

## Indices (Index of Notation, General Index), 717-735

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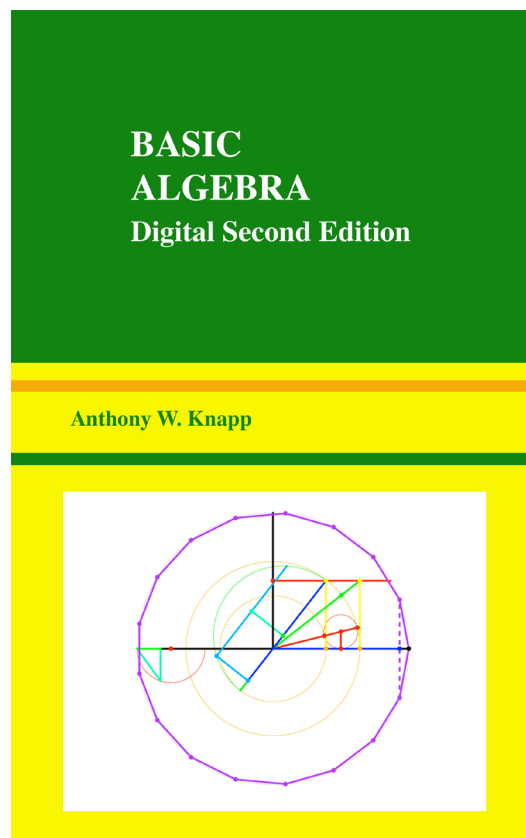
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Anthony W. Knapp

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## INDEX OF NOTATION

This list indexes recurring symbols introduced in Chapters I through X (pages 1–591). For other recurring symbols, including set-theoretic notation introduced in the appendix (pages 593–613), see the list of Standard Notation on page xx.

In the list below, each piece of notation is regarded as having a key symbol. The first group consists of those items for which the key symbol is a fixed Latin letter, and the items are arranged roughly alphabetically by that key symbol. The next group consists of those items for which the key symbol is a Greek letter. The final group consists of those items for which the key symbol is a variable or a nonletter, and these are arranged by type.

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