f)$ whenever P_1 and P_2 are partitions of A,
(c) $\underline{\int}_A f\,dx \le \overline{\int}_A f\,dx$,
(d) $\overline{\int}_A f\,dx - \underline{\int}_A f\,dx \le (M - m)|A|$,
(e) the function f is Riemann integrable on A if and only if for each $\epsilon > 0$, there exists a partition P of A with $U(P, f) - L(P, f) < \epsilon$.

Theorem 3.24. If $f : A \to \mathbb{R}$ is continuous on A, then f is Riemann integrable on A.

Next we argue as in Proposition 1.30 and Theorem 1.31 to obtain two more generalizations to several variables. The several-variable version of uniform continuity is needed in the proof of Proposition 3.25d.

Proposition 3.25. If f_1 and f_2 are Riemann integrable on A, then

(a) $f_1 + f_2$ is in $\mathcal{R}(A)$ and $\int_A (f_1 + f_2)\,dx = \int_A f_1\,dx + \int_A f_2\,dx$,
(b) cf_1 is in $\mathcal{R}(A)$ and $\int_A cf_1\,dx = c\int_A f_1\,dx$ for any real number c,
(c) $f_1 \le f_2$ on A implies $\int_A f_1\,dx \le \int_A f_2\,dx$,
(d) $m \le f_1 \le M$ on A and $\varphi : [m, M] \to \mathbb{R}$ continuous imply that $\varphi \circ f_1$ is in $\mathcal{R}(A)$,
(e) $|f_1|$ is in $\mathcal{R}(A)$, and $\left|\int_A f_1\,dx\right| \le \int_A |f_1|\,dx$,
(f) f_1^2 and $f_1 f_2$ are in $\mathcal{R}(A)$,
(g) $\sqrt{f_1}$ is in $\mathcal{R}(A)$ if $f_1 \ge 0$ on A.

Theorem 3.26. If $\{f_n\}$ is a sequence of Riemann integrable functions on A and if $\{f_n\}$ converges uniformly to f on A, then f is Riemann integrable on A, and $\lim_n \int_A f_n\,dx = \int_A f\,dx$.

There is also a several-variable version of Theorem 1.35, which says that Riemann integrability can be detected by convergence of Riemann sums as the mesh of the partition gets small. Relative to our standard partition $P = (P_1, \ldots, P_n)$, select a member t_R of each component rectangle R relative to P, and define

$$S(P, \{t_R\}, f) = \sum_R f(t_R)|R|.$$

This is called a **Riemann sum** of f.

Theorem 3.27. If f is Riemann integrable on A, then

$$\lim_{\mu(P)\to 0} S(P, \{t_R\}, f) = \int_A f\,dx.$$

Conversely if f is bounded on A and if there exists a real number r such that for any $\epsilon > 0$, there exists some $\delta > 0$ for which $|S(P, \{t_R\}, f) - r| < \epsilon$ whenever $\mu(P) < \delta$, then f is Riemann integrable on A.

REMARK. The proof of the direct part is more subtle in the several-variable case than in the one-variable case, and we therefore include it. The proof of the converse part closely imitates the proof of the converse part of Theorem 1.35, and we omit that.

PROOF. For the direct part the function f is assumed bounded; suppose $|f(x)| \le M$ on A. Let $\epsilon > 0$ be given. Choose a partition $P^* = (P_1^*, \ldots, P_n^*)$ of A with $U(P^*, f) \le \int_A f\,dx + \epsilon$. Fix an integer k such that the number of component intervals of P_j^* is $\le k$ for $1 \le j \le n$. Put

$$\delta_1 = \frac{\epsilon}{Mk \sum_{j=1}^n \prod_{i\ne j} |A_i|},$$

and suppose that $P = (P_1, \ldots, P_n)$ is any partition of $A = A_1 \times \cdots \times A_n$ with $\mu(P) \le \delta_1$. For each j with $1 \le j \le n$, we separate the component intervals of P_j into two kinds, the ones in $\mathcal{F}^{(j)}$ being the component intervals of P_j that do not lie completely within a single component interval of P_j^* and the ones in $\mathcal{G}^{(j)}$ being the rest. Similarly we separate the component rectangles of P into two kinds, the ones in $\mathcal{F}$ being the component rectangles that do not lie completely within a single component rectangle of P^* and the ones in $\mathcal{G}$ being the rest.

If $R = R_1 \times \cdots \times R_n$ is a member of $\mathcal{F}$, then R_j is in $\mathcal{F}^{(j)}$ for some j with $1 \le j \le n$; let $j = j(R)$ be the first such index. Let $\mathcal{F}_j$ be the subset of R's in $\mathcal{F}$ with $j(R) = j$, so that $\mathcal{F} = \bigcup_{j=1}^n \mathcal{F}_j$ disjointly. Then we have

$$U(P, f) = \sum_{j=1}^n \sum_{R\in\mathcal{F}_j} M_R(f)\,|R| + \sum_{R\in\mathcal{G}} M_R(f)\,|R|. \qquad (*)$$

For the first term on the right side,

$$\begin{aligned}\Big|\sum_{j=1}^{n}\sum_{R\in\mathcal{F}_j} M_R(f)\,|R|\Big| &\le M\sum_{j=1}^{n}\sum_{R\in\mathcal{F}_j}|R|\\ &= M\sum_{j=1}^{n}\sum_{R_1\times\cdots\times R_n\in\mathcal{F}_j}|R_1|\times\cdots\times|R_n|\\ &\le M\sum_{j=1}^{n}\sum_{R_j\in\mathcal{F}^{(j)}}|R_j|\textstyle\prod_{i\ne j}|A_i|.\end{aligned}$$

Each member R_j of $\mathcal{F}^{(j)}$ contains some point of the partition P_j^* in its interior, and two distinct R_j's cannot contain the same point. Thus the number of R_j's in $\mathcal{F}^{(j)}$ is $\le k$. Also, $|R_j|\le\mu(P)$. Consequently we have

$$\Big|\sum_{j=1}^{n}\sum_{R\in\mathcal{F}_j} M_R(f)\,|R|\Big| \le Mk\mu(P)\sum_{j=1}^{n}\textstyle\prod_{i\ne j}|A_i| \le Mk\delta_1\sum_{j=1}^{n}\prod_{i\ne j}|A_i| = \epsilon.$$

The contribution to $U(P,f)$ of the second term on the right side of $(*)$ is

$$\sum_{R\in\mathcal{G}} M_R(f)|R| = \sum_{R^*}\sum_{R\subseteq R^*} M_{R^*}(f)|R| \le \sum_{R^*} M_{R^*}(f)|R^*| \le U(P^*,f).$$

Thus

$$U(P,f)\le\epsilon+U(P^*,f)\le\int_A f\,dx+2\epsilon.$$

Similarly we can define δ_2 such that $\mu(P)\le\delta_2$ implies

$$L(P,f)\ge\int_A f\,dx-2\epsilon.$$

If $\delta=\min\{\delta_1,\delta_2\}$ and $\mu(P)\le\delta$, then

$$\int_A f\,dx-2\epsilon\le L(P,f)\le S(P,\{t_R\},f)\le U(P,f)\le\int_A f\,dx+2\epsilon$$

for any choice of points t_R, and hence $\big|S(P,\{t_R\},f)-\int_A f\,dx\big|\le 2\epsilon$. This completes the proof of the direct part of the theorem. $\square$

Finally we include one simple interchange-of-limits result that is handy in working with integrals involving derivatives.

Proposition 3.28. Let f be a complex-valued C^1 function defined on an open set U in $\mathbb{R}^m$, and let K be a compact subset of U. Then

(a) the convergence of $\frac{1}{h}[f(x+he_j)-f(x)]$ to $\frac{\partial f}{\partial x_j}(x)$, as h tends to 0, is uniform for x in K,

(b) the function $g(x_2,\dots,x_n) = \int_a^b f(x_1,\dots,x_n)\,dx_1$ is of class C^1 on the set of all points $y = (x_2,\dots,x_n)$ for which $[a,b]\times\{y\}$ lies in U, and $\frac{\partial}{\partial x_j}\int_a^b f(x)\,dx_1 = \int_a^b \frac{\partial f}{\partial x_j}(x)\,dx_1$ for $j \neq 1$ as long as the set $[a,b]\times\{(x_2,\dots,x_n)\}$ lies in U.

PROOF. In (a), we may assume that f is real-valued. The Mean Value Theorem gives

$$\frac{1}{h}\big[f(x+he_j)-f(x)\big]-\frac{\partial f}{\partial x_j}(x)=\frac{\partial f}{\partial x_j}(x+te_j)-\frac{\partial f}{\partial x_j}(x)$$

for some t between 0 and h, and then (a) follows from the uniform continuity of $\partial f\big/\partial x_j$ on K. Conclusion (b) follows by combining (a) and Theorem 1.31. □

As we did in the one-variable case in Sections 3 and I.5, we can extend our results concerning integration in several variables to functions with values in $\mathbb{R}^m$ or $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Integration of a vector-valued function is defined entry by entry, and then all the results from Theorem 3.24 through Proposition 3.28 extend. The one thing that needs separate proof is the inequality $\big|\int_A f_1\,dx\big| \le \int_A |f_1|\,dx$ of Proposition 3.25e, and a proof can be carried out in the same way as at the end of Section 3 in the one-variable case.

8. Riemann Integrable Functions

Let E be a subset of $\mathbb{R}^n$. We say that E is of **measure 0** if for any $\epsilon > 0$, E can be covered by a finite or countably infinite set of closed rectangles in the sense of Section 7 of total volume less than ϵ. It is equivalent to require that E can be covered by a finite or countably infinite set of open rectangles of total volume less than ϵ. In fact, if a system of open rectangles covers E, then the system of closures covers E and has the same total volume; conversely if a system of closed rectangles covers E, then the system of open rectangles with the same centers and with sides expanded by a factor $1+\delta$ covers E as long as $\delta > 0$.

Several properties of sets of measure 0 are evident: a set consisting of one point is of measure 0, a face of a closed rectangle is a set of measure 0, and any subset of a set of measure 0 is of measure 0. Less evident is the fact that the countable union of sets of measure 0 is of measure 0. In fact, if $\epsilon > 0$ is given and if $E_1, E_2, \ldots$ are sets of measure 0, find finite or countably infinite systems $\mathcal{R}_j$ of closed rectangles for $j \geq 1$ such that the total volume of the members of $\mathcal{R}_j$ is $< \epsilon/2^n$. Then $\mathcal{R} = \bigcup_j \mathcal{R}_j$ is a system of closed rectangles covering $\bigcup_j E_j$ and having total volume $< \epsilon$.

The goal of this section is to prove the following theorem, which gives a useful necessary and sufficient condition for a function of several variables to be Riemann integrable. The theorem immediately extends from the scalar-valued case as stated to the case that f has values in $\mathbb{R}^m$ or $\mathbb{C}^m$.

Theorem 3.29. Let A be a finite closed rectangle in $\mathbb{R}^n$ of positive volume, and let $f : A \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if the set

$$B = \{x \mid f \text{ is not continuous at } x\}$$

has measure 0.

Theorem 3.29 supplies the reassurance that a finite closed rectangle of positive volume cannot have measure 0. In fact, the function f on A that is 1 at every point with all coordinates rational and is 0 elsewhere is discontinuous everywhere on A. By inspection every $U(P, f)$ is $|A|$ for this f, and every $L(P, f)$ is 0; thus f is not Riemann integrable. The theorem then implies that A is not of measure 0.

The proof of the theorem will make use of an auxiliary notion, that of "content 0," in order to simplify the process of checking whether a given compact set has measure 0. A subset E of $\mathbb{R}^n$ has **content 0** if for any $\epsilon > 0$, E can be covered by a *finite* set of closed rectangles in the sense of Section 7 of total volume less than ϵ. It is equivalent to require that E can be covered by a finite set of open rectangles of total volume less than ϵ. A set consisting of one point is of content 0, a face of a closed rectangle is a set of content 0, any subset of a set of content 0 is of content 0, and the union of *finitely* many sets of content 0 is of content 0.

Every set of content 0 is certainly of measure 0, but the question of any converse relationship is more subtle. Consider the set E of rationals in $[0, 1]$ as a subset of $\mathbb{R}^1$. Since this set is a countable union of one-point sets, it has measure 0. However, it does not have content 0. In fact, if we were to have $E \subseteq \bigcup_{n=1}^N [a_j, b_j]$ with $\sum_{n=1}^N (b_j - a_j) < \epsilon$, then we would have $E^{\mathrm{cl}} \subseteq \bigcup_{n=1}^N [a_j, b_j]$ by Proposition 2.10, since $\bigcup_{n=1}^N [a_j, b_j]$ is closed. Then E^{cl} would have content 0 and necessarily measure 0. This contradicts the fact observed after the statement of the theorem—that a closed rectangle of positive volume, such as $E^{\mathrm{cl}} = [0, 1]$ in $\mathbb{R}^1$, cannot have measure 0. We conclude that a bounded set of measure 0 need not have content 0.

Lemma 3.30. If E is a compact subset of $\mathbb{R}^n$ of measure 0, then E is of content 0.

PROOF. Let E be of measure 0, and let $\epsilon > 0$ be given. Choose open rectangles E_j with $E \subseteq \bigcup_{j=1}^{\infty} E_j$ and $\sum_{j=1}^{\infty} |E_j| < \epsilon$. By compactness, $E \subseteq \bigcup_{j=1}^{N} E_j$ for some N. Then $\sum_{j=1}^{N} |E_j| < \epsilon$. Since ϵ is arbitrary, E has content 0. $\square$

Recall from Section II.9 that the **oscillation** at x_0 of a function $f : A \to \mathbb{R}$ is given by

$$\operatorname{osc}_f(x_0) = \lim_{\delta \downarrow 0} \sup_{x \in B(\delta; x_0)} |f(x) - f(x_0)|.$$

The oscillation is 0 at x_0 if and only if f is continuous there. Lemma 2.55 tells us that

$$\left\{x \in U \,\middle|\, \operatorname{osc}_g(x) \geq 2\epsilon\right\}^{\mathrm{cl}} \subseteq \left\{x \in U \,\middle|\, \operatorname{osc}_g(x) \geq \tfrac{\epsilon}{2}\right\}$$

for any $\epsilon > 0$.

Lemma 3.31. Let A be a nontrivial closed rectangle in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a bounded function with $\operatorname{osc}_f(x) < \epsilon$ for all x in A. Then there is a partition P of A with $U(P, f) - L(P, f) \leq 2\epsilon|A|$.

PROOF. For each x_0 in A, there is an open rectangle U_{x_0} centered at x_0 such that $|f(x) - f(x_0)| \leq \epsilon$ on $A \cap U_{x_0}^{\mathrm{cl}}$. Then $M_{U_{x_0}^{\mathrm{cl}}}(f) - m_{U_{x_0}^{\mathrm{cl}}}(f) \leq 2\epsilon$. These open rectangles cover A. By compactness a finite number of them suffice to cover A. Write $A \subseteq U_{x_1} \cup \cdots \cup U_{x_m}$ accordingly. Let P be the partition of A generated by the endpoints in each coordinate of A and the endpoints of the closed rectangles $U_{x_j}^{\mathrm{cl}}$; we discard endpoints that lie outside A. Each component rectangle R of P then lies completely within some $U_{x_j}^{\mathrm{cl}}$, and we have $M_R(f) - m_R(f) \leq 2\epsilon$ for each component rectangle R of P. Therefore

$$U(P, f) - L(P, f) = \sum_R \big(M_R(f) - m_R(f)\big)\,|R| \leq 2\epsilon \sum_R |R| = 2\epsilon|A|. \quad \square$$

PROOF OF THEOREM 3.29. Define $B_\epsilon = \left\{x \,\middle|\, \operatorname{osc}_f(x) \geq \epsilon\right\}$ for each $\epsilon > 0$, so that $B = \bigcup_{n=1}^{\infty} B_{1/n}$. For the easy direction of the proof, suppose that f is Riemann integrable. We show that $B_{1/n}$ has content 0 for all n. Since content 0 implies measure 0, B_n will have measure 0 for all n. So will the countable union, and therefore B will have measure 0.

Given $\epsilon > 0$ and n, use Lemma 3.23e to choose a partition P of A with $U(P, f) - L(P, f) \leq \epsilon/n$. Let

$$\mathcal{R} = \left\{\text{component rectangles } R \text{ of } P \,\middle|\, R^o \cap B_{1/n} \neq \varnothing\right\},$$

where R^o is the interior of R. Then $B_{1/n}$ is covered by the closed rectangles in $\mathcal{R}$ and the boundaries of the component rectangles of P. The latter are of content 0.

For R in $\mathcal{R}$, let us see that $M_R(f) - m_R(f) \geq 1/n$. In fact, if x_0 is in $R^o \cap B_{1/n}$, then $\operatorname{osc}_f(x_0) \geq 1/n$, so that $\lim_{\delta\downarrow 0} \sup_{|x-x_0|<\delta} |f(x) - f(x_0)| \geq 1/n$ and

$$\sup_{\substack{|x-x_0|<\delta,\\ x\in R^o}} |f(x) - f(x_0)| \geq 1/n \qquad \text{for all } \delta > 0.$$

Therefore $M_R(f) - m_R(f) \geq 1/n$. Summing on $R \in \mathcal{R}$ gives

$$\begin{aligned}\frac{1}{n}\sum_{R\in\mathcal{R}} |R| \leq \sum_{R\in\mathcal{R}} \big(M_R(f) - m_R(f)\big)\,|R| &\leq \sum_{\text{all } R} \big(M_R(f) - m_R(f)\big)\,|R| \\ &= U(P, f) - L(P, f) \leq \epsilon/n,\end{aligned}$$

and thus $\sum_{R\in\mathcal{R}} |R| \leq \epsilon$. Consequently $B_{1/n}$ has content 0, as asserted.

For the converse direction of the proof, suppose that B has measure 0. We are to prove that f is Riemann integrable. Let $\epsilon > 0$ be given. The inclusion of Lemma 2.55 gives $B_\epsilon^{\text{cl}} \subseteq B_{\epsilon/4} \subseteq B$, and thus B_ϵ^{cl} has measure 0. The set B_ϵ^{cl} is compact, and Lemma 3.30 shows that it has content 0. Hence the subset B_ϵ has content 0. Choose open rectangles $U_1, \dots, U_m$ such that $B_\epsilon \subseteq \bigcup_{j=1}^m U_j$ and $\sum_{j=1}^m |U_j| < \epsilon$. Form the partition P of A generated by the endpoints in each coordinate of A and the endpoints of the closed rectangles $U_{x_j}^{\text{cl}}$; we discard endpoints that lie outside A.

Then every component closed rectangle R of P is in one of the two classes

$$\begin{aligned}\mathcal{R}_1 &= \big\{R \bigm| R \subseteq U_j^{\text{cl}} \text{ for some } j\big\},\\ \mathcal{R}_2 &= \big\{R \bigm| R \cap B_\epsilon = \varnothing\big\}.\end{aligned}$$

In fact, our definition is such that $R \cap U_j \neq \varnothing$ implies $R \subseteq U_j^{\text{cl}}$. If $R \cap B_\epsilon \neq \varnothing$, let x_0 be in $R \cap B_\epsilon$. Then x_0 is in some U_j, and $R \cap U_j \neq \varnothing$ for that j. Hence R is in $\mathcal{R}_1$.

We shall construct a particular refinement P' of P in a moment. Let R' be a typical component rectangle of P'. For any refinement P' of P, we have

$$\begin{aligned}&U(P', f) - L(P', f)\\ &\quad\leq \sum_{R\in\mathcal{R}_1}\sum_{R'\subseteq R} \big(M_{R'}(f) - m_{R'}(f)\big)\,|R'| + \sum_{R\in\mathcal{R}_2}\sum_{R'\subseteq R} \big(M_{R'}(f) - m_{R'}(f)\big)\,|R'|\\ &\quad\leq 2\big(\sup_A |f|\big)\sum_{R\in\mathcal{R}_1}\sum_{R'\subseteq R} |R'| + \sum_{R\in\mathcal{R}_2}\sum_{R'\subseteq R} \big(M_{R'}(f) - m_{R'}(f)\big)\,|R'|\\ &\quad\leq 2\big(\sup_A |f|\big)\epsilon + \sum_{R\in\mathcal{R}_2}\sum_{R'\subseteq R} \big(M_{R'}(f) - m_{R'}(f)\big)\,|R'|\end{aligned}$$

since $\sum_{j=1}^{m} |U_j| < \epsilon$. For R in $\mathcal{R}_2$, we have $\operatorname{osc}_f(x) < \epsilon$ for all x in R. Lemma 3.31 shows that there is a partition P_R of R such that $U(P_R, f) - L(P_R, f) \le 2\epsilon|R|$. In other words, $\sum_{R' \subseteq R} \big(M_{R'}(f) - m_{R'}(f)\big)\,|R'| \le 2\epsilon|R|$ if P' is fine enough to include all the n-tuples of P_R. If P' is fine enough so that this happens for all R in $\mathcal{R}_2$, then we obtain

$$U(P', f) - L(P', f) \le 2\big(\sup_A |f|\big)\epsilon + \sum_{R \in \mathcal{R}_2} 2\epsilon|R| \le 2\epsilon\big(\sup_A |f| + |A|\big),$$

and the theorem follows. $\square$

9. Fubini's Theorem for the Riemann Integral

Fubini's Theorem is a result asserting that a double integral is equal to an iterated integral in either order. An unfortunate feature of the Riemann integral is that when an integrable function $f(x, y)$ is restricted to one of the two variables, then the resulting function of that variable need not be integrable. Thus a certain amount of checking is often necessary in using the theorem. This feature is corrected in the Lebesgue integral, and that, as we shall see in Chapter V, is one of the strengths of the Lebesgue integral.

Theorem 3.32 (Fubini's Theorem). Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles, and let $f : A \times B \to \mathbb{R}$ be Riemann integrable. For x in A let f_x be the function $y \mapsto f(x, y)$ for y in B, and define

$$\mathcal{L}(x) = \underline{\int}_B f_x(y)\,dy = \underline{\int}_B f(x, y)\,dy,$$
$$\mathcal{U}(x) = \overline{\int}_B f_x(y)\,dy = \overline{\int}_B f(x, y)\,dy,$$

as functions on A. Then $\mathcal{L}$ and $\mathcal{U}$ are Riemann integrable on A and

$$\int_{A \times B} f\,dx\,dy = \int_A \mathcal{L}(x)\,dx = \int_A \Big[\underline{\int}_B f(x, y)\,dy\Big]\,dx,$$
$$\int_{A \times B} f\,dx\,dy = \int_A \mathcal{U}(x)\,dx = \int_A \Big[\overline{\int}_B f(x, y)\,dy\Big]\,dx.$$

PROOF. Let P be a partition of the form (P_A, P_B), and let $R = R_A \times R_B$ be a typical component rectangle of P. Then

$$L(P, f) = \sum_R m_R(f)\,|R| = \sum_{R_A} \Big(\sum_{R_B} m_{R_A \times R_B}(f)\,|R_B|\Big)|R_A|.$$

For x in R_A, $m_{R_A\times R_B}(f)\le m_{R_B}(f_x)$. Hence x in R_A implies

$$\sum_{R_B} m_{R_A\times R_B}(f)\,|R_B| \le \sum_{R_B} m_{R_B}(f_x)\,|R_B| \le \underline{\int}_B f_x\,dy = \mathcal{L}(x).$$

Taking the infimum over x in R_A and summing over R_A gives

$$L(P,f)=\sum_{R_A}\Big(\sum_{R_B} m_{R_A\times R_B}(f)\,|R_B|\Big)|R_A| \le \sum_{R_A} m_{R_A}(\mathcal{L})\,|R_A| = L(P_A,\mathcal{L}).$$

Similarly

$$U(P_A,\mathcal{U})\le U(P,f).$$

Thus

$$L(P,f)\le L(P_A,\mathcal{L})\le U(P_A,\mathcal{L})\le U(P_A,\mathcal{U})\le U(P,f).$$

Since f is Riemann integrable, the ends of the above display can be made close together by choosing P appropriately. The second and third members of the display will then be close, and hence

$$\int_{A\times B} f\,dx\,dy = \underline{\int}_A \mathcal{L}\,dx = \overline{\int}_A \mathcal{L}\,dx.$$

The result for $\mathcal{L}$ follows. The result for $\mathcal{U}$ follows in similar fashion immediately from the inequalities

$$L(P,f)\le L(P_A,\mathcal{L})\le L(P_A,\mathcal{U})\le U(P_A,\mathcal{U})\le U(P,f).$$

This proves the theorem. $\square$

REMARKS.

(1) Equality of the double integral with the iterated integral in the other order is the same theorem. Thus the iterated integrals in the two orders are equal.

(2) If f is continuous on $A\times B$, then f_x is continuous on B as a consequence of Corollary 2.27, so that $\underline{\int}_B f(x,y)\,dy = \overline{\int}_B f(x,y)\,dy$. Hence

$$\int_{A\times B} f\,dx\,dy = \int_A\Big[\int_B f(x,y)\,dy\Big]\,dx$$

when f is continuous on $A\times B$. This result is isolated as Corollary 3.33 below. Evidently it immediately extends to continuous functions with values in $\mathbb{R}^k$ or $\mathbb{C}^k$.

(3) In practice one often considers integrals of the form $\int_U f(x, y)\,dx\,dy$ for some open set U, where f is continuous on some closed rectangle $A \times B$ containing U. Then the double integral equals $\int_{A\times B} f(x, y) I_U(x, y)\,dx\,dy$, where I_U is the **indicator function**[3] of U equal to 1 on U and 0 off U. In many applications the functions $(I_U)_x$ have harmless discontinuities for each x, and $(f I_U)_x$ is therefore Riemann integrable as a function of y. In this case, the upper and lower integrals can again be dropped in the statement of Theorem 3.32.

Corollary 3.33 (Fubini's Theorem for continuous integrand). Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles, and let $f : A \times B \to \mathbb{R}$ be continuous. Then

$$\int_{A\times B} f\,dx\,dy = \int_A \Big[\int_B f(x, y)\,dy\Big]\,dx = \int_B \Big[\int_A f(x, y)\,dx\Big]\,dy.$$

10. Change of Variables for the Riemann Integral

The goal in this section is to prove a several-variables generalization of the one-variable formula

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\varphi'(y)\,dy$$

given in Theorem 1.34. In the one-variable case we assumed in effect that φ was a strictly increasing function of class C^1 on $[A, B]$ and that f was merely Riemann integrable. The several-variables theorem in this section will be only a preliminary result, with a final version stated and proved in Chapter VI in the context of the Lebesgue integral. In particular we shall assume in the present section that f is continuous and that it vanishes near the boundary of the domain, and we shall make strong assumptions about φ. To capture succinctly the notion that f vanishes near the boundary of its domain, we introduce the notion of the **support** of f, which is the closure of the set where f is nonzero.

Theorem 3.34 (change-of-variables formula). Let φ be a one-one function of class C^1 from an open subset U of $\mathbb{R}^n$ onto an open subset $\varphi(U)$ of $\mathbb{R}^n$ such that $\det \varphi'(x)$ is nowhere 0. Then

$$\int_{\varphi(U)} f(y)\,dy = \int_U f(\varphi(x))|\det \varphi'(x)|\,dx$$

for every continuous function $f : \varphi(U) \to \mathbb{R}$ whose support is a compact subset of $\varphi(U)$.

[3]Indicator functions are called "characteristic functions" by many authors, but the term "characteristic function" has another meaning in probability theory and is best avoided as a substitute for "indicator function" in any context where probability might play a role.

Before a discussion of the sense in which this result has to regarded as preliminary, a few remarks are in order. The function φ' is the usual derivative of φ, and $\varphi'(x)$ is therefore a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$ that depends on x. The matrix of the linear function $\varphi'(x)$ is the Jacobian matrix $[\partial\varphi_i/\partial x_j]$, and $\det\varphi'(x)$ is the determinant of this matrix. In classical notation, this determinant is often written as $\dfrac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}$, and then the effect on the integral of changing variables can be summarized by the formula $dy = \left|\dfrac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}\right| dx$. The absolute value signs did not appear in the one-variable formula in Theorem 1.34, but the assumption that φ was strictly increasing made them unnecessary, $\varphi'(x)$ being > 0. Had we worked with strictly decreasing φ, we would have assumed $\varphi'(x) < 0$ everywhere, and the limits of integration on one side of the formula would have been reversed from their natural order. The minus sign introduced by putting the limits of integration in their natural order would have compensated for a minus sign introduced in changing $\varphi'(x)$ to $|\varphi'(x)|$.

The hypotheses on φ make the Inverse Function Theorem (Theorem 3.17) applicable at every x in U. Consequently $\varphi(U)$ is automatically open, and φ has a locally defined C^1 inverse function about each point $\varphi(x)$ of the image. Since φ has been assumed to be one-one, $\varphi : U \to \varphi(U)$ has a global inverse function φ^{-1} of class C^1.

We can use φ^{-1} to verify that $f\circ\varphi$ has compact support in U: To the equality $\varphi\big(\{x\in U \mid f(\varphi(x))\neq 0\}\big) = \{y\in\varphi(U)\mid f(y)\neq 0\}$, we apply φ^{-1} and obtain $\{x\in U\mid f(\varphi(x))\neq 0\} = \varphi^{-1}\big(\{y\in\varphi(U)\mid f(y)\neq 0\}\big)$. Hence

$$\{x\in U\mid f(\varphi(x))\neq 0\}^{\mathrm{cl}} = \big(\varphi^{-1}\big(\{y\in\varphi(U)\mid f(y)\neq 0\}\big)\big)^{\mathrm{cl}}.$$

The identity $F(E^{\mathrm{cl}}) \subseteq (F(E))^{\mathrm{cl}}$ holds whenever F is a continuous function between two metric spaces, by Proposition 2.25. When E^{cl} is compact, equality actually holds. The reason is that Propositions 2.34 and 2.38 show $F(E^{\mathrm{cl}})$ to be closed; since $F(E^{\mathrm{cl}})$ is a closed set containing $F(E)$, it contains $(F(E))^{\mathrm{cl}}$. Applying this fact to the displayed equation above, we obtain

$$\{x\in U\mid f(\varphi(x))\neq 0\}^{\mathrm{cl}} = \varphi^{-1}\big(\{y\in\varphi(U)\mid f(y)\neq 0\}^{\mathrm{cl}}\big).$$

In other words,

$$\operatorname{support}(f\circ\varphi) = \varphi^{-1}(\operatorname{support}(f)).$$

Applying Proposition 2.38 a second time, we see that $f\circ\varphi$ has compact support.

As a result, we can rewrite the formula to be proved in Theorem 3.34 as

$$\int_{\mathbb{R}^n} f(y)\,dy = \int_{\mathbb{R}^n} f(\varphi(x))|\det\varphi'(x)|\,dx,$$

and the supports will take care of themselves in the proof.

The result of Theorem 3.34 has to be regarded as preliminary. To understand the sense in which the result is limited, consider the case of polar coordinates in $\mathbb{R}^2$. In this case we can take

$$U = \left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} \;\middle|\; 0 < r < +\infty \quad \text{and} \quad 0 < \theta < 2\pi \right\},$$

$$\varphi \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and we have

$$\varphi(U) = \mathbb{R}^2 - \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; x \geq 0 \right\}.$$

We readily compute that $\det \varphi' \left(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\right) = r$, and the desired formula is

$$\int_{\mathbb{R}^2} f(x, y)\, dx\, dy = \int_{0 \leq r < \infty,\ 0 \leq \theta < 2\pi} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

At first glance this formula seems fine. But if we refer to the precise hypotheses, we see that f is assumed to vanish in a neighborhood of the set of points $(x, 0)$ with $x \geq 0$, as well as when (x, y) is sufficiently far from the origin. Without some sort of passage to the limit, the theorem therefore settles few cases of interest. This passage to the limit will be accomplished easily with the Lebesgue integral, and we therefore postpone the final form of the change-of-variables formula to Chapter VI.

In any event, we shall use Theorem 3.34 in proving the final change-of-variables formula, and thus a proof is warranted now. Before coming to the formal proof, it is well to understand the mechanism of the theorem. The proof will then flow easily from the analysis that is done for motivation.

The motivation for the theorem comes from taking f to be the constant function 1 and from thinking of φ as of the form $\varphi(y) = y_0 + L(y - y_0)$ with L linear. In $\mathbb{R}^3$, if we take U to be the cube $\{y = (y_1, y_2, y_3) \mid 0 \leq y_i \leq 1 \text{ for all } i\}$, along with $f = 1$, the formula asserts that $\varphi(U)$ has volume $|\det L|$. This is just the well-known fact about 3-by-3 matrices that the volume of the parallelepiped with sides u, v, w is the scalar $|(u \times v) \cdot w|$. For a corresponding result in $\mathbb{R}^n$, where vector product is not available, the relationship between the determinant and a volume has to be argued differently. One way of proceeding in $\mathbb{R}^n$ is to use row or column reduction to write the given matrix as the product of **elementary matrices** (those corresponding to the effect of a single step in the reduction), to check the change of variables for each factor, and to use the multiplication formula $\det(AB) = \det A \det B$ to obtain the result. This argument can be adjusted so as to work with a function f in place; the elementary matrices that interchange two variables are handled by Fubini's Theorem (Theorem 3.32 or Corollary 3.33), and the other elementary matrices are handled by the one-variable change-of-variables formula (Theorem 1.34).

That being the case, one can envision a proof of Theorem 3.34 that proceeds by approximation, using Taylor's Theorem (Theorem 3.11), at least if f is of class C^2. The contribution to the integrand from the integral remainder term in the Taylor expansion of φ is to be estimated as an error term. The approximation generates an additional error term because the image of U under φ does not match the image of U under the approximating first-order expansion of φ. Of course, one cannot expect the approximation to be very good far away from the point where the Taylor expansion is centered, and thus the argument needs to be carried out locally. The local contributions can then be pieced together by using a partition of unity. Such an argument can actually be carried out, but the argument is lengthy.

A more economical argument comes by finding a nonlinear analog of row or column reduction. The Inverse Function Theorem will allow us to prove that a general φ decomposes into suitably defined nonlinear elementary transformations, but the decomposition is valid only locally. A partition of unity is used to piece together the local results and obtain the theorem. We introduce two kinds of nonlinear elementary transformations:

(i) a **flip** β, which interchanges two coordinates. This is a linear function, and it satisfies $|\det \beta'(x)| = 1$ for all x. Application of Fubini's Theorem in the form of Corollary 3.33 shows that Theorem 3.34 is valid when φ is a flip.

(ii) a **primitive mapping**

$$\psi(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ g(x_1, \dots, x_n) \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix},$$

where g is real-valued and occurs in a single entry. If that entry is the i^{th} entry, then the Jacobian matrix of ψ is the identity matrix except in the i^{th} row, where the entries are $\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}$. Hence $|\det \psi'(x)| = \left|\frac{\partial g}{\partial x_i}\right|$.

To prove Theorem 3.34 for a primitive mapping as in (ii), it is enough to handle $i = 1$. If we write $x = (x_1, x')$ and $y = (y_1, x')$ with x' in $\mathbb{R}^{n-1}$, Fubini's

Theorem (Corollary 3.33) reduces matters to showing that

$$\int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}} f(y_1,x')\,dy_1\Big]\,dx' \\ = \int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}} f(g(x_1,x'),x')\Big|\frac{\partial g}{\partial x_1}(x_1,x')\Big|\,dx_1\Big]\,dx'$$

under suitable hypotheses on g, and it is enough to prove that the inner integrals are equal for all x'. Theorem 1.34 yields the equality of the inner integrals if g is a C^1 function for which $g(x_1, x')$ is defined for x_1 in an interval for any relevant x', and if $\big|\frac{\partial g}{\partial x_1}(x_1,x')\big|$ is everywhere positive at the points in question.

In the linear case a primitive mapping ψ for which $g(x)$ appears in the i^{th} entry is given by a matrix that is the identity except in the i^{th} row. For ψ' to be nonvanishing, the diagonal entry in the i^{th} row must be nonzero. This kind of matrix is not always elementary but is the product of n elementary matrices.

What needs to be proved for Theorem 3.34 is that apart from translations, any nonlinear φ as in Theorem 3.34 can be decomposed into the product of primitive transformations and flips, at least locally. The argument will peel primitive mappings from the right side of φ and flips from the left side. In that sense it will be a nonlinear version of column reduction with primitive mappings and row reduction with flips. The decomposition will be forced to be local because it uses the Inverse Function Theorem, which guarantees the existence of an inverse function only locally.

Lemma 3.35. Suppose that E is an open neighborhood of 0 in $\mathbb{R}^n$ and that $\varphi : E \to \mathbb{R}^n$ is a C^1 function such that $\varphi(0) = 0$ and $\varphi'(0)^{-1}$ exists. Then there is a subneighborhood of 0 in $\mathbb{R}^n$ in which φ factors as

$$\varphi = \beta_1 \circ \cdots \circ \beta_{n-1} \circ \psi_n \circ \cdots \circ \psi_1,$$

where each β_j is a flip or the identity and each ψ_j is a primitive C^1 function in some open neighborhood of 0 such that $\psi_j(0) = 0$ and $\psi_j'(0)^{-1}$ exists.

PROOF. Let us set up an inductive procedure by assuming at the start that

$$\varphi(x_1,\dots,x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_{i_0-1} \\ \varphi_{i_0}(x_1,\dots,x_n) \\ \vdots \\ \varphi_n(x_1,\dots,x_n) \end{pmatrix} \qquad (*)$$

with $1 \le i_0 \le n$. We shall make use of the following formula for multiplying two matrices A and B when B has the property that it is equal to the identity matrix except possibly in row i_0. The formula is

$$(AB)_{ij} = \begin{cases} A_{ii_0}B_{i_0j} + A_{ij}B_{jj} & \text{if } j \ne i_0, \\ A_{ii_0}B_{i_0i_0} & \text{if } j = i_0. \end{cases} \tag{$**$}$$

It will be convenient to identify linear functions like $\varphi'(x)$ with their matrices, so that the $(i, j)^{\text{th}}$ entry $\varphi'(x)_{ij}$ of $\varphi'(x)$ is meaningful.

Let $j = j_0$ be the least row index for which the $(j, i_0)^{\text{th}}$ entry of $\varphi'(0)$ is nonzero. The index j_0 exists because $\varphi'(0)$ is nonsingular, and j_0 is $\ge i_0$ since the top $i_0 - 1$ rows of $\varphi'(x)$ match the corresponding rows of the identity matrix. Let

$$\beta_{i_0} = \begin{cases} \text{identity function} & \text{if } j_0 = i_0, \\ \text{flip of entries } j_0 \text{ and } i_0 & \text{if } j_0 > i_0. \end{cases}$$

Then $\beta_{i_0} \circ \varphi$ has the general form of $(*)$ except that the i_0^{th} and j_0^{th} entries have been interchanged. By inspection the Jacobian matrix at 0 of $\beta_{i_0} \circ \varphi$ equals the identity matrix in rows 1 through $i_0 - 1$ and has $(i_0, i_0)^{\text{th}}$ entry nonzero.

Thus if we possibly incorporate a composition with a flip into the definition of φ, we may assume that $\varphi'(0)_{i_0i_0}$ is nonzero. Put

$$\psi(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_{i_0-1} \\ \varphi_{i_0}(x_1, \dots, x_n) \\ x_{i_0+1} \\ \vdots \\ x_n \end{pmatrix}.$$

Then $\psi'(x)$ is an n-by-n matrix with

$$\psi'(x)_{ij} = \begin{cases} \delta_{ij} & \text{if } i \ne i_0, \\ \varphi'(x)_{i_0j} & \text{if } i = i_0, \end{cases}$$

where δ_{ij} is the Kronecker delta. Since $\det \psi'(0) = \varphi'(0)_{i_0i_0} \ne 0$, we can apply the Inverse Function Theorem (Theorem 3.17) to ψ, obtaining a C^1 inverse function ψ^{-1} that carries an open neighborhood of 0 onto an open subset of the domain of φ, has $\psi^{-1}(0) = 0$, and has derivative $(\psi^{-1})'(y) = \psi'(x)^{-1}$, where x and y are related by $y = \psi(x)$ and $x = \psi^{-1}(y)$. Using $(**)$, we readily verify that

$$(\psi'(x)^{-1})_{ij} = \begin{cases} \delta_{ij} & \text{if } i \ne i_0, \\ -(\varphi'(x)_{i_0i_0})^{-1}\varphi'(x)_{i_0j} & \text{if } i = i_0 \ne j, \\ (\varphi'(x)_{i_0i_0})^{-1} & \text{if } i = j = i_0. \end{cases}$$

Therefore

$$((\psi^{-1})'(y))_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq i_0, \\ -(\varphi'(x)_{i_0 i_0})^{-1}\varphi'(x)_{i_0 j} & \text{if } i = i_0 \neq j, \\ (\varphi'(x)_{i_0 i_0})^{-1} & \text{if } i = j = i_0. \end{cases}$$

Form $\eta = \varphi \circ \psi^{-1}$. By the chain rule (Theorem 3.10), we have $\eta'(x) = \varphi'(x)(\psi^{-1})'(y)$, and this is nonsingular for x close enough to 0. Combining the formula for $((\psi^{-1})'(y))_{ij}$ with the chain rule and $(**)$ gives

$$\begin{aligned} \eta'(x)_{ij} &= (\varphi'(x)(\psi^{-1})'(y))_{ij} \\ &= \begin{cases} \varphi'(x)_{ii_0}((\psi^{-1})'(y))_{i_0 j} + \varphi'(x)_{ij}((\psi^{-1})'(y))_{jj} & \text{if } j \neq i_0, \\ \varphi'(x)_{ii_0}((\psi^{-1})'(y))_{i_0 i_0} & \text{if } j = i_0, \end{cases} \\ &= \begin{cases} \varphi'(x)_{ii_0}(-(\varphi'(x)_{i_0 i_0})^{-1}\varphi'(x)_{i_0 j}) + \varphi'(x)_{ij} & \text{if } j \neq i_0, \\ \varphi'(x)_{ii_0}(\varphi'(x)_{i_0 i_0})^{-1} & \text{if } j = i_0. \end{cases} \end{aligned}$$

Since $\varphi'(x)_{ii_0}$ is 0 for $i < i_0$, the above formula shows that $\eta'(x)_{ij} = \delta_{ij}$ for $i < i_0$. For $i = i_0$, the formula shows first that $\eta'(x)_{i_0 j}$ is 0 for $j \neq i_0$ and then that $\eta'(x)_{i_0 j}$ is 1 for $j = i_0$. Thus $\eta'(x)_{ij} = \delta_{ij}$ for $i \leq i_0$. Consequently the i^{th} entry of $\eta(x)$ is $x_i + c_i$ if $i \leq i_0$, where c_i is a constant. Evaluating η at $x = 0$, we see that $c_i = 0$. Thus $\eta(x)$ has the same general shape as $(*)$ except that the i_0^{th} entry is now x_{i_0}.

Following this argument inductively for $i = 1, \dots, n-1$ leads us to a decomposition

$$\eta = \beta_{n-1} \circ \cdots \circ \beta_1 \circ \varphi \circ \psi_1^{-1} \circ \cdots \circ \psi_{n-1}^{-1}, \tag{†}$$

where each β_j is a flip or the identity and where each ψ_j is primitive. The function η has $\eta(0) = 0$ and $\eta'(0)$ nonsingular, and η has the form

$$\eta(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \xi(x_1, \dots, x_n) \end{pmatrix}.$$

Therefore η is primitive. Solving (†) for φ thus exhibits φ as decomposed into the required form. $\square$

PROOF OF THEOREM 3.34. We are to prove that

$$\int_{\varphi(U)} f(y)\,dy = \int_U f(\varphi(x))|\det \varphi'(x)|\,dx \tag{$*$}$$

whenever $\varphi : U \to \varphi(U)$ is a C^1 function between open sets with a C^1 inverse and $f : \varphi(U) \to \mathbb{R}$ is continuous and has compact support lying in $\varphi(U)$. In the argument we shall work with several functions in place of φ, and the set U may be different for each. We have seen that $(*)$ holds if φ is a flip or an invertible primitive function. Let us observe also that $(*)$ holds if φ is a translation $\varphi(x) = x + x_0$ for some x_0 in $\mathbb{R}^n$; the reason is that $(*)$ in this case can be reduced via successive uses of Fubini's Theorem (Corollary 3.33) to the 1-dimensional case, where we know it to be true by Theorem 1.34.

If $(*)$ holds when φ is either $\alpha : U \to \alpha(U)$ or $\beta : \alpha(U) \to \beta(\alpha(U))$, then $(*)$ holds when φ is the composition $\gamma = \beta \circ \alpha : U \to \beta(\alpha(U))$ because

$$\begin{aligned}\int_{\mathbb{R}^n} f(z)\,dz &= \int_{\mathbb{R}^n} f(\beta(y))|\det \beta'(y)|\,dy \\ &= \int_{\mathbb{R}^n} f(\beta(\alpha(x))|\det \beta'(\alpha(x))||\det \alpha'(x)|\,dx \\ &= \int_{\mathbb{R}^n} f(\gamma(x))|\det(\beta'(\alpha(x))\alpha'(x))|\,dx \\ &= \int_{\mathbb{R}^n} f(\gamma(x))|\det \gamma'(x)|\,dx,\end{aligned}$$

the last two steps holding by the formula $\det(BA) = \det B \det A$ and the chain rule (Theorem 3.10).

For any a in the given set U, Lemma 3.35 applies to the function φ_a carrying $U - a$ to $\varphi(U) - \varphi(a)$ and defined by $\varphi_a(x) = \varphi(x + a) - \varphi(a)$ because $\varphi_a(0) = 0$ and $\varphi_a'(0) = \varphi'(a)$. The lemma produces an open neighborhood E_a of 0 on which φ_a factors as a composition of flips and invertible primitive functions. If τ_{x_0} denotes the translation $\tau_{x_0}(x) = x + x_0$, then $\varphi_a = \tau_{-\varphi(a)} \circ \varphi \circ \tau_a$ shows that $\varphi = \tau_{\varphi(a)} \circ \varphi_a \circ \tau_{-a}$. Therefore φ factors on the open neighborhood $E_a + a$ as the composition of translations, flips, and invertible primitive functions. From the previous paragraph we conclude for each $a \in U$ that $(*)$ holds for φ if f is continuous and is compactly supported in the open neighborhood $\varphi(E_a + a)$ of $\varphi(a)$.

As a varies through U, the subsets $V_a = \varphi(E_a + a)$ of $\varphi(U)$ form an open cover of $\varphi(U)$. Fix f continuous with compact support K in $\varphi(U)$. By compactness a finite subfamily of the family $\{V_a\}$ forms an open cover of K. Applying Proposition 3.14, we obtain a finite family $\Psi = \{\psi\}$ of continuous functions defined on $\varphi(U)$ and taking values in [0, 1] with the properties that

(i) each ψ is 0 outside of some compact set contained in some V_a,
(ii) $\sum_{\psi \in \Psi} \psi$ is identically 1 on K.

Then property (i) and the conclusion of the previous paragraph show that $(*)$ holds for ψf. From (ii), we have $\sum_\psi \psi f = f$ on $\varphi(U)$. Since there are only finitely many terms in the sum, we can interchange sum and integral and conclude that $(*)$ holds for f. This completes the proof. $\square$

One final remark is appropriate: Theorem 3.34 immediately extends from the scalar-valued case as stated to the case that f takes values in $\mathbb{R}^m$ or $\mathbb{C}^m$.

11. Arc Length and Integrals with Respect to Arc Length

This section gives a careful treatment of arc length and of integrals of scalar-valued functions on simple arcs in $\mathbb{R}^n$. Most readers will already have seen some form of this material in a calculus course, and the point here will be to give precise definitions and to make the proofs rigorous.

The term "curve" is used in various contexts in mathematics and has not yet been defined in this book. In this chapter we shall be interested in a **parametrically defined curve** in $\mathbb{R}^n$, a set given as the image of a continuous function from a closed bounded interval of the line into $\mathbb{R}^n$. Such a function was called a "path" in Section II.8, but the term "path" is not commonly used in the present context.[4] Curves can also be defined implicitly as the set of simultaneous solutions to a system of (nonlinear) equations, and this kind of curve will play a role in Chapter IV.

For parametrically defined curves such as $t \mapsto c(t)$, with $c(t)$ in $\mathbb{R}^n$ for each t, the interplay between the function c and its image will be of the utmost importance in the theory, and we shall pay attention to what notions concerning a parametrically defined curve are defined by the geometry of the image and what notions depend on the actual parametrization.

EXAMPLE. The quarter of the unit circle in the first quadrant of the (x, y) plane is given in three standard ways:

(i) as the image of $x \mapsto (x, \sqrt{1-x^2})$ for $0 \le x \le 1$, i.e., as part of the graph of $y = \sqrt{1-x^2}$,

(ii) as the image of $y \mapsto (\sqrt{1-y^2}, y)$ for $0 \le y \le 1$, i.e., as part of the graph of $x = \sqrt{1-y^2}$,

(iii) as the image of $t \mapsto (\cos t, \sin t)$ for $0 \le t \le \pi/2$, with the angle t as the parameter.

There are, of course, many other ways that are less standard. When we get to Green's Theorem in Section 13, it will be essential to be able to view this set as given both by (i) and by (ii). In making computations, such as for the length of the quarter circle, it will often be convenient to view the set as given by (iii).

[4]This book will resist any temptation to come into conflict with longstanding traditions in terminology.

If we think of the function giving a parametrization as tracing out its image as the domain variable t increases from one endpoint to the other, we realize that we cannot tell from the image whether particular points have been traced out more than once. Thus in order to have an easy time isolating useful geometric notions that are independent of the parametrization, we should assume that this retracing does not occur. We build that condition into a definition.

A **simple arc** in $\mathbb{R}^n$ is a one-one function γ from a closed bounded interval of the line into $\mathbb{R}^n$. Let γ_1 and γ_2 be simple arcs in $\mathbb{R}^n$ with respective domains $[a_1, b_1]$ and $[a_2, b_2]$. We say that γ_2 is a **reparametrization** of γ_1 if there exists a continuous function $\varphi : [a_1, b_1] \to [a_2, b_2]$ with a continuous inverse such that $\gamma_1 = \gamma_2 \circ \varphi$. The relation "is a reparametrization of" is an equivalence relation. The three parametrizations of the quarter circle in the example above are reparametrizations of one another; one can check this fact by direct computation, or one can appeal to Proposition 3.36 below. A reparametrization φ must carry a_1 to an endpoint of $[a_2, b_2]$ because the complement of $\varphi(a_1)$ in the image has to be connected, and we introduce terminology to distinguish these two cases. A reparametrization is **orientation-preserving** if $\varphi(a_1) = a_2$ and **orientation-reversing** if $\varphi(a_1) = b_2$.

When two simple arcs are reparametrizations of one another, they have the same image. The virtue of considering *simple* arcs is that there is a converse.

Proposition 3.36. If γ_1 and γ_2 are simple arcs in $\mathbb{R}^n$ with the same image, then they are reparametrizations of one another.

PROOF. Let E be the common image of γ_1 and γ_2, and let $[a_1, b_1]$ and $[a_2, b_2]$ be the respective domains. The function $\gamma_2 : [a_2, b_2] \to E$ is continuous one-one and onto, and its domain $[a_2, b_2]$ is compact. Corollary 2.40 shows that it has a continuous inverse $\eta : E \to [a_2, b_2]$. Define $\varphi = \eta \circ \gamma_1$. Then φ is continuous and one-one from $[a_1, b_1]$ onto $[a_2, b_2]$, and it has a continuous inverse and satisfies $\gamma_1 = \gamma_2 \circ \varphi$ because $\gamma_2 \circ \varphi = \gamma_2 \circ (\eta \circ \gamma_1) = (\gamma_2 \circ \eta) \circ \gamma_1 = \gamma_1$. Thus γ_2 is exhibited as a reparametrization of γ_1. $\square$

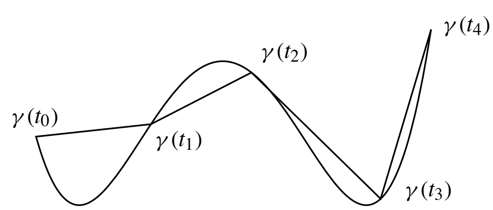

FIGURE 3.1. Polygonal approximation for estimating arc length.

The **arc length** of a simple arc is defined to be the least upper bound of the lengths of all inscribed polygonal approximations. See Figure 3.1. Specifically let

$\gamma : [a, b] \to \mathbb{R}^n$ be a simple arc. As in Section I.4, let $P = \{t_j\}_{j=0}^m$ be a partition of $[a, b]$. We write $\ell(\gamma(P))$ for the sum of the lengths of the line segments connecting the consecutive points $\gamma(t_j)$, namely $\ell(\gamma(P)) = \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})|$, and we put

$$\ell(\gamma) = \sup_P \ell(\gamma(P)),$$

the supremum being taken over all partitions P of $[a, b]$. We say that γ is **rectifiable** if $\ell(\gamma)$ is finite. Simple arcs that are rectifiable will be the ones of interest to us. Obtaining a usable formula for their length is a question that we shall address later in this section.

Observe that the length of a simple arc is a geometric property in that it depends only on the image. In fact, any two simple arcs with the same image are reparametrizations of one another, according to Proposition 3.36. Thus we may assume that the two arcs whose lengths are to be compared are γ and $\gamma \circ \varphi$. Then

$$\begin{aligned}\ell(\gamma(P)) &= \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})| \\ &= \sum_{j=1}^m |\gamma \circ \varphi(\varphi^{-1}(t_j)) - \gamma \circ \varphi(\varphi^{-1}(t_{j-1}))| = \ell(\gamma \circ \varphi)(\varphi^{-1}(P)).\end{aligned}$$

Taking the supremum over P, we obtain $\ell(\gamma) = \ell(\gamma \circ \varphi)$, as asserted.

A sufficient condition for rectifiability appears in Proposition 3.37. Unfortunately that condition will be too strong for our purposes, as two examples after the proposition will illustrate.

Proposition 3.37. A sufficient condition for a simple arc $\gamma : [a, b] \to \mathbb{R}^n$ to be rectifiable is that the derivative $\gamma'(t)$ exist for $a < t < b$ and be bounded.

PROOF. Let M be an upper bound for the absolute value of the derivative of each entry $\gamma_i(t)$ of $\gamma(t)$, and let a partition $P = \{t_i\}_{i=0}^m$ of $[a, b]$ be given. Applying the Mean Value Theorem to the i^{th} entry $\gamma_i(t_j) - \gamma_i(t_{j-1})$ of $\gamma(t_j) - \gamma(t_{j-1})$ shows that

$$|\gamma_i(t_j) - \gamma_i(t_{j-1})| \le M(t_j - t_{j-1}).$$

Squaring both sides, summing on i, and taking the square root gives

$$|\gamma(t_j) - \gamma(t_{j-1})| \le n^{1/2} M(t_j - t_{j-1}).$$

Hence

$$\ell(\gamma(P)) = \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})| \le n^{1/2} M \sum_{j=1}^m (t_j - t_{j-1}) = n^{1/2} M(b - a).$$

Taking the supremum over all partitions P, we obtain $\ell(\gamma) \le n^{1/2} M(b - a)$. $\square$

EXAMPLES.

(1) The quarter circle in the example earlier in this section. Here one parametrization of the arc is $\gamma(t) = (t, \sqrt{1-t^2}\,)$ for $0 \le t \le 1$. We definitely want to have this arc fit within our theory, since we know perfectly well that the length of one quarter of the unit circle ought to be $\pi/2$. Yet the derivative of the second entry is unbounded as t increases to 1. Fortunately the length of a simple arc does not depend on the parametrization, and we can reparametrize the quarter circle with angle as the parameter. Then Proposition 3.37 applies and shows the rectifiablility. Later in this section we shall see that the arc length is indeed $\pi/2$ as expected.

(2) The simple arc given by $\gamma(0) = 0$ and $\gamma(t) = (t, t^2\sin(t^{-2}))$ for $0 < t \le 1$. The derivative of the second entry exists everywhere. (At $t = 0$, use of the definition of derivative shows that the derivative is 0.) A little computation shows that the derivative is unbounded as t decreases to 0. In this respect this example is nicer than the previous one because the derivative exists everywhere this time. But in fact this example is not nice at all: the arc in question is not rectifiable. To see this, we use a partition that includes as many of the points $t = \sqrt{2/(\pi k)}$ as we please. The corresponding point in the plane is $p_k = \big(\sqrt{2/(\pi k)}, 2\sin(\pi k/2)/(\pi k)\big)$, and

$$|p_{k+1} - p_k| \ge \left|\frac{2\sin(\pi(k+1)/2)}{\pi(k+1)} - \frac{2\sin(\pi k/2)}{\pi k}\right|.$$

The expression on the right collapses to $2/(\pi k)$ if k is odd and to $2/(\pi(k+1))$ if k is even. Since $\sum 1/k$ diverges, the sum over k of the expressions on the right can be made as large as we want by taking enough terms. Thus γ is not rectifiable.

Before proceeding, let us make some observations about the definitions of simple arcs and arc length. Throughout let us suppose that $\gamma : [a, b] \to \mathbb{R}^n$ is a simple arc. When $a \le a' \le b' \le b$, we write $\gamma_{[a',b']}$ for the restriction of γ to the domain $[a', b']$; this too is a simple arc. Also we write $-\gamma$ for the **reverse simple arc** with domain $[-b, -a]$ and with values given by $(-\gamma)(t) = \gamma(-t)$.

(i) If P' is a partition obtained by including one additional point in the partition P, then $\ell(\gamma(P)) \le \ell(\gamma(P'))$. In fact, if the new point is t', what is happening is that a term of the form $|\gamma(t_j) - \gamma(t_{j-1})|$ in the sum for $\ell(\gamma(P))$ gets replaced by an expression $|\gamma(t_j) - \gamma(t')| + |\gamma(t') - \gamma(t_{j-1})|$ in the sum for $\ell(\gamma(P'))$, and the term in the sum for $\ell(\gamma(P))$ is $\le$ the expression in the sum for is $\ell(\gamma(P'))$ by the triangle inequality.

(ii) If $a < b$, then $\ell(\gamma) > 0$. In fact, use of the partition P_0 with $t_0 = a$ and $t_1 = b$ already has $\ell(\gamma(P_0)) = |\gamma(b) - \gamma(a)|$, and this is positive since

γ is one-one. Adding further points to the partition cannot decrease the sum of lengths, by (i), and thus $\ell(\gamma) > 0$.

(iii) If $a \leq c \leq b$ and if c is a point in a partition P of $[a, b]$, we can regard P as the union of the set P_1 of members of the partition $\leq c$ and the set P_2 of members of the partition $\geq c$, and we evidently have $\ell(\gamma(P)) = \ell(\gamma_{[a,c]}(P_1)) + \ell(\gamma_{[c,b]}(P_2))$. Observation (i) implies that if we are computing lengths, we can disregard partitions of $[a, b]$ not containing c, and thus

$$\ell(\gamma) = \ell(\gamma_{[a,c]}) + \ell(\gamma_{[c,b]}).$$

(iv) The simple arc $-\gamma$ is an orientation-reversing reparametrization of γ, and thus $\ell(-\gamma) = \ell(\gamma)$.

We shall use these observations to show that any rectifiable simple arc $\gamma : [a, b] \to \mathbb{R}^n$ can be reparametrized in such a way that the new parameter is the **cumulative arc length** starting from $\gamma(a)$. For this purpose define a real-valued function on $[a, b]$ by $s(t) = \ell(\gamma_{[a,t]})$. The function $t \mapsto s(t)$ has $s(a) = 0$ and $s(b) = \ell(\gamma)$, and it is strictly increasing by observations (ii) and (iii).

Proposition 3.38. If $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable simple arc, then the function $s : [a, b] \to [0, \ell(\gamma)]$ giving the cumulative arc length starting from $\gamma(a)$ is continuous.

PROOF. We prove continuity from the left, and then we prove continuity from the right. For continuity from the left, let $\{t_k\}$ be any increasing sequence in $[a, b]$ with limit t in $[a, b]$. Since s is an increasing function, we certainly have

$$\limsup s(t_k) \leq s(t). \tag{$*$}$$

Choose a sequence of partitions P_r of $[a, t]$ with $\lim_r \ell(\gamma(P_r)) = \ell(\gamma_{[a,t]})$. By observation (i), we may assume that the P_r form an increasing sequence of partitions. Let the last interval of P_r be $[u_r, t]$, and let $P_r^\flat$ be P_r with the last interval omitted. By observation (i), there is no loss of generality in assuming that $\lim u_r = t$. Then

$$\ell(\gamma_{[a,t]}(P_r)) = \ell(\gamma_{[a,u_r]}(P_r^\flat)) + |\gamma(t) - \gamma(u_r)| \leq s(u_r) + |\gamma(t) - \gamma(u_r)|.$$

The lim sup of this inequality on r gives

$$s(t) \leq \limsup_r s(u_r) + \limsup_r |\gamma(t) - \gamma(u_r)| = \limsup_r s(u_r), \tag{$**$}$$

the equality holding since γ is continuous and $\{u_r\}$ increases to t. Continuity of s from the left follows by combining $(*)$ and $(**)$.

For right continuity let $\{t_k\}$ be a decreasing sequence in $[a, b]$ with limit t. We are to show that $\lim s(t_k) = s(t)$. To do so, we make use of the reverse arc $-\gamma$ and take into account that $\{-t_k\}$ is an increasing sequence in $[-b, -a]$ with limit $-t$. The observations before the proposition show that

$$s(t_k) = \ell(\gamma_{[a,t_k]}) = \ell((-\gamma)_{[-t_k,-a]}) = \ell((-\gamma)_{[-b,-a]} - \ell((-\gamma)_{[-b,-t_k]}).$$

The first half of the proof, applied to $-\gamma$, shows that $\lim_k \ell((-\gamma)_{[-b,-t_k]}) = \ell((-\gamma)_{[-b,-t]})$. Therefore

$$\lim_k s(t_k) = \ell((-\gamma)_{[-b,-a]}) - \ell((-\gamma)_{[-b,-t]}) = \ell((-\gamma)_{[-t,-a]} = \ell(\gamma_{[a,t]}),$$

as required. $\square$

As a result of Proposition 3.38 and Corollary 2.40, $s : [a, b] \to [0, \ell(\gamma)]$ has a continuous inverse function. Classically this inverse is written as $s \mapsto t(s)$, but let us call it ω in order to be careful. Then $\widetilde{\gamma} = \gamma \circ \omega$ is a simple arc with domain $[0, \ell(\gamma)]$ and with the same image as γ; it is an orientation-preserving reparametrization of γ, and its parameter is s. The result is that γ has been reparametrized so that the new parameter is the cumulative arc length starting from $\gamma(a)$. With this parameter in place, we can do integration. As in Section I.4, we define the **mesh** of the partition $P = \{t_j\}_{j=0}^m$ of $[a, b]$ to be the number $\mu(P) = \max_{j=1}^m (t_j - t_{j-1})$.

Theorem 3.39 (Existence Theorem). If $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable simple arc and f is a continuous complex-valued function on the image of γ, then there exists a unique complex number, denoted $\int_\gamma f\,ds$, with the following property. For any $\epsilon > 0$, there exists a $\delta > 0$ such that any partition $P = \{t_j\}_{j=0}^m$ of $[a, b]$ with $\mu(P) < \delta$ has

$$\Big| \int_\gamma f\,ds - \sum_{j=1}^m f(\gamma(t_{j-1}))\big|\gamma(t_j) - \gamma(t_{j-1})\big| \Big| < \epsilon.$$

Moreover,

$$\int_\gamma f\,ds = \int_0^{\ell(\gamma)} f(\widetilde{\gamma}(s))\,ds,$$

where $\widetilde{\gamma}$ is the reparametrization of γ by the cumulative arc length starting from $\gamma(a)$.

REMARKS. The number $\int_\gamma f\,ds$ is called the **integral of** f **over** γ **with respect to arc length**. When γ is an arc in the plane and when a nonnegative f is graphed in $\mathbb{R}^3$ with the z axis vertical, the number has a geometric interpretation as the area under the fence determined by the graph. The proof of the theorem will be completed after two preliminary lemmas.

Lemma 3.40. Suppose that $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable simple arc. Let $\omega : [0, \ell(\gamma)] \to [a, b]$ be the inverse of the cumulative arc length function $t \mapsto s(t)$ from $\gamma(a)$, and define $\widetilde{\gamma} = \gamma \circ \omega$. Whenever s and s' are members of $[0, \ell(\gamma)]$ with $s < s'$, then

$$|\widetilde{\gamma}(s') - \widetilde{\gamma}(s)| \le |s' - s|.$$

PROOF. Define $t = \omega(s)$ and $t' = \omega(s')$. Then we have

$$\begin{aligned} s' - s &= \ell(\gamma_{[a,t']}) - \ell(\gamma_{[a,t]}) = \ell(\gamma_{[t,t']}) \\ &= \ell(\gamma_{[\omega(s),\omega(s')]}) = \sup_R \ell(\gamma_{[\omega(s),\omega(s')]}(R)), \end{aligned} \tag{$*$}$$

the supremum being taken over all partitions R of $[\omega(s), \omega(s')]$. The expression $\ell(\gamma_{[\omega(s),\omega(s')]}(R))$ is the length of a polygonal path connecting $\widetilde{\gamma}(s) = \gamma(\omega(s))$ to $\widetilde{\gamma}(s') = \gamma(\omega(s'))$, and the triangle inequality shows that

$$|\widetilde{\gamma}(s') - \widetilde{\gamma}(s)| \le \ell(\gamma_{[\omega(s),\omega(s')]}(R)).$$

Since $(*)$ shows that the right side can be made arbitrarily close to $|s' - s|$ by choosing R suitably, the inequality $|\widetilde{\gamma}(s') - \widetilde{\gamma}(s)| \le |s' - s|$ follows. $\square$

Lemma 3.41. If $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable simple arc and if $\epsilon > 0$ is given, then there exists $\delta > 0$ such that any partition P of $[a, b]$ with $\mu(P) < \delta$ has $|\ell(\gamma) - \ell(\gamma(P))| < \epsilon$.

REMARK. This lemma is the special case of Theorem 3.39 in which f is the constant function 1. We shall see that the special case implies the general case because of Lemma 3.40.

PROOF. Let $\omega : [0, \ell(\gamma)] \to [a, b]$ be the inverse of the cumulative arc length function $t \mapsto s(t)$ from $\gamma(a)$, and let $\epsilon > 0$ be given. Choose by definition of $\ell(\gamma)$ a partition P^* of $[a, b]$ with the property that $|\ell(\gamma) - \ell(\gamma(P^*))| \le \epsilon/3$. Say that P^* has $k + 1$ points and therefore determines k subintervals of $[a, b]$. We shall say that these subintervals are the "intervals of P^*." Put $\eta = \frac{\epsilon}{6(k+1)^2}$. Theorem 1.10 shows that ω is uniformly continuous; choose $\delta > 0$ small enough so that $|s - s'| \le \delta$ implies $|t - t'| \le \eta$, where $t = \omega(s)$ and $t' = \omega(s')$.

Let P be any partition of $[a, b]$ with $\mu(P) \le \delta$. Then $Q = s(P) = \omega^{-1}(P)$ is a partition of $[0, \ell(\gamma)]$, and the choice of η makes $\mu(Q) \le \eta$. Define $\widetilde{\gamma} = \gamma \circ \omega$ as above. Since $\widetilde{\gamma}$ is a reparametrization of γ, $\ell(\widetilde{\gamma}) = \ell(\gamma)$. Thus Q is a partition of $[0, \ell(\widetilde{\gamma})]$ with $\mu(Q) \le \eta$.

Let Q^* be the partition of $[0, \ell(\widetilde{\gamma})]$ given by $Q^* = \omega^{-1}(P^*)$, and let $Q^\#$ be the common refinement of Q and Q^*. Since $Q^\#$ is a refinement of Q^*,

$$\begin{aligned} |\ell(\widetilde{\gamma}) - \ell(\widetilde{\gamma}(Q^\#))| &\le |\ell(\widetilde{\gamma}) - \ell(\widetilde{\gamma}(Q^*))| \\ &= |\ell(\gamma) - \ell(\gamma(\omega(Q^*))| = |\ell(\gamma) - \ell(\gamma(P^*))| \le \epsilon/3. \end{aligned}$$

Therefore

$$
\begin{aligned}
|\ell(\widetilde{\gamma})-\ell(\widetilde{\gamma}(Q))| &\le |\ell(\widetilde{\gamma})-\ell(\widetilde{\gamma}(Q^\#))|+|\ell(\widetilde{\gamma}(Q^\#))-\ell(\widetilde{\gamma}(Q))| \\
&\le |\ell(\widetilde{\gamma}(Q^\#))-\ell(\widetilde{\gamma}(Q))|+\epsilon/3. \qquad (*)
\end{aligned}
$$

Many of the terms that contribute to the polygonal length $\ell(\widetilde{\gamma}(Q))$ contribute also to $\ell(\widetilde{\gamma}(Q^\#))$ and therefore cancel. Such a cancellation occurs except when the interval of Q fails to lie in a single interval of Q^*. In the exceptional case an interval I of Q is the union of two or more intervals $I^\#$ of $Q^\#$, and I contains a point of Q^* in its interior. For these exceptional cases we shall regard the contribution from I to $\ell(\widetilde{\gamma}(Q))$ as one kind of error term, and we shall regard the contribution to $\ell(\widetilde{\gamma}(Q^\#))$ from the two or more intervals $I^\#$ as a second kind of error term.

Since Q^* has $k+1$ points, at most $k+1$ such points are involved in exceptional intervals. Hence there at most $k+1$ such intervals I. Each such is of the form $[s, s']$ and contributes to $\ell(\widetilde{\gamma}(Q))$ an amount $|\widetilde{\gamma}(s')-\widetilde{\gamma}(s)|$ with

$$
|\widetilde{\gamma}(s')-\widetilde{\gamma}(s)| \le |s'-s| \le \mu(Q) \le \eta
$$

by Lemma 3.40. The first error term, coming from their total contribution to $\ell(\widetilde{\gamma}(Q))$, is thus $\le (k+1)\eta$.

Similarly each of the constituent intervals $I^\#$ of I contributes to $\ell(\widetilde{\gamma}(Q^\#))$ an amount $\le \eta$. Each such interval $I^\#$ contains a point of Q^* at one end or the other or possibly in its interior. The number of constituent intervals is $\le 2(k+1)$, and the total contribution to $\ell(\widetilde{\gamma}(Q^\#))$ from the constituents of the exceptional I is $\le 2(k+1)\eta$. Since at most $k+1$ intervals I are exceptional, the second error term, coming from the total contribution to $\ell(\widetilde{\gamma}(Q^\#))$, is $\le 2(k+1)^2\eta$.

Taking the two error terms into account and using $(*)$, we see that

$$
|\ell(\widetilde{\gamma})-\ell(\widetilde{\gamma}(Q))| \le \epsilon/3+(k+1)\eta+2(k+1)^2\eta \le \epsilon/3+\epsilon/3+\epsilon/3=\epsilon.
$$

Since $|\ell(\widetilde{\gamma})-\ell(\widetilde{\gamma}(Q))| = |\ell(\gamma)-\ell(P)|$, this inequality proves the lemma. $\square$

PROOF OF THEOREM 3.39. Let M be an upper bound for $|f|$ on the image of γ. Let $s : [a, b] \to [0, \ell(\gamma)]$ be the cumulative arc length function from $\gamma(a)$ given by $s(t) = \ell(\gamma|_{[a,t]})$, and let $\omega : [0, \ell(\gamma)] \to [a, b]$ be its inverse function. Define $\widetilde{\gamma} = \gamma \circ \omega$. If $P = \{t_j\}_{j=0}^m$ is a partition of $[a, b]$, then $\omega^{-1}(P) = \{s_j\}_{j=0}^m$

is a partition of $[0, \ell(\gamma)]$, and

$$\begin{aligned}
\Big| \int_0^{\ell(\gamma)} f(\widetilde{\gamma}(s))\,ds - \sum_{j=1}^m f(\gamma(t_{j-1}))|\gamma(t_j) - \gamma(t_{j-1})|\Big| \\
&= \Big| \int_0^{\ell(\gamma)} f(\widetilde{\gamma}(s))\,ds - \sum_{j=1}^m f(\widetilde{\gamma}(s_{j-1}))|\widetilde{\gamma}(s_j) - \widetilde{\gamma}(s_{j-1})|\Big| \\
&\le \Big| \int_0^{\ell(\gamma)} f(\widetilde{\gamma}(s))\,ds - \sum_{j=1}^m f(\widetilde{\gamma}(s_{j-1}))|s_j - s_{j-1}|\Big| \\
&\quad + \Big| \sum_{j=1}^m f(\widetilde{\gamma}(s_{j-1}))\big(|s_j - s_{j-1}| - |\widetilde{\gamma}(s_j) - \widetilde{\gamma}(s_{j-1})|\big)\Big|.
\end{aligned}$$

The first of the two terms on the right side of the inequality is the error term in approximating a Riemann integral by a Riemann sum, and Theorem 1.35 shows that it tends to 0 as the mesh tends to 0. By Lemma 3.40 the second of the two terms is

$$\begin{aligned}
&\le M \sum_{j=1}^m \big(|s_j - s_{j-1}| - |\widetilde{\gamma}(s_j) - \widetilde{\gamma}(s_{j-1})|\big) = M\big(s_m - s_0 - \ell(\widetilde{\gamma}(\omega^{-1}(P)))\big) \\
&= M\big(\ell(\widetilde{\gamma}) - \ell(\gamma\omega(\omega^{-1}P))\big) = M\big(\ell(\gamma) - \ell(\gamma(P))\big),
\end{aligned}$$

and Lemma 3.41 shows that this expression tends to 0 as the mesh tends to 0. □

To be able to make calculations, we introduce a niceness condition on rectifiable arcs. In Section 2 we said that an $\mathbb{R}^n$ valued function on an open set is of class C^1 if it is everywhere differentiable and if its derivative is continuous. We need to extend this definition to allow the domain to be a closed interval. To do so, we say that a simple arc $\gamma : [a, b] \to \mathbb{R}^n$ is **tamely behaved** if it is of class C^1 on (a, b) and if near each endpoint, each entry of γ' has the property of being either bounded below or bounded above (or both).[5]

Theorem 3.42. If $\gamma : [a, b] \to \mathbb{R}^n$ is a tamely behaved simple arc, then γ is rectifiable, and

$$\ell(\gamma) = \lim_{\substack{a'\downarrow a,\ b'\uparrow b,\\ a<a'<b'<b}} \int_{a'}^{b'} |\gamma'(t)|\,dt$$

[5]Other authors use other concepts here, and the names for them vary. The notion of "tamely behaved" on $[a, b]$ is emphatically different from the notion of having a continuous derivative on $[a, b]$ in the sense of Section A2 of Appendix A, and the extra generality here is vital. The exact notion that is needed is that $|\gamma'|$ is "Lebesgue integrable" on $[a, b]$, as will be shown in Section V.10, but "tamely behaved" is sufficient for the theory in this chapter. Example 1 following Proposition 3.37 shows that it would be too restrictive to assume as in Section A2 of Appendix A that γ is of class C^1 on (a, b) and that the derivative has a *finite* one-sided limit at each endpoint, and Example 2 shows that we encounter nonrectfiable arcs if we assume instead that γ is of class C^1 on (a, b) and extends beyond the endpoints so as to be everywhere differentiable.

REMARKS. The Riemann integral is well defined for each a' and b' since $|\gamma'(t)|$ is continuous, and the limit indicates that the length is obtained by passing to the limit as a' and b' tend to a and b. The limits as a' decreases to a and b' increases to b can be taken in either order or in any joint fashion, according to Theorem 1.13. One frequently writes this formula in shortcut language as

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

even though Riemann integrals are not defined for unbounded functions.[6] Recall that the cumulative arc length function is given by $s(t) = \ell(\gamma_{[a,t]})$. Then the above formula shows that

$$\boxed{s(t) = \int_a^t |\gamma'(u)|\,du}$$

PROOF. With a' and b' fixed such that $a < a' < b' < b$, Proposition 3.37 shows that $\gamma_{[a',b']}$ is rectifiable. Let $\epsilon > 0$ be given. Choose $\delta_1 > 0$ small enough by Lemma 3.41 so that any partition P of $[a', b']$ with $\mu(P) < \delta$ has $|\ell(\gamma_{[a',b']}) - \ell(\gamma_{[a',b']}(P))| < \epsilon$, choose $\delta_2 > 0$ small enough by Theorem 1.35 so that $\big|\sum_{j=1}^m |\gamma'(t_j)|(t_j - t_{j-1}) - \int_{a'}^{b'} |\gamma'(t)\,dt\big| < \epsilon$, and choose $\delta_3 > 0$ small enough by uniform continuity (Theorem 1.10) so that $|t' - t| < \delta_3$ implies $|\gamma'(t') - \gamma'(t)| < \epsilon$. Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then any partition P of $[a', b']$ with $\mu(P) < \delta$ satisfies all three of the following conditions:

$$\left.\begin{aligned} |\ell(\gamma_{[a',b']}) - \ell(\gamma_{[a',b']}(P))| &< \epsilon \\ \Big|\sum_{j=1}^m |\gamma'(t_j)|(t_j - t_{j-1}) - \textstyle\int_{a'}^{b'} |\gamma'(t)\,dt\Big| &< \epsilon \\ |\gamma'(t') - \gamma'(t)| < \epsilon \ \text{ whenever } t \text{ and } t' &\text{ are in the same interval of } P. \end{aligned}\right\} \qquad (*)$$

Let $P = \{t_j\}_{j=0}^m$ be any such partition of $[a', b']$. Then

$$\ell(\gamma_{[a',b']}(P)) = \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})| \qquad (**)$$

by definition. By the Mean Value Theorem, to each i and j corresponds a real number $t_{i,j}^{\#}$ with $t_{j-1} < t_{i,j}^{\#} < t_j$ such that the i^{th} component of $\gamma(t_j) - \gamma(t_{j-1})$

[6] Lebesgue integrals are introduced in Chapter V. The integral in this theorem can be interpreted as a Lebesgue integral, and then no limit sign is needed.

is of the form

$$\begin{aligned}\gamma_i(t_j) - \gamma_i(t_{j-1}) &= \gamma_i'(t_{i,j}^{\#})(t_j - t_{j-1}) \\ &= \big(\gamma_i'(t_j) + (\gamma_i'(t_{i,j}^{\#}) - \gamma_i'(t_j))\big)(t_j - t_{j-1}) \\ &= (\gamma_i'(t_j) + \varphi_i^{(j)})(t_j - t_{j-1}),\end{aligned}$$

say, with $|\varphi_i^{(j)}| \le \epsilon$. If $\varphi^{(j)}$ denotes the vector whose i^{th} entry is $\varphi_i^{(j)}$, then the triangle inequality gives

$$|\gamma(t_j) - \gamma(t_{j-1}) - \gamma'(t_j)(t_j - t_{j-1})| \le |\varphi^{(j)}|(t_j - t_{j-1}) \le n^{1/2}\epsilon(t_j - t_{j-1}),$$

n being the dimension. By the triangle inequality,

$$\big||\gamma(t_j) - \gamma(t_{j-1})| - |\gamma'(t_j)|(t_j - t_{j-1})\big| \le n^{1/2}\epsilon(t_j - t_{j-1}). \tag{†}$$

Summing (†) on j and again using the triangle inequality, we obtain

$$\begin{aligned}&\Big|\sum_{j=1}^{k}|\gamma(t_j) - \gamma(t_{j-1})| - \sum_{j=1}^{k}|\gamma'(t_j)|(t_j - t_{j-1})\Big| \\ &\qquad \le \sum_{j=1}^{k}\big||\gamma(t_j) - \gamma(t_{j-1})| - |\gamma'(t_j)|(t_j - t_{j-1})\big| \le n^{1/2}\epsilon(b' - a').\end{aligned} \tag{††}$$

Therefore

$$\begin{aligned}\big|\ell(\gamma_{[a',b']}) - \textstyle\int_{a'}^{b'}|\gamma'(t)\,dt\big| \le{}& \big|\ell(\gamma_{[a',b']}) - \ell(\gamma_{[a',b']}(P))| \\ &+ \Big|\ell(\gamma_{[a',b']}(P)) - \sum_{j=1}^{m}|\gamma(t_j) - \gamma(t_{j-1})|\Big| \\ &+ \Big|\sum_{j=1}^{m}|\gamma(t_j) - \gamma(t_{j-1})| - \sum_{j=1}^{m}|\gamma'(t_j)|(t_j - t_{j-1})\Big| \\ &+ \Big|\sum_{j=1}^{m}|\gamma'(t_j)|(t_j - t_{j-1}) - \textstyle\int_{a'}^{b'}|\gamma'(t)\,dt\Big|.\end{aligned}$$

The first line on the right side of the above display is $\le \epsilon$ by the first condition in $(*)$, the second line is 0 by $(**)$, the third line is $\le n^{1/2}\epsilon(b'-a')$ as a consequence of (††), and the fourth line is $\le \epsilon$ by the second condition in $(*)$. Thus the left side is $\le (2 + n^{1/2}(b' - a'))\epsilon$, and the proof is complete.

This proves the formula $\ell(\gamma|_{[a',b']}) = \int_{a'}^{b'}|\gamma'(t)|\,dt$. We are left with proving that γ is rectifiable on $[a, b]$ as a consequence of the fact that γ is tamely behaved, since the limit formula for $\ell(\gamma)$ will then follow from Proposition 3.38. Theorem 1.13, which is an interchange-of-limits result, shows that the two endpoints a and b operate independently of each other, and it will be enough by symmetry to treat b. Thus we want to see that γ is rectifiable on $[a', b]$, and the relevant assumption is that each entry of $\gamma'(t)$ is bounded below near b, or is bounded above near b, or both.

Imagine a fixed partition and the computation of the polygonal length from it. A typical term is of the form $|\gamma(t_j) - \gamma(t_{j-1})|$, where t_{j-1} and t_j are consecutive points of the partition. If an entry of the column vector $\gamma(t)$ is replaced by its negative, the value of the term in the computation of the partition does not change. Thus we may assume that each entry of $\gamma'(t)$ is bounded below.

Next we can replace γ by the sum of it and any rectifiable arc $\zeta(t)$, and the effect on the computation will be harmless, as a consequence of the triangle inequality. The rectifiable arc we choose is one of the form $\zeta(t) = tc$, where c is a vector of constants. (This is rectifiable by Proposition 3.37, for example.) With the entries of c chosen large enough, the effect will be to make all the entries of $\gamma'(t)$ be everywhere positive on $[a', b]$. Choosing the vector of constants suitably, we can arrange that every entry of $\gamma'(t)$ is ≥ 0.

Thus we may assume that $\gamma'(t)$ is continuous on $[a', b)$ and that every entry $\gamma_i(t)$ of $\gamma(t)$ is positive there. Hence every entry $\gamma_i(t)$ is a nondecreasing function. We make use of the inequality

$$\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{1/2} \leq \sum_{i=1}^{n} |c_i|,$$

which follows by squaring both sides, canceling the squared terms, and observing that the left side reduces to 0 while the right side reduces to the sum of nonnegative terms. Using this inequality we compute that

$$\begin{aligned}
\sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})| &= \sum_{j=1}^{k} \Big(\sum_{i=1}^{n} (\gamma_i(t_j) - \gamma_i(t_{j-1}))^2\Big)^{1/2} \\
&\leq \sum_{j=1}^{k} \sum_{i=1}^{n} |\gamma_i(t_j) - \gamma_i(t_{j-1})| \\
&\leq \sum_{j=1}^{k} \sum_{i=1}^{n} (\gamma_i(t_j) - \gamma_i(t_{j-1})) \quad \text{since } \gamma_i \text{ is nondecreasing} \\
&= \sum_{i=1}^{n} (\gamma_i(b) - \gamma_i(a')),
\end{aligned}$$

and this is bounded independently of the partition. Thus γ is rectifiable. □

Corollary 3.43. If $\gamma : [a, b] \to \mathbb{R}^n$ is a tamely behaved simple arc and f is a continuous complex-valued function on the image of γ, then the integral of f over γ with respect to arc length is given by

$$\int_\gamma f\,ds = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

PROOF. Theorem 3.39 and the boxed formula in the remarks with Theorem 3.42 give

$$\int_\gamma f\,ds = \int_0^{\ell(\gamma)} f(\widetilde{\gamma}(s))\,ds = \int_a^b f(\widetilde{\gamma}(s(t)))\tfrac{ds}{dt}\,dt = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt. \quad \square$$

EXAMPLE. Cycloid. A **cycloid** is the locus of points swept out by a point on the circumference of a circle when the circle rolls without slipping along a straight line. When the radius of the circle is r and the circle rolls along the x-axis starting from the origin, the parametric equations are

$$\begin{aligned} x &= r(t - \sin t) \\ y &= r(1 - \cos t). \end{aligned}$$

A plot appears in Figure 3.2. The cusps occur on the x-axis when y equals 0, hence when $t = 2\pi m$ for some integer m, and the corresponding value of x is $2\pi m r$.

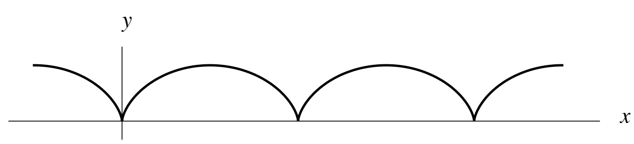

FIGURE 3.2. Cycloid.

Theorem 3.42 says that the length $s(t)$ of the locus swept out from 0 to t satisfies

$$\begin{aligned} \tfrac{ds}{dt} &= \sqrt{\left(\tfrac{d}{dt}(r(t - \sin t))\right)^2 + \left(\tfrac{d}{dt}(r(1 - \cos t))\right)^2} \\ &= r\sqrt{2 - 2\cos t} \\ &= 2r|\sin(t/2)|. \end{aligned}$$

Therefore $$s(t) = 4r(1 - \cos(t/2)) \qquad \text{for } 0 \le t \le 2\pi.$$

The end of the first arch corresponds to $t = 2\pi$, and the length of the arch is $s(2\pi) = 8r$. If, say, we build a fence over the first arch of the cycloid with height $(1 - \cos t)$ in the z direction above $(x(t), y(t))$, then Corollary 3.43 allows us to use the formula $\frac{ds}{dt} = 2r\,\sin(t/2)$ to compute the area of the fence as

$$\text{area} = \int_{\substack{\text{first}\\ \text{arch}}} (\text{height})\,ds = \int_0^{2\pi} (1 - \cos t)(2r\,\sin(t/2))\,dt.$$

The first cusp occurs at the point $(x, y) = (2\pi r, 0)$, which corresponds to $t = 2\pi$. Notice that the simultaneous C^1 behavior of $x(t)$ and $y(t)$ at $t = 2\pi$ is insufficient to rule out corners and other irregular behavior for the curve. However, there is a sufficient condition to rule out such irregular behavior: if $(x(t), y(t))$ is a parametrically defined C^1 curve in $\mathbb{R}^2$ and at least one of $x'(t_0)$ and $y'(t_0)$ is nonzero, as is the case here at $t = 2\pi$, then irregular behavior can be ruled out near $t = t_0$. The reason is that the Inverse Function Theorem in principle allows us locally to solve one of $x(t)$ and $y(t)$ for t near $t = t_0$ and to substitute the result into the other of $x(t)$ and $y(t)$. The result is that one of x and y is exhibited as a C^1 function of the other.

12. Line Integrals and Conservative Vector Fields

While integrals with respect to arc length are motivated by the geometric problem of calculating the area under the graph of a numerical-valued function on a parametrically defined curve in the plane, line integrals are motivated by physical considerations. In physics the work done (energy expended) by a constant force in moving an object is the product of the force by the displacement.[7] More precisely force is given as a vector quantity, say F; the displacement is given by another vector quantity, say $\mathbf{s}$; and the product in question is the dot product $F \cdot \mathbf{s}$. In particular, no work is done when the motion is perpendicular to the force.

When the force varies from point to point and the object moves along a curve, it is natural to think of replacing the product $F \cdot \mathbf{s}$ by a sum of contributions from successive small displacements $\sum F_j \cdot (\mathbf{s}_j - \mathbf{s}_{j-1})$ and to hope for a realistic answer in the limit as the displacements tend to 0. This situation arises in the case of motion in an electrical, gravitational, or magnetic field. The mathematical object that abstracts this kind of field in physics is a "vector field."

A **vector field** on a subset U of $\mathbb{R}^n$ is a function $F : U \to \mathbb{R}^n$. The vector field is **continuous** if F is a continuous function.[8] The traditional geometric interpretation of F, particularly when $n = 2$ or $n = 3$, is to attach to each point p of U the vector $F(p)$ as an arrow based at p. This interpretation is appropriate, for example, if F represents the velocity vector at each point in space of a time-independent fluid flow. It is appropriate also for an electrical, gravitational, or magnetic field. Let an object (or particle) move along a parametrically defined curve $\gamma(t)$ in $\mathbb{R}^n$, starting at $t = a$ and ending at $t = b$.

For the moment we suppose that the curve is a simple rectifiable arc. It will be important eventually to allow for more general parametrically defined curves, such as ones that retrace themselves and ones whose value at a equals the value at

[7]The exact wording is important. Careless wording can lead to an answer that differs in a sign from the usual notion of work.

[8]In Section IV.4 we shall consider "smooth vector fields." In this case we shall require U to be open in $\mathbb{R}^n$, so that the notion of a smooth function is meaningful.

b. But for now, we stick with rectifiable simple arcs. We think of computing an approximation to the work done by the force as a sum of amounts involving small displacements. If $P = \{t_j\}_{j=1}^m$ is a partition of $[a, b]$, the vector $F(\gamma(t_{j-1}))$ plays the role of F_j in the above formula, and the displacements $\mathbf{s}_j - \mathbf{s}_{j-1}$ above are the differences $\gamma(t_j) - \gamma(t_{j-1})$. Then the work done by the force field in moving the particle along the curve is to be approximately given by a sum

$$\sum_{j=1}^{m} F(\gamma(t_{j-1})) \cdot (\gamma(t_j) - \gamma(t_{j-1})).$$

The hope is that a suitable limit of this quantity exists and can be computed.

One virtue of formulating work as a limit of a sum of this kind is that we can see by inspection that the answer is independent of the parametrization as long as a reparametrization is orientation-preserving. In fact, let $\gamma_1 : [a_1, b_1] \to \mathbb{R}^n$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^n$ be simple arcs related by $\gamma_1 = \gamma_2 \circ \varphi$, where $\varphi : [a_1, b_1] \to [a_2, b_2]$ is continuous, has a continuous inverse, and has $\varphi(a_1) = a_2$. Then the same kind of computation as for arc length before Proposition 3.37 shows that the approximating sum for γ_1 using the partition $P = \{t_j\}_{j=0}^m$ is equal to the approximating sum for γ_2 using the partition $\varphi(P) = \{u_j\}_{j=0}^k$, where $u_j = \varphi(t_j)$. That being said, the existence theorem is as follows.

Theorem 3.44 (Existence Theorem). If $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable simple arc and F is a continuous vector field on the image of γ, then there exists a unique number, denoted $\int_\gamma F \cdot d\mathbf{s}$, with the following property. For any $\epsilon > 0$, there exists a $\delta > 0$ such that any partition $P = \{t_j\}_{j=0}^m$ of $[a, b]$ with $\mu(P) < \delta$ has

$$\Big| \int_\gamma F \cdot d\mathbf{s} - \sum_{j=1}^{m} F(\gamma(t_{j-1})) \cdot \big(\gamma(t_j) - \gamma(t_{j-1})\big) \Big| < \epsilon.$$

REMARKS. The number $\int_\gamma F \cdot d\mathbf{s}$ is called the **line integral of F over γ**. In this generality and unlike in the case of integration with respect to arc length, a line integral is not given in terms of a Riemann integral. Instead it is given in terms of a generalization of the Riemann integral called a "Stieltjes integral." Stieltjes integrals are not developed in this book other than in problems at the end of this chapter,[9] and accordingly we omit the proof of Theorem 3.44.

[9]The defining properties of "Stieltjes integration" as in the proof are simple enough that they can be summarized here. The first relevant fact is that for any partition $P = \{t_j\}_{j=0}^m$ of $[a, b]$, each component $\gamma_i(t)$ of $\gamma(t)$ satisfies an inequality $\sum_{j=1}^m |\gamma_i(t_j) - \gamma_i(t_{j-1})| \le \sum_{j=1}^m |\gamma(t_j) - \gamma(t_{j-1})| \le \ell(\gamma)$, a condition summarized in the language of Section VI.9 below by saying that γ_i is of "bounded variation" on $[a, b]$. Proposition 6.54 below will show that such a function is the difference of two

Let us observe from the approximation formula in Theorem 3.44 that if $-\gamma$ denotes the reverse simple arc of γ, then the line integral of F over $-\gamma$ is the *negative* of the line integral of F over γ.

We turn now to the question of obtaining a useful formula for the value of a line integral. We make the same assumption as in the case of arc length: that the simple arc γ is tamely behaved. Theorem 3.42 ensures that γ is rectifiable.

Theorem 3.45. If $\gamma : [a, b] \to \mathbb{R}^n$ is a tamely behaved simple arc and if F is a continuous vector field on the image of γ, then the line integral of F over γ, which exists by Theorem 3.44, is given by

$$\int_\gamma F \cdot d\mathbf{s} = \lim_{\substack{a'\downarrow a,\ b'\uparrow b,\\ a<a'<b'<b}} \int_{a'}^{b'} F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

REMARKS.

(1) The proof will follow these remarks and an example.

(2) The limit sign is present in the formula because we are allowing $\gamma'(t)$ to be unbounded near either endpoint. It is customary to write the integral as $\int_a^b$ in every case even though Riemann integrals are not defined for unbounded functions,[10] and we shall follow this convention once Theorem 3.45 has been proved. Then the displayed formula in the theorem becomes

$$\boxed{\int_\gamma F \cdot d\mathbf{s} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt}$$

(3) Before this theorem the expression $d\mathbf{s}$ was only symbolic; it meant nothing mathematically. Now we can see that it provides a handy reminder that equality in the boxed formula comes by taking $d\mathbf{s}/dt = \gamma'(t)$. (For comparison we know from Theorem 3.42 that the derivative of the cumulative arc length of a tamely behaved simple arc is $ds/dt = |\gamma'(t)|$.)

(4) The Schwarz inequality gives

$$\left| \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt \right| \le \int_a^b |F(\gamma(t)) \cdot \gamma'(t)|\,dt \le \int_a^b |F(\gamma(t))||\gamma'(t)|\,dt,$$

nondecreasing functions. There is even a natural such decomposition as a difference. (Problem 23 at the end of Chapter VI will show for this case that the two nondecreasing functions are continuous, and this fact simplifies the relevant theory somewhat.) Stieljes integration is defined relative to a nondecreasing function α in a way similar to that for Riemann integration. The new ingredient is that whenever the length of an interval $[a', b']$ appears in an argument, it is to be replaced by $\alpha(b') - \alpha(a')$. For more details see Problems 24–26 at the end of this chapter.

[10]Lebesgue integrals are introduced in Chapter V. The integral in this theorem can be interpreted as a Lebesgue integral, and then no limit sign is needed.

which translates into a way of estimating a line integral in terms of an integral with respect to arc length:

$$\Big|\int_\gamma F \cdot d\mathbf{s}| \le \int_\gamma |F|\, ds$$

(5) The traditional way of writing the line integral is as

$$\int_\gamma F \cdot d\mathbf{s} = \int_\gamma F_1\, dx_1 + \cdots + F_n\, dx_n$$

with just the one integral sign for all n terms. This too is handy notation, since it points to an evaluation procedure for the line integral that correctly gives the formula of the theorem. Namely for each argument x_i of F, we substitute $x_i = \gamma_i(t)$, and for each dx_i we use the formula $dx_i = \frac{dx_i}{dt}\, dt = \gamma_i'(t)\, dt$. Here is an example.

EXAMPLE. To evaluate the line integral of the vector field $F(x, y, z) = (x, x^2y^2, x^3z^3)$ over the simple arc $\gamma(t) = (t^5, t^4, 1)$, defined for $0 \le t \le 1$, we use $\gamma'(t) = (5t^4, 4t^3, 0)$ and compute

$$\begin{aligned}\textstyle\int_\gamma x\, dx + x^2y^2\, dy + x^3z^3\, dz \\ &= \textstyle\int_0^1 t^5(5t^4\, dt) + (t^5)^2(t^4)^2(4t^3\, dt) + (t^5)^3(1)^3 0\, dt \\ &= \textstyle\int_0^1 (5t^9 + 4t^{21} + 0)\, dt = \frac{5}{10} + \frac{4}{22} = \frac{1}{2} + \frac{2}{11} = \frac{15}{22}.\end{aligned}$$

PROOF OF THEOREM 3.45. For the moment fix a' and b' such that $a < a' < b' < b$. We prove the formula of the theorem on $[a', b']$. Write the values of F in terms of the standard basis $\{e_i\}$ of $\mathbb{R}^n$ as $F(t) = \sum_{i=1}^n F_i(t)e_i$. By linearity it is enough to handle a single F_i. Fix that i.

Let $\epsilon > 0$ be given. Choose $\delta_1 > 0$ by Theorem 3.44 to be small enough so that any partition $P = \{t_j\}_{j=0}^m$ of $[a', b']$ with $\mu(P) < \delta_1$ has

$$\Big| \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\, dt - \sum_{j=1}^m F_i(\gamma(t_{j-1}))\big(\gamma_i(t_j) - \gamma_i(t_{j-1})\big)\Big| < \epsilon. \qquad (*)$$

Choose $\delta_2 > 0$ small enough by Theorem 1.35 so that any partition $P = \{t_j\}_{j=0}^m$ of $[a', b']$ with $\mu(P) < \delta_2$ has

$$\Big| \sum_{j=1}^m F_i(\gamma(t_{j-1}))\gamma_i'(t_{j-1})(t_j - t_{j-1}) - \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\, dt\Big| < \epsilon. \qquad (**)$$

Let C be an upper bound for $|F_i|$ on the image of γ. Choose $\delta_3 > 0$ by uniform continuity of $\gamma'(t)$ on $[a', b']$ (Theorem 1.10) so that $|\gamma'(t') - \gamma'(t)| < C^{-1}(b' - a')^{-1}\epsilon$ whenever t' and t are members of $[a', b']$ with $|t' - t| < \delta_3$.

Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then any partition P of $[a', b']$ with $\mu(P) < \delta$ satisfies $(*)$, $(**)$, and

$$|\gamma_i'(t^{\#}_{i,j}) - \gamma_i'(t_{j-1})| < C^{-1}(b' - a')^{-1}\epsilon \quad \text{whenever } t_{j-1} \le t^{\#}_{i,j} \le t_j. \qquad (\dagger)$$

Let $P = \{t_j\}_{j=0}^m$ be any such partition. By the Mean Value Theorem the i^{th} component of $\gamma(t_j) - \gamma(t_{j-1})$ is of the form

$$\begin{aligned}\gamma_i(t_j) - \gamma_i(t_{j-1}) &= \gamma_i'(t^{\#}_{i,j})(t_j - t_{j-1}) \\ &= \big(\gamma_i'(t_{j-1}) + (\gamma_i'(t^{\#}_{i,j}) - \gamma_i'(t_{j-1}))\big)(t_j - t_{j-1}) \\ &= (\gamma_i'(t_{j-1}) + \varphi_i^{(j)})(t_j - t_{j-1}),\end{aligned}$$

say, and $(\dagger)$ shows that $|\varphi_i^{(j)}| \le C^{-1}(b' - a')^{-1}\epsilon$. We estimate

$$\begin{aligned}&\Big|\int_{\gamma_{[a',b']}} F_i e_i \cdot d\mathbf{s} - \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\,dt\Big| \\ &\quad\le \Big|\int_{\gamma_{[a',b']}} F_i e_i \cdot d\mathbf{s} - \sum_{j=1}^m F_i(\gamma(t_{j-1}))e_i \cdot \big(\gamma(t_j) - \gamma(t_{j-1})\big)\Big| \\ &\qquad + \Big|\sum_{j=1}^m F_i(\gamma(t_{j-1}))\big(\gamma_i(t_j) - \gamma_i(t_{j-1})\big) - \sum_{j=1}^m F_i(\gamma(t_{j-1}))\gamma'(t_{j-1})(t_j - t_{j-1})\Big| \\ &\qquad + \Big|\sum_{j=1}^m F_i(\gamma(t_{j-1}))\gamma'(t_{j-1})(t_j - t_{j-1}) - \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\,dt\Big|.\end{aligned}$$

The first line of the right side of the above inequality is $< \epsilon$ by $(*)$, and the third line is $< \epsilon$ by $(**)$. The second line on the right side is equal to

$$\begin{aligned}&\Big|\sum_{j=1}^m F_i(\gamma(t_{j-1}))\big(\gamma_i(t_j) - \gamma_i(t_{j-1}) - \gamma'(t_{j-1})(t_j - t_{j-1})\big)\Big| \\ &\qquad = \Big|\sum_{j=1}^m F_i(\gamma(t_{j-1}))(\varphi_i^{(j)})(t_j - t_{j-1})\Big| \\ &\qquad \le \sum_{j=1}^m |F_i(\gamma(t_{j-1}))||\varphi_i^{(j)}|(t_j - t_{j-1}) \\ &\qquad \le \sum_{j=1}^m C(C^{-1}(b' - a')^{-1}\epsilon)(t_j - t_{j-1}) \\ &\qquad = \epsilon.\end{aligned}$$

The whole right side is therefore $< 3\epsilon$.

Since ϵ is arbitrary, $\big|\int_{\gamma_{[a',b']}} F_i e_i \cdot d\mathbf{s} - \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\,dt\big| = 0$. Hence $\int_{\gamma_{[a',b']}} F_i e_i \cdot d\mathbf{s} = \int_{a'}^{b'} F_i(\gamma(t))\gamma_i'(t)\,dt$, and the proof is complete for the interval $[a', b']$.

To complete the proof, we need to consider the effects from near the endpoints. For the right endpoint it is enough to show that

$$\limsup_{b'\uparrow b} \Big| \sum_{\substack{1\le j\le m,\\ b'\le t_j\le b}} F_i(\gamma(t_{j-1}))e_i \cdot \big(\gamma(t_j) - \gamma(t_{j-1})\big)\Big| = 0 \tag{††}$$

and

$$\limsup_{b'\uparrow b,\ b'\le b''} \Big| \int_{b'}^{b''} F_i(\gamma(t))\gamma_i'(t)\,dt \Big| = 0. \tag{‡}$$

Still with C as an upper bound for $|F_i|$ on the image of γ, we have

$$\begin{aligned}\Big| \sum_{\substack{1\le j\le m,\\ b'\le t_j\le b}} F_i(\gamma(t_{j-1}))e_i \cdot \big(\gamma(t_j) - \gamma(t_{j-1})\big)\Big| &\le C \sum_{\substack{1\le j\le m,\\ b'\le t_j\le b}} \big|\gamma(t_j) - \gamma(t_{j-1})\big| \\ &\le C\ell(\gamma|_{[b',b]}),\end{aligned}$$

and the right side has limit 0 as $b' \uparrow b$ by Proposition 3.38. This proves (††). For (‡) we have

$$\Big| \int_{b'}^{b''} F_i(\gamma(t))\gamma_i'(t)\,dt \Big| \le C \int_{b'}^{b''} |\gamma'(t)|\,dt,$$

and Theorem 3.42 shows that the right side is $= C\ell(\gamma|_{[b',b'']})$. In turn this is $\le C\ell(\gamma|_{[b',b]})$, which has limit 0 as $b' \uparrow b$ by Proposition 3.38.

This proves (‡). A similar argument applies to handle the left endpoint of $[a, b]$ and completes the proof of the theorem. □

Now we enlarge the definition of the kind of parametrically defined curve we consider, no longer restricting ourselves to simple rectifiable arcs. A continuous function $\gamma : [a, b] \to \mathbb{R}^n$ is said to be a **piecewise C^1 curve** if there is a partition $P_0 = \{c_j\}_{j=0}^m$ of $[a, b]$ such that each $\gamma|_{[c_{j-1},c_j]}$ for $1 \le j \le m$ is a tamely behaved simple arc in the sense of the previous section. Piecewise C^1 curves can cross themselves and can even retrace their steps. Most parametrically defined curves that arise in practice are piecewise C^1. The piecewise C^1 curve $\gamma : [a, b] \to \mathbb{R}^n$ is said to be **closed** if $\gamma(a) = \gamma(b)$. The adjective **simple** is sometimes used in connection with closed curves; it means that $\gamma(a) = \gamma(b)$ but that otherwise γ is one-one.

If $\gamma : [a, b] \to \mathbb{R}^n$ is a piecewise C^1 curve given relative to a partition P_0 as above and if F is a vector field on the image of γ, then the definition of the **line integral of F over γ** extends to this situation by the formula

$$\int_\gamma F \cdot d\mathbf{s} = \sum_{j=1}^{m} \int_{\gamma_{[c_{j-1}, c_j]}} F \cdot d\mathbf{s}.$$

As far as line integrals go, the value of a line integral over a single constituent simple arc of the piecewise C^1 curve γ is unchanged by any orientation-preserving reparametrization, as we know. However, the value changes in sign if the reparametrization is orientation-reversing. For this reason it is common in diagrams of piecewise C^1 curves to indicate the direction that such a curve is traced out.

One often encounters line integrals over piecewise C^1 curves of the kind shown in Figure 3.3, in which geometrically one of the segments is a reparametrization of the reverse of another. Then the contributions to the line integral from the two segments cancel.

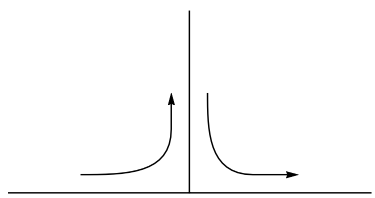

FIGURE 3.3. A piecewise C^1 curve that retraces part of itself.

In electrostatics the (vector) field resulting from a configuration of charges has an accompanying potential, a scalar-valued function whose value at a point gives the change in potential energy of moving a unit charge from infinity to that point. The existence of that potential imposes a condition on the vector field, and we now study that condition.

Any connected open subset of $\mathbb{R}^n$ is called a **region**.

Lemma 3.46. Any two points in a region U of $\mathbb{R}^n$ can be connected by a piecewise C^1 curve.

PROOF. Fix p in U, and let E be the set E of points in U that can be connected to p by a piecewise C^1 curve. The set E is nonempty, and it is open since it certainly contains an open ball about any of its members. To see that it is relatively closed in U, suppose that $\{x_k\}$ is a sequence in E with a limit point q in U. Choose an open ball about q that is contained in U. Some x_k lies in the ball, and x_k can be connected to q by a line segment within the ball. Since x_k can be connected to

q by a piecewise C^1 curve in U, the extension of the curve by the straight line segment is a piecewise C^1 curve connecting p to q. Therefore E is relatively closed. Since U is connected, $E = U$. $\square$

If f is a C^1 numerical-valued function on an open subset U of $\mathbb{R}^n$, the **gradient** of f, denoted ∇f, is the vector field given by the transpose of the row vector $[f'(x)]$ for the derivative $f'(x)$, namely[11]

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Proposition 3.47. If F is a continuous vector field on a region U of $\mathbb{R}^n$, then F is the gradient of a C^1 numerical-valued function on U if and only if the values of the line integrals $\int_\gamma F \cdot d\mathbf{s}$ over piecewise C^1 curves $\gamma : [a, b] \to U$ depend only on the endpoints $\gamma(a)$ and $\gamma(b)$ and not on the values of $\gamma(t)$ for $a < t < b$.

REMARKS. Briefly F is a gradient if and only if "line integrals of F in U are independent of the path." In this case we say that the vector field F is **conservative**.

PROOF OF NECESSITY. Suppose $F = \nabla f$. We first give the argument under the assumption that $\gamma : [a, b] \to \mathbb{R}^n$ is a tamely behaved simple arc. In this case the chain rule gives

$$\begin{aligned} \int_\gamma F \cdot d\mathbf{s} &= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(\gamma(t))\gamma_i'(t)\,dt \\ &= \int_a^b \tfrac{d}{dt}(f \circ \gamma)(t)\,dt = (f \circ \gamma)(b) - (f \circ \gamma)(a), \end{aligned}$$

and the right side depends only on $\gamma(a)$ and $\gamma(b)$. For a general piecewise C^1 curve γ, we write $\int_\gamma F \cdot d\mathbf{s}$ as a sum of terms $\int_{\gamma|_{[c_{j-1}, c_j]}} F(\gamma(t)) \cdot \gamma'(t)\,dt$ and go through the above argument. The sum of all the terms is then

$$\sum_{j=1}^m [(f \circ \gamma)(c_j) - (f \circ \gamma)(c_{j-1})] = (f \circ \gamma)(b) - (f \circ \gamma)(a),$$

and again the right side depends only on $\gamma(a)$ and $\gamma(b)$. $\square$

PROOF OF SUFFICIENCY. Suppose $\int_\gamma F \cdot d\mathbf{s}$ depends only on $\gamma(a)$ and $\gamma(b)$, and fix a point p in U. If x is given in U, define $f(x) = \int_\gamma F \cdot d\mathbf{s}$ for any

[11]The symbol ∇ is called "nabla."

piecewise C^1 curve γ connecting p to x. Let us see that f is of class C^1 in U and that $F = \nabla f$. Fix attention on points in U in a closed ball centered at x_0. On the closed ball, $|F|$ is bounded, say by M. Such a point x can be connected to x_0 by a straight line segment γ_0, and $|f(x) - f(x_0)| = |\int_{\gamma_0} F \cdot d\mathbf{s}| \le \int_{\gamma_0} |F|\,ds \le M\ell(\gamma_0) = M|x - x_0|$. Hence f is continuous at x_0.

Similarly let us connect x_0 to $x_0 + he_i$ by the straight line $\gamma_0(t) = x_0 + the_i$ defined for t in [0, 1]. Then $\gamma_0'(t) = he_i$, and

$$f(x_0 + he_i) - f(x_0) = \int_{\gamma_0} F \cdot d\mathbf{s} = \int_0^1 F(x_0 + the_i) \cdot he_i\,dt.$$

So

$$\tfrac{1}{h}\big(f(x_0 + he_i) - f(x_0)\big) - F_i(x_0) = \int_0^1 [F_i(x_0 + the_i) - F_i(x_0)]\,dt,$$

and

$$\big|\tfrac{1}{h}\big(f(x_0 + he_i) - f(x_0)\big) - F_i(x_0)\big| \le \int_0^1 |F_i(x_0 + the_i) - F_i(x_0)|\,dt,$$

Given $\epsilon > 0$, choose $\delta > 0$ by continuity of F_i at x_0 so that $|F_i(x) - F_i(x_0)| \le \epsilon$ whenever $|x - x_0| \le \delta$. If $|h| \le \delta$, then the integrand on the right is $\le \epsilon$, and hence so is the integral. Thus $\frac{\partial f}{\partial x_i}(x_0) = F_i(x_0)$, and the sufficiency follows. $\square$

Proposition 3.48. Let F be a C^1 vector field on a region U of $\mathbb{R}^n$, and suppose that $F = \nabla f$ for some scalar-valued f of class C^2 on U. Then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for all } i \text{ and } j.$$

Conversely for $U = \mathbb{R}^n$ if F is a C^1 vector field such that $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all i and j everywhere on $\mathbb{R}^n$, then there exists a C^2 scalar-valued function f on $\mathbb{R}^n$ with $F = \nabla f$.

REMARKS. The converse part depends on the global geometry of the region where F is defined. It holds for $\mathbb{R}^n$, as is asserted, and it extends to any **star-shaped** open set, i.e, a set U having a point p such that for each point x of U, the straight line segment from p to x lies completely in U. It fails in dimension two if U is an **annulus**, i.e., the open set lying between two concentric circles, as is shown in Problem 29 at the end of the chapter.

PROOF. We proceed by induction on the dimension n. If $n = 1$, then a C^1 vector field F is just an ordinary scalar-valued function, and the condition on the partial derivatives of F is vacuous. The function $f(x) = \int_0^x F(u)\,du$ has the required property that $F = \nabla f = \frac{df}{dx}$.

Inductively assume that the result holds for dimension $n-1$. The functions $G_j(x_2,\dots,x_n) = F_j(0,x_2,\dots,x_n)$ have the property that $\frac{\partial G_i}{\partial x_j} = \frac{\partial G_j}{\partial x_i}$ for $i \geq 2$ and $j \geq 2$, and the inductive hypothesis produces a C^2 function $g(x_2,\dots,x_n)$ such that $\frac{\partial g}{\partial x_j} = G_j$ for all $j \geq 2$. Define

$$f(x_1,\dots,x_n) = \int_0^{x_1} F_1(u_1,x_2,\dots,x_n)\,du_1 + g(x_2,\dots,x_n).$$

The $\frac{\partial f}{\partial x_1} = F_1$ by Theorem 1.32a. Also for $j > 1$, we have

$$\begin{aligned}
&\frac{\partial f}{\partial x_j}(x_1,\dots,x_n)\\
&\quad = \frac{\partial}{\partial x_j}\int_0^{x_1} F_1(u_1,x_2,\dots,x_n)\,du_1 + \frac{\partial g}{\partial x_j}(x_2,\dots,x_n)\\
&\quad = \int_0^{x_1} \frac{\partial F_1}{\partial x_j}(u_1,x_2,\dots,x_n)\,du_1 + \frac{\partial g}{\partial x_j}(x_2,\dots,x_n) \qquad \text{by Proposition 3.28b}\\
&\quad = \int_0^{x_1} \frac{\partial F_j}{\partial x_1}(u_1,x_2,\dots,x_n)\,du_1 + \frac{\partial g}{\partial x_j}(x_2,\dots,x_n) \qquad \text{by hypothesis}\\
&\quad = F_j(x_1,x_2,\dots,x_n) - F_j(0,x_2,\dots,x_n) + G_j(x_2,\dots,x_n)\\
&\quad = F_j(x_1,x_2,\dots,x_n),
\end{aligned}$$

as required. □

13. Green's Theorem in the Plane

Green's Theorem in the plane relates a line integral over the boundary of certain kinds of regions to a double integral over the region. The core idea is visible in the case of a closed geometric rectangle, which we discuss in the first example. There the theorem reduces to the Fundamental Theorem of Calculus.

EXAMPLE 1. Green's Theorem for a closed rectangle. Suppose we are given the closed rectangle R with $a \leq x \leq b$ and $c \leq y \leq d$. Let $P(x,y)$ and $Q(x,y)$ be C^1 functions on a region containing R, and let γ denote the boundary of R regarded as a piecewise C^1 curve that is traversed counterclockwise. Then

$$\int_\gamma P\,dx + Q\,dy = \iint_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y}\Big)\,dx\,dy.$$

To see this equality, we start from

$$\int_c^d \Big[\int_a^b \tfrac{\partial Q}{\partial x}\,dx\Big]\,dy = \int_c^d (Q(b,y) - Q(a,y))\,dy$$

and

$$\int_a^b \Big[\int_c^d \tfrac{\partial P}{\partial y}\,dy\Big]\,dx = \int_a^b (P(x,d) - P(x,c))\,dx.$$

The difference of the two left sides equals the double integral by Theorem 3.32, and the difference of the right sides equals the line integral if we understand γ to consist of four parts: the bottom, parametrized by $t \mapsto (t, c)$ for $a \le t \le b$; the right side, parametrized by $t \mapsto (b, t)$ for $c \le t \le d$; the top, parametrized by $t \mapsto (-t, d)$ for $-b \le t \le -a$; and the left side, parametrized by $t \mapsto (a, -t)$ for $-d \le t \le -c$.

Notice in Example 1 that we did not actually write γ as a single piecewise C^1 curve but instead wrote it as four curves. This is an artificial distinction; we could have followed the definition literally by merely using a translation of the parameter for each interval. For example, we could have parametrized the bottom as $(a+t, c)$ for $0 \le t \le b-a$, the right side as $(b, c+t-(b-a))$ for $b-a \le t \le (b-a)+(d-c)$, and so on.

To adapt our definition to be able to handle these matters automatically, we can introduce the notion of a **piecewise C^1 chain**. This is a formal sum of piecewise C^1 curves, say $\gamma = \gamma_1 + \cdots + \gamma_r$ with $r \ge 0$, without regard to the order of the terms. We regard two chains as **equal** if they can be obtained from each other by a sequence of operations of the form

(i) subdivision of an arc,
(ii) fusion of subarcs into a single arc,
(iii) reparametrization of an arc,
(iv) cancellation of a pair of opposite arcs,
(v) insertion of a pair of opposite arcs,
(vi) dropping a one-point arc (with domain of the form $[a, a]$), or
(vii) insertion of a one-point arc.

A line integral over γ is defined as the corresponding sum of line integrals over the constituent piecewise C^1 curves:

$$\int_\gamma F \cdot d\mathbf{s} = \sum_{k=1}^{r} \int_{\gamma_k} F d\mathbf{s}.$$

If two such chains are equal, then all line integrals defined on both are equal.

We denote the reverse of γ by $-\gamma$. If $\gamma = \gamma_1 + \cdots + \gamma_r$ and $\sigma = \sigma_1 + \cdots + \sigma_s$ are chains, let $\gamma + \sigma = \gamma_1 + \cdots + \gamma_r + \sigma_1 + \ldots \sigma_s$. Then $\int_{\gamma+\sigma} F \cdot d\mathbf{s} = \int_\gamma F \cdot d\mathbf{s} + \int_\sigma F \cdot d\mathbf{s}$. We shall write $n\gamma$ for $\gamma + \cdots + \gamma$ (n times) and $-n(\gamma) = n(-\gamma)$ and $0(\gamma) = (\text{empty arc})$. Then every chain can be written as $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$ with the a_j positive integers and the γ_j distinct, and if we allow some coefficients to be 0, then any two chains can be expressed as sums of the same γ_j's.

If we look carefully at Example 1, we see that it admits a generalization.

EXAMPLE 2. The set between two graphs. A certain amount of the argument for Example 1 works if the rectangle is replaced by the set between two graphs. Namely if we replace the y limits c and d in the formula

$$\int_a^b \left[\int_c^d \tfrac{\partial P}{\partial y}\, dy \right] dx = \int_a^b (P(x,d) - P(x,c))\, dx$$

by two functions $f(x)$ and $g(x)$ with $f(x) \le g(x)$, then the integration formula is still meaningful when written as

$$\int_a^b \left[\int_{f(x)}^{g(x)} \tfrac{\partial P}{\partial y}\, dy \right] dx = \int_a^b (P(x,g(x)) - P(x,f(x)))\, dx.$$

At first glance it looks as if the full argument will go through for this more general situation, but there is a difficulty: the corresponding argument for the Q term works for the set between two graphs of functions with x given in terms of y, not y in terms of x. To handle both P and Q this way, the set of integration must look like the set between two graphs in both directions.[12] The closed unit disk $x^2 + y^2 \le 1$ in $\mathbb{R}^2$ provides an example. This is the set between the graphs of $y = -\sqrt{1-x^2}$ and $y = +\sqrt{1-x^2}$, and also it is the set between the graphs of $x = -\sqrt{1-y^2}$ and $x = +\sqrt{1-y^2}$. Notice that our ability to get the argument to go through this way for a disk depends crucially on two points:

(i) Handling the P term and handling the Q term involved two different parametrizations of the boundary circle $x^2 + y^2 = 1$, and it was important that these two parametrizations were related by an orientation-preserving reparametrization.

(ii) The functions whose graphs were involved had unbounded first derivatives. This behavior had to show up for a curve with a well-defined tangent line at every point. Thus the definition of piecewise C^1 curve had to allow for an unbounded derivative at the endpoints of each piece.

Theorem 3.49 (Green's Theorem, first form). Suppose that a region U in $\mathbb{R}^2$ can be described as the set between two graphs of y as a continuous function of x and also as the set between two graphs of x as a continuous function of y. Suppose further that all four graphs are piecewise C^1 curves.[13] Write γ for the chain consisting of the four graphs, and assume that each piece of γ is oriented so that U is on the left. If P and Q are C^1 functions on an open set containing U and its boundary, then

$$\int_\gamma P\, dx + Q\, dy = \iint_U \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y} \Big)\, dx\, dy.$$

PROOF. The argument has already been given in Example 1, as amended in Example 2, and there is no need to repeat it. $\square$

[12] Some authors refer to such a set as a "Type III region." We shall not use this term.

[13] The hypothesis "tamely behaved" has been built into the definition of a piecewise C^1 curve.

The theorem admits several useful generalizations, and we say something about those now. The first such is that we can get more scope from the theorem by piecing together regions for which it holds. The example of an annulus (washer) will illustrate.

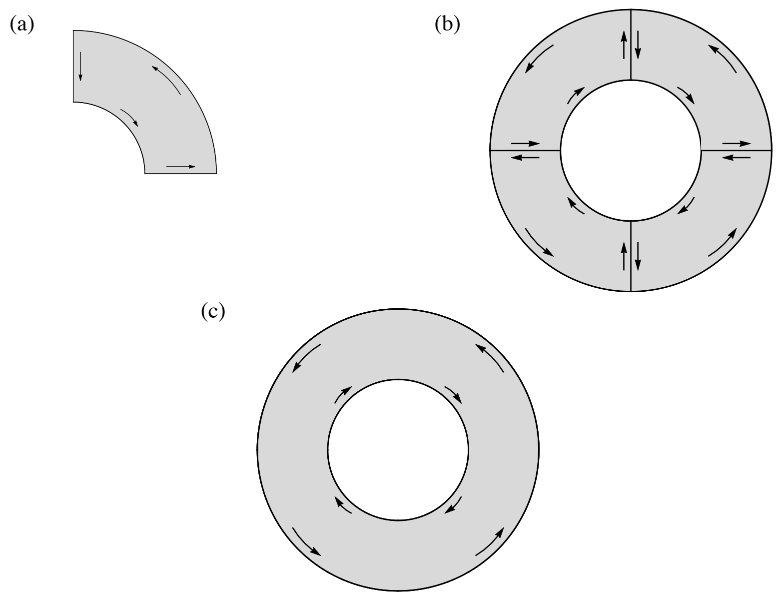

FIGURE 3.4. Green's Theorem for an annulus.

EXAMPLE 3. Annulus or washer, the set between two concentric circles. Figure 3.4 shows how we can handle this set by applying Theorem 3.49 four times, adding the results, and canceling contributions from arcs where the curve retraces itself. A single quarter of the annulus is a set to which Theorem 3.49 applies, provided the boundary is traversed with the region on the left. The boundary chain is the sum of four arcs. See Figure 3.4a. But we can equally well handle any other quarter of the annulus, as in Figure 3.4b. If we look at all the quarters together, the straight line segments cancel in pairs, as in Figure 3.4c, and the result is a annulus U with two boundary components, the outer circle γ_1 and the inner circle γ_2. The outer circle is traversed counterclockwise, as it is in a simple application of the theorem to a disk and its boundary circle. But the inner circle is traversed clockwise. The formula of Green's Theorem applies with this understanding of how the pieces of the boundary are oriented. If P and Q are C^1 functions on an open set containing the annulus and the boundary, then Green's Theorem applies to those functions on each quarter annulus and hence on the whole annulus. For example, let the outer and inner circles have respective radii 1 and ρ, and let them be centered at the origin. The outer circle can be parametrized by $t \mapsto (\cos t, \sin t)$ for $0 \le t \le 2\pi$, and then it is traversed counterclockwise. The inner circle can be taken as parametrized by $t \mapsto (\rho\cos t, -\rho\sin t)$. The line integral over the outer circle with clockwise orientation equals

$$\int_0^{2\pi} \begin{pmatrix} P(\cos t, \sin t) \\ Q(\cos t, \sin t) \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} dt.$$

To this is to be *added* the line integral over the inner circle with counterclockwise orientation, which equals

$$\int_0^{2\pi} \begin{pmatrix} P(\rho\cos t, -\rho\sin t) \\ Q(\rho\cos t, -\rho\sin t) \end{pmatrix} \cdot \begin{pmatrix} \rho\cos t \\ \rho\sin t \end{pmatrix} dt.$$

The sum equals the double integral, which we can conveniently write in polar coordinates as

$$\int_{\substack{\rho \le r \le 1, \\ 0 \le \theta \le 2\pi}} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) r\, dr\, d\theta.$$

It is fairly clear that the technique of Example 3 applies to more complicated bounded regions of $\mathbb{R}^2$ with finitely many holes in them. Every component of the boundary has to be taken into account in the line integral. We shall not try to formulate a general result.

A variation on this technique handles any "simple closed polygon." By a **simple closed polygon** in $\mathbb{R}^2$ is meant a piecewise C^1 simple closed curve whose component arcs are straight line segments. The following result says how the ingredients of such a polygon fit into the context of Green's Theorem, and the formula of Green's Theorem is a corollary.

Proposition 3.50 (Jordan Curve Theorem for polygons). If γ is a simple closed polygon with image A, then the open complement of A in $\mathbb{R}^2$ has exactly two connected components, A is the boundary of each, and exactly one of the components is a bounded set.

REMARK. The bounded component is called the **inside** of the polygon, and the unbounded component is called the **outside**.

SKETCH OF PROOF. Since A is a bounded set, all points sufficiently far from the origin are connected to one another by paths, and there can be only one unbounded component. Fix a line L in $\mathbb{R}^2$ going in a direction not parallel to any edge of A. We divide the complement of A into two open subsets, U and V. U consists of those points p not in A such that the line through p parallel to L intersects A an odd number of times, and V consists of those points p not in A such that the line through p parallel to L intersects A an even number of times. In counting the number of times the line intersects A, one has to take the vertices of A into account; the vertex is to be counted if the adjacent edges lie on opposite sides of the line but not if the adjacent edges lie on the same side of the line. Then U and V are open and disjoint, and $U \cup V$ equals the complement of A. Some checking is needed that A is the boundary of U and of V and that U and V are actually connected, and we omit those steps. □

Corollary 3.51 (Green's Theorem for a simple closed polygon). Let γ be a simple closed polygon in $\mathbb{R}^2$, let A be its image, and let U be its inside. Assume that γ is traversed in such a way that V is always on the left. If P and Q are scalar-valued C^1 functions on an open set containing A and U, then

$$\int_\gamma P\,dx + Q\,dy = \iint_U \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y}\Big)\,dx\,dy.$$

REMARK. The meaning of the phrase "always on the left" is intuitively clear, but the mathematical meaning is subtle and its details are omitted. Accurate use of the word "left" depends on having the coordinate axes oriented in the usual way so that the positive y axis is on the left of the positive x axis. If the x and y axes are interchanged, for example, then "left" and "right" get interchanged.

SKETCH OF PROOF. The idea is to decompose A into nonoverlapping triangles, each regarded as a simple closed polygon. For each triangle we apply Theorem 3.49. The sum of the double integrals over the insides of the triangles equals the double integral over U because the edges of the triangles contribute nothing to the double integral. In the sum of the line integrals over the triangles, the contributions from the edges that lie in the inside of A cancel in pairs, and the contributions from the remaining edges add to the contribution from A. The formula of Corollary 3.51 therefore results.

What needs proof is that the decomposition into nonoverlapping triangles is possible. Fix a line L in $\mathbb{R}^2$ going in a direction not parallel to any edge of A, and adjoin to A all lines parallel to L and passing through vertices of A. One readily checks that U gets decomposed into nonoverlapping triangles and trapezoids. Each trapezoid decomposes into two nonoverlapping triangles, and the desired decomposition into triangles results. □

The above techniques amount to the classical method for approaching Green's Theorem. The modern geometric approach uses a partition of unity, first handling matters locally and referring them to standard sets in $\mathbb{R}^2$. Its details are written out in the book by M. Spivak entitled *Calculus on Manifolds*. Two notions that play a role are those of "C^1 manifold-with-boundary" and "singular n-cube." Smooth manifolds are not defined in the present volume but instead appear in Chapter VIII of *Advanced Real Analysis*. Spivak defines a subset M of $\mathbb{R}^n$ to be a two-dimensional **smooth manifold** if for each point p of M there exist an open set U containing p, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = \{x \in V \mid x_{k+1} = \cdots = x_n = 0\}.$$

He defines a subset M of $\mathbb{R}^n$ to be a two-dimensional **smooth manifold-with-boundary** if for each point p of M, either the above manifold condition holds or there exist an open set U containing p, an open set $V \subseteq \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = \{x \in V \mid x_k \geq 0 \text{ and } x_{k+1} = \cdots = x_n = 0\}.$$

He states Green's Theorem for subsets M of $\mathbb{R}^2$ that are smooth manifolds-with-boundary. The set U of Green's Theorem will be the manifold part of the smooth-manifold-with-boundary, and the image of γ will be the boundary part of the smooth manifold-with-boundary. In our situation this assumption will forbid the "piecewise" aspect of the boundary and insist on no corners. It will also have the minor effect of replacing the assumption of C^1 behavior on the boundary by C^∞. The machinery of singular 2-cubes in effect examines matters locally and refers local sets to the plane or the upper half plane, where Example 1 applies directly. The local results are assembled into a final theorem by means of a smooth partition of unity.[14]

This completes our discussion of Green's Theorem in the plane. We conclude with some comments about generalizations to other dimensions. In the first place the idea of computing arc length by taking the supremum of inscribed polygonal arcs does not generalize well. If one takes a finite part of a right circular cylinder in $\mathbb{R}^3$ and defines the surface area to be the supremum of the sum of the areas of inscribed filled triangles, the result is infinity. Figure 3.5 illustrates.[15] It assumes that the axis of the cylinder is vertical, that the height is h, and that the radius is r. One tries to estimate a part of the area by using inscribed triangles. The large rectangle in the picture has height h and width b, with b less than the horizontal diameter $2r$ of the cylinder. Regard the rectangle as placed inside the back half of the cylinder so that its left and right edges lie in the surface of the cylinder and its top and bottom edges are chords of the top and bottom circles of the cylinder. One

[14] As was mentioned earlier, the partitions of unity in use in this chapter involve only continuous functions, but smooth partitions of unity will be constructed and used in *Advanced Real Analysis*.

[15] This figure is based on the one by Spivak on page 129 of *Calculus on Manifolds*.

takes a positive integer m, which is 5 in Figure 3.5, and introduces m rectangular pyramids, each turned on its side so the apex is at the back of the cylinder. The four triangles in each pyramid are each inscribed in the cyclinder, and we get $4m$ triangles in this way. If a limit of the sum of the areas of inscribed triangles is to have any hope of giving the surface area of the cylinder, then the sum of the areas of these particular triangles had better be at most the surface area of the cylinder.

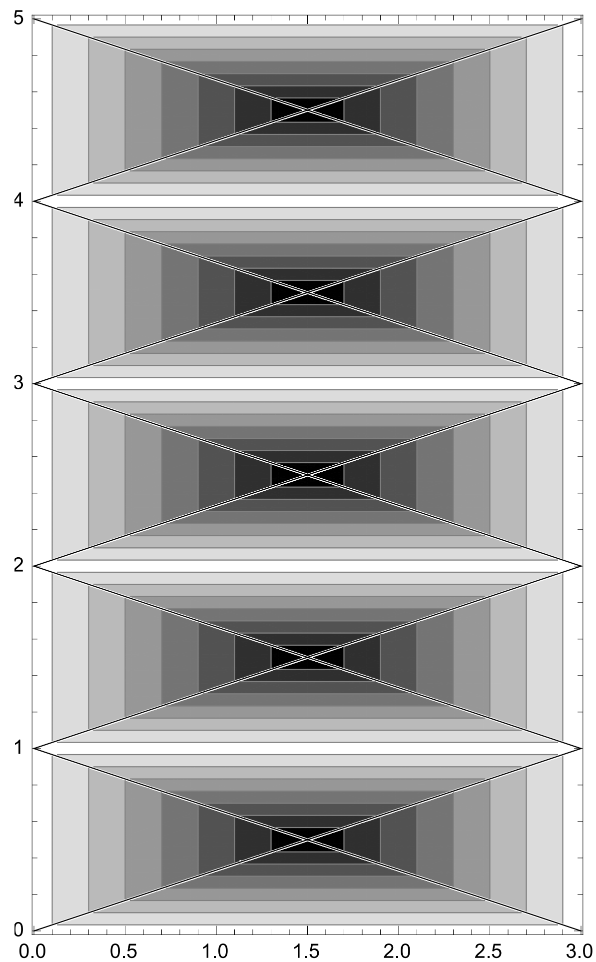

FIGURE 3.5. Failure of inscribed triangles to give a useful notion of surface area.

For each pyramid we can give a lower bound for the areas of the top and bottom triangles. The top and bottom triangles are isosceles, each with base b, and their common height is at least what it would be if the face of the triangle were perpendicular to the cylinder. A little computation shows that this amount is $r - \sqrt{r^2 - b^2/4}$, which is some positive number independent of m. Since there are $2m$ such triangles, the sum of the areas of these triangles is unbounded. The sum of the areas of the m left and right triangles is some number ≥ 0, and thus the sum of the areas of all $4m$ triangles is $\geq m(r - \sqrt{r^2 - b^2/4})$, which is unbounded.

Thus surface area cannot be defined by using inscribed polygons. It is worth examining how the above example defies intuition. For parametrically defined curves we inscribed line segments, and we were guided by the fact that the length of each line segment was at most the length of the corresponding part of the curve. In the above example, we inscribed triangles, and we would have expected the area of each triangle to be at most the area of a certain part of the surface. But what part of the surface is relevant? The difficulty is that our intuition is working with some projection of the triangle onto the surface, and there is no canonical such projection in this case. To get something canonical, it would be really helpful to have a notion of perpendicularity.

For this reason the area of a bounding surface in $\mathbb{R}^3$ is defined by taking advantage of the direction perpendicular to the surface, and good behavior of the surface becomes essential. A desire to make use of perpendicularity is the reason Spivak's book works with smooth manifolds-with-boundary. The notion of a "normal" to the surface is then available. We shall not elaborate except to observe that the introduction of normals means that geometry now plays a much more significant role in the higher-dimensional theory than it did in the theory for curves.

There is one higher-dimensional situation where everything is accessible without a whole new theory, and this particular situation happens to be an especially useful one for analysis. This is the case of a closed ball in $\mathbb{R}^n$, whose boundary is a sphere. Section III.3 of *Advanced Real Analysis* gives a direct proof of a theorem relating a volume integral over the ball and a surface integral over the sphere.

Green's Theorem in the plane admits a generalization for smooth manifolds-with-boundary of dimension k in $\mathbb{R}^n$ for every pair (k, n) with $2 \leq k \leq n$. The result is that an integral on the boundary "surface" is related to a "volume" integral over the set. The results of this kind are collectively known as Stokes's Theorem. See Spivak's book for details. The classical results that fit into this framework are Green's Theorem when $k = n = 2$, the Divergence Theorem[16] when $k = n = 3$, and Stokes's Theorem when $k = 2$ and $n = 3$.

In the usual modern approach to the general Stokes's Theorem, the mechanism of proof is the same as the one in Spivak's book, using "smooth manifolds-with-

[16] The Divergence Theorem is known also as Gauss's Theorem.

boundary" and "singular n-cubes.". The reader is referred to that book for the details.

14. Problems

1. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Prove that the Hilbert–Schmidt norm satisfies
   (a) $|TS| \le |T|\,|S|$ if S is in $L(\mathbb{F}^n, \mathbb{F}^m)$ and T is in $L(\mathbb{F}^m, \mathbb{F}^k)$,
   (b) $|1| = \sqrt{n}$ if $n = m$ and 1 denotes the identity function on $\mathbb{F}^n$.

2. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function with Jacobian matrix A. What is $f'(x_0)$?

3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^1$ has $|f(x)| \le |x|^2$ for all x. Prove that f is differentiable at $x = 0$.

4. Let $x = (x_1, \dots, x_n)$ and $u = (u_1, \dots, u_n)$ be in $\mathbb{R}^n$. For $f : \mathbb{R}^n \to \mathbb{R}$ differentiable at x, use the chain rule to derive a formula for $\frac{d}{dt} f(x + tu)\big|_{t=0}$.

5. Compute $\exp tX$ from the definition for $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

6. It was observed in Section 6 in the context of polar coordinates that the Implicit Function Theorem implies the Inverse Function Theorem. Namely, the pair of polar-coordinate formulas $(u, v) = (r\cos\theta, r\sin\theta)$ was inverted by applying the Implicit Function Theorem to the system of equations
$$r\cos\theta - u = 0, \qquad r\sin\theta - v = 0.$$
   Using this example as a model, derive the Inverse Function Theorem in the general case from the Implicit Function Theorem in the general case.

7. Define $\int_1^\infty$ to mean $\lim_{N\to\infty} \int_1^N$ when the integrand is continuous. Prove or disprove:
$$\int_0^1 \Big[\int_1^\infty (e^{-xy} - 2e^{-2xy})\,dx \Big]\,dy = \int_1^\infty \Big[\int_0^1 (e^{-xy} - 2e^{-2xy})\,dy \Big]\,dx.$$

Problems 8–9 use Fubini's Theorem to supplement the theory of Fourier series as given in Section I.10.

8. Let f and g be continuous complex-valued periodic functions of period 2π, and define their **convolution** to be the function

$$f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\,dt.$$

(a) Show that $f * g$ is continuous periodic and that $f * g = g * f$.
(b) Let $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and $g(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}$. Prove that $(f * g)(x) \sim \sum_{n=-\infty}^{\infty} c_n d_n e^{inx}$.
(c) Prove that the Fourier series of $f * g$ converges uniformly.

9. Let f, g, and h be continuous complex-valued periodic functions of period 2π. Prove that $f * (g * h) = (f * g) * h$.

Problems 10–13 deal with homogeneous functions. If $f : \mathbb{R}^n - \{0\} \to \mathbb{R}$ is a function not identically 0 such that $f(rx) = r^d f(x)$ for all x in $\mathbb{R}^n - \{0\}$ and all $r > 0$, we say that f is **homogeneous** of degree d. For example, the function in the first problem below is homogeneous of degree 0.

10. On $\mathbb{R}^2$, define

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere in $\mathbb{R}^2$ and that f is not continuous at $(0, 0)$.

11. Let $f : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be smooth and homogeneous of degree d.
(a) Prove that if $d = 0$, then $f(x)$ is bounded on $\mathbb{R}^n - \{0\}$ and that f extends to be continuous at 0 only if it is constant.
(b) Prove that if $d > 0$, then the definition $f(0) = 0$ makes f continuous for all x in $\mathbb{R}^n$, while if $d < 0$, then no definition of $f(0)$ makes f continuous at 0.
(c) Prove that $\frac{\partial f}{\partial x_j}$ is homogeneous of degree $d - 1$ unless it is identically 0.
(d) If f is homogeneous of degree 1 and satisfies $f(-x) = -f(x)$ and $f(0) = 0$, prove that each $\frac{\partial f}{\partial x_j}$ exists at 0 but that $\frac{\partial f}{\partial x_j}$ is not continuous at 0 unless it is constant.

12. On $\mathbb{R}^2$, let f be the function homogeneous of degree 1 given by

$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Prove that f is continuous at $(0, 0)$.
(b) Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0, 0)$ but are not continuous there.

(c) Calculate $\frac{d}{dt} f(t+tu)\big|_{t=0}$ for $x = 0$ and $u = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Show that the formula in Problem 4 fails, and conclude that f is not differentiable at $(0, 0)$.

13. On $\mathbb{R}^2$, let f be the function homogeneous of degree 2 given by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Prove that f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are continuous on all of $\mathbb{R}^2$.

(b) Prove that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at $(0, 0)$ but are not continuous there.

(c) Prove that $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$ and $\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$.

Problems 14–15 concern "harmonic functions" in $\{(x, y) \in \mathbb{R}^2 \mid |(x, y)| < 1\}$, the **open unit disk** of the plane. A **harmonic function** is a complex-valued C^2 function satisfying the **Laplace equation** $\Delta u(x, y) = 0$, where Δ is the **Laplacian** $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

14. If (r, θ) are regarded as polar coordinates, prove for all integers n that each function $r^{|n|}e^{in\theta}$ is a C^∞ function in the open unit disk and is harmonic there. Deduce that if $\{c_n\}$ is a doubly infinite sequence such that $\sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ converges absolutely for each r with $0 \leq r < 1$, then the sum is a C^∞ function in the open unit disk and is harmonic there.

15. Prove that if u is harmonic in the unit disk, then so is the function $u \circ R$, where R is the rotation about the origin given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Problems 16–20 illustrate the Inverse and Implicit Function Theorems.

16. Verify that the equations $u = x^4 y + x$ and $v = x + y^3$ define a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose derivative at $(1, 1)$ is given by the matrix $\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$. This matrix being invertible, the Inverse Function Theorem applies. Let the locally defined C^1 inverse function be given by $x = F(u, v)$ and $y = G(u, v)$ in an open neighborhood of $(u, v) = (2, 2)$, the point $(2, 2)$ having the property that $F(2, 2) = 1$ and $G(2, 2) = 1$. Find $\frac{\partial F}{\partial u}(2, 2)$.

17. Show that the equations

$$\begin{aligned} x^2 - y\cos(uv) + z^2 &= 0, \\ x^2 + y^2 - \sin(uv) + 2z^2 &= 2, \\ xy - \sin u \cos v + z &= 0, \end{aligned}$$

implicitly define x, y, z as C^1 functions of (u, v) near $x = 1$, $y = 1$, $u = \pi/2$, $v = 0$, and $z = 0$, and find $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ for the function $x(u, v)$. Is the function $x(u, v)$ of class C^∞?

18. Regard the operation of squaring an n-by-n matrix as a function from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$, and show that this mapping is invertible on some open set of the domain that contains the identity matrix.

19. **(Lagrange multipliers)** Let f and g be real-valued C^1 functions defined on an open subset U of $\mathbb{R}^n$, and let $S = \{x \in U \mid g(x) = 0\}$. Prove that if $f\big|_S$ has a local maximum or minimum at a point x_0 of S, then either $g'(x_0) = 0$ or there exists a number λ such that $f'(x_0) + \lambda g'(x_0) = 0$.

20. **(Arithmetic-geometric mean inequality)** Using Lagrange multipliers, prove that any n real numbers $a_1, \ldots, a_n$ that are ≥ 0 satisfy
$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Problems 21–23 concern arc length and integrals with respect to arc length.

21. Sometimes a parametrically defined curve is given in polar coordinates by an equation $r = r(\theta)$. Show that the arc length of a simple arc of the form $r = r(\theta)$ from θ_1 to θ_2 is $\int_{\theta_1}^{\theta_2} \sqrt{r(\theta)^2 + \big(\frac{dr}{d\theta}\big)^2}\, d\theta$.

22. Only a few tamely behaved simple arcs are known for which arc length can be expressed in terms of elementary functions. These include the straight line, the circle, the cycloid, the helix, the catenary, the semicubical parabola, the parabola, and the logarithmic spiral. The first two are part of Euclidean geometry, and the cycloid was treated as an example in Section 11. This problem treats the last five. In each case, do not necessarily go through all the steps of evaluating the integral, but carry out enough of the computation to show that the result can be expressed in terms of elementary functions.
    (a) Express as a function of t the cumulative arc length of the **helix** $(x, y, z) = (\cos t, \sin t, t)$ starting from the origin.
    (b) Express as a function of x the cumulative arc length of the **catenary** $y = \frac{1}{2}(e^t + e^{-t})$ starting from the origin.
    (c) Express as a function of x the cumulative arc length of the **semicubical parabola** $y = x^{3/2}$ starting from the origin.
    (d) Express as a function of x the cumulative arc length of the **parabola** $y = x^2$ starting from the origin.
    (e) Express as a function of θ with $0 < \theta \leq 2\pi$ the cumulative arc length from $\theta = \theta_0$ of the **logarithmic spiral** $r(\theta) = \theta$, using the result of Problem 21.
    (f) Is the logarithmic spiral in (d) tamely behaved as θ tends down to 0?

23. Let γ be the piecewise C^1 curve defined for $t \in [0, 3]$ given by
$$\gamma(t) = \begin{cases} (t^2, t) & \text{for } 0 \leq t \leq 1 \\ (t, t) & \text{for } 1 \leq t \leq 2 \\ (t, 2 + (t-2)^2) & \text{for } 2 \leq t \leq 3. \end{cases}$$
Find the total length $\ell(\gamma)$.

Problems 24–26 elaborate on the remarks in a footnote connected with Theorem 3.44 that explained that the line integral $\int_\gamma F \cdot d\mathbf{s}$ of the theorem always has a meaning in terms of "Stieltjes integrals." No assumption that γ be piecewise C^1 is needed, only that γ is a rectifiable simple arc. Let $\alpha : [a, b] \to \mathbb{R}$ be a continuous nondecreasing function. (Continuity of α is not needed in the theory but will be assumed here to simplify the statements.) If $f : [a, b] \to \mathbb{R}$ is a continuous function, it is desired to define the **Stieltjes integral** $\int_{[a,b]} f \, d\alpha$.

24. For any partition $P = \{x_j\}_{j=1}^m$ of $[a, b]$, define the upper and lower sums of f relative to P and α by

$$U(P, f, \alpha) = \sum_{j=1}^m \big(\max\nolimits_{x_{j-1} \le x \le x_j} f(x)\big)\big(\alpha(x_j) - \alpha(x_{j-1})\big)$$
$$L(P, f, \alpha) = \sum_{j=1}^m \big(\min\nolimits_{x_{j-1} \le x \le x_j} f(x)\big)\big(\alpha(x_j) - \alpha(x_{j-1})\big).$$

Show that if P' is a refinement of P, then

$$U(P, f, \alpha) \ge U(P', f, \alpha) \ge L(P', f, \alpha)) \ge L(P, f, \alpha).$$

Explain how it follows that

$$\inf_P U(P, f, \alpha) \ge \sup_P L(P, f, \alpha).$$

25. With $\mu(P)$ equal to the mesh of P, prove that

$$\lim_{\mu(P) \to 0} \big(U(P, f, \alpha) - L(P, f, \alpha)\big) = 0.$$

26. Conclude from Problems 24 and 25 that $U(P, f, \alpha)$ and $L(P, f, \alpha)$ tend to a common limit as $\mu(P)$ tends to 0. This common limit is what is taken as the definition of $\int_{[a,b]} f \, d\alpha$.

Problems 27–30 concern line integrals and conservative vector fields.

27. Let (x_1, y_1) and (x_2, y_2) be two points in $\mathbb{R}^2$, and let γ be the line segment from (x_1, y_1) to (x_2, y_2). Parametrize γ, and compute $\int_\gamma x \, dy - y \, dx$.

28. Let $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ be the vector field on $\mathbb{R}^2 - \{(0, 0)\}$ with $P(x, y) = \frac{x}{x^2+y^2}$ and $Q(x, y) = \frac{y}{x^2+y^2}$.
    (a) Check that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

(b) Exhibit a C^1 function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ such that $\begin{pmatrix} P \\ Q \end{pmatrix} = \nabla f$.

29. Let $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ be the vector field on $\mathbb{R}^2 - \{(0,0)\}$ with $P(x,y) = \frac{y}{x^2+y^2}$ and $Q(x,y) = \frac{-x}{x^2+y^2}$.
(a) Check that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
(b) Evaluate $\int_\gamma F \cdot d\mathbf{s}$ counterclockwise around the unit circle, thus over the curve $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ for $0 \le t \le 2\pi$.
(c) Show that there is no C^1 function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ such that $\begin{pmatrix} P \\ Q \end{pmatrix} = \nabla f$.

30. Let $F(x,y,z) = (x, y^2, z^3)$. Evaluate $\int_\gamma F \cdot d\mathbf{s}$ over the curve $\gamma(t) = (t, t^2, t^3)$ for $0 \le t \le 1$.

Problems 31–33 concern Green's Theorem in the plane.

31. With γ as in Problem 29b, evaluate $\int_\gamma \begin{pmatrix} y+e^x \cos y \\ -x-e^x \sin y \end{pmatrix} \cdot d\mathbf{s}$.

32. Let U be any bounded open subset of $\mathbb{R}^2$ to which Green's Theorem applies, and let γ be the boundary of U oriented so that U is always on the left. Prove that $\frac{1}{2}\big(\int_\gamma x\,dy - y\,dx\big)$ equals the area of U.

33. **(Shoelace formula)** Combine Problems 27 and 32 with Corollary 3.51 to prove that the area of the inside of any simple closed polygon whose m consecutive vertices are $\{(x_j, y_j)\}_{j=1}^m$ is given by

$$\text{Area} = \Big| \sum_{j=0}^{m} (x_j y_{j+1} - y_j x_{j+1}) \Big|,$$

where by convention (x_0, y_0) is defined to be (x_m, y_m). In fact, the absolute value signs are not needed if the polygon is traversed with the inside always on the left. (Educational notes: This formula is of historical importance in the transfer of ownership of pieces of land; traditional surveying tools easily allow rather accurate measurements of distances and angles, and this formula gives a comparably accurate measurement of area. The name of the formula derives from the criss-cross pattern made if one forms an $(n+1)$-by-2 matrix with rows $(x_j \quad y_j)$ and then indicates the pairs of entries that are to be multiplied.)