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Anthony W. Knapp

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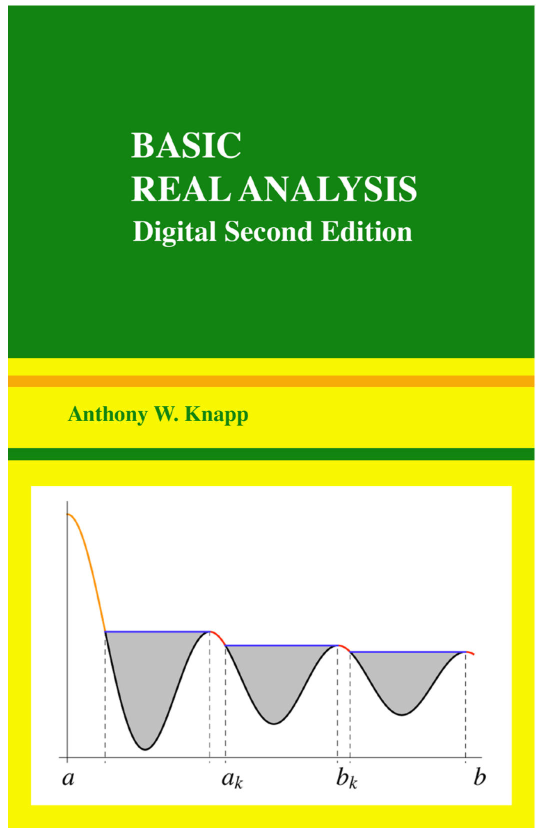

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/~aknapp

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CHAPTER XI

Integration on Locally Compact Spaces

Abstract. This chapter deals with the special features of measure theory when the setting is a locally compact Hausdorff space and when the measurable sets are the Borel sets, those generated by the compact sets.

Sections 1–2 establish the basic theorem, the Riesz Representation Theorem, which says that any positive linear functional on the space $C_{\text{com}}(X)$ of continuous scalar-valued functions of compact support on the underlying space X is given by integration with respect to a unique Borel measure having a property called regularity. The steps in the construction of the measure run completely parallel to those for Lebesgue measure if one regards the geometric information about lengths of intervals as being encoded in the Riemann integral. The Extension Theorem of Chapter V is the main technical tool.

Section 3 studies more closely the nature of regularity of Borel measures. One direct generalization of a Euclidean theorem is that the space of continuous functions of compact support in an open set is dense in every L^p space on that open set for $1 \le p < \infty$. A new result is the Helly–Bray Theorem—that any sequence of Borel measures of bounded total measure in a locally compact separable metric space has a weak-star convergent subsequence whose limit is a Borel measure.

Section 4 regards $C_{\text{com}}(X)$ as a normed linear space under the supremum norm and identifies the space of continuous linear functionals, with its norm, as a space of signed or complex Borel measures with a regularity property, the norm being the total-variation norm for the signed or complex Borel measure.

1. Setting

This chapter brings together the measure theory of Chapters V–VI and the theory of topological spaces of Chapter X in a way that takes many of our earlier most interesting examples into account. Specifically we shall study the special features of measure theory when the underlying space is a locally compact Hausdorff space. Our primary example from earlier is that of Lebesgue measure, first on $\mathbb{R}^1$ and then in $\mathbb{R}^N$. In $\mathbb{R}^1$ we considered also the class of all Stieltjes measures and showed how they are classified by monotone functions satisfying certain properties. We introduced Borel measures in $\mathbb{R}^N$ but did not attempt to classify them.

Along the way we saw glimpses of some other examples: The unit circle of $\mathbb{C}$ can be regarded as $[-\pi, \pi]$ if we identify $-\pi$ and π, and we obtained Lebesgue measure on the circle. As we saw, any open set or any compact set in $\mathbb{R}^N$ has a theory of Borel measures associated with it. Most of our concrete examples of such measures when $N > 1$ came about as a consequence of the change-of-variables formula for multiple integrals. Of particular interest is what we anticipated in Section VI.5 would ultimately come to be regarded as a "rotation-invariant measure on the sphere," the sphere S^{N-1} being a compact metric space. This measure corresponds to the expression $d\omega$ when Lebesgue measure dx on $\mathbb{R}^N$ is written in spherical coordinates and the factor $r^{N-1}\,dr$ is dropped. In the concrete case of $\mathbb{R}^3$, in which r is the radius, θ_1 is the latitude from the north pole, and θ_2 is the longitude, Lebesgue measure is given by $dx = r^2 \sin\theta_1\, d\theta_2\, d\theta_1\, dr$ and we have $d\omega = \sin\theta_1\, d\theta_2\, d\theta_1$. The change-of-variables formula in the N-variable case then reads

$$\int_{\mathbb{R}^N} f(x)\,dx = \int_{r=0}^{\infty}\int_{\omega\in S^{N-1}} f(r\omega)\,r^{N-1}\,d\omega\,dr$$

for every Borel measurable function $f \geq 0$ on $\mathbb{R}^N$. We shall be making sense of $d\omega$ as a genuine measure on S^{N-1} in the course of the present chapter.

In the opposite direction it is important not to get the idea that all important measure-theoretic examples in mathematics arise from locally compact Hausdorff spaces. Examples that arise from probability theory need not fit this pattern. This fact becomes clearer after one encounters some specific measure spaces that arise in the theory.[1]

Let us turn to the setting of this chapter, a locally compact Hausdorff space X. In order that the measure theory have some connection with the topological-space structure, we shall build our σ-algebra out of topologically significant sets. There will be a choice for how to do so, and we come to that point in a moment.

We shall follow as much as possible the pattern of the development of Lebesgue measure on an interval of $\mathbb{R}^1$ or on all of $\mathbb{R}^1$, as occurred in Chapter V, in order to construct measures on X. The thing that is missing for general X occurs right at the start: it is the kind of geometric information that goes into regarding the length of an interval as a quantity worthy of study. That is where an ingenious idea comes into play, that of studying linear functionals on the vector space $C_{\text{com}}(X)$ of continuous scalar-valued functions on X that vanish off a compact subset of X. As in earlier chapters, it will not be important whether the scalars for $C_{\text{com}}(X)$ are real or complex, and the reader may fix attention on either of these.

On an interval $[a, b]$, we thus consider the space $C([a, b])$ of scalar-valued continuous functions on the interval. The particular linear functional of interest is the Riemann integral $\ell(f) = \mathcal{R}\int_a^b f(x)\,dx$, the notation with the $\mathcal{R}$ being as in Section VI.4. This kind of integral is a fairly simple object analytically; it was

[1]The measure-theoretic foundations of probability theory are discussed in the companion volume, *Advanced Real Analysis*.

quickly shown to make sense in Theorem 1.26. Our point of view will be that the Riemann integral encodes information about the lengths of all intervals.

Why might one consider linear functionals? In the subject of linear algebra, linear functionals play an important role. Two important ways of realizing subsets of Euclidean space are parametric form and implicit form. In the case of a vector subspace of $\mathbb{R}^n$, the idea of parametric form leads us to represent the subspace as all linear combinations of members of a spanning set. If we use implicit form instead, the subspace is realized as all vectors satisfying a set of homogeneous linear equations, thus as the kernel of some linear function. The most primitive case of the latter is that there is just one nontrivial equation. Then the linear function has range the scalars, and the linear function is a linear functional. When there are several equations, the subspace is in effect described as the intersection of the kernels of several linear functionals.

Thus linear functionals in linear algebra arise in describing vector subspaces, specifically in describing subspaces by limiting their size from the outside. In analysis we have occasionally needed this kind of control of a subspace in proving theorems by an approximation argument. Two nontrivial examples were the proofs in Chapter VI of differentiation of integrals and the proof in Chapter IX of the boundedness of the Hilbert transform. In each case we proved a theorem for "nice" functions, and we obtained some estimate for all functions of interest. To connect the one conclusion with the other, we needed to know that the subspace of "nice" functions is dense. Corollary 6.4 was a result of this kind, saying that $C_{\text{com}}(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$ and in $L^2(\mathbb{R}^N)$. The proof given for Corollary 6.4 was more like an argument using spanning sets, showing that we can pass from $C_{\text{com}}(\mathbb{R}^N)$ to simple functions and then recalling that simple functions are dense as a consequence of basic properties of the Lebesgue integral.

However, we can visualize another argument of this kind, one with continuous linear functionals. If one could prove, for any proper closed vector subspace of our total space of functions (L^1 or L^2 or something else), that there is a nonzero continuous linear functional on the total space vanishing on the closed subspace, then we could test whether a given vector subspace is dense by examining the effect of continuous linear functionals when restricted to the subspace. Historically this idea began to be applied in analysis in the early part of the twentieth century at about the same time that people began thinking frequently about spaces of functions and not just individual functions. The key general existence tool for such continuous linear functionals was the Hahn–Banach Theorem, which we shall take up in Chapter XII.

In any event, out of this confluence of ideas arose the idea of considering continuous linear functionals on $C_{\text{com}}(X)$ as capturing enough information about X to make measure theory possible. The continuity of a linear functional will actually be somewhat concealed in what we do for most of this chapter, and instead we impose on the linear functional the natural condition that it needs to satisfy in order to provide a notion of integration—that it be ≥ 0 on functions ≥ 0.

Let us be more precise about the definitions. Let X be a locally compact Hausdorff space, and let $C_{\text{com}}(X)$ be the vector space of scalar-valued functions on X that vanish outside some compact set. For a specific function f, the **support** of f is the closure of the set where f is not zero. The members of $C_{\text{com}}(X)$ are then the continuous scalar-valued functions on X having compact support. A linear functional ℓ on $C_{\text{com}}(X)$ is said to be **positive** if $\ell(f) \geq 0$ whenever $f \geq 0$. The Riesz Representation Theorem, to be stated formally in Section 2 with all details in place, will say that to any such ℓ corresponds a measure μ on a certain σ-algebra of "topologically significant" sets such that

$$\ell(f) = \int_X f\,d\mu \qquad \text{for all } f \in C_{\text{com}}(X).$$

The "topologically significant" sets have to include the sets necessary to make each f in $C_{\text{com}}(X)$ measurable. At first glance it might seem that the smallest σ-algebra containing the open sets is the right object. But in fact this σ-algebra is unnecessarily large. In an uncountable discrete space, we do not need to have every subset measurable in order to have all the functions of compact support be measurable. Accordingly we define the σ-algebra $\mathcal{B}(X)$ of **Borel sets** of X to be the smallest σ-algebra containing all compact subsets of X.

The plan of attack now follows the steps in the construction of Lebesgue measure. We take the compact subsets of X to be the analog of the bounded intervals in $\mathbb{R}^1$, and we thus define the **elementary sets** in X to be the sets in the smallest ring $\mathcal{K}(X)$ containing all the compact sets. In the case of $\mathbb{R}^1$, every set in the ring generated by the bounded intervals is a finite disjoint union of sets that are the difference of two bounded intervals. We shall prove for X in Section 2 that every member of $\mathcal{K}(X)$ is a finite disjoint union of sets that are the difference of two compact sets.

For $\mathbb{R}^1$, we defined the measure of the difference of two bounded intervals to be the difference of their lengths as soon as the second interval is contained in the first; this was no loss of generality because the intersection of two bounded intervals is a bounded interval. The measure of a finite disjoint union was defined as the sum of the measures. We showed that this was well defined, and then we had a finite-valued nonnegative additive set function on a ring of sets.

For X, we define the measure of a compact set K by the natural formula

$$\mu(K) = \inf_{\substack{f \in C_{\text{com}}(X),\\ I_K \leq f}} \ell(f),$$

where I_K as usual is the indicator function of K. The intersection of two compact sets is compact, and thus we can define the measure of $K_1 - K_2$ for K_1 and K_2

compact, to be $\mu(K_1) - \mu(K_1 \cap K_2)$. We define the measure of the disjoint union of such sets $K_1 - K_2$ to be the sum of the measures. We have to prove that this is well defined, and then we have a finite-valued nonnegative additive set function μ on the ring $\mathcal{K}(X)$.

The next step for $\mathbb{R}^1$ was to prove complete additivity on the ring generated by the bounded intervals. With X, the problem is the same; we are to prove complete additivity on the ring $\mathcal{K}(X)$. Suppose that this has been done. Since μ is everywhere finite-valued on $\mathcal{K}(X)$, we can apply the Extension Theorem (Theorem 5.5) to extend μ to the generated σ-ring. Either this σ-ring is already the generated σ-algebra $\mathcal{B}(X)$, or Proposition 5.37 supplies a canonical extension to a measure on the generated σ-algebra $\mathcal{B}(X)$. This completes the construction of the measure μ on $\mathcal{B}(X)$. It is then a fairly easy matter to see that $\ell(f)$ is recovered as the integral of f if f is in $C_{\text{com}}(X)$: In the case of $\mathbb{R}^1$, we carried out this step by first establishing the Fundamental Theorem of Calculus for the Lebesgue integral of a continuous function; the argument appears at the end of Section V.3. A more direct argument would have been possible, and that direct argument works for general X.

Thus the problem comes down to proving that the set function, as defined on the ring of sets, is actually completely additive on that ring. In the case of $\mathbb{R}^1$, that complete additivity was an easy consequence of "regularity" of Lebesgue measure on the ring generated by the bounded intervals; in other words, the measure of any set in the ring could be approximated from within by the measure of compact sets in the ring and from without by the measure of open sets in the ring. Exactly the same approach works for general X, but the regularity has to be established.

Quantitatively the construction of the measure comes down to defining $\mu(K)$ for K compact as above and then proving three identities:

(i) $\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2) + \mu(K_1 \cap K_2)$ if K_1 and K_2 are compact,

(ii) $\sup\limits_{\substack{f \in C_{\text{com}}(X), \\ 0 \le f \le I_U}} \ell(f) = \mu(K) - \mu(K - U)$ if U is any open set contained in some compact set K,

(iii) $\sup\limits_{\substack{K \subseteq U, \\ K \text{ compact}}} \mu(K) = \sup\limits_{\substack{f \in C_{\text{com}}(X), \\ 0 \le f \le I_U}} \ell(f)$ if U is open and has compact closure.

Identity (i) and an elementary but lengthy computation in elementary set theory together allow us to prove that μ is well defined on the ring $\mathcal{K}(X)$ under the definitions above. Once μ has been so extended, the right side of (ii) is just $\mu(U)$ if U is open with compact closure. Thus (iii) says that $\mu(U)$ is the supremum of $\mu(K)$ over compact sets K contained in U, provided U is open and has compact closure. Since $\mu(U)$ is trivially the infimum of $\mu(V)$ for open sets V in $\mathcal{K}(X)$ containing U, this is the regularity conclusion for U. It is easy to see that the subclass of $\mathcal{K}(X)$ for which regularity holds is a ring and contains the compact sets, and hence regularity is established for $\mathcal{K}(X)$.

When the locally compact Hausdorff space X is a metric space, the three identities above are fairly easy to prove. When X is metric, any indicator function I_K for K compact is the pointwise decreasing limit of members of $C_{\text{com}}(X)$ that are ≥ 0. In fact, if $D(\,\cdot\,, K)$ is the distance to K, then the sequence $\{f_n\}$ with $f_n(x) = \max\{0,\ 1-nD(x, K)\}$ has the required properties. A little trick proves in this case that $\mu(K) = \lim_n \ell(f_n)$. To prove (i), we choose such sequences $\{f_n\}$ and $\{g_n\}$ for K_1 and K_2. If φ is a member of $C_{\text{com}}(X)$ that is identically 1 on the union of the supports of f_1 and g_1, then $f_n + g_n = \min\{f_n + g_n, \varphi\} + \big(\max\{f_n + g_n, \varphi\} - \varphi\big)$ decomposes $f_n + g_n$ into the sum of such sequences for $K_1 \cup K_2$ and $K_1 \cap K_2$, and identity (i) follows from linearity of ℓ and a passage to the limit. Identities (ii) and (iii) follow from equally simple arguments.

The difficulty for a general locally compact Hausdorff space X is that the indicator function of a compact set need not be a pointwise decreasing limit of a sequence of continuous functions. The technicalities introduced by this fact have the effect of making the proofs of (i), (ii), and (iii) be more complicated, but these complications need not obscure the line of argument that is so clear in the metric case.

2. Riesz Representation Theorem

Throughout this section we fix the locally compact Hausdorff space X. We continue to let $C_{\text{com}}(X)$ be the space of continuous functions of compact support, $\mathcal{K}(X)$ be the ring of elementary sets, and $\mathcal{B}(X)$ be the σ-algebra of Borel sets.

A subset E of X is said to be **bounded** if it is contained in a compact set, hence if E^{cl} is compact; it is σ**-bounded** if it is contained in the countable union of compact sets. The class of all σ-bounded Borel sets is a σ-ring containing $\mathcal{K}(X)$, and it is therefore the smallest σ-ring containing $\mathcal{K}(X)$.

A measure on the Borel sets of X is called a **Borel measure** if it is finite on every compact set. A Borel measure μ is said to be **regular** if it satisfies

$$\mu(E) = \sup_{\substack{K\subseteq E,\\ K \text{ compact}}} \mu(K) \qquad \text{for every set } E \text{ in } \mathcal{B}(X)$$

$$\mu(E) = \inf_{\substack{U\supseteq E,\\ U \text{ open } \sigma\text{-bounded}}} \mu(U) \qquad \text{for every } \sigma\text{-bounded set } E \text{ in } \mathcal{B}(X).$$

Theorem 11.1 (Riesz Representation Theorem). If ℓ is a positive linear functional on $C_{\text{com}}(X)$, then there exists a unique regular Borel measure μ on X such that

$$\ell(f) = \int_X f\,d\mu \qquad \text{for all } f \in C_{\text{com}}(X).$$

EXAMPLES.

(1) If X is the line $\mathbb{R}^1$ and ℓ is given by Riemann integration $l(f) = \mathcal{R}\int_a^b f(x)\,dx$ whenever $[a, b]$ contains the support of f, then ℓ is a positive linear functional on $C_{\text{com}}(\mathbb{R}^1)$ and the corresponding μ is Lebesgue measure.

(2) If $X = S^2$ is the unit sphere in $\mathbb{R}^3$, parametrized by latitude θ_1 from 0 to π and by longitude θ_2 from 0 to 2π, then $\ell(f) = \mathcal{R}\int_0^\pi \int_0^{2\pi} f(\theta_1, \theta_2)\sin\theta_1\,d\theta_2\,d\theta_1$ is a positive linear functional on $C(S^2)$, and the corresponding measure, which is written $d\omega$ in the same way that Lebesgue measure is written as dx, is a rotation-invariant measure on the sphere such that $\int_{\mathbb{R}^3} F(x)\,dx = \int_0^\infty \int_{S^2} F(r\omega)r^2\,d\omega\,dr$ for every nonnegative Borel function on $\mathbb{R}^N$. The proof of this identity and of the rotation invariance will be indicated in Problem 5 at the end of the chapter.

(3) If X is general and if μ is a regular Borel measure on X, then $\ell(f) = \int_X f\,d\mu$ is a positive linear functional on $C_{\text{com}}(X)$.

The proof of Theorem 11.1 will occupy the remainder of this section. We begin with some lemmas clarifying the nature of the ring $\mathcal{K}(X)$, the linear functional ℓ, and general compact and open subsets of X. Then we recall the definition of $\mu(K)$ for compact sets and establish the identities (i), (ii), and (iii) in Section 1. Finally we give the details of how the three identities imply the theorem.

We begin with information about the ring $\mathcal{K}(X)$.

Lemma 11.2. The members of the ring $\mathcal{K}(X)$ are exactly all finite disjoint unions of subsets V of X of the form $K - L$ with K and L compact and $L \subseteq K$. The ring $\mathcal{K}(X)$ may be characterized also as the smallest ring containing all bounded open subsets of X.

PROOF. If $K_1 - L_1$ and $K_2 - L_2$ are two sets of the same kind as V in the statement of the lemma, then the identity

$$\begin{aligned}&(K_1 - L_1) \cup (K_2 - L_2)\\ &= ((K_1 \cup K_2) - (L_1 \cup L_2)) \cup ((K_2 \cap L_1) - (L_1 \cap L_2)) \cup ((K_1 \cap L_2) - (L_1 \cap L_2))\end{aligned}$$

shows that a union of two such sets is a disjoint union, and the identity

$$(K_1 - L_1) - (K_2 - L_2) = ((K_1 \cap L_2) - (L_1 \cap L_2)) \cup (K_1 - (L_1 \cup (K_1 \cap K_2)))$$

shows that the difference of two such sets is such a set. Therefore the collection of all such sets is a ring of subsets of X. This ring contains all compact sets because any compact set K is of the form $K - \varnothing$, and hence this ring equals $\mathcal{K}(X)$.

Any open bounded set U is the difference of the compact sets U^{cl} and $U^{\text{cl}} - U$, and hence it lies in $\mathcal{K}(X)$. In the reverse direction Corollary 10.23 shows that any compact set K is contained in the interior L^o of some compact set L. Thus K is the difference of the bounded open sets L^o and $L^o - K$, and $\mathcal{K}(X)$ is contained in the smallest ring containing all bounded open sets. □

Next we observe some properties of the linear functional ℓ. It is to be understood throughout the section that ℓ is a positive linear functional on $C_{\mathrm{com}}(X)$. The positivity implies that $\ell(f-g) \geq 0$ if $f-g \geq 0$; the linearity therefore gives $\ell(f) \geq \ell(g)$ for $f \geq g$. The linear functional has a kind of continuity property, according to the following lemma.

Lemma 11.3. Let K be a compact set, and let $\{f_n\}$ be a sequence in $C_{\mathrm{com}}(X)$ converging uniformly to a member f of $C_{\mathrm{com}}(X)$ in such a way that support$(f_n) \subseteq K$ for all n. Then $\lim_n \ell(f_n)$ exists and equals $\ell(f)$.

PROOF. Corollaries 10.23 and 10.44 show that there exists a function F in $C_{\mathrm{com}}(X)$ such that F takes values in [0, 1] and is 1 on K. Since $f_n - f \leq |f_n - f|$ and $-(f_n - f) \leq |f_n - f|$, we have

$$|\ell(f_n) - \ell(f)| = |\ell(f_n - f)| \leq \ell(|f_n - f|) \leq \ell(c_n F) = c_n \ell(F),$$

where $c_n = \|f_n - f\|_{\mathrm{sup}}$. The assumed uniform convergence means that c_n tends to 0. Since $\ell(F)$ is some fixed constant, the asserted convergence of $\ell(f_n)$ follows. □

Lemma 11.4 (Dini's Theorem). If $\{f_n\}$ is a sequence of functions in $C_{\mathrm{com}}(X)$ decreasing pointwise to 0, then $\{f_n\}$ converges uniformly to 0.

PROOF. Because of the pointwise decrease to 0, all the functions f_n have support contained in the compact set $K = \mathrm{support}(f_1)$. Let $\epsilon > 0$ be given, and let U_n be the open set where the continuous function f_n is $< \epsilon$. The pointwise decrease implies that the U_n are increasing with n, and the limit of 0 implies that each x in K is in some U_n. Thus the open sets U_n form an open cover of K. By compactness, there is a finite subcover. Since the sets U_n are increasing, some particular U_N covers K. Then $\|f_n\|_{\mathrm{sup}} \leq \epsilon$ for $n \geq N$. □

The final step of preparation is to observe some properties of compact and open sets. A bounded subset of X is said to be a G_δ if it is the countable intersection of bounded open sets. It is said to be an F_σ if it is the countable union of compact sets. We shall be especially interested in compact G_δ's and in open bounded F_σ's.

Lemma 11.5. Let f be a member of $C_{\mathrm{com}}(X)$ with values in [0, 1]. If $r > 0$, then the set where f is $\geq r$ is a compact G_δ. If $r \geq 0$, then the set where f is $> r$ is a bounded open F_σ.

PROOF. The set where f is $\geq r$ is closed because of continuity, and this closed set is a subset of the compact support. Hence the set is compact. Similarly the set where f is $> r$ is open because of continuity, and this open set is a subset of the compact support. Hence the set is bounded.

When $r \geq 0$, the set where f is $> r$ is the union, for $n \geq 1$, of the sets where f is $\geq r + \frac{1}{n}$. For $r > 0$ when N is large enough so that $r - \frac{1}{N} > 0$, the set where f is $\geq r$ is the intersection, for $n \geq N$, of the sets where f is $> r - \frac{1}{n}$. The lemma follows. $\square$

Lemma 11.6.

(a) If K is a compact G_δ, then there exists a decreasing sequence of bounded open sets U_n such that $U_n \supseteq U_{n+1}^{\text{cl}}$ for all n and $\cap_{n=1}^\infty U_n = K$.

(b) If U is a bounded open F_σ, then there exists an increasing sequence of compact sets K_n such that $K_n \subseteq K_{n+1}^o$ for all n and $\cup_{n=1}^\infty K_n = U$.

PROOF. For (a), let $\{V_n\}$ be a sequence of bounded open sets with intersection K. This is possible since K is a G_δ. Without loss of generality we may assume that the V_n decrease with n. We define the sequence $\{U_n\}$ inductively on n. Put $U_1 = V_1$. If U_n has been constructed, use Corollary 10.22 to find an open set V_n' such that $K \subseteq V_n'$ and $V_n'^{\text{cl}} \subseteq U_n$, and then define $U_{n+1} = V_n' \cap V_{n+1}$. Then the sets U_n have the required properties.

For (b), let $\{L_n\}$ be a sequence of compact sets with union U. This is possible since U is an F_σ. Without loss of generality we may assume that the L_n increase with n. We define the sequence $\{K_n\}$ inductively on n. Put $K_1 = L_1$. If K_n has been constructed, use Corollary 10.22 to find an open set V_n' such that $U \supseteq V_n'^{\text{cl}}$ and $V_n' \supseteq K_n$. The compact set $L_n' = V_n'^{\text{cl}}$ has $(L_n')^o \supseteq V_n'$. If we define $K_{n+1} = L_n' \cup L_{n+1}$, then the sets K_n have the required properties. $\square$

Lemma 11.7.

(a) If K is a compact G_δ, then there exists a decreasing sequence of functions f_n in $C_{\text{com}}(X)$ with values in $[0, 1]$ such that each f_n is 1 on some neighborhood of K and $\lim f_n = I_K$ pointwise.

(b) If U is a bounded open F_σ, then there exists an increasing sequence of functions f_n in $C_{\text{com}}(X)$ with values in $[0, 1]$ such that each f_n has compact support contained in U and $\lim f_n = I_U$ pointwise.

PROOF. For (a), apply Lemma 11.6a to choose a sequence of bounded open sets U_n with intersection K such that $U_n \supseteq U_{n+1}^{\text{cl}}$ for all n. Using Corollary 10.44, let g_n be a member of $C_{\text{com}}(X)$ with values in $[0, 1]$ such that g_n is 1 on U_{n+1}^{cl} and is 0 off U_n, and put $f_n = \min\{g_1, \dots, g_n\}$. Then the functions f_n have the required properties.

For (b), apply Lemma 11.6b to choose a sequence of compact sets K_n with union U such that $K_n \subseteq K_{n+1}^o$ for all n. Using Corollary 10.44, let g_n be a member of $C_{\text{com}}(X)$ with values in $[0, 1]$ such that g_n is 1 on K_n and is 0 off K_{n+1}^o, and put $f_n = \max\{g_1, \dots, g_n\}$. Then the functions f_n have the required properties. $\square$

Now we begin the proofs of the three identities in Section 1. If K is compact, let
$$\mu(K) = \inf\ \ell(f),$$
the infimum being taken over all f in $C_{\text{com}}(X)$ such that $f \geq I_K$. Since $\ell(\min\{f, 1\}) \leq \ell(f)$, there is no harm in considering only those f's taking values in [0, 1]. It is immediate from this definition and the positivity of ℓ that μ is nonnegative and **monotone** in the sense that $K' \subseteq K$ implies $\mu(K') \leq \mu(K)$. The next lemma is the key to being able to prove the three identities in Section 1.

Lemma 11.8. If K is a compact subset of X, then the infimum of $\ell(f)$ over all f in $C_{\text{com}}(X)$ such that $f \geq I_K$ equals the infimum of $\ell(f)$ over all f in $C_{\text{com}}(X)$ with values in [0, 1] such that $f \geq I_N$ for some neighborhood N of K depending on f.

REMARK. In particular, $\mu(K)$ can be computed by using only functions $f \geq I_K$ that are equal to 1 in some neighborhood of K.

PROOF. The problem is to show that the first infimum I_1 is not less than the second infimum I_2. Let $\epsilon > 0$ be given. Choose f in $C_{\text{com}}(X)$ with values in [0, 1] such that $f \geq I_K$ and $\ell(f) \leq I_1 + \epsilon$, and let L be the set where f is ≥ 1. Lemma 11.5 shows that L is a compact G_δ, and Lemma 11.7a produces a decreasing sequence of functions f_n in $C_{\text{com}}(X)$ with values in [0, 1] such that each f_n is 1 on some neighborhood of L and $\lim f_n = I_L$ pointwise. Then the sequence $\{\max\{f_n, f\}\}$ is pointwise decreasing with limit $\max\{I_L, f\} = f$, and hence $\{\max\{f_n, f\} - f\}$ is a pointwise decreasing sequence in $C_{\text{com}}(X)$ with limit 0. By Dini's Theorem (Lemma 11.4), the sequence $\{\max\{f_n, f\} - f\}$ converges uniformly to 0, and hence $\ell(\max\{f_n, f\})$ decreases to $\ell(f)$. For some sufficiently large n_0, we therefore have $\ell(\max\{f_{n_0}, f\}) \leq I_1 + 2\epsilon$. The function $\max\{f_{n_0}, f\}$ is one of the functions that figures into I_2, and thus $I_2 \leq I_1 + 2\epsilon$. Since ϵ is arbitrary, $I_2 \leq I_1$. □

Lemma 11.8 puts us in a position to prove identity (i) in Section 1 and to deduce that μ extends in a well-defined fashion to a nonnegative additive set function on $\mathcal{K}(X)$. We make use of the formula $a + b = \min\{a, b\} + \max\{a, b\}$, from which it follows that $a = \min\{a, b\} + (\max\{a, b\} - b)$.

Lemma 11.9. If K_1 and K_2 are any two compact subsets of X, then
$$\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2) + \mu(K_1 \cap K_2).$$

REMARK. The argument in Lemma 11.8 adapts to give a quick proof of the present lemma when X is a metric space. In the metric case we can find a decreasing sequence $\{f_n\}$ of functions ≤ 1 in $C_{\text{com}}(X)$ with pointwise limit I_{K_1}. If

$f \geq I_{K_1}$, then the proof of Lemma 11.8 shows that $f_n f$ converges uniformly to f and hence $\ell(f_n f)$ decreases to $\ell(f)$. It follows that $\ell(f_n)$ decreases to $\mu(K_1)$ whenever f_n decreases to I_{K_1}. If we similarly choose $\{g_n\}$ decreasing to I_{K_2} and choose, by Corollary 10.44, a function $\varphi \in C_{\text{com}}(X)$ with values in $[0, 1]$ that is identically 1 on the support of $f_1 + g_1$, then the formula stated just above shows that $f_n + g_n = \min\{f_n + g_n, \varphi\} + \big(\max\{f_n + g_n, \varphi\} - \varphi\big)$. The first term on the right side decreases pointwise to $I_{K_1 \cup K_2}$, and the second term decreases to $I_{K_1 \cap K_2}$. Thus a passage to the limit in the formula $\ell(f_n) + \ell(g_n) = \ell(\min\{f_n + g_n, \varphi\}) + \ell\big(\big(\max\{f_n + g_n, \varphi\} - \varphi\big)\big)$ immediately yields the result of the present lemma.

PROOF. Let f and g be functions in $C_{\text{com}}(X)$ with values in $[0, 1]$ such that $f \geq I_{K_1}$ and $g \geq I_{K_2}$, and choose, by Corollary 10.44, $\varphi \in C_{\text{com}}(X)$ with values in $[0, 1]$ that is identically 1 on the support of $f + g$. Then we have $f + g = \min\{f + g, \varphi\} + (\max\{f + g, \varphi\} - \varphi)$. The first term on the right side is $\geq I_{K_1 \cup K_2}$, and the second term is $\geq I_{K_1 \cap K_2}$. Therefore

$$\begin{aligned} \ell(f) + \ell(g) &= \ell(\min\{f + g, \varphi\}) + \ell((\max\{f + g, \varphi\} - \varphi)) \\ &\geq \mu(K_1 \cup K_2) + \mu(K_1 \cap K_2). \end{aligned}$$

Taking the infimum over f and then over g, we obtain

$$\mu(K_1) + \mu(K_2) \geq \mu(K_1 \cup K_2) + \mu(K_1 \cap K_2).$$

For the reverse direction let F be a member of $C_{\text{com}}(X)$ with values in $[0, 1]$ that is $\geq I_{K_1 \cup K_2}$ and is equal to 1 at least on some open set U containing $K_1 \cup K_2$. Similarly let G be a member of $C_{\text{com}}(X)$ with values in $[0, 1]$ that is $\geq I_{K_1 \cap K_2}$ and is equal to 1 at least on some open set V containing $K_1 \cap K_2$. Lemma 11.8 shows that F and G are the most general functions of a kind needed for the computation of $\mu(K_1 \cup K_2)$ and $\mu(K_1 \cap K_2)$. The sets U and V have compact closure in X since they are subsets of the supports of F and G. Choose, by Corollary 10.44, $\varphi \in C_{\text{com}}(X)$ with values in $[0, 1]$ that is identically 1 on the support of $F + G$. Let V_0 be an open set with $K_1 \cap K_2 \subseteq V_0 \subseteq V_0^{\text{cl}} \subseteq V$. Then $(K_2 - V_0) \cap K_1 = K_2 \cap V_0^c \cap K_1 \subseteq V_0 \cap V_0^c = \varnothing$. So there exists an open set W such that $K_2 - V_0 \subseteq W \subseteq W^{\text{cl}} \subseteq K_1^c$.

We define f and g to be members of $C_{\text{com}}(X)$ having compact support contained in U and having values in $[0, 1]$ such that

$$f = \begin{cases} 1 & \text{on } K_1, \\ 0 & \text{on } W^{\text{cl}}, \end{cases}$$

and

$$g = \begin{cases} 1 & \text{on } K_2, \\ 0 & \text{on support}(f) - V. \end{cases}$$

The functions f and g exist by Corollary 10.44 if it is shown that the closed sets K_1 and W^{cl} are disjoint and the closed sets K_2 and $\operatorname{support}(f) - V$ are disjoint. The sets K_1 and W^{cl} are disjoint since $W^{\mathrm{cl}} \subseteq K_1^c$. For K_2 and $\operatorname{support}(f) - V$, we observe that $\operatorname{support}(f) \subseteq ((W^{\mathrm{cl}})^c)^{\mathrm{cl}} \subseteq (W^c)^{\mathrm{cl}} = W^c \subseteq (K_2 - V_0)^c = V_0 \cup K_2^c \subseteq V \cup K_2^c$. Therefore

$$\begin{aligned}(\operatorname{support}(f) - V) \cap K_2 &\subseteq (V \cup K_2^c) \cap V^c \cap K_2 \\ &= (V \cap V^c \cap K_2) \cup (K_2^c \cap V^c \cap K_2) = \varnothing.\end{aligned}$$

We conclude that f and g exist.

By inspection, $f \geq I_{K_1}$ and $g \geq I_{K_2}$, from which $f + g \geq I_{K_1} + I_{K_2}$. Then $\min\{f + g, \varphi\}$ is 1 on $K_1 \cup K_2$ and is 0 off U. Since F is 1 on U, we obtain

$$\min\{f + g, \varphi\} \leq F. \tag{$*$}$$

Since $f + g \geq I_{K_1} + I_{K_2} = I_{K_1 \cup K_2} + I_{K_1 \cap K_2}$, the function $\max\{f + g, \varphi\} - \varphi$ equals $f + g - 1$ on $K_1 \cup K_2$, and this in turn is ≤ 1 everywhere. Let us see that

$$\max\{f + g, \varphi\} - \varphi \leq G \tag{$**$}$$

everywhere. The only points x at which $(**)$ could possibly fail are those where $G(x) < 1$, hence points of V^c. At such points the definition of g shows that $f(x) + g(x) \leq 1$. If also x is in U, then $\varphi(x) = 1$ and we compute that $\max\{f(x) + g(x), \varphi(x)\} - \varphi(x) = 1 - 1 = 0$. Thus $(**)$ holds at points of $U \cap V^c$. At points of $U^c \cap V^c$, the equality $f(x) = g(x) = 0$ implies that $\max\{f(x) + g(x), \varphi(x)\} - \varphi(x) = \varphi(x) - \varphi(x) = 0$. Thus again $(**)$ holds, and hence $(**)$ holds at every point of V^c, therefore everywhere.

Addition of $(*)$ and $(**)$ gives $f + g \leq F + G$ everywhere. Therefore

$$\ell(F) + \ell(G) = \ell(F + G) \geq \ell(f + g) = \ell(f) + \ell(g) \geq \mu(K_1) + \mu(K_2).$$

Taking the infimum over F and then over G gives $\mu(K_1 \cup K_2) + \mu(K_1 \cap K_2) \geq \mu(K_1) + \mu(K_2)$ and completes the proof of the lemma. $\square$

Lemma 11.9 yields by iteration a corresponding formula with the sum of n terms on each side. This extension of Lemma 11.9 is a computation in Boolean algebra involving no analysis at all—only the fact that the collection of compact sets is closed under finite unions and intersections. The details are carried out in the next lemma.

Lemma 11.10. If $K_1 \dots, K_n$ are compact subsets of X, then

$$\sum_{l=1}^{n} \mu(K_l) = \sum_{k=1}^{n} \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_k \le n} \Big(\bigcap_{j=1}^{k} K_{i_j} \Big)\Big).$$

PROOF. The argument is by induction on n, the base case of the induction being the case $n = 2$ that was settled by Lemma 11.9. Thus let $n > 2$, and assume the identity for the case $n - 1$. The inductive hypothesis gives

$$\sum_{l=1}^{n} \mu(K_l) = \sum_{k=1}^{n-1} \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_k < n} \Big(\bigcap_{j=1}^{k} K_{i_j} \Big)\Big) + \mu(K_n). \tag{$*$}$$

We shall prove by induction on $r \ge 1$ that

$$\begin{aligned}\sum_{l=1}^{n} \mu(K_l) &= \sum_{k=1}^{r-1} \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_k \le n} \Big(\bigcap_{j=1}^{k} K_{i_j} \Big)\Big) \\ &\quad + \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_r = n} \Big(\bigcap_{j=1}^{r} K_{i_j} \Big)\Big) + \sum_{k=r}^{n-1} \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_k < n} \Big(\bigcap_{j=1}^{k} K_{i_j} \Big)\Big),\end{aligned}$$

the base case of this induction being $r = 1$, where this identity reduces to $(*)$. The proof for the case $r = n$ will complete the inductive step for the outer induction and thereby will complete the proof of the lemma. To pass from r to $r + 1$ in the inner induction, the question is whether

$$\begin{aligned}&\mu\Big(\bigcup_{1 \le i_1 < \cdots < i_r = n} \Big(\bigcap_{j=1}^{r} K_{i_j} \Big)\Big) + \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_r < n} \Big(\bigcap_{j=1}^{r} K_{i_j} \Big)\Big) \\ &\qquad \overset{?}{=} \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_r \le n} \Big(\bigcap_{j=1}^{r} K_{i_j} \Big)\Big) + \mu\Big(\bigcup_{1 \le i_1 < \cdots < i_{r+1} = n} \Big(\bigcap_{j=1}^{r+1} K_{i_j} \Big)\Big).\end{aligned}$$

The union of the two sets on the left here is the first set on the right side. In view of Lemma 11.9, this formula will follow if it is shown that the second set on the right side is the intersection of the two sets on the left. The intersection of the two sets on the left side is equal to

$$\bigcup_{\substack{1 \le i_1 < \cdots < i_r = n, \\ 1 \le i'_1 < \cdots < i'_r < n}} \Big(\Big(\bigcap_{j=1}^{r} K_{i_j} \Big) \cap \Big(\bigcap_{j=1}^{r} K_{i'_j} \Big)\Big). \tag{$**$}$$

A term in the union in this expression is an intersection of at least $r + 1$ of the sets $K_1, \dots, K_n$, the last of which is K_n, namely the ones corresponding to indices $i'_1, \dots, i'_r$ and $i_r = n$. Every intersection of exactly $r + 1$ of the sets $K_1, \dots, K_n$ occurs if the last one is K_n because we can take $i_1 = i'_1, \dots, i_{r-1} = i'_{r-1}$. Any intersection of more than $r + 1$ sets is contained in one with exactly $r + 1$ sets, and thus $(**)$ equals $\bigcup_{1 \le i_1 < \cdots < i_{r+1} = n} \Big(\bigcap_{j=1}^{r+1} K_{i_j} \Big)$, as asserted. $\square$

A further formality is the derivation from these results that μ extends in a well-defined fashion to a nonnegative additive set function on the ring $\mathcal{K}(X)$. Again no analysis is involved, only the one additional fact that the intersection of two sets of the form $K-L$ with K and L compact is again of this form, specifically that $(K-L)\cap(K'-L')=(K\cap K')-(L\cup L')$.

Lemma 11.11. The set function μ extends in a well-defined fashion to a nonnegative additive set function on $\mathcal{K}(X)$ under the definition

$$\mu\Big(\bigcup_{j=1}^{n}(K_j-L_j)\Big)=\sum_{j=1}^{n}\big(\mu(K_j)-\mu(L_j)\big)$$

whenever K_j and L_j are compact with $L_j\subseteq K_j$ for each j with $1\le j\le n$ and the sets $K_1-L_1,\dots,K_n-L_n$ are pairwise disjoint.

REMARKS. Lemma 11.2 assures us that every member of $\mathcal{K}(X)$ is of the form in this lemma. The subtlety of the lemma arises from the fact that the sets K_j need not be disjoint.

PROOF. First let us see that μ is well defined in the case $j=1$, i.e., that $K'-L'=K-L$ with $L'\subseteq K'$ and $L\subseteq K$ implies $\mu(K')-\mu(L')=\mu(K)-\mu(L)$. We are to show that $\mu(K')+\mu(L)=\mu(K)+\mu(L')$, and Lemma 11.9 shows that it is enough to show that $K'\cup L=K\cup L'$ and $K'\cap L=K\cap L'$. Suppose x is in $K'\cup L$. If x is in L, then x is in K, hence in $K\cup L'$. If x is in K' instead, then either x has to be in L' in the case that x is not in $K'-L'$ or x has to be in K in the case that x is in $K'-L'=K-L$. So $K'\cup L\subseteq K\cup L'$. If x is in $K'\cap L$, then x is not in $K-L$ and must be in L' in order to avoid being in $K'-L'$. So x is in $L\cap L'\subseteq K\cap L'$. Reversing the roles of $K'-L'$ and $K-L$, we see that $K'\cup L=K\cup L'$ and $K'\cap L=K\cap L'$.

Next suppose that $K'-L'=\bigcup_{j=1}^{n}(K_j-L_j)$ with $L'\subseteq K'$, $L_j\subseteq K_j$ for each j, and the sets K_j-L_j disjoint. We are to show that $\mu(K')-\mu(L')=\sum_{j=1}^{n}(\mu(K_j)-\mu(L_j))$, i.e., that $\mu(K')+\sum_{j=1}^{n}\mu(L_j)=\mu(L')+\sum_{j=1}^{n}\mu(K_j)$. The argument will generalize that in the previous paragraph: The set $K'-L'$ has complement $L'\cup K'^c$, and therefore the given condition of disjointness means that

$$X=(L'\cup K'^c)\cup\bigcup_{j=1}^{n}(K_j-L_j)\qquad(*)$$

disjointly. Put $L_{n+1}=K'$ and $K_{n+1}=L'$, so that we are asking whether

$$\sum_{j=1}^{n+1}\mu(L_j)\stackrel{?}{=}\sum_{j=1}^{n+1}\mu(K_j).$$

In view of Lemma 11.10, it would be enough to show that

$$\bigcup_{1\le i_1<\cdots<i_k\le n+1}\Big(\bigcap_{j=1}^{k} L_{i_j}\Big)=\bigcup_{1\le i_1<\cdots<i_k\le n+1}\Big(\bigcap_{j=1}^{k} K_{i_j}\Big)$$

for $1\le k\le n+1$. The left side is the set of x lying in at least k of the sets L_i, and the right side is the corresponding set for the K_i's. Thus it is enough to prove that the set of x lying in exactly r sets K_i is contained in the set of x lying in exactly r sets L_i, for $1\le r\le n+1$.

We check this condition separately for the three cases $x\in L'$, $x\notin K'$, and $x\in K'-L'$. From $(*)$ we see that x in $L'\cup K'^c$ implies that x is not in any K_j-L_j for $1\le j\le n$. Hence for the first two cases, x is in L_j with $1\le j\le n$ if and only if x is in K_j.

Case 1. $x\in L'$. For x to be in r of the sets $K_1,\dots,K_{n+1}$, x must be in $r-1$ of the sets $K_1,\dots,K_n$, hence in $r-1$ of the sets $L_1,\dots,L_n$. Since x is in L', it is in $K'=L_{n+1}$. Therefore x is in r of the sets $L_1,\dots,L_{n+1}$.

Case 2. $x\notin K'$. For x to be in r of the sets $K_1,\dots,K_{n+1}$, x must be in r of the sets $K_1,\dots,K_n$, hence in r of the sets $L_1,\dots,L_n$. Since x is not in K', it is not in L_{n+1}. Therefore x is in r of the sets $L_1,\dots,L_{n+1}$.

Case 3. $x\in K'-L'$. Since x is not in $L'\cup K'^c$, $(*)$ shows that x is in exactly one K_j-L_j with $1\le j\le n$. For x to be in r of the sets $K_1,\dots,K_{n+1}$, x must be in r of the sets $K_1,\dots,K_n$, hence in $r-1$ of the sets $L_1,\dots,L_n$. Since x is in $K'=L_{n+1}$, it is in r of the sets $L_1,\dots,L_{n+1}$.

For the general case, suppose that $\bigcup_{j=1}^m(K_j'-L_j')=\bigcup_{j=1}^n(K_j-L_j)$. Intersecting both sides with $K_i'-L_i'$, we obtain

$$K_i'-L_i'=\bigcup_{j=1}^{n}((K_j\cap K_i')-((L_j\cup L_i')\cap(K_j\cap K_i'))).$$

The case just proved shows that

$$\mu(K_i'-L_i')=\sum_{j=1}^{n}\big(\mu(K_j\cap K_i')-\mu((L_j\cup L_i')\cap(K_j\cap K_i'))\big)$$

and hence

$$\sum_{i=1}^{m}\mu(K_i'-L_i')=\sum_{i=1}^{m}\sum_{j=1}^{n}\big(\mu(K_j\cap K_i')-\mu((L_j\cup L_i')\cap(K_j\cap K_i'))\big).$$

Similarly

$$\sum_{j=1}^{n}\mu(K_j-L_j)=\sum_{j=1}^{n}\sum_{i=1}^{m}\big(\mu(K_j\cap K_i')-\mu((L_j\cup L_i')\cap(K_j\cap K_i'))\big).$$

Therefore $\sum_{i=1}^m\mu(K_i'-L_i')=\sum_{j=1}^n\mu(K_j-L_j)$, and the proof is complete. $\square$

In short order, we can now prove identities (ii) and (iii). Lemma 11.12 will prove (iii), and Lemma 11.13 will prove (ii).

Lemma 11.12. If U is any bounded open subset of X, then

$$\sup_{\substack{g\in C_{\text{com}}(X),\\ 0\le g\le I_U,\\ \text{support } g\subseteq U}} \ell(g) = \sup_{\substack{K\subseteq U,\\ K \text{ compact}}} \mu(K) = \sup_{\substack{f\in C_{\text{com}}(X),\\ 0\le f\le I_U}} \ell(f).$$

PROOF. Let S_1, S_2, S_3 be the three suprema in question. We first check that $S_1 \le S_2 \le S_3$. If g contributes to S_1, then $g \le I_{\text{support } g} \le I_U$. If $h \in C_{\text{com}}(X)$ has $I_{\text{support } g} \le h$, then $g \le h$ and hence $\ell(g) \le \ell(h)$. Taking the infimum over all such h, we obtain $\ell(g) \le \mu(\text{support } g) \le S_2$. Taking the supremum over all g therefore gives $S_1 \le S_2$. Next if K is compact with $K \subseteq U$, Corollary 10.44 allows us to find $f \in C_{\text{com}}(X)$ with values in $[0, 1]$ such that f is equal to 1 on K and equal to 0 on U^c. Then $I_K \le f \le I_U$. The definitions of $\mu(K)$ and S_3 yield $\mu(K) \le \ell(f) \le S_3$. Taking the supremum over all K therefore gives $S_2 \le S_3$.

To complete the proof, we show that $S_1 \ge S_3$. Let $\epsilon > 0$ be given. Choose f in $C_{\text{com}}(X)$ such that $0 \le f \le I_U$ and $\ell(f) \ge S_3 - \epsilon$, and let V be the set where f is > 0. Lemma 11.5 shows that V is a bounded open F_σ, and Lemma 11.7b produces an increasing sequence of functions f_n in $C_{\text{com}}(X)$ with values in $[0, 1]$, each with support some compact subset of V, such that $\lim f_n = I_V$ pointwise. Then the sequence $\{\min\{f_n, f\}\}$ is pointwise increasing with limit $\min\{I_V, f\}$. If x is a point where $I_V(x) < f(x)$, then $f(x) > 0$, x is in V, and $I_V(x) = 1$, contradiction. So there is no such point, and $\min\{I_V, f\} = f$. Therefore the sequence $\{f - \min\{f_n, f\}\}$ is a pointwise decreasing sequence in $C_{\text{com}}(X)$ with limit 0. By Dini's Theorem (Lemma 11.4), the sequence $\{f - \min\{f_n, f\}\}$ converges uniformly to 0, and hence $\ell(\min\{f_n, f\})$ increases to $\ell(f)$. For some sufficiently large n_0, we therefore have $\ell(\min\{f_{n_0}, f\}) \ge S_3 - 2\epsilon$. The function $\min\{f_{n_0}, f\}$ is one of the functions that figures into S_1, and thus $S_1 \ge \ell(\min\{f_{n_0}, f\}) \ge S_3 - 2\epsilon$. Since ϵ is arbitrary, $S_1 \ge S_3$. $\square$

Lemma 11.13. Let μ be extended to a nonnegative additive set function on $\mathcal{K}(X)$ as in Lemma 11.11. If U is a bounded open subset of X, then $\mu(U) = \sup_{K\subseteq U,\, K \text{ compact}} \mu(K)$.

PROOF. For the bounded open set U, let S_1, S_2, S_3 be the three equal suprema of Lemma 11.12. By definition, $\mu(U) = \mu(L) - \mu(L - U)$ for any compact set L containing U, and we are to prove that $\mu(U) = S_2$. If K is a compact subset of U, then $K \cup (L - U)$ is a disjoint union contained in L, and we have $\mu(K) + \mu(L - U) = \mu(K \cup (L - U)) \le \mu(L)$. Taking the supremum over all such K, we obtain $S_2 + \mu(L - U) \le \mu(L)$, i.e., $S_2 \le \mu(U)$.

Let h be any member of $C_{\text{com}}(X)$ with values in $[0, 1]$ such that $h \geq I_{L-U}$ and such that h is 1 on an open neighborhood N of $L - U$. Then $L \subseteq N \cup U$. For each point x of U, find an open neighborhood U_x of x with $U_x^{\text{cl}} \subseteq U$. Then N and the U_x's form an open cover of L, and there is a finite subcover. Let us say that $L \subseteq N \cup U_{x_1} \cup \cdots \cup U_{x_n}$. The set $K = U_{x_1}^{\text{cl}} \cup \cdots \cup U_{x_n}^{\text{cl}}$ is a compact subset of U, and $L \subseteq N \cup K$. Choose, by Corollary 10.44, a function $f \in C_{\text{com}}(X)$ with values in $[0, 1]$ such that f is 1 on K and is 0 off U. This function has $0 \leq f \leq I_U$. Since f is 1 on K and h is 1 on N, $h + f$ is ≥ 1 on L. Hence $\mu(L) \leq \ell(h + f) = \ell(h) + \ell(f) \leq \ell(h) + S_3$. Thus $\mu(L) \leq \mu(L - U) + S_3$ and $\mu(U) \leq S_3$. Since $S_3 = S_2$ by Lemma 11.12, $\mu(U) = S_2$ as required. □

PROOF OF EXISTENCE IN THEOREM 11.1. If K is compact, we define $\mu(K)$, just as we did earlier in this section, to be the infimum of $\ell(f)$ over all f in $C_{\text{com}}(X)$ such that $f \geq I_K$. Lemma 11.11 shows that μ extends, necessarily in a unique fashion, to a well-defined nonnegative additive set function on $\mathcal{K}(X)$.

Consider the set $\mathcal{C}$ of all members E of $\mathcal{K}(X)$ satisfying the following regularity property: for each $\epsilon > 0$, there exist compact K and open bounded U with $K \subseteq E \subseteq U$ and $\mu(U - K) < \epsilon$. Lemma 11.13 shows that every open bounded set is in $\mathcal{C}$. We show closure of $\mathcal{C}$ under finite unions. If E_1 and E_2 are in $\mathcal{C}$, then we can choose K_1 and K_2 compact and U_1 and U_2 bounded open such that $K_1 \subseteq E_1 \subseteq U_1$, $K_2 \subseteq E_2 \subseteq U_2$, $\mu(U_1 - K_1) < \epsilon/2$, and $\mu(U_2 - K_2) < \epsilon/2$. Then $K_1 \cup K_2 \subseteq E_1 \cup E_2 \subseteq U_1 \cup U_2$ and $(U_1 \cup U_2) - (K_1 \cup K_2) \subseteq (U_1 - K_1) \cup (U_2 - K_2)$. It follows that $\mu((U_1 \cup U_2) - (K_1 \cup K_2)) \leq \mu((U_1 - K_1)) + \mu((U_2 - K_2)) < \epsilon$, and $\mathcal{C}$ is closed under finite unions.

We show closure of $\mathcal{C}$ under differences. If E_1 and E_2 are in $\mathcal{C}$, then we again choose K_1 and K_2 compact and U_1 and U_2 bounded open such that $K_1 \subseteq E_1 \subseteq U_1$, $K_2 \subseteq E_2 \subseteq U_2$, $\mu(U_1 - K_1) < \epsilon/2$, and $\mu(U_2 - K_2) < \epsilon/2$. Then $K_1 - U_2 \subseteq E_1 - E_2 \subseteq U_1 - K_2$, and $(U_1 - K_2) - (K_1 - U_2) \subseteq (U_1 - K_1) \cup (U_2 - K_2)$. Hence $\mu((U_1 - K_2) - (K_1 - U_2)) \leq \mu(U_1 - K_1) + \mu(U_2 - K_2) < \epsilon$, and $\mathcal{C}$ is closed under differences. By Lemma 11.2, $\mathcal{C}$ equals $\mathcal{K}(X)$. Thus every set in $\mathcal{K}(X)$ satisfies the regularity property.

Next let us see that μ is completely additive on $\mathcal{C}$. Let E_n be a disjoint sequence of sets in $\mathcal{K}(X)$ with union E in $\mathcal{K}(X)$. For every N, we have $\sum_{n=1}^{N} \mu(E_n) = \mu(E_1 \cup \cdots \cup E_N) \leq \mu(E)$. Hence $\sum_{n=1}^{\infty} \mu(E_n) \leq \mu(E)$. For the reverse inequality, let $\epsilon > 0$ be given. Choose, by the regularity property, K compact and U_n open bounded with $K \subseteq E$, $E_n \subseteq U_n$, $\mu(E - K) < \epsilon$, and $\mu(U_n - E_n) < \epsilon/2^n$. Then $K \subseteq E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} U_n$. In other words, the sets U_n form an open cover of the compact set K. Some finite subcollection is a cover, and thus $K \subseteq U_1 \cup \cdots \cup U_N$ for some N. Then we have

$$\begin{aligned}\mu(E) &= \mu(E - K) + \mu(K) \leq \epsilon + \mu(U_1 \cup \cdots \cup U_N) \\ &\leq \epsilon + \textstyle\sum_{n=1}^{N} \mu(U_n) \leq \epsilon + \sum_{n=1}^{N} (\mu(E_n) + \epsilon/2^n) \leq \sum_{n=1}^{\infty} \mu(E_n) + 2\epsilon.\end{aligned}$$

Since ϵ is arbitrary, $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$. Therefore $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$, and μ is completely additive on $\mathcal{K}(X)$.

The Extension Theorem (Theorem 5.5) shows that μ extends uniquely to a measure on the smallest σ-ring containing $\mathcal{K}(X)$, i.e., the σ-ring of σ-bounded Borel sets. Proposition 5.37 shows further that μ extends canonically to a measure on the σ-algebra of all Borel sets under the definition

$$\mu(E) = \sup_{\substack{F \subseteq E,\ F \in \mathcal{B}(X), \\ F\ \sigma\text{-bounded}}} \mu(F).$$

This defines μ on $\mathcal{B}(X)$. We are left with showing that μ is regular and that $\ell(f) = \int_X f\,d\mu$ for every $f \in C_{\text{com}}(X)$.

In showing that $\ell(f) = \int_X f\,d\mu$ for every $f \in C_{\text{com}}(X)$, it is enough to handle an arbitrary $f \geq 0$. Fix $\epsilon > 0$, and fix an integer N such that $\|f\|_{\text{sup}} < N\epsilon$. For $0 \leq n \leq N$, define $f_n = \min\{f, n\epsilon\}$. Each f_n is in $C_{\text{com}}(X)$, the function f_0 is 0, and the function f_N is f. For $0 \leq n < N$, define $g_n = f_{n+1} - f_n$. We can recover f from the g_n's as $f = \sum_{n=0}^{N-1} g_n$. For $n \geq 1$, define $K_n = \{x \mid f(x) \geq n\epsilon\}$, and let $K_0 = \text{support}(f)$. All the sets K_n are compact, and they decrease in size with n. In this notation the formula for g_n is

$$g_n(x) = \begin{cases} 0 & \text{if } x \notin K_n, \\ f(x) - n\epsilon & \text{if } x \in K_n - K_{n+1}, \\ \epsilon & \text{if } x \in K_{n+1}. \end{cases}$$

Consequently
$$\epsilon I_{K_{n+1}} \leq g_n \leq \epsilon I_{K_n}. \tag{$*$}$$

Integration therefore gives

$$\epsilon\mu(K_{n+1}) \leq \int_X g_n\,d\mu \leq \epsilon\mu(K_n). \tag{$\dagger$}$$

The inequality given as $I_{K_{n+1}} \leq \epsilon^{-1} g_n$ in $(*)$ implies that $\mu(K_{n+1}) \leq \epsilon^{-1}\ell(g_n)$. The other inequality $\epsilon^{-1} g_n \leq I_{K_n}$ in $(*)$ says that any $h \in C_{\text{com}}(X)$ with $I_{K_n} \leq h$ has $\epsilon^{-1} g_n \leq h$. Taking the infimum over h yields $\epsilon^{-1}\ell(g_n) \leq \mu(K_n)$. Thus we have

$$\epsilon\mu(K_{n+1}) \leq \ell(g_n) \leq \epsilon\mu(K_n). \tag{$\dagger\dagger$}$$

Subtracting $(\dagger)$ and $(\dagger\dagger)$, we obtain

$$-\epsilon(\mu(K_n) - \mu(K_{n+1})) \leq \int_X g_n\,d\mu - \ell(g_n) \leq \epsilon(\mu(K_n) - \mu(K_{n+1})).$$

Since $f = \sum_{n=0}^{N-1} g_n$, summing from $n = 0$ to $n = N - 1$ gives

$$\left| \int_X f\,d\mu - \ell(f) \right| \leq \epsilon \sum_{n=0}^{N-1} (\mu(K_n) - \mu(K_{n+1})) = \epsilon\mu(\text{support}(f)).$$

Since ϵ is arbitrary, $\left|\int_X f\,d\mu - \ell(f)\right| = 0$. Thus $\ell(f) = \int_x f\,d\mu$.

Fix a compact subset K_0 of X, form the σ-ring $\mathcal{B}(X) \cap K_0$, and let $\mathcal{A}(K_0)$ be the collection of members E of $\mathcal{B}(X) \cap K_0$ such that $\mu(E)$ is the supremum of $\mu(K)$ over all compact subsets K of E and $\mu(E)$ is the infimum of $\mu(U)$ over all bounded open sets in X that contain E; the open sets in question need not lie within K_0. Since the sets in $\mathcal{A}(K_0)$ all have finite measure, the regularity condition on E is that there exist, for each $\epsilon > 0$, K compact and U bounded open with $K \subseteq E \subseteq U$ and $\mu(U - K) < \epsilon$. The same arguments as at the beginning of the present proof show that $\mathcal{A}(K_0)$ is closed under finite unions and differences. To see closure under countable disjoint unions, let $\{E_n\}$ be a disjoint sequence in $\mathcal{A}(K_0)$ with union E, let ϵ be given, and choose K_n compact and U_n bounded open with $K_n \subseteq E_n \subseteq U_n$ and $\mu(U_n - K_n) < \epsilon/2^n$. Applying Corollary 10.23, let L be a compact subset of X with $K_0 \subseteq L^o$. The sets K_n are disjoint, and thus $\sum_{n=1}^\infty \mu(K_n)$ converges. Choose N such that $\sum_{n=N+1}^\infty \mu(K_n) < \epsilon$. Define $U = L^o \cap \bigcup_{n=1}^\infty U_n$, $K = \bigcup_{n=1}^N K_n$, $K_\infty = \bigcup_{n=1}^\infty K_n$, and $F = \bigcup_{n=N+1}^\infty K_n$. Then K is compact, U is bounded open, and $K \subseteq E \subseteq U$. Since $K_\infty = K \cup F$, we have

$$\begin{aligned}\mu(U-K) &\le \mu(U-K_\infty) + \mu(F) \le \mu\Big(\bigcup_{n=1}^\infty (U_n - K_n)\Big) + \mu\Big(\bigcup_{n=N+1}^\infty K_n\Big)\\ &\le \sum_{n=1}^\infty \mu(U_n - K_n) + \sum_{n=N+1}^\infty \mu(K_n) \le \sum_{n=1}^\infty \epsilon/2^n + \epsilon = 2\epsilon.\end{aligned}$$

Thus $\mathcal{A}(K_0)$ is closed under countable disjoint unions and is a σ-ring. Since the compact subsets of K_0 are in $\mathcal{A}(K_0)$, we conclude that $\mathcal{A}(K_0) = \mathcal{B}(K_0)$.

This proves regularity for all bounded sets. If E is σ-bounded, we can choose an increasing sequence $\{L_n\}$ of compact sets whose union contains E. Put $E_n = E \cap L_n$. Given $\epsilon > 0$, we apply the previous step to choose K_n compact and U_n bounded open such that $K_n \subseteq E_n \subseteq U_n$ and $\mu(U_n - K_n) < \epsilon/2^n$. Taking $U = \bigcup_{n=1}^\infty U_n$ and $K_\infty = \bigcup_{n=1}^\infty K_n$, we have $K_\infty \subseteq E \subseteq U$ and $\mu(U - K_\infty) < \epsilon$. Thus $\mu(U) \le \mu(E) + \epsilon$, and $\mu(E) \le \mu(K_\infty) + \epsilon$. The first of these inequalities, being possible for any ϵ, shows that $\mu(E)$ is the infimum of the measures of open σ-bounded sets containing E. Since $\mu(K_\infty) = \lim_N \mu\big(\bigcup_{n=1}^N K_n\big)$ by complete additivity, the second of these inequalities, being possible for any ϵ, shows that $\mu(E)$ is the supremum of the measures of compact sets contained in E.

This proves regularity for all σ-bounded sets. If E is a Borel set that is not σ-bounded, we know that $\mu(E)$ is the supremum of the measures of $\mu(F)$ for σ-bounded Borel subsets F of E, and we know that $\mu(F)$ is the supremum of the measures of $\mu(K)$ for compact subsets K of F. Therefore $\mu(E)$ is the supremum of the measures of $\mu(K)$ for compact subsets K of E. This completes the proof of regularity of μ. $\square$

PROOF OF UNIQUENESS IN THEOREM 11.1. Let μ be the constructed measure, and let ν be a second measure satisfying the properties of the theorem. The assumed regularity of ν implies that it is enough to prove that $\nu(K) = \mu(K)$ for every compact subset K of X. Fix K, and let α be the infimum defining $\mu(K)$, namely the infimum of $\ell(f)$ over all $f \in C_{\text{com}}(X)$ with values in [0, 1] such that $I_K \leq f$. Integrating this inequality with respect to ν, we see that $\nu(K) \leq \int_X f\,d\nu$ and therefore $\nu(K) \leq \alpha$. Suppose that $\nu(K) < \alpha$. By Corollary 10.23 and the assumed regularity of ν, we can find a bounded open set U with $U \supseteq K$ and $\nu(U) < \alpha$. By Corollary 10.44 we can find a function $g \in C_{\text{com}}(X)$ with values in [0, 1] such that g is 1 on K and is 0 off U. Then $I_K \leq g \leq I_U$. Hence $\ell(g) = \int_X g\,d\mu = \int_X g\,d\nu \leq \int_X I_U\,d\nu = \nu(U) < \alpha \leq \ell(g)$, and we obtain a contradiction. We conclude that $\nu(K) = \alpha = \mu(K)$, and the uniqueness follows. $\square$

3. Regular Borel Measures

The fact that compact sets for a general locally compact Hausdorff X need not be countable intersections of open sets suggests a look at the ring of sets generated by the compact sets that are indeed such intersections, as well as the associated σ-algebra. The sets in this σ-algebra are known as "Baire sets," and it turns out that the members of $C_{\text{com}}(X)$ are measurable with respect to this σ-algebra. The σ-algebra of Baire sets can be strictly smaller than the σ-algebra of Borel sets, and thus one can make a case for limiting oneself to Baire sets all along. This would be a fine point, one not worth pursuing here, but for one fact: the σ-algebra of Baire sets for $X \times Y$ is a correct σ-algebra to use in Fubini's Theorem for changing iterated integrals over X and Y to a double integral—and this may not be true when Borel sets are used.

This fact about Fubini's Theorem might seem to be a telling argument for replacing Borel sets by Baire sets everywhere in the theory. The difficulty is that it is a little tedious to check constantly whether sets are Baire sets—for example, whether one-point sets are Baire sets. Thus the normal practice is to work with Borel sets and to resort to Baire sets only when Fubini's Theorem comes into play in a way that makes the distinction important. The most frequent case that arises in applications of Fubini's Theorem in this theory is that a function on $X \times Y$ is continuous with compact support, in which case only Baire sets are involved anyway.

Thus let X be a locally compact Hausdorff space. The sets in the smallest σ-algebra $\mathcal{B}(X)$ containing the compact sets are the **Borel sets**, and the sets in the smallest σ-algebra $\mathcal{B}_0(X)$ containing the compact G_δ's are the **Baire sets**. Measurable functions in the first case will be called **Borel measurable functions** or **Borel functions**, and measurable functions in the second case will be called

Baire measurable functions or **Baire functions**. We shall observe in Corollary 11.16 below that every member of $C_{\text{com}}(X)$ is a Baire function.

If the locally compact Hausdorff space X is a metric space, then any closed set F is the intersection of the sets $U_n = \{x \mid D(x, F) < \frac{1}{n}\}$, where $D(\cdot, F)$ is the distance to the set F. Consequently every compact subset of X is a G_δ, and every Borel set is a Baire set.

Proposition 11.14. If K and U are subsets of X with K compact, U open, and $K \subseteq U$, then there exist a compact G_δ, say K_0, and an open bounded F_σ, say U_0, such that $K \subseteq U_0 \subseteq K_0 \subseteq U$.

PROOF. Choose by Corollary 10.44 a member f of $C_{\text{com}}(X)$ with values in $[0, 1]$ such that f is 1 on K and is 0 on U^c. If K_0 is the set where f is $\geq \frac{1}{2}$ and U_0 is the set where f is $> \frac{1}{2}$, then Lemma 11.5 shows that K_0 and U_0 have the required properties.

Corollary 11.15. Any σ-compact open subset of X is a Baire set.

PROOF. If $U = \bigcup_{n=1}^\infty K_n$ is open with each K_n compact, we can apply Proposition 11.14 to the inclusion $K_n \subseteq U$ and find a set $(K_n)_0$ that is a compact G_δ and has $K_n \subseteq (K_n)_0 \subseteq U$. Then $U = \bigcup_{n=1}^\infty (K_n)_0$ exhibits U as the countable union of compact G_δ's, hence as a Baire set.

Corollary 11.16. Every member of $C_{\text{com}}(X)$ is a Baire function.

PROOF. This is immediate from Lemma 11.5 and Corollary 11.15.

Proposition 11.17. If X and Y are σ-compact, then the product σ-algebra for $X \times Y$ obtained from the Baire sets of X and Y is the σ-algebra of Baire sets of $X \times Y$.

PROOF. If K_X and K_Y are compact G_δ's in X and Y, then $K_X \times K_Y$ is a compact G_δ in $X \times Y$, and it follows that $\mathcal{B}_0(X) \times \mathcal{B}_0(Y) \subseteq \mathcal{B}_0(X \times Y)$. For the reverse inclusion let K be a compact G_δ in $X \times Y$, and write K as $K = \bigcap_{n=1}^\infty U_n$ with each U_n open. We construct open sets S_n in $\mathcal{B}_0(X) \times \mathcal{B}_0(Y)$ with $K \subseteq S_n \subseteq U_n$, and then it follows that $K = \bigcap_{n=1}^\infty S_n$ and K is a Baire set.

To do so, it is enough to show that if $K \subseteq W$ with W open, then there is an open set S in $\mathcal{B}_0(X) \times \mathcal{B}_0(Y)$ with $K \subseteq S \subseteq W$. For each (x, y) in K, find open neighborhoods U_x of x and V_y of y such that $U_x \times V_y \subseteq W$. Proposition 11.14, applied to the inclusion $\{x\} \subseteq U_x$ and then to the inclusion $\{y\} \subseteq V_y$, shows that we may assume that U_x and V_y are open F_σ's. In view of Corollary 11.15, they are then Baire sets. Hence $U_x \times V_y$ is in $\mathcal{B}_0(X) \times \mathcal{B}_0(Y)$. As (x, y) varies, the sets $U_x \times V_y$ form an open cover of K, and there is a finite subcover. We can take S to be the union of the elements in the finite subcover, and then S has the required properties.

Now we turn our attention to measures. A **Baire measure** on X is a measure on the Baire sets that is finite on every compact G_δ. The restriction of a Borel measure to the Baire sets is a Baire measure. We are going to prove that Baire measures are automatically regular in the same sense that Borel measures in $\mathbb{R}^N$ are automatically regular.

Proposition 11.18. Every Baire measure μ is regular in the following sense:

$$\mu(E) = \sup_{\substack{K \subseteq E,\\ K \text{ compact } G_\delta}} \mu(K) \qquad \text{for every set } E \text{ in } \mathcal{B}_0(X),$$

$$\mu(E) = \inf_{\substack{U \supseteq E,\\ U \text{ open } F_\sigma}} \mu(U) \qquad \text{for every } \sigma\text{-bounded set } E \text{ in } \mathcal{B}_0(X).$$

REMARK. Since Baire sets and Borel sets are the same in a metric space, this proposition generalizes the known regularity of Borel measures on any open subset of $\mathbb{R}^n$, as given in Theorem 6.25.

PROOF. If L is a compact G_δ, then $\mu(L)$ is certainly the supremum of $\mu(K)$ for the compact G_δ's contained in L. Suppose that U is σ-bounded open with $L \subseteq U$. Proposition 11.14 produces a bounded open set U_0 that is an F_σ and has $L \subseteq U_0 \subseteq U$. Consequently $\mu(L)$ is the infimum of $\mu(U_0)$ for the open F_σ's containing L. Thus every compact G_δ satisfies the stated regularity condition.

The remainder of the proof runs parallel to the proof of regularity at the end of the proof of existence for Theorem 11.1, and we shall be brief. Fix a compact G_δ in X, say K_0. Form the σ-ring $\mathcal{B}_0(X) \cap K_0$, and let $\mathcal{A}_0(K_0)$ be the collection of members E of $\mathcal{B}_0(X) \cap K_0$ such that $\mu(E)$ is the supremum of $\mu(K)$ over all compact subsets K of E that are G_δ's and $\mu(E)$ is the infimum of $\mu(U)$ over all open supersets U of E that are F_σ's; the open sets in question need not lie within K_0. Since the sets in $\mathcal{A}_0(K_0)$ all have finite measure, the regularity condition on E is that there exist, for each $\epsilon > 0$, K compact and U open of the correct kind with $K \subseteq E \subseteq U$ and $\mu(U - K) < \epsilon$. The same arguments as earlier show that $\mathcal{A}_0(K_0)$ is closed first under finite unions and differences, then under countable disjoint unions. Thus $\mathcal{A}_0(K_0)$ is a σ-ring containing all compact G_δ's, and we conclude that $\mathcal{A}(K_0) = \mathcal{B}(K_0)$.

This proves regularity for all bounded Baire sets. If the Baire set E is σ-bounded, we can choose an increasing sequence $\{L_n\}$ of compact G_δ's whose union contains E. Put $E_n = E \cap L_n$. Then the same argument as earlier, using the sets E_n, shows that the regularity condition holds for E.

Finally if E is a Baire set that is not σ-bounded, we know that $\mu(E)$ is the supremum of the measures of $\mu(F)$ for σ-bounded Baire subsets F of E, and we know that $\mu(F)$ is the supremum of the measures of $\mu(K)$ for compact subsets K of F that are G_δ's. Therefore $\mu(E)$ is the supremum of the measures of $\mu(K)$ for compact subsets K of E that are G_δ's. $\square$

Proposition 11.19. If ν is a Baire measure on X, then there is one and only one regular Borel measure μ on X whose restriction to the Baire sets is μ.

PROOF. Since the members of $C_{\mathrm{com}}(X)$ are Baire functions (Corollary 11.16), we can define a positive linear functional ℓ on $C_{\mathrm{com}}(X)$ by $\ell(f) = \int_X f\,d\nu$. The uniqueness of the extending μ follows from the uniqueness part of Theorem 11.1. For existence we take μ to be the regular Borel measure given by the existence part of Theorem 11.1. We are to prove that μ and ν agree on Baire sets. The measures μ and ν agree on compact G_δ's by Lemma 11.7a and dominated convergence. By regularity of Baire measures (Proposition 11.18), μ and ν agree on all Baire sets. $\square$

Proposition 11.20. Suppose that X is compact and that μ and ν are Borel measures on X with μ regular. If ν is absolutely continuous with respect to μ, then ν is regular.

PROOF. Let $\epsilon > 0$ be given. The Radon–Nikodym Theorem (Theorem 9.16) and Corollary 5.24 together show that there exists $\delta > 0$ such that any Borel set A with $\mu(A) < \delta$ has $\nu(A) < \epsilon$. Let E be a Borel set to be tested for regularity under ν. Since μ is regular, we can choose K compact and U open with $K \subseteq E \subseteq U$ and $\mu(U - K) < \delta$. Then $\nu(U - K) < \epsilon$, and it follows that $\nu(E)$ is approximated within ϵ by $\nu(K)$ and $\nu(U)$. $\square$

Proposition 11.21. If μ is a regular Borel measure on X and if $1 \le p < \infty$, then

(a) $C_{\mathrm{com}}(X)$ is dense in $L^p(X, \mu)$,
(b) the smallest closed subspace of $L^p(X, \mu)$ containing all indicator functions of compact G_δ's in X is $L^p(X, \mu)$ itself.

REMARK. This generalizes conclusions (a) and (b) of Proposition 9.9 from open subsets of $\mathbb{R}^N$ to all locally compact Hausdorff spaces.

PROOF. If E is a Borel set of finite μ measure and if ϵ is given, the regularity of μ allows us to choose a compact set K with $K \subseteq E$ and $\mu(E - K) < \epsilon$. Then we can find a bounded open set U with $K \subseteq U$ and $\mu(U - K) < \epsilon$, and Proposition 11.14 gives us a compact G_δ set K_0 such that $K \subseteq K_0 \subseteq U$. We have $\int_X |I_E - I_K|^p\,d\mu = \mu(E - K) < \epsilon$, $\int_X |I_U - I_K|^p\,d\mu = \mu(U - K) < \epsilon$, and $\int_X |I_U - I_{K_0}|^p\,d\mu = \mu(U - K_0) < \epsilon$. Consequently we see in succession that the closure in $L^p(X, \mu)$ of the set of all indicator functions of compact sets contains all indicator functions of Borel sets of finite μ measure, the closure in $L^p(X, \mu)$ of the set of all indicator functions of bounded open sets contains all indicator functions of Borel sets of finite μ measure, and the closure in $L^p(X, \mu)$ of the set of all indicator functions of compact G_δ's contains all indicator functions of Borel sets of finite μ measure. Proposition 5.56 shows consequently that the smallest closed subspace of $L^p(X,\mu)$ containing all indicator functions of compact Baire sets is $L^p(X,\mu)$ itself. This proves (b).

For (a), let K_0 be a compact G_δ, and use Lemma 11.7a to choose a decreasing sequence $\{f_n\}$ of real-valued members of $C_{\text{com}}(X)$ with pointwise limit I_{K_0}. Since f_1^p is integrable, dominated convergence yields $\lim_n \int_X |f_n - I_{K_0}|^p\,d\mu = 0$. Hence the closure of $C_{\text{com}}(X)$ in $L^p(X,\mu)$ contains all indicator functions of compact G_δ's. By Proposition 5.55d this closure contains the smallest closed subspace of $L^p(X,\mu)$ containing all indicator functions of compact G_δ's. Conclusion (b) shows that the latter subspace is $L^p(X,\mu)$ itself. This proves (a). $\square$

Corollary 11.22. Suppose that X is a locally compact separable metric space. If μ is a Borel measure on X and if $1 \le p < \infty$, then

(a) $C_{\text{com}}(X)$, as a normed linear space under the supremum norm, is separable,

(b) $L^p(X,\mu)$ is separable.

REMARK. This generalizes Corollary 6.27c and Proposition 9.9c from open subsets of $\mathbb{R}^N$ to all locally compact separable metric spaces. The measure μ is automatically regular by Proposition 11.8 since Baire measures and Borel measures coincide in any locally compact metric space.

PROOF. Part (a) is proved by the same argument as for Corollary 6.27c. What is required is a substitute for Lemma 6.22a in order to obtain a sequence $\{F_n\}_{n=1}^\infty$ of compact subsets of X with union X such that $F_n \subseteq F_{n+1}^o$ for all n. It was observed at the beginning of Section X.3 that separable implies Lindelöf, and it follows from Proposition 10.24 that X is consequently σ-compact. Application of Proposition 10.25 then gives the sequence $\{F_n\}_{n=1}^\infty$. Corollary 2.59 is still to be applied to $C(F_n)$; since F_n is a compact metric space, the corollary shows that $C(F_n)$ is separable, and the argument goes through.

Part (b) follows from (a) and Proposition 11.21a in the same way that Corollary 6.27d follows from parts (a) and (c) of that corollary. The sequence $\{F_n\}_{n=1}^\infty$ of the previous paragraph is to be used in the argument. $\square$

Theorem 11.23 (Helly–Bray Theorem). Let X be a locally compact separable metric space. If $\{\mu_n\}$ is a sequence of Borel measures on X with $\{\mu_n(X)\}$ bounded, say by M, then there exist a Borel measure μ on X and a subsequence $\{\mu_{n_k}\}$ such that $\mu(X) \le M$ and $\lim_n \int_X f\,d\mu_{n_k} = \int_X f\,d\mu$ for all f in $C_{\text{com}}(X)$.

REMARKS. In the terminology of Section V.9, the measures μ_n are continuous linear functionals on the normed linear space $C_{\text{com}}(X)$, and the norm of the linear functional corresponding to μ_n is $\mu_n(X)$. The convergence is weak-star convergence, and the limiting linear functional is given by a Borel measure μ with $\mu(X) \le M$. The theorem amounts to an application of the preliminary form of Alaoglu's Theorem (Theorem 5.58) and the identification of the limit as a measure.

PROOF. The proof consists of filling in the details in the remarks above. We regard $Y = C_{\text{com}}(X)$ as a normed linear space with the supremum norm. Any Borel measure ν on X defines by integration a linear functional on Y with norm given by $\|\nu\| = \sup_{f \in C_{\text{com}}(X),\ \|f\| \le 1} \left| \int_X f\,d\nu \right|$. The right side is certainly $\le \|f\|_{\sup} \nu(X)$. In the reverse direction, let $\{K_n\}$ be an increasing sequence of compact subsets of X with union X, so that $\lim_n \nu(K_n) = \nu(X)$. Choose functions $f_n : X \to [0, 1]$ in $C_{\text{com}}(X)$ by Corollary 10.44 such that f_n is 1 on K_n. Then $\|f_n\|_{\sup} \le 1$ for all n, and $\int_X f_n\,d\nu \ge \int_{K_n} d\nu = \nu(K_n)$. Hence $\|\nu\| \ge \limsup_n \nu(K_n) = \nu(X)$, and we conclude that $\|\nu\| = \nu(X)$.

Thus the given sequence $\{\mu_n\}$ corresponds to a sequence in Y^* with $\|\mu_n\| \le M$ for all n. Corollary 11.22 shows that Y is separable. Theorem 5.58 therefore applies and yields a subsequence $\{\mu_{n_k}\}$ and a member ℓ of Y^* with $\|\ell\| \le M$ such that $\lim_k \int_X f\,d\mu_{n_k} = \ell(f)$ for all f in $C_{\text{com}}(X)$. If $f \ge 0$, $\lim_k \int_X f\,d\mu_{n_k}$ is certainly ≥ 0, and thus ℓ is a positive linear functional on $C_{\text{com}}(X)$. The Riesz Representation Theorem (Theorem 11.1) produces a Borel measure μ on X with $\ell(f) = \int_X f\,d\mu$ for all f in $C_{\text{com}}(X)$. Since $\|\ell\| \le M$, we have $\mu(X) \le M$. $\square$

4. Dual to Space of Finite Signed Measures

We continue in this section with X as a locally compact Hausdorff space. We now change the point of view a little and regard $C_{\text{com}}(X)$ as a normed linear space under the supremum norm $\|f\|_{\sup} = \sup_{x \in X} |f(x)|$. The problem is to identify all continuous linear functionals on this normed linear space. We shall see shortly that it is enough to handle the case that X is compact.

If X^* is the one-point compactification of X, then two spaces to be considered in conjunction with $C_{\text{com}}(X)$ are $C(X^*)$, the space of continuous scalar-valued functions on X^*, and $C_0(X)$, the space of continuous scalar-valued functions on X that "vanish at infinity." When applied to a function f, the term **vanishes at infinity** means that for any $\epsilon > 0$, there is some compact set with the property that $|f(x)| \le \epsilon$ outside that set. It is equivalent to say that f extends to a member of $C(X^*)$ that is 0 at ∞.

The three spaces $C_{\text{com}}(X)$, $C_0(X)$, and $C(X^*)$ are related. In the first place, $C_{\text{com}}(X)$ is dense in $C_0(X)$. In fact, if f is in $C_0(X)$ and if $\epsilon > 0$ is given, we find K compact with $|f(x)| \le \epsilon$ outside K. Corollary 10.44 supplies a member g of $C_{\text{com}}(X)$ with values in [0, 1] that is 1 on K. Then the product fg is in $C_{\text{com}}(X)$, and $\|f - fg\|_{\sup} \le \epsilon$. Thus $C_{\text{com}}(X)$ is dense in $C_0(X)$. Any continuous linear functional on $C_{\text{com}}(X)$ is uniformly continuous by Proposition 5.57, and Proposition 2.47 shows that it extends uniquely to a continuous linear functional on $C_0(X)$. Thus the continuous linear functionals on $C_0(X)$ and $C_{\text{com}}(X)$ are in one-one correspondence by restriction.

If we identify $C_0(X)$ as the subspace of $C(X^*)$ of functions equal to 0 at ∞, then every continuous linear functional on $C(X^*)$ restricts to a continuous linear functional on $C_0(X)$. In the reverse direction every continuous linear functional on $C_0(X)$ extends (nonuniquely) to a continuous linear functional on $C(X^*)$. In fact, let ℓ_0 be a continuous linear functional on $C_0(X)$, and fix a member f_0 of $C(X^*)$ with $f_0(\infty) = 1$. If f is any member of $C(X^*)$, then $f - f(\infty) f_0$ is in $C_0(X)$ and it makes sense to define $\ell(f) = \ell_0(f - f(\infty) f_0)$. Since

$$\begin{aligned}|\ell(f)| = |\ell_0(f - f(\infty) f_0)| &\leq \|\ell_0\| \|f - f(\infty) f_0\|_{\text{sup}} \\ &\leq \|\ell_0\|(\|f\|_{\text{sup}} + |f(\infty)| \|f_0\|_{\text{sup}}) \leq \|\ell_0\|(1 + \|f_0\|_{\text{sup}}) \|f\|_{\text{sup}},\end{aligned}$$

ℓ is bounded on $C(X^*)$ and is therefore continuous. Thus the study of continuous linear functionals on $C_{\text{com}}(X)$ reduces to the case that X is compact.

The first result below shows that any continuous linear functional on $C(X)$ with X compact is a finite linear combination of positive linear functionals. In view of Theorem 11.1, it is therefore given as a finite linear combination of integrations with respect to regular Borel measures. The remainder of the section will be devoted to making this result look tidier and seeing what happens to various norms under the correspondence.

Proposition 11.24. Let X be a compact Hausdorff space, and let ℓ be a continuous linear functional on $C(X)$. If ℓ takes real values on real-valued functions, define, for $f \geq 0$ in $C(X)$,

$$\ell^+(f) = \sup_{0 \leq g \leq f} \ell(g) \qquad \text{and} \qquad \ell^-(f) = \ell^+(f) - \ell(f);$$

then ℓ^+ and ℓ^- extend to positive linear functionals on $C(X)$ such that $\ell = \ell^+ - \ell^-$. If ℓ does not necessarily take real values on real-valued functions, then ℓ is a complex linear combination of positive linear functionals on $C(X)$.

PROOF. The functions f and g in this argument will all be in $C(X)$. For general ℓ not necessarily taking real values on real-valued functions, define $\bar{\ell}(f) = \overline{\ell(\bar{f})}$. We readily check that $\bar{\ell}$ is a continuous linear functional on $C(X)$, that $\ell_R = \frac{1}{2}(\ell + \bar{\ell})$ and $\ell_I = \frac{1}{2i}(\ell - \bar{\ell})$ are continuous linear functionals on $C(X)$ taking real values on real-valued functions, and that $\ell = \ell_R + i\ell_I$ exhibits ℓ as a complex linear combination of continuous linear functionals taking real values on real-valued functions. This reduces the proposition to the case that ℓ takes real values on real-valued functions.

In this case, for $f \geq 0$, inspection gives the following: $\ell(f) = \ell^+(f) - \ell^-(f)$, $\ell^+(0) = \ell^-(0) = 0$, $\ell^+(cf) = c\ell^+(f)$ for $c \geq 0$, and $\ell^-(cf) = c\ell^-(f)$ for $c \geq 0$. In addition, $\ell^+(f) \geq 0$ for $f \geq 0$ because

$$\ell^+(f) = \sup_{0 \leq g \leq f} \ell(g) \geq \ell(0) = 0,$$

and $\ell^-(f) \geq 0$ for $f \geq 0$ because

$$\ell^-(f) = \ell^+(f) - \ell(f) = \sup_{0 \leq g \leq f} \ell(g) - \ell(f) \geq \ell(f) - \ell(f) = 0.$$

To complete the proof, all that we have to do is show that $\ell^+(f_1 + f_2) = \ell^+(f_1) + \ell^+(f_2)$ whenever $f_1 \geq 0$ and $f_2 \geq 0$. The argument for $\geq$ is that

$$\begin{aligned} \ell^+(f_1 + f_2) &= \sup_{0 \leq g \leq f_1 + f_2} \ell(g) \geq \sup_{\substack{g_1, g_2, \\ 0 \leq g_1 \leq f_1, \\ 0 \leq g_2 \leq f_2}} \ell(g_1 + g_2) \\ &= \sup_{0 \leq g_1 \leq f_1} \ell(g_1) + \sup_{0 \leq g_2 \leq f_2} \ell(g_2) = \ell^+(f_1) + \ell^+(f_2). \end{aligned}$$

For the reverse direction, let g be arbitrary with $0 \leq g \leq f_1 + f_2$, and set $g_1 = \min\{g, f_1\}$ and $g_2 = g - g_1$. Certainly $0 \leq g_1 \leq f_1$. Let us show that $0 \leq g_2 \leq f_2$. In fact,

$$\begin{aligned} g_2 = g - g_1 &= (g + f_1) - (f_1 + g_1) = \max\{g, f_1\} + \min\{g, f_1\} - (f_1 + g_1) \\ &= \max\{g, f_1\} + g_1 - (f_1 + g_1) = \max\{g, f_1\} - f_1. \end{aligned}$$

Thus g_2 is certainly ≥ 0. In addition, the computation

$$g_2 = \max\{g, f_1\} - f_1 \leq \max\{f_1 + f_2,\ f_1\} - f_1 = (f_1 + f_2) - f_1 = f_2$$

shows that g_2 is $\leq f_2$. Thus any g with $0 \leq g \leq f_1 + f_2$ gives us a corresponding decomposition

$$\begin{aligned} \ell(g) &= \ell(g_1 + g_2) = \ell(g_1) + \ell(g_2) \\ &\leq \sup_{0 \leq g_1 \leq f_1} \ell(g_1) + \sup_{0 \leq g_2 \leq f_2} \ell(g_2) = \ell^+(f_1) + \ell^+(f_2). \end{aligned}$$

Taking the supremum over g, we obtain $\ell^+(f_1 + f_2) \leq \ell^+(f_1) + \ell^+(f_2)$, and the proof is complete. $\square$

Let us reinterpret matters in terms of Borel measures. We begin with the real-valued case. Recall from Section IX.3 that a real-valued completely additive set function ρ on a σ-algebra is called a signed measure. It is bounded if $|\rho(E)| \le C$ for all E in the algebra. In this case Theorem 9.14 shows that it has a Jordan decomposition $\rho = \rho^+ - \rho^-$, where ρ^+ and ρ^- are uniquely determined finite measures such that any decomposition $\rho = \nu^+ - \nu^-$ as the difference of finite measures has $\rho^+ \le \nu^+$ and $\rho^- \le \nu^-$. We say that a bounded signed measure ρ on the Borel sets of the compact Hausdorff space X is a **regular Borel signed measure** if its Jordan decomposition is into regular Borel measures. If $\rho = \nu^+ - \nu^-$ is any decomposition of a bounded signed measure ρ on the Borel sets as the difference of regular Borel measures, then the equalities $\rho^+ \le \nu^+$ and $\rho^- \le \nu^-$ that compare the decomposition with the Jordan decomposition force ρ^+ and ρ^- to be regular, in view of Proposition 11.20. Hence ρ is a regular Borel signed measure.

The regular Borel signed measures form a real vector space $M(X, \mathbb{R})$. To see closure under vector space operations, we observe from the definition of regularity that the sum of two (nonnegative) regular Borel measures is a regular Borel measure. From this fact we can see that the sum of two regular Borel signed measures is regular and hence that $M(X, \mathbb{R})$ is closed under addition: in fact, if $\rho = \rho^+ - \rho^-$ and $\sigma = \sigma^+ - \sigma^-$ are given in their Jordan decompositions, then the formula $(\rho+\sigma)^+ - (\rho+\sigma)^- = (\rho^+ + \sigma^+) - (\rho^- + \sigma^-)$ shows that $\rho + \sigma$ is the difference of two regular Borel measures and hence is regular. Thus $M(X, \mathbb{R})$ is a real vector space.

Proposition 11.25. The real vector space $M(X, \mathbb{R})$ becomes a real normed linear space under the definition $\|\rho\| = \rho^+(X) + \rho^-(X)$, where $\rho = \rho^+ - \rho^-$ is the Jordan decomposition of ρ.

PROOF. Certainly $\|\rho\| \ge 0$ with equality if and only if $\rho = 0$. Also, if ρ has the Jordan decomposition $\rho = \rho^+ - \rho^-$, then $-\rho = \rho^- - \rho^+$ is the Jordan decomposition of $-\rho$, and it follows that $\|c\rho\| = |c|\|\rho\|$ for any real scalar c.

Finally consider $\|\rho + \sigma\|$. If $\rho = \rho^+ - \rho^-$ and $\sigma = \sigma^+ - \sigma^-$ are Jordan decompositions, then the formula $(\rho+\sigma)^+ - (\rho+\sigma)^- = (\rho^+ + \sigma^+) - (\rho^- + \sigma^-)$ shows that $(\rho + \sigma)^+ \le \rho^+ + \sigma^+$ and hence $(\rho + \sigma)^+(X) \le \rho^+(X) + \sigma^+(X)$. Similarly $(\rho + \sigma)^-(X) \le \rho^- X) + \sigma^-(X)$. Adding these inequalities, we obtain $\|\rho + \sigma\| \le \|\rho\| + \|\sigma\|$. □

Returning to the statement of Proposition 11.24, let us write $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ for the space of continuous scalar-valued functions when the field of scalars is important, reserving the expression $C(X)$ for situations in which the scalars do not matter. Suppose that ℓ is a continuous linear function on $C(X)$ that takes real values on real-valued functions. The proposition shows that ℓ is the difference of two positive linear functionals. By Theorem 11.1, ℓ operates as the difference of two integrations: $\ell(f) = \int_X f\,d\nu^+ - \int_X f\,d\nu^-$, where ν^+ and ν^- are the regular Borel measures corresponding to ℓ^+ and ℓ^-. Then ℓ corresponds to a regular Borel signed measure ρ and is given by integration: $\ell(f) = \int_X f\,d\rho$, the integral with respect to the signed measure being interpreted as the difference of two integrals with respect to measures. Conversely any regular Borel signed measure ρ yields a continuous linear functional ℓ on $C(X)$ by the definition $\ell(f) = \int_X f\,d\rho$.

In particular the passage to integration gives us a real-linear mapping of $M(X,\mathbb{R})$ onto the space $C(X,\mathbb{R})^*$ of continuous linear functionals on the real vector space $C(X,\mathbb{R})$. Both of these spaces are normed linear spaces, and the theorem is that the map is one-one and that the norms match.

Theorem 11.26. The real-linear map of $M(X,\mathbb{R})$ onto $C(X,\mathbb{R})^*$ given by $\rho \mapsto \ell$ with $\ell(f) = \int_X f\,d\rho$ is one-one and norm preserving.

REMARK. As in Section V.9 the norm $\|\ell\|$ of ℓ is the least constant C such that $|\ell(f)| \le C\|f\|_{\sup}$ for all f. The constant C equals the supremum of $|\ell(f)|$ over all f with $\|f\|_{\sup} \le 1$.

PROOF. To see that the map is one-one, suppose that $\int_X f\,d\rho = 0$ for all f in $C(X,\mathbb{R})$. Then $\int_X f\,d\rho^+ = \int_X f\,d\rho^-$, and the uniqueness part of Theorem 11.1 shows that $\rho^+ = \rho^-$. Hence $\rho = \rho^+ - \rho^- = 0$.

Now suppose that ℓ and ρ correspond. Then we have

$$\begin{aligned} |\ell(f)| &= \big|\textstyle\int_X f\,d\rho^+ - \int_X f\,d\rho^-\big| \\ &\le \textstyle\int_X |f|\,d\rho^+ + \int_X |f|\,d\rho^- \\ &\le \rho^+(X)\|f\|_{\sup} + \rho^-(X)\|f\|_{\sup}. \end{aligned}$$

Taking the supremum over all f with $\|f\|_{\sup} \le 1$, we obtain

$$\|\ell\| \le \rho^+(X) + \rho^-(X) = \|\rho\|.$$

For the inequality in the reverse direction, let $\epsilon > 0$ be given, and let $X = P \cup N$ be a Hahn decomposition (Theorem 9.15) for ρ. By regularity of ρ^+ on P and ρ^- on N, choose compact subsets K_P and K_N with $K_P \subseteq P$, $K_N \subseteq N$, $\rho^+(P - K_P) < \epsilon$, and $\rho^-(N - K_N) < \epsilon$. Since $\rho^+(N) = 0$ and $\rho^-(P) = 0$,

$$\rho^+(X - K_P) < \epsilon \qquad \text{and} \qquad \rho^-(X - K_N) < \epsilon. \tag{$*$}$$

By Urysohn's Lemma (Corollary 10.43), we can find a continuous function $f : X \to [-1,1]$ such that f is 1 on K_P and is -1 on K_N. Then

$$\begin{aligned} \big|\ell(f) - \|\rho\|\big| &\le \big|\textstyle\int_{K_P} f\,d\rho - \|\rho^+\|\big| + \big|\int_{K_N} f\,d\rho - \|\rho^-\|\big| + \big|\int_{K_P^c \cap K_N^c} f\,d\rho\big| \\ &\le \big|\rho^+(K_P) - \rho^+(X)\big| + \big|\rho^-(K_N) - \rho^-(X)\big| + \big|\textstyle\int_{K_P^c \cap K_N^c} f\,d\rho\big|. \end{aligned}$$

By $(*)$ the first two terms on the right side are each $< \epsilon$. Since $\rho^+(K_P^c \cap K_N^c) = \rho^+(P - K_P) < \epsilon$ and $\rho^-(K_P^c \cap K_N^c) = \rho^-(N - K_N) < \epsilon$, and since $\|f\|_{\sup} \leq 1$, the third term on the right side is $\leq 2\epsilon$. Therefore $\big|\ell(f) - \|\rho\|\big| < 4\epsilon$, and our function f has the property that $|\ell(f)| \geq (\|\rho\| - 4\epsilon)\|f\|_{\sup}$. In other words, $\|\ell\| \geq \|\rho\| - 4\epsilon$. Since ϵ is arbitrary, $\|\ell\| \geq \|\rho\|$. This completes the proof. $\square$

Now let us consider the case in which the values are complex. A **regular Borel complex measure** on the compact Hausdorff space X is an expression $\rho = \rho_R + i\rho_I$ in which ρ_R and ρ_I are regular Borel signed measures. In other words, it is a complex-valued set function whose real and imaginary parts are regular Borel signed measures. The space $M(X, \mathbb{C})$ of these is a complex vector space, and we shall make it into a normed linear space shortly. Meanwhile, the space $C(X, \mathbb{C})^*$ of continuous linear functionals on $C(X, \mathbb{C})$ is a complex normed linear space. Extending the definition of $\int_X f\,d\rho$ to handle members of $M(X, \mathbb{C})$, we see from Proposition 11.24 that the complex-linear map of $M(X, \mathbb{C})$ into $C(X, \mathbb{C})^*$ given by $\rho \mapsto \ell$ with $\ell(f) = \int_X f\,d\rho$ is one-one and onto.

To have a theorem in this case that parallels Theorem 11.26, we need to define the norm on $M(X, \mathbb{C})$. Doing so on an element ρ is not just a matter of combining the norms of the real and imaginary parts of ρ any more than writing the norm of a complex-valued L^1 function can be done in terms of the L^1 norms of the real and imaginary parts. A more subtle definition is needed.

We define the **total variation** $|\rho|$ of a member ρ of $M(X, \mathbb{C})$ to be the nonnegative set function whose value on a Borel set E is the supremum of all finite sums $\sum_{j=1}^n |\rho(E_j)|$ with $E = \bigcup_{j=1}^n E_j$ disjointly. The **total-variation norm** of the member ρ of $M(X, \mathbb{C})$ is defined to be $\|\rho\| = |\rho|(X)$. It is a simple matter to verify that the total-variation norm is indeed a norm.

Proposition 11.27. The total variation $|\rho|$ of a member ρ of $M(X, \mathbb{C})$ is a regular Borel measure, there exists a Borel function h with $\|h\|_{\sup} \leq 1$ such that $\rho = h\,d|\rho|$, and the total-variation norm on $M(X, \mathbb{C})$ makes $M(X, \mathbb{C})$ into a normed linear space in such a way that $\big|\int_X f\,d\rho\big| \leq \|\rho\|\|f\|_{\sup}$ for every bounded Borel function f. Moreover, $|\rho|$ equals $\rho^+ + \rho^-$ if ρ is real valued and has $\rho = \rho^+ - \rho^-$ as its Jordan decomposition.

REMARK. It follows that if ρ is real valued and if $X = P \cup N$ is a Hahn decomposition (Theorem 9.15) for ρ, then the corresponding function h may be taken to be $+1$ on P and -1 on N.

PROOF. To see that $|\rho|$ is additive, let E and F be disjoint Borel sets. If $E = \bigcup_{i=1}^m E_i$ disjointly and $F = \bigcup_{j=1}^n F_j$ disjointly, then $E \cup F = \big(\bigcup_{i=1}^m E_i\big) \cup \big(\bigcup_{j=1}^n F_j\big)$ disjointly, and hence $\sum_{i=1}^m |\rho(E_i)| + \sum_{j=1}^n |\rho(F_j)| \leq |\rho|(E \cup F)$. Taking the supremum over systems $\{E_i\}$ and then over systems

$\{F_j\}$, we obtain $|\rho|(E) + |\rho|(F) \le |\rho|(E \cup F)$. In the reverse direction let $E \cup F = \bigcup_{k=1}^{p} G_k$ disjointly. Then $E = \bigcup_{k=1}^{p}(E \cap G_k)$ disjointly, and $F = \bigcup_{k=1}^{p}(F \cap G_k)$ disjointly. Hence

$$\begin{aligned}\sum_{k=1}^{p} &|\rho(G_k)| \\ &= \sum_{k=1}^{p} |\rho(E \cap G_k) + \rho(F \cap G_k)| \le \sum_{k=1}^{p} |\rho(E \cap G_k)| + \sum_{k=1}^{p} |\rho(F \cap G_k)|,\end{aligned}$$

and this is $\le |\rho|(E) + |\rho|(F)$. Taking the supremum over systems $\{G_k\}$, we obtain $|\rho|(E \cup F) \le |\rho|(E) + |\rho|(F)$. Thus $|\rho|$ is additive.

To prove that $|\rho|$ is completely additive, let $E = \bigcup_{n=1}^{\infty} E_n$ disjointly. For every N, $\sum_{n=1}^{N} |\rho|(E_n) = |\rho|(E_1 \cup \cdots \cup E_N) \le |\rho|(E)$, and hence $\sum_{n=1}^{\infty} |\rho|(E_n) \le |\rho|(E)$. For the reverse inequality let $\{G_k\}_{k=1}^{p}$ be a finite collection of disjoint Borel sets with union E. Then $E_n = \bigcup_{k=1}^{p}(E_n \cap G_k)$ disjointly, and hence

$$\begin{aligned}\sum_{k=1}^{p} |\rho(G_k)| = \sum_{k=1}^{p} |\rho(E \cap G_k)| &= \sum_{k=1}^{p} \Big| \sum_{n=1}^{\infty} \rho(E_n \cap G_k)\Big| \\ &\le \sum_{k=1}^{p} \sum_{n=1}^{\infty} |\rho(E_n \cap G_k)| = \sum_{n=1}^{\infty} \sum_{k=1}^{p} |\rho(E_n \cap G_k)| \le \sum_{n=1}^{\infty} |\rho|(E_n).\end{aligned}$$

Thus $|\rho|(E) \le \sum_{n=1}^{\infty} |\rho|(E_n)$, and $|\rho|$ is completely additive.

The measure $|\rho|$ is certainly finite on X and hence on all compact sets. To see regularity, we write $\rho = \rho_R + i\rho_I = \rho_R^+ - \rho_R^- + i\rho_I^+ - i\rho_I^-$. Writing a set E as the disjoint union of n sets E_i and writing out $\rho(E_i)$ according to this expansion of ρ, we see that $|\rho|(E) \le (\rho_R^+ + \rho_R^- + \rho_I^+ + \rho_I^-)(E)$. Each measure on the right side is regular, and Proposition 11.20 therefore shows that $|\rho|$ is regular.

For the existence of h, let us write ρ in terms of its real and imaginary parts as $\rho = \rho_R + i\rho_I$. If E is a Borel set, then the definitions give $|\rho|(E) \ge |\rho(E)| \ge |\rho_R(E)|$ and similarly $|\rho|(E) \ge |\rho_I(E)|$. Hence $\rho_R \ll |\rho|$ and $\rho_I \ll |\rho|$. By the Radon–Nikodym Theorem (Corollary 9.17), there exist functions h_R and h_I integrable $[d|\rho|]$ such that $\rho_R = h_R\,d|\rho|$ and $\rho_I = h_I\,d|\rho|$. Thus the $|\rho|$ integrable complex-valued function $h = h_R + ih_I$ has $\rho = h\,d|\rho|$. We shall show that h has $|h(x)| \le 1$ a.e. $[d|\rho|]$. If the contrary were the case, then there would exist a constant c with $|c| = 1$ and an $\epsilon > 0$ such that $\operatorname{Re}(ch) \ge 1+\epsilon$ on a set E of positive $|\rho|$ measure and we would have

$$\begin{aligned}\Big|\int_E h\,d|\rho|\Big| = \Big|\int_E ch\,d|\rho|\Big| &\ge \operatorname{Re}\int_E ch\,d|\rho| = \int_E \operatorname{Re}(ch)\,d|\rho| \\ &\ge (1+\epsilon)|\rho|(E) \ge (1+\epsilon)|\rho(E)| = (1+\epsilon)\Big|\int_E h\,d|\rho|\Big|,\end{aligned}$$

a contradiction. Thus h exists as asserted.

The inequality $\left|\int_X f\,d\rho\right| \le \|\rho\|\|f\|_{\sup}$ follows from the existence of h since $\left|\int_X f\,d\rho\right| = \left|\int_X fh\,d|\rho|\right| \le \|fh\|_{\sup}\int_X d|\rho| \le \|f\|_{\sup}|\rho|(X) = \|f\|_{\sup}\|\rho\|$.

Finally if ρ is real valued, then any Borel set E satisfies $|\rho(E)| = |\rho^+(E) - \rho^-(E)| \le \rho^+(E) + \rho^-(E)$. If E is the disjoint union of Borel sets $E_1, \dots, E_n$, we consequently have

$$\sum_{j=1}^n |\rho(E \cap E_j)| \le \sum_{j=1}^n (\rho^+(E \cap E_j) + \rho^-(E \cap E_j)) = \rho^+(E) + \rho^-(E).$$

Taking the supremum over all decompositions of E of this kind gives $|\rho|(E) \le \rho^+(E) + \rho^-(E)$. For the reverse inequality let $X = P \cup N$ be a Hahn decomposition (Theorem 9.15) for ρ, so that $\rho^+(E) = \rho(P \cap E)$ and $\rho^-(E) = -\rho(N \cap E)$. Then E is the disjoint union of $E \cap P$ and $E \cap N$, and thus $\rho^+(E) + \rho^-(E) = |\rho(E \cap P)| + |\rho(E \cap N)| \le |\rho|(E)$. In other words, $|\rho| = \rho^+ + \rho^-$ as asserted. $\square$

Theorem 11.28. The one-one complex-linear map of $M(X, \mathbb{C})$ onto $C(X, \mathbb{C})^*$ given by $\rho \mapsto \ell$ with $\ell(f) = \int_X f\,d\rho$ is norm preserving.

PROOF. If f is in $C(X)$, then Proposition 11.27 gives $|\ell(f)| = \left|\int_X f\,d\rho\right| \le \|\rho\|\|f\|_{\sup}$. Taking the supremum over all f with $\|f\|_{\sup} \le 1$, we obtain $\|\ell\| \le \|\rho\|$.

For the reverse inequality let $\epsilon > 0$ be given, and choose a finite disjoint collection of Borel sets $E_1, \dots, E_n$ with union X such that $\sum_{i=1}^n |\rho(E_i)| \ge \|\rho\| - \epsilon$. Since $|\rho|$ is regular, we can find compact sets $K_i \subseteq E_i$ such that $|\rho|(E_i - K_i)| \le \epsilon/n$ for each i.

We shall define disjoint open sets U_i with $K_i \subseteq U_i$ for all i. We do so by making an inductive construction as follows. For $i = 1$, Corollary 10.22 produces disjoint open sets U_1 and V_1 with $K_1 \subseteq U_1$ and $K_2 \cup \cdots \cup K_n \subseteq V_1$. Suppose that the construction has been carried out for stage i with $1 \le i < n$. Using Corollary 10.22 for the locally compact Hausdorff space V_i and taking into account that $K_{i+1} \cup \cdots \cup K_n \subseteq V_i$, we choose disjoint open sets U_{i+1} and V_{i+1} of V_i with $K_{i+1} \subseteq U_{i+1}$ and $K_{i+2} \cup \cdots \cup K_n \subseteq V_{i+1}$. At the end of the construction, we have obtained open sets U_i with $K_i \subseteq U_i$ for all i, and we have obtained auxiliary open sets V_i with $V_{i+1} \subseteq V_i$ for all i. Let us see that the sets U_i are disjoint. In fact, if $j > i$, then $U_j \subseteq V_{j-1} \subseteq V_i$. Since V_i is disjoint from U_i, its subset U_j is disjoint from U_i. This proves the required disjointness and completes the construction of $U_1, \dots, U_n$.

For $1 \le i \le n$, choose $f_i \in C(X)$ with values in $[0, 1]$ such that f_i is 1 on K_i and is 0 off U_i. Choose $c_i \in \mathbb{C}$ for each i such that $c_i\rho(E_i) = |\rho(E_i)|$, and define $f_0 = \sum_{i=1}^n c_i f_i$. The function f_0 has $\|f_0\|_{\sup} = 1$ since the sets U_i are disjoint. Then

$$\ell(f_0) = \int_X f_0\,d\rho = \sum_{i=1}^n \int_{E_i} f_0\,d\rho = \sum_{i=1}^n \big(\int_{E_i} c_i\,d\rho + \int_{E_i} (f_0 - c_i)\,d\rho \big)$$
$$= \sum_{i=1}^n |\rho(E_i)| + \sum_{i=1}^n \int_{E_i - K_i} (f_0 - c_i)\,d\rho.$$

Hence

$$\Big|\ell(f_0) - \sum_{i=1}^n |\rho(E_i)|\Big| \le \sum_{i=1}^n \int_{E_i - K_i} |f_0 - c_i|\,d|\rho|$$
$$\le 2\sum_{i=1}^n |\rho|(E_i - K_i)| \le 2\sum_{i=1}^n \epsilon/n = 2\epsilon$$

and

$$\big|\ell(f_0) - \|\rho\|\big| \le \Big|\ell(f_0) - \sum_{i=1}^n |\rho(E_i)|\Big| + \Big|\sum_{i=1}^n |\rho(E_i)| - \|\rho\|\Big| \le 3\epsilon.$$

Therefore

$$\|\ell\| = \|\ell\|\|f_0\|_{\sup} \ge |\ell(f_0)| \ge \|\rho\| - \big|\ell(f_0) - \|\rho\|\big| \ge \|\rho\| - 3\epsilon.$$

Since ϵ is arbitrary, $\|\ell\| \ge \|\rho\|$. □

5. Problems

In all problems for this chapter, X is assumed to be a locally compact Hausdorff space. Sometimes additional hypotheses are imposed on X.

1. (a) Prove that if X is σ-compact, then the σ-algebra of Borel subsets of X coincides with the σ-algebra of intersections of X with the Borel subsets of the one-point compactification X^*.
   (b) Prove that if X is an uncountable discrete space, then the σ-algebra of Borel subsets of X is strictly smaller than the σ-algebra of intersections of X with the Borel subsets of the one-point compactification X^*.
2. Prove that if X is σ-compact and $f : X \to \mathbb{C}$ is continuous, then f is a Borel function.
3. Suppose that X is σ-compact. Prove that if μ is a regular Borel measure on X and if f is Borel measurable, then there exists a Baire measurable function g such that $f = g$ except on a Borel set of μ measure 0.
4. **(Lusin's Theorem)** Let X be compact, let μ be a regular Borel measure on X, let f be a Borel function on X, and let $\epsilon > 0$ be given. By first considering simple functions and then passing to the limit via Egoroff's Theorem, prove that there exists a compact subset K of X with $\mu(K^c) < \epsilon$ such that $f\big|_K$ is continuous.

5. This problem establishes the rotation invariance of the Borel measure $d\omega$ on the sphere $S^2 \subseteq \mathbb{R}^3$ obtained from Riemann integration with respect to $\sin\theta_1\,d\theta_1\,d\theta_2$, where θ_1 and θ_2 are latitude and longitude with $0 \le \theta_1 \le \pi$ and $0 \le \theta_2 \le 2\pi$. The measure $d\omega$ was constructed by means of the Riesz Representation Theorem as one of the examples in Section 2.
   (a) A rotation in $\mathbb{R}^3$ is the linear function L determined by a matrix A with $AA^{\mathrm{tr}} = 1$ and $\det A = 1$. For $0 < a < 1 < b < \infty$, let S_{ab} be the subset of $\mathbb{R}^3$ given in spherical coordinates by $a < r < b$, $0 \le \theta_1 \le \pi$, $0 \le \theta_2 \le 2\pi$. Show that S_{ab} is carried to itself by any such rotation L.
   (b) For any bounded Borel function $F : S_{ab} \to \mathbb{C}$, let $(LF)(x) = F(L^{-1}x)$ if x is in S_{ab} and L is a rotation. Prove that $\int_{S_{ab}} LF\,dx = \int_{S_{ab}} F\,dx$.
   (c) Let $f : S^2 \to \mathbb{C}$ be any continuous function, and define $(Lf)(\omega) = f(L^{-1}\omega)$. Extend f to a function F defined on S_{ab} by the definition $F(r\omega) = f(\omega)$. Prove that $\int_{S_{ab}} F\,dx = \big(\int_a^b r^2\,dr\big)\big(\int_{S^2} f(\omega)\,d\omega\big)$ and deduce that $\int_{S^2} Lf\,d\omega = \int_{S^2} f\,d\omega$.
   (d) Deduce from (c) that $d\omega(L(E)) = d\omega(E)$ for every Borel subset E of S^2.

6. Let X be compact.
   (a) Let $\{K_\alpha\}$ be a collection of compact subsets of X closed under finite intersections, and let $K = \bigcap_\alpha K_\alpha$. Prove that every regular Borel measure μ on X has the property that $\mu(K) = \inf_\alpha \mu(K_\alpha)$.
   (b) If μ is a nonzero regular Borel measure on X assuming only the values 0 and 1, prove that μ is a point mass.
   (c) If μ is a nonzero regular Borel measure on X with
   $$\int_X fg\,d\mu = \Big(\int_X f\,d\mu\Big)\Big(\int_X g\,d\mu\Big)$$
   for all f and g in $C(X)$, prove that μ is a point mass.
   (d) If ℓ is a positive linear functional on $C(X)$ that is multiplicative in the sense that $\ell(fg) = \ell(f)\ell(g)$ for all f and g in $C(X)$, prove that ℓ is zero or ℓ is evaluation at some point of X.

7. This problem continues the investigation of harmonic functions and Poisson integrals in the unit disk of $\mathbb{R}^2$, following up on Problems 7–8 at the end of Chapter IX. Problem 8 in that series provides orientation. The new ingredient for the present problem is weak-star convergence of sequences in $M(S^1, \mathbb{C})$ against $C(S^1)$, where S^1 is the unit circle.
   (a) State and prove a characterization of the harmonic functions $u(r,\theta)$ on the open unit disk such that $\sup_{0 \le r < 1} \|u(r,\,\cdot\,)\|_1$ is finite.
   (b) **(Herglotz's Theorem)** Prove that if $u(r,\theta)$ is a nonnegative harmonic function on the open unit disk, then there is a Borel measure μ on the circle such that $u(r,\theta) = \int_{(-\pi,\pi]} P_r(\theta - \varphi)\,d\mu(\varphi)$.

Problems 8–10 construct a Borel measure μ on a compact space such that μ is not regular. The totally ordered set Ω of countable ordinals was introduced in Problems 25–33 at the end of Chapter V. Let $\Omega^* = \Omega \cup \{\infty\}$, totally ordered so that every element of Ω is less than $\{\infty\}$. Give Ω^* the order topology, as discussed in Problems 25–32 at the end of Chapter X.

8. Prove that Ω^* is compact Hausdorff.

9. Prove that the class of all relatively closed uncountable subsets of Ω is closed under the formation of countable intersections.

10. Define μ on the Borel sets of Ω^* to be 1 on those sets E such that $E - \{\infty\}$ contains a relatively closed uncountable subset of Ω, and put $\nu(E) = 0$ otherwise. Prove that μ is a Borel measure that is not regular.

Problems 11–14 concern decomposing any finite Borel measure on a compact X into a regular Borel measure and a "purely irregular" Borel measure. They make use of Zorn's Lemma (Section A9 of Appendix A). A Borel measure μ will be called **purely irregular** if there is no nonzero regular Borel measure ν such that $0 \le \nu(E) \le \mu(E)$ for every Borel set E.

11. Use Zorn's Lemma to show that any Borel measure on X is the sum of a regular Borel measure and a purely irregular Borel measure.

12. Prove that if ν is a regular Borel measure, if μ is purely irregular, and if $0 \le \mu \le \nu$, then $\mu = 0$.

13. Deduce from the Jordan decomposition (Theorem 9.14) that the decomposition of Problem 11 is unique.

14. Prove that the irregular Borel measure constructed in Problem 10 is purely irregular.

Problems 15–19 concern extension of measures from finite products of compact metric spaces to countably infinite such products. Let X be a compact metric space, and for each integer $n \ge 1$, let X_n be a copy of X. Define $\Omega^{(N)} = \bigtimes_{n=1}^{N} X_n$, and let $\Omega = \bigtimes_{n=1}^{\infty} X_n$. Each of $\Omega^{(N)}$ and Ω is given the product topology. If E is a Borel subset of $\Omega^{(N)}$, we can regard E as a subset of Ω by identifying E with $E \times \big(\bigtimes_{n=N+1}^{\infty} X_n\big)$. In this way any Borel measure on $\Omega^{(N)}$ can be regarded as a measure on a certain σ-subalgebra $\mathcal{F}_n$ of $\mathcal{B}(\Omega)$.

15. Prove that $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{F}$ is an algebra.

16. Let ν_n be a (regular) Borel measure on $\Omega^{(n)}$ with $\nu(\Omega^{(n)}) = 1$, and regard ν_n as defined on $\mathcal{F}_n$. Suppose for each n that ν_n agrees with ν_{n+1} on $\mathcal{F}_n$. Define $\nu(E)$ for E in $\mathcal{F}$ to be the common value of $\nu_n(E)$ for n large. Prove that ν is nonnegative additive, and prove that in a suitable sense ν is regular on $\mathcal{F}$.

17. Using the kind of regularity established in the previous problem, prove that ν is completely additive on $\mathcal{F}$.

18. In view of Problems 16 and 17, ν extends to a measure on the smallest σ-algebra for Ω containing $\mathcal{F}$. Prove that this σ-algebra is $\mathcal{B}(\Omega)$.

19. Let X be a 2-point space, and let ν_n be 2^{-n} on each one-point subset of $\Omega^{(n)}$. Exhibit a homeomorphism of Ω onto the standard Cantor set in $[0, 1]$ that carries ν to the Cantor measure defined in Problems 17–20 at the end of Chapter VI.