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Anthony W. Knapp

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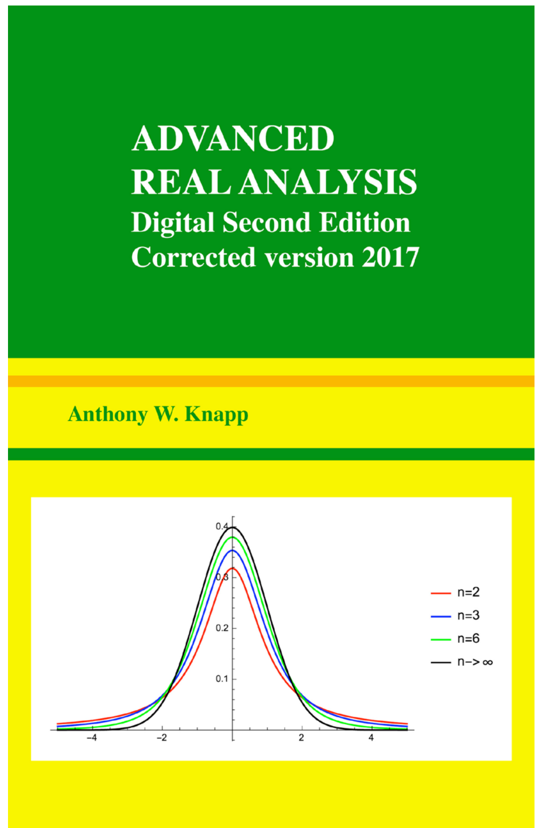

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/∼aknapp

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CHAPTER IV

Topics in Functional Analysis

Abstract. This chapter pursues three lines of investigation in the subject of functional analysis—one involving smooth functions and distributions, one involving fixed-point theorems, and one involving spectral theory.

Section 1 introduces topological vector spaces. These are real or complex vector spaces with a Hausdorff topology in which addition and scalar multiplication are continuous. Examples include normed linear spaces, spaces given by a separating family of countably many seminorms, and weak and weak-star topologies in the context of Banach spaces. Various general properties of topological vector spaces are proved, and it is proved that the quotient of a topological vector space by a closed vector subspace is Hausdorff and is therefore a topological vector space.

Section 2 introduces a topology on the space $C^\infty(U)$ of smooth functions on an open subset of $\mathbb{R}^N$. The support of a continuous linear functional on $C^\infty(U)$ is defined and shown to be a compact subset of U. Accordingly, the continuous linear functionals are called distributions of compact support.

Section 3 studies weak and weak-star topologies in more detail. The main result is Alaoglu's Theorem, which says that the closed unit ball in the weak-star topology on the dual of a normed linear space is compact. In an earlier chapter a preliminary form of this theorem was used to construct elements in a dual space as limits of weak-star convergent subsequences.

Section 4 follows Alaoglu's Theorem along a particular path, giving what amounts to a first example of the Gelfand theory of Banach algebras. The relevant theorem, known as the Stone Representation Theorem, says that conjugate-closed uniformly closed subalgebras containing the constants in $B(S)$ are isomorphic via a norm-preserving algebra isomorphism to the space of all continuous functions on some compact Hausdorff space. The compact space in question is the space of multiplicative linear functionals on the subalgebra, and the proof of compactness uses Alaoglu's Theorem.

Sections 5–6 return to the lines of study toward distributions and fixed-point theorems. Section 5 studies the relationship between convexity and the existence of separating linear functionals. The main theorem makes use of the Hahn–Banach Theorem. Section 6 introduces locally convex topological vector spaces. Application of the basic separation theorem from the previous section shows the existence of many continuous linear functionals on such a space.

Section 7 specializes to the line of study via smooth functions and distributions. The topic is the introduction of a certain locally convex topology on the space $C^\infty_{\text{com}}(U)$ of smooth functions of compact support on U. This is best characterized by a universal mapping property introduced in the section.

Sections 8–9 pursue locally convex spaces along the other line of study that split off in Section 5. Section 8 gives the Krein–Milman Theorem, which asserts the existence of a supply of extreme points for any nonempty compact convex set in a locally convex topological vector space. Section 9 relates compact convex sets to the subject of fixed-point theorems.

Section 10 takes up the abstract theory of Banach algebras, with particular attention to commutative C^* algebras with identity. Three examples are the algebras characterized by the Stone Representation Theorem, any L^∞ space, and any adjoint-closed commutative Banach algebra consisting of bounded linear operators on a Hilbert space and containing the identity.

Section 11 continues the investigation of the last of the examples in the previous section and derives the Spectral Theorem for bounded self-adjoint operators and certain related families of operators. Powerful applications follow from a functional calculus implied by the Spectral Theorem. The section concludes with remarks about the Spectral Theorem for unbounded self-adjoint operators.

1. Topological Vector Spaces

In this section we shall work with vector spaces over $\mathbb{R}$ or $\mathbb{C}$, and the distinction between the two fields will not be very important. We write $\mathbb{F}$ for this field of scalars. A **topological vector space** or **linear topological space** is a vector space X over $\mathbb{F}$ with a Hausdorff topology such that addition, as a mapping $X \times X \to X$, and scalar multiplication, as a mapping $\mathbb{F} \times X \to X$, are continuous. The mappings that we study between topological vector spaces are the continuous linear functions, which may be referred to as "continuous linear operators." An **isomorphism** of topological vector spaces over $\mathbb{F}$ is a continuous linear operator with a continuous inverse.

The simplest examples of topological vector spaces are the spaces $\mathbb{F}^N$ of column vectors with the usual metric topology. Since the topologies of $\mathbb{F}^N$, $\mathbb{F}^N \times \mathbb{F}^N$, and $\mathbb{F} \times \mathbb{F}^N$ are given by metrics, continuity of functions defined on any of these spaces may be tested by sequences. In particular, continuity of the vector-space operations on $\mathbb{F}^N$ reduces to the familiar results about limits of sums of vectors and limits of scalars times vectors. Moreover, if $L : \mathbb{F}^N \to Y$ is any linear function from $\mathbb{F}^N$ into a topological vector space over $\mathbb{F}$, then L is continuous. To see this, let $\{e_1, \dots, e_N\}$ be the standard basis of column vectors, and let $(\cdot\,,\,\cdot)$ be the standard inner product on $\mathbb{F}^N$, namely the dot product if $\mathbb{F} = \mathbb{R}$ and the usual Hermitian inner product if $\mathbb{F} = \mathbb{C}$. Write $y_j = L(e_j)$. For any x in $\mathbb{F}^N$, we have

$$L(x) = \sum_{j=1}^{N} (x, e_j) L(e_j) = \sum_{j=1}^{N} (x, e_j) y_j.$$

If $\{x_n\}$ is a sequence converging to x in $\mathbb{F}^N$, then the continuity of the inner product forces $(x_n, e_j) \to (x, e_j)$ for each j. Then $L(x_n)$ tends to $L(x)$ in Y since the vector space operations are continuous in Y. Hence L is continuous.

A second class of examples is the class of normed linear spaces. These were defined in *Basic*, and the continuity of the operations was established there.[1] The spaces $\mathbb{F}^N$ of column vectors are examples. Further examples include the space $B(S)$ of all bounded scalar-valued functions on a nonempty set S with the supremum norm, the vector subspace $C(S)$ of continuous members of $B(S)$ when S is a topological space, the vector subspaces $C_{\text{com}}(S)$ and $C_0(S)$ of continuous functions of compact support and of continuous functions vanishing at infinity when S is locally compact Hausdorff, the space $L^p(X,\mu)$ for $1 \le p \le \infty$ when (X,μ) is a measure space, and the space $M(S)$ of finite regular Borel complex measures on a locally compact Hausdorff space with the total variation norm.

A wider class of examples, which includes the normed linear spaces, is the class of topological vector spaces defined by seminorms. Seminorms were defined in Section III.1. If we have a family $\{\|\cdot\|_s\}$ of seminorms on a vector space X over $\mathbb{F}$, with indexing given by s in some nonempty set S, the corresponding topology on X is defined as the weak topology determined by all functions $x \mapsto \|x-y\|_s$ for $s \in S$ and $y \in X$. A base for the open sets of X is obtained as follows: For each triple (y,s,r), with y in X, with s one of the seminorm indices, and with $r>0$, the set $\{x \mid \|x-y\|_s < r\}$ is to be in the base, and the base consists of all finite intersections of these sets as (y,s,r) varies.

In order to obtain a topological vector space from a system of seminorms, we must ensure the Hausdorff property, and we do so by insisting that the only f in X with $\|f\|_s = 0$ for all s is $f=0$. In this case the family of seminorms is called a **separating family**. Let us go through the argument that a space defined by a separating family of seminorms is a topological vector space.

Proposition 4.1. Let X be a vector space over $\mathbb{F}$ endowed with a separating family $\{\|\cdot\|_s\}$ of seminorms. Then the weak topology determined by all functions $x \mapsto \|x-y\|_s$ makes X into a topological vector space.

PROOF. To see that X is Hausdorff, let x_0 and y_0 be distinct points of X. By assumption, there exists some s such that $\|x_0-y_0\|_s$ is a positive number r. The sets $\{x \mid \|x-x_0\|_s < r/2\}$ and $\{y \mid \|y-y_0\|_s < r/2\}$ are disjoint and open, and they contain x_0 and y_0, respectively. Hence X is Hausdorff.

To see that addition is continuous, we are to show that if a net $\{(x_\alpha, y_\alpha)\}$ is convergent in $X \times X$ to (x_0,y_0), then $\{x_\alpha + y_\alpha\}$ converges to x_0+y_0. This means that if $\|x_\alpha - x_0\|_s + \|y_\alpha - y_0\|_s$ tends to 0 for each s, then $\|(x_\alpha+y_\alpha)-(x_0+y_0)\|_s$ tends to 0 for each s. This is immediate from the triangle inequality for the seminorm $\|\cdot\|_s$, and hence addition is continuous. The proof that scalar multiplication is continuous is similar. $\square$

[1] The definition appears in Section V.9 of *Basic*, and the continuity of the operations is proved in Proposition 5.55.

We have encountered two distinctly different kinds of examples of topological vector spaces defined by families of seminorms. In the first kind a countable family of seminorms suffices to define the topology. Normed linear spaces are examples. So is the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, consisting of all smooth scalar-valued functions on $\mathbb{R}^N$ such that the product of any polynomial with any iterated partial derivative of the function is bounded. The defining seminorms for the Schwartz space are

$$\|f\|_{P,Q} = \sup_{x\in\mathbb{R}^N} |P(x)(Q(D)f)(x)|,$$

where P and Q are arbitrary polynomials. We saw in Section III.1 that the same topology arises if we use only the countably many seminorms for which P is some monomial x^α and Q is some monomial x^β. This family of seminorms is a separating family because if $\|f\|_{1,1} = 0$, then $f = 0$.

Another example of a topological vector space whose topology can be defined by countably many seminorms is the space $C^\infty(U)$ of smooth scalar-valued functions on a nonempty open set U of $\mathbb{R}^N$ with the topology of uniform convergence on compact sets of all derivatives. The family of seminorms is indexed by pairs (K, P) with K a compact subset of U and with P a polynomial, the corresponding seminorm being $\|f\|_{K,P} = \sup_{x\in K} |(P(D)f)(x)|$. The Hausdorff condition is satisfied because if $\|f\|_{K,1} = 0$ for all K, then $f = 0$. We shall see in the next section that the topology can be defined by a countable subfamily of these seminorms.

Still a third space of smooth scalar-valued functions, besides $\mathcal{S}(\mathbb{R}^N)$ and $C^\infty(U)$, will be of interest to us. This is the space $C^\infty_{\text{com}}(U)$ of smooth functions on a nonempty open U with compact support contained in U. The useful topology on this space is more complicated than the topologies considered so far. In particular, it cannot be given by countably many seminorms. Describing the topology requires some preparation, and we come back to the details in Section 7.

The examples we have encountered of topological vector spaces defined by an uncountable family of seminorms, but not definable by a countable family, are qualitatively different from the examples above. Indeed, they lead along a different theoretical path, as we shall see—one that takes us in the direction of spectral theory rather than distribution theory.

The first class of such examples is the class of normed linear spaces X with the "weak topology," as contrasted with the norm topology. Let X^* be the set of linear functionals of X that are continuous in the norm topology. The **weak topology** on X was defined in Chapter X of *Basic* as the weakest topology that makes all members of X^* continuous. Of course, any set that is open in the weak topology on X is open in the norm topology. A base for the open sets in the weak topology on X is obtained as follows: For each triple (x_0, x^*, r), with x_0 in X, x^* in X^*, and $r > 0$, the set $\{x \mid |x^*(x - x_0)| < r\}$ is to be in the base, and the base consists of all finite intersections of these sets as (x_0, x^*, r) varies. The weak topology is given by the family of seminorms $\|\cdot\|_{x^*} = |x^*(\,\cdot\,)|$. The proof that the weak topology is Hausdorff requires the fact, for each $x \neq 0$ in X, that there is some member x^* with $x^*(x) \neq 0$; this fact is one of the standard corollaries of the Hahn–Banach Theorem. Examples of weak topologies will be discussed in Section 3.

Similarly the weak-star topology on X^*, when X is a normed linear space, was defined in *Basic* as the weakest topology on X^* that makes all members of X continuous. This is given by the family of seminorms $\|\cdot\|_x = |\cdot(x)|$. Here the relevant fact for seeing that the topology is Hausdorff is that for each $x^* \neq 0$ in X^*, there is some x in X with $x^*(x) \neq 0$. This is just a matter of the definition of $x^* \neq 0$ and depends on no theorem. Examples of weak-star topologies will be discussed in Section 3.

The above classes of examples by no means exhaust the possibilities for topological vector spaces. Let us mention briefly one example that is not even close to being definable by seminorms. It is the space $L^p([0, 1])$ with $0 < p < 1$. This is the vector space of all real-valued Borel functions on $[0, 1]$ with $\int_{[0,1]} |f|^p\,dx$ finite, except that we identify two functions if they differ only on a set of measure 0. Let us see that $d(f, g) = \int_{[0,1]} |f - g|^p\,dx$ is a metric. We need only verify the triangle inequality in the form $\int_{[0,1]} |f + g|^p\,dx \le \int_{[0,1]} |f|^p\,dx + \int_{[0,1]} |g|^p\,dx$. To check this, we observe for nonnegative r that $(1+r)^p - (1+r^p)$ is 0 at $r = 0$ and has negative derivative $p((1+r)^{p-1} - r^{p-1})$ since $p - 1$ is negative. Thus $(1+r)^p \le 1+r^p$ for $r \ge 0$, and consequently $|a+b|^p \le (|a|+|b|)^p \le |a|^p+|b|^p$ for all real a and b. Taking $a = f(x)$ and $b = g(x)$ and integrating, we obtain the desired triangle inequality. One readily shows that $L^p([0, 1])$ with this metric is a topological vector space. On the other hand, this topological vector space is rather pathological, as is shown in Problem 8 at the end of the chapter. For example it has no nonzero continuous linear functionals, whereas nonzero topological vector spaces whose topologies are given by seminorms always have enough continuous linear functionals to separate points.[2]

Now we turn our attention to a few results valid for arbitrary topological vector spaces.

Proposition 4.2. In any topological vector space, the closure of any vector subspace is a vector subspace.

PROOF. Let V be a vector subspace of the topological vector space X. If x and y are in V^{cl}, then (x, y) is in $V^{\mathrm{cl}} \times V^{\mathrm{cl}} = (V \times V)^{\mathrm{cl}}$. Any continuous function

[2] More precisely it will be observed in Section 6 that topological vector spaces whose topologies are given by seminorms are "locally convex," and it will be proved in that same section that locally convex spaces always have enough continuous linear functionals to separate points.

f has the property for any set S that $f(S^{\mathrm{cl}}) \subseteq f(S)^{\mathrm{cl}}$. Applying this fact to the addition function, we see that $x+y$ is in V^{cl} since V is the image of $V \times V$ under addition. Thus V^{cl} is closed under addition. Similarly V^{cl} is closed under scalar multiplication. $\square$

Lemma 4.3. If X is a real or complex vector space in which addition and scalar multiplication are continuous and if $\{0\}$ is a closed subset of X, then X is Hausdorff and hence is a topological vector space.

PROOF. Since translations are homeomorphisms, it is enough to separate 0 and an arbitrary $x \neq 0$ by disjoint open neighborhoods. Since $X - \{0\}$ is open, so is $V = X - \{x\}$. By continuity of subtraction, choose an open neighborhood U of 0 such that the set of differences satisfies $U - U \subseteq V$. Then U and $x + U$ are open neighborhoods of 0 and x. If y is in their intersection, then y is in U, and y is of the form $x + u$ for some u in U. Hence $x = y - u$ exhibits x as in $U - U \subseteq V = X - \{x\}$, contradiction. Thus we can take U and $x + U$ as the required disjoint open neighborhoods of 0 and x. $\square$

Proposition 4.4. If X is a topological vector space, if Y is a closed vector subspace, and if the quotient vector space X/Y is given the quotient topology,[3] then X/Y is a topological vector space, and the quotient map $q : X \to X/Y$ carries open sets to open sets.

PROOF. If U is open in X, then $q^{-1}(q(U)) = \bigcup_{y \in Y}(y + U)$ exhibits $q^{-1}(q(U))$ as the union of open sets and hence as an open set. By definition of the topology on X/Y, $q(U)$ is open in X/Y. Hence q carries open sets in X to open sets in X/Y.

To see that addition is continuous in X/Y, let x_1 and x_2 be in X, and let E be an open neighborhood of the member $x_1 + x_2 + Y$ of X/Y. Then $q^{-1}(E)$ is an open neighborhood of $x_1 + x_2$ in X. By continuity of addition in X, there exist open neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 + U_2 \subseteq q^{-1}(E)$. The map q is open and linear, and hence $q(U_1)$ and $q(U_2)$ are open subsets of X/Y with $q(U_1) + q(U_2) \subseteq q(q^{-1}(E)) = E$. Thus addition is continuous in X/Y.

To see that scalar multiplication is continuous in X/Y, let c be a scalar, let x be in X, and let E be an open neighborhood of cx in X/Y. Then $q^{-1}(E)$ is an open neighborhood of cx in X. By continuity of scalar multiplication in X, there exist open neighborhoods A of c in the scalars and U of x in X such that $AU \subseteq q^{-1}(E)$. Then $q(U)$ is an open subset of X/Y such that $Aq(U) \subseteq q(q^{-1}(E)) = E$. Hence scalar multiplication is continuous in X/Y.

Applying Lemma 4.3, we see that X/Y is Hausdorff. Therefore X/Y is a topological vector space. $\square$

[3]If $q : X \to X/Y$ is the quotient mapping, the open sets E of X/Y are defined as all subsets such that $q^{-1}(E)$ is open in X.

Proposition 4.5. If Y is an n-dimensional topological vector space over $\mathbb{F}$, then Y is isomorphic to $\mathbb{F}^n$.

PROOF. Let $y_1, \dots, y_n$ be a vector-space basis of Y, and let $(\,\cdot\,,\,\cdot\,)$ and $|\,\cdot\,|$ be the usual inner product and norm on $\mathbb{F}^n$. If $e_1, \dots, e_n$ is the standard basis of $\mathbb{F}^n$, define $L\big(\sum_{j=1}^n c_j e_j\big) = \sum_{j=1}^n c_j y_j$. Then L is one-one and hence is onto Y. We saw earlier in this section that L is continuous. We shall prove that L^{-1} is continuous, and it is enough to do so at 0 in Y.

Assuming on the contrary that L^{-1} is not continuous at 0, we can find some $\epsilon > 0$ such that no open neighborhood U of 0 in Y maps under L^{-1} into the open neighborhood $\{|x| < \epsilon\}$ of 0 in $\mathbb{F}^n$. For each such U, find y_U in U with $|L^{-1}(y_U)| \geq \epsilon$. Define $z_U = |L^{-1}(y_U)|^{-1} y_U$. The net $\{y_U\}$ tends to 0 in Y by construction, and the numbers $|L^{-1}(y_U)|^{-1}$ are bounded by ϵ^{-1}. By continuity of scalar multiplication in Y, z_U has limit 0 in Y. On the other hand, the members of $\mathbb{F}^n$ defined by $x_U = L^{-1}(z_U) = |L^{-1}(y_U)|^{-1} L^{-1}(y_U)$ have $|x_U| = 1$ for all U. The unit sphere in $\mathbb{F}^n$ is compact, and it follows that $\{x_U\}$ has a convergent subnet, say $\{x_{U_\mu}\}$, with some limit x_0 such that $|x_0| = 1$. We have $L(x_U) = z_U$, and passage to the limit gives $L(x_0) = \lim_\mu L(x_{U_\mu}) = \lim_\mu z_{U_\mu} = 0$. On the other hand, L is one-one, and hence the equality $L(x_0) = 0$ for some x_0 with $|x_0| = 1$ is a contradiction. We conclude that L^{-1} is continuous. $\square$

Corollary 4.6. Every finite-dimensional vector subspace of a topological vector space is closed.

PROOF. Let V be an n-dimensional subspace of a topological vector space X, and suppose that V^{cl} properly contains V. Choose x_0 in $V^{\mathrm{cl}} - V$, and form the vector subspace $W = V + \mathbb{F}x_0$. Then the closure of V in W, being a vector subspace (Proposition 4.2), is W. The vector subspace W has dimension $n+1$, and Proposition 4.5 shows that W is isomorphic to $\mathbb{F}^{n+1}$. All vector subspaces of $\mathbb{F}^{n+1}$ are closed in $\mathbb{F}^{n+1}$, and hence V is closed in W, contradiction. $\square$

Lemma 4.7. If X is a topological vector space, K is a compact subset of X, and V is an open neighborhood of 0, then there exists $\epsilon > 0$ such that $\delta K \subseteq V$ whenever $|\delta| \leq \epsilon$.

PROOF. For each $k \in K$, choose $\epsilon_k > 0$ and an open neighborhood U_k of k such that $\delta U_k \subseteq V$ whenever $|\delta| \leq \epsilon_k$; this is possible since scalar multiplication is continuous at the point where the scalar is 0 and the vector is k. The open sets U_k cover K, and the compactness of K implies that there is a finite subcover: $K \subseteq U_{k_1} \cup \cdots \cup U_{k_m}$. Then $\delta K \subseteq V$ whenever $|\delta| \leq \min_{1 \leq j \leq m} \epsilon_{k_j}$. $\square$

Proposition 4.8. Every locally compact topological vector space is finite dimensional.

PROOF. Let X be a locally compact topological vector space, let K be a compact neighborhood of 0, and let U be its interior. Suppose that we have a sequence $\{y_m\}$ in X with the property that for any $\delta > 0$, there is an integer M such that $m \geq M$ implies y_m lies in δK. Then the result of Lemma 4.7 implies that $\{y_m\}$ tends to 0.

The sets $\{k+\frac{1}{2}U \mid k \in K\}$ form an open cover of K. If $\{k_1+\frac{1}{2}U, \dots, k_n+\frac{1}{2}U\}$ is a finite subcover, we prove that $\{k_1, \dots, k_n\}$ spans X. It is enough to prove that $S = \{k_1, \dots, k_n\}$ spans U. If x is in U, then x is in one of the sets of the finite subcover, say $k_{j_1} + \frac{1}{2}U$. Write $x = k_{j_1} + \frac{1}{2}u_1$ accordingly. The finite subcover covers K and hence its interior U, and thus $\frac{1}{2}U$ is covered by $\frac{1}{2}(k_1 + \frac{1}{2}U), \dots, \frac{1}{2}(k_n + \frac{1}{2}U)$. Applying this observation to the element $\frac{1}{2}u_1$ of $\frac{1}{2}U$, we see that x is in $k_{j_1} + \frac{1}{2}(k_{j_2} + \frac{1}{2}U)$ for some k_{j_2}. Write $x = k_{j_1} + \frac{1}{2}k_{j_2} + \frac{1}{4}u_2$ accordingly. Continuing in this way, we see that

$$x \qquad \text{is in} \qquad k_{j_1} + \tfrac{1}{2}\,k_{j_2} + \cdots + \tfrac{1}{2^{r-1}}\,k_{j_r} + \tfrac{1}{2^r}\,U \qquad \text{for each } r.$$

Put $x_r = k_{j_1} + \frac{1}{2}\,k_{j_2} + \cdots + \frac{1}{2^{r-1}}\,k_{j_r}$. This is an element of the finite-dimensional subspace spanned by S, which is closed by Corollary 4.6; thus if $\{x_r\}$ converges, it must converge to a member x_0 of this subspace. Using the result of the previous paragraph, we shall show that $x - x_r$ converges to 0. Then we can conclude that x_r converges to x, hence that x is in the span of S. To see that $x - x_r$ converges to 0, choose l such that $|\delta_0| \leq 2^{-l}$ implies $\delta_0 K \subseteq U$. Applying the criterion of the previous paragraph, let $\delta > 0$ be given. Choose M such that $2^{-M}\delta^{-1} \leq 2^{-l}$. Then $m \geq M$ implies that $2^{-m}\delta^{-1} \leq 2^{-M}\delta^{-1} \leq 2^{-l}$. Thus $2^{-m}\delta^{-1}$ is an allowable choice of δ_0, and we therefore obtain $2^{-m}\delta^{-1}K \subseteq U$ and $2^{-m}K \subseteq \delta U$. For $m \geq M$, the element $x - x_m$ lies in $2^{-m}U \subseteq 2^{-m}K$, and we have just proved that $2^{-m}K \subseteq \delta U$. Thus $x - x_m$ lies in δU, and the criterion of the previous paragraph applies. Hence $x - x_m$ tends to 0. This completes the proof. $\square$

2. $C^\infty(U)$, Distributions, and Support

As was mentioned in Section III.1, distributions are continuous linear functionals on vector spaces of smooth functions. Their properties are deceptively simple-looking and enormously helpful in working with linear partial differential equations. We considered tempered distributions in Section III.1; these are the continuous linear functionals on the space $\mathcal{S}(\mathbb{R}^N)$ of Schwartz functions on $\mathbb{R}^N$. In this section we study the topology on the space $C^\infty(U)$ of arbitrary scalar-valued smooth functions on an open subset U of $\mathbb{R}^N$, together with the associated space of distributions.

To topologize $C^\infty(U)$, we use the family of seminorms indexed by pairs (K, P) with K a compact subset of U and with P a polynomial, the $(K, P)^{\text{th}}$ seminorm being $\|f\|_{K,P} = \sup_{x\in K} |(P(D)f)(x)|$. The resulting topology is Hausdorff, and $C^\infty(U)$ becomes a topological vector space.

Let us see that this topology is given by a countable subfamily of these seminorms and is therefore implemented by a metric. It is certainly sufficient to consider only the monomials D^α instead of all polynomials $P(D)$, and thus the P index of (K, P) can be assumed to run through a countable set. We make use of a notion already used in Section III.2. An **exhausting sequence** of compact subsets of U is an increasing sequence of compact sets with union U such that each set is contained in the interior of the next set. An exhausting sequence exists in any locally compact separable metric space. If $\{K_n\}$ is an exhausting sequence for U and if K is a compact subset of U, then the interiors K_n^o of the K_n's form an open cover of K, and there is a a finite subcover; since the members of the open cover are nested, K is contained in some single K_n^o and hence in K_n. Therefore $\|f\|_{K,P} \le \|f\|_{K_n,P}$ for every P, and we can discard all the seminorms except the ones from some K_n. In short, the countably many seminorms $\|f\|_{K_n,x^\alpha} = \sup_{x\in K_n} |(D^\alpha f)(x)|$ suffice to determine the topology of $C^\infty(U)$. In particular, the topology is independent of the choice of exhausting sequence.

After the statement of Theorem 3.9, we constructed a smooth partition of unity $\{\psi_n\}_{n\ge1}$ associated to an exhausting sequence $\{K_n\}_{n\ge1}$ of an open subset U of $\mathbb{R}^N$. Such a partition of unity is sometimes useful, and Problem 9 at the end of the chapter illustrates this fact. The functions ψ_n are in $C^\infty(U)$ and have the properties that $\sum_{n=1}^\infty \psi_n(x) = 1$ on U, $\psi_1(x) > 0$ on K_3, $\psi_1(x) = 0$ on $(K_4^o)^c$, and for $n \ge 2$,

$$\psi_n(x) \begin{cases} > 0 & \text{for } x \in K_{n+2} - K_{n+1}^o, \\ = 0 & \text{for } x \in (K_{n+3}^o)^c \cup K_n. \end{cases}$$

Since $C^\infty(U)$ is a metric space, its topology may be characterized in terms of convergence of sequences: a sequence of functions converges in $C^\infty(U)$ if and only if the functions converge uniformly on each compact subset of U and so do each of their iterated partial derivatives

If a particular metric for $C^\infty(U)$ is specified as constructed in Section III.1 from an enumeration of some determining countable family of seminorms, then it is apparent that a sequence of functions is Cauchy in $C^\infty(U)$ if and only if the functions and all their iterated partial derivatives are uniformly Cauchy on each compact subset of U. As a consequence we can see that $C^\infty(U)$ is complete as a metric space: in fact, let us extract limits from each uniformly Cauchy sequence of derivatives and use the standard theorem on derivatives of convergent sequences whose derivatives converge uniformly; the result is that we obtain a member of $C^\infty(U)$ to which the Cauchy sequence converges.

It is unimportant which particular metric is used for this completeness argument. The relevant consequence is that the Baire Category Theorem[4] is applicable to $C^\infty(U)$, and the statement of the Baire Category Theorem makes no reference to a particular metric.

In similar fashion one checks that $\mathcal{S}(\mathbb{R}^N)$, whose topology is likewise given by countably many seminorms, is complete as a metric space.

The vector space of continuous linear functionals on $C^\infty(U)$, i.e., its **continuous dual**, is called the space of all **distributions of compact support** on U and is traditionally[5] denoted by $\mathcal{E}'(U)$. The words "of compact support" require some explanation and justification, which we come back to after giving an example.

EXAMPLE. Take finitely many complex Borel measures ρ_α of compact support on U, the indexing being by the set of n-tuples α of nonnegative integers with $|\alpha| \leq m$, and define

$$T(\varphi) = \sum_{|\alpha|\leq m} \int_U D^\alpha \varphi(x)\, d\rho_\alpha(x).$$

It is easy to check that T is a distribution of compact support on U. A theorem in Chapter V will provide a converse, saying essentially that every continuous linear functional on $C^\infty(U)$ is of this form.

Let us observe that the vector subspace $C^\infty_{\text{com}}(U)$ is dense in $C^\infty(U)$. In fact, let $\{K_j\}$ be an exhausting sequence of compact sets in U, and choose $\psi_j \in C^\infty_{\text{com}}(\mathbb{R}^n)$ by Proposition 3.5f to be 1 on K_j and 0 off K_{j+1}. If f is in $C^\infty(U)$, then $\psi_j f$ is in $C^\infty_{\text{com}}(U)$ and tends to f in every seminorm on $C^\infty(U)$.

To obtain a useful notion of "support" for a distribution, we need the following lemma.

Lemma 4.9. If U_1 and U_2 are nonempty open sets in $\mathbb{R}^N$ and if φ is in $C^\infty_{\text{com}}(U_1 \cup U_2)$, then there exist $\varphi_1 \in C^\infty_{\text{com}}(U_1)$ and $\varphi_2 \in C^\infty_{\text{com}}(U_2)$ such that $\varphi = \varphi_1 + \varphi_2$.

PROOF. Let L be the compact support of φ, and choose a compact set K such that $L \subseteq K^o \subseteq K \subseteq U_1 \cup U_2$. Then $\{U_1, U_2\}$ is a finite open cover of K, and Lemma 3.15b of *Basic* produces an open cover $\{V_1, V_2\}$ of K such that V_1^{cl} is a compact subset of U_1 and V_2^{cl} is a compact subset of U_2. Proposition 3.5f produces functions $g_1 \in C^\infty_{\text{com}}(U_1)$ and $g_2 \in C^\infty_{\text{com}}(U_2)$ with values in [0, 1] such that g_1 is 1 on V_1^{cl} and g_2 is 1 on V_2^{cl}. Then $g = g_1 + g_2$ is in $C^\infty_{\text{com}}(U_1 \cup U_2)$ and

[4] Theorem 2.53 of *Basic*.

[5] The tradition dates back to Laurent Schwartz's work, in which $\mathcal{E}(U)$ was the notation for $C^\infty(U)$ and $\mathcal{E}'(U)$ was the space of continuous linear functionals.

is 1 on K. If W is the open set where $g \neq 0$, then Proposition 3.5f produces a function h in $C^\infty_{\text{com}}(W)$ with values in [0, 1] such that h is 1 on K. The function $1-h$ is smooth, has values in [0, 1], is 1 where $g \neq 0$, and is 0 on K. Hence $g + (1-h)$ is a smooth function that is everywhere positive on $\mathbb{R}^N$ and equals g on K. Therefore the functions $g_1/(g+1-h)$ and $g_2/(g+1-h)$ are smooth functions on $\mathbb{R}^N$ compactly supported in U_1 and U_2, respectively, with sum equal to 1 on K. If we define $\varphi_1 = g_1\varphi$ and $\varphi_2 = g_2\varphi$, then φ_1 and φ_2 have the required properties. □

Proposition 4.10. If T is an arbitrary linear functional on $C^\infty_{\text{com}}(U)$ and if U' is the union of all open subsets U_γ of U such that T vanishes on $C^\infty_{\text{com}}(U_\gamma)$, then T vanishes on $C^\infty_{\text{com}}(U')$.

PROOF. Let φ be in $C^\infty_{\text{com}}(U')$, and let K be the support of φ. The open sets U_γ form an open cover of K, and some finite subcollection must have $K \subseteq U_{\gamma_1} \cup \cdots \cup U_{\gamma_p}$. Lemma 4.9 applied inductively shows that φ is the sum of functions in $C^\infty_{\text{com}}(U_j)$, $1 \leq j \leq p$. Since T is 0 on each of these, it is 0 on the sum. □

If T is in $\mathcal{E}'(U)$, the **support** of T is the complement of the set U' in Proposition 4.10, i.e., the complement of the union of all open sets U_γ such that T vanishes on $C^\infty_{\text{com}}(U_\gamma)$. If T has empty support, then $T = 0$ because T vanishes on $C^\infty_{\text{com}}(U)$ and $C^\infty_{\text{com}}(U)$ is dense in $C^\infty(U)$.

Proposition 4.11. Every member T of $\mathcal{E}'(U)$ has compact support.

REMARKS. For the moment this proposition justifies using the name "distributions of compact support" for the continuous linear functionals on $C^\infty(U)$. After we define general distributions in Section V.1, we shall have to return to this matter.

PROOF. Let $\{K_n\}$ be an exhausting sequence of compact sets in U. If T is not supported in any K_n, then there is some f_n in $C^\infty_{\text{com}}(U - K_n)$ with $T(f_n) \neq 0$. Put $g_n = f_n/T(f_n)$, so that $T(g_n) = 1$. If K is any compact subset of U, then $K \subseteq K_n$ for large n, and $g_n\big|_K = 0$ for such n. Thus g_n tends to 0 in $C^\infty(U)$ while $T(g_n)$ tends to $1 \neq 0 = T(0)$, in contradiction to continuity of T. □

Similarly we can use Proposition 4.10 to define the **support** of a tempered distribution T in $\mathcal{S}'(\mathbb{R}^N)$ as the complement of the union of all open sets U_γ such that T vanishes on $C^\infty_{\text{com}}(U_\gamma)$. Tempered distributions need not have compact support; for example, the function 1 defines a tempered distribution whose support is $\mathbb{R}^N$.

In the case of tempered distributions, a little argument is required to show that the only tempered distribution with empty support is the 0 distribution. What is needed is the following fact.

Proposition 4.12. $C^\infty_{\text{com}}(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$.

REMARKS. If T in $\mathcal{S}'(\mathbb{R}^N)$ has empty support, then T vanishes on $C^\infty_{\text{com}}(\mathbb{R}^N)$. Proposition 4.12 and the continuity of T imply that $T = 0$ on $\mathcal{S}(\mathbb{R}^N)$. Thus the only tempered distribution with empty support is the 0 distribution.

PROOF. Fix h in $C^\infty_{\text{com}}(\mathbb{R}^N)$ with values in $[0, 1]$ such that $h(x)$ is 1 for $|x| \le 1$ and is 0 for $|x| \ge 2$. Define $h_R(x) = h(R^{-1}x)$. If φ is in $\mathcal{S}(\mathbb{R}^N)$, we shall show that $\lim_{R\to\infty} h_R\varphi = \varphi$ in the metric space $\mathcal{S}(\mathbb{R}^N)$, and then the proposition will follow. Thus we want $\lim_{R\to\infty} \sup_{x\in\mathbb{R}^N} |x^\gamma D^\alpha(\varphi - h_R\varphi)(x)| = 0$. By the Leibniz rule, $D^\alpha(h_R\varphi) = h_R D^\alpha\varphi + \sum_{\beta<\alpha} c_\beta (D^{\alpha-\beta}h_R)(D^\beta\varphi)$. Hence it is enough to prove that

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}^N} |x^\gamma(1 - h_R)D^\alpha\varphi| = 0$$

and

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}^N} |x^\gamma(D^{\alpha-\beta}h_R)(D^\beta\varphi)| = 0 \quad \text{for } \beta < \alpha.$$

The first of these limit formulas is a consequence of the fact that $x^\gamma D^\alpha\varphi$ vanishes at infinity, which in turn follows from the fact that $x^\gamma(1 + |x|^2)D^\alpha\varphi$ is bounded, i.e., that $\|\varphi\|_{x^\gamma(1+|x|^2),x^\alpha}$ is finite. For the second of these limit formulas, we observe from the chain rule that $D^{\alpha-\beta}h_R(x) = R^{-|\alpha-\beta|}D^{\alpha-\beta}h(R^{-1}x)$. For $\beta < \alpha$, this function is dominated in absolute value by $c_\alpha R^{-1}$. Hence $\sup_{x\in\mathbb{R}^N} |x^\gamma(D^{\alpha-\beta}h_R)(D^\beta\varphi)| \le c_\alpha R^{-1}\sum_{\beta<\alpha}\|\varphi\|_{x^\gamma,x^\beta}$, and the limit on R is 0. $\square$

3. Weak and Weak-Star Topologies, Alaoglu's Theorem

Let X be a normed linear space, and let X^* be its dual, which we know to be a Banach space. We have defined the **weak topology** on X to be the weakest topology on X making all members of X^* continuous, i.e., making $x \mapsto x^*(x)$ continuous for each x^* in X^*. This topology is given by the family of seminorms $\|x\|_{x^*} = |x^*(x)|$ indexed by X^*. The **weak-star topology** on X^* relative to X is the weakest topology on X^* making all members of $\iota(X)$ continuous,[6] i.e., making $x^* \mapsto x^*(x)$ continuous for each x in X. This topology is given by the family of seminorms $\|x^*\|_x = |x^*(x)|$ indexed by X. In this section we

[6]The symbol ι denotes the canonical map $X \to X^{**}$ given by $\iota(x)(x^*) = x^*(x)$.

study these topologies[7] in more detail, proving an important theorem about the weak-star topology.

We shall discuss some examples in a moment. The space X^* is a normed linear space in its own right, and therefore it has a well-defined weak topology. The definitions make the weak topology on X^* the same as the weak-star topology on X^* relative to X if X is reflexive, but we cannot draw this conclusion in general.

The weak topology on X is of less importance to real analysis than the weak-star topology on X^*, and thus the main interest in the weak topology on X will be in the case that X is reflexive. It is also true that exact conditions that interpret the weak or weak-star topology in a particular example tend not to be useful. Nevertheless, it may still be helpful to consider examples in order to get a better sense of what these topologies do.

We shall discuss the examples in terms of convergence. However, the convergence will involve only convergence of sequences, not convergence of general nets. A difficulty with nets is that one cannot draw familiar conclusions from convergence of nets even in the case of nets in the real numbers; for example, a convergent net of real numbers need not be bounded, just eventually bounded.

In order to have it available in the discussion, we prove one fact about convergence of sequences in weak and weak-star topologies before coming to the examples.

Proposition 4.13. Let X be a normed linear space, and let X^* be its dual space.

(a) If $\{x_n\}$ is a sequence in X converging to x_0 in the weak topology on X, then $\{\|x_n\|\}$ is a bounded sequence in $\mathbb{R}$ and $\|x_0\| \leq \liminf_n \|x_n\|$.

(b) If X is a Banach space and if $\{x_n^*\}$ is a sequence in X^* converging to x_0^* in the weak-star topology on X^* relative to X, then $\{\|x_n^*\|\}$ is a bounded sequence in $\mathbb{R}$ and $\|x_0^*\| \leq \liminf_n \|x_n^*\|$.

PROOF. For the first half of (a), let $\iota : X \to X^{**}$ be the canonical map. Since the sequence $\{\iota(x_n)(x^*)\}$ converges to $x^*(x_0)$ for each x^* in X^*, $\{\iota(x_n)\}$ is a set of bounded linear functionals on the Banach space X^* with $\{\iota(x_n)(x^*)\}$ bounded for each x^* in X^*. By the Uniform Boundedness Theorem the norms $\|\iota(x_n)\|$ are bounded. Since ι preserves norms as a consequence of the Hahn–Banach Theorem, the norms $\|x_n\|$ are bounded. For the second half of (a), let x^* be arbitrary in X^* with $\|x^*\| \leq 1$. Then

$$|x^*(x_0)| = \lim |x^*(x_n)| \leq \liminf \|x^*\|\|x_n\| \leq \liminf \|x_n\|.$$

Taking the supremum over x^* with $\|x^*\| \leq 1$ and applying the formula $\|x_0\| = \sup_{\|x^*\|\leq 1} |x^*(x_0)|$, which is known from the Hahn–Banach Theorem, we obtain $\|x_0\| \leq \liminf \|x_n\|$.

[7]The weak topology on X is also called the X^* topology of X, and the weak-star topology on X^* is also called the X topology of X^*.

For the first half of (b), $\{x_n^*\}$ is a set of bounded linear functionals on the Banach space X with $\{x_n^*(x)\}$ bounded for each x in X. Then the Uniform Boundedness Theorem shows that the norms $\|x_n^*\|$ are bounded. For the second half of (b), let x be arbitrary in X with $\|x\| \le 1$. Then

$$|x_0^*(x)| = \lim |x_n^*(x)| \le \liminf \|x_n^*\| \|x\| \le \liminf \|x_n^*\|.$$

Taking the supremum over x and applying the definition of $\|x_0^*\|$, we obtain $\|x_0^*\| \le \liminf \|x_n^*\|$. $\square$

EXAMPLES OF CONVERGENCE IN WEAK TOPOLOGIES.

(1) $X = L^p(S,\mu)$ when $1 < p < \infty$. Then $X^* \cong L^{p'}(X,\mu)$, where p' is the dual index[8] of p. The assertion is that a sequence $\{f_n\}$ tends weakly to f in L^p if and only if $\{\|f_n\|_p\}$ is bounded and $\lim \int_E f_n\,d\mu = \int_E f\,d\mu$ for every measurable subset E of S of finite measure. The necessity is immediate from Proposition 4.13a and from taking the member of X^* to be the indicator function of E. Let us prove the sufficiency. From $\lim \int_E f_n\,d\mu = \int_E f\,d\mu$, we see that $\lim \int_S f_n t\,d\mu = \int_S f t\,d\mu$ for t simple if t is 0 off a set of finite measure. Let g be given in $L^{p'}(S,\mu)$, and choose a sequence $\{t_m\}$ of simple functions equal to 0 off sets of finite measure such that $\lim_m t_m = g$ in the norm topology of $L^{p'}$. For all m and n, we have

$$\begin{aligned}
&\left|\int_S f_n g\,d\mu - \int_S fg\,d\mu\right| \\
&\qquad \le \left|\int_S f_n(g-t_m)\,d\mu\right| + \left|\int_S f_n t_m\,d\mu - \int_S f t_m\,d\mu\right| \\
&\qquad\quad + \left|\int_S f(t_m-g)\,d\mu\right| \\
&\qquad \le \|f_n\|_p \|g-t_m\|_{p'} + \left|\int_S f_n t_m\,d\mu - \int_S f t_m\,d\mu\right| + \|f\|_p \|g-t_m\|_{p'}.
\end{aligned}$$

The first and third terms on the right tend to 0 as m tends to infinity, uniformly in n. If $\epsilon > 0$ is given, choose m such that those two terms are $< \epsilon$, and then, with m fixed, choose n large enough to make the middle term $< \epsilon$.

(2) $X = C(S)$ with S compact Hausdorff, $C(S)$ being the space of continuous scalar-valued functions on S. Then X^* may be identified with the space $M(S)$ of (signed or) complex regular Borel measures on S, with the total-variation norm.[9] The assertion is that a sequence $\{f_n\}$ tends weakly to f in $C(S)$ if and only if $\{\|f_n\|\}$ is bounded and $\lim f_n = f$ pointwise. The necessity is immediate from Proposition 4.13a and from taking the member of X^* to be any point mass at a point

[8] The index p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. This duality was proved in Theorem 9.19 of *Basic* when μ is σ-finite, but it holds without this restrictive assumption on μ.

[9] This identification was obtained in *Basic* in Theorem 11.24 for real scalars and in Theorem 11.26 for complex scalars. The starting point for the identification is the Riesz Representation Theorem.

of S. For the sufficiency we simply observe that any member of $M(S)$ is a finite linear combination of regular Borel measures μ on S and $\lim \int_S f_n\,d\mu = \int_S f\,d\mu$ for any Borel measure μ by dominated convergence.

(3) $X = C_0(S)$ with S locally compact separable metric, $C_0(S)$ being the space of continuous scalar-valued functions vanishing at infinity. Again the dual X^* may be identified with the space $M(S)$ of complex regular Borel measures on S, with the total-variation norm. This example can be handled by applying the previous example to the one-point compactification of S. All signed or complex Borel measures are automatically regular in this case. A sequence $\{f_n\}$ tends weakly to f in $C_0(S)$ if and only if $\{\|f_n\|\}$ is bounded and $\lim f_n = f$ pointwise.

EXAMPLES OF CONVERGENCE IN WEAK-STAR TOPOLOGIES.

(1) $X = L^p(S,\mu)$ and $X^* \cong L^{p'}(S,\mu)$ when $1 < p < \infty$, p' being the dual index of p. This X is reflexive. Therefore the first example of convergence in weak topologies shows that $\{f_n\}$ converges weak-star in $L^{p'}(S,\mu)$ relative to $L^p(S,\mu)$ if and only if $\{\|f_n\|_{p'}\}$ is bounded and $\lim \int_E f_n\,d\mu = \int_E f\,d\mu$ for every measurable subset E of S of finite measure.

(2) $X = L^1(S,\mu)$ and $X^* \cong L^\infty(S,\mu)$ when μ is σ-finite. This X is usually not reflexive. However, the condition for weak-star convergence is the same as in the previous example: $\{f_n\}$ converges weak-star in $L^\infty(S,\mu)$ relative to $L^1(S,\mu)$ if and only if $\{\|f_n\|_\infty\}$ is bounded and $\lim \int_E f_n\,d\mu = \int_E f\,d\mu$ for every measurable subset E of S of finite measure. The argument in the first example of convergence in weak topologies can easily be modified to prove this.

(3) $X = C(S)$ with S compact Hausdorff, and $X = C_0(S)$ with S locally compact separable metric. Weak-star convergence of complex regular Borel measures does not have a useful necessary and sufficient condition beyond the definition. The notion of weak-star convergence in this situation is, nevertheless, quite helpful as a device for producing new complex measures out of old ones.[10]

A theorem about the weak topology, due to Banach, is that the vector subspaces that are closed in the weak topology are the same as the vector subspaces that are closed in the norm topology. More generally the closed convex sets coincide in the weak and norm topologies. We shall not have occasion to use this theorem or mention any of its applications, and we therefore omit the proof.

The weak-star topology has results of more immediate interest, and we turn our attention to those. Theorem 5.58 of *Basic* established for any separable normed linear space X that any bounded sequence in the dual X^* has a weak-star convergent subsequence; this was called a "preliminary form of Alaoglu's Theorem."

[10] *Warning*. Many probabilists and some other people use the unfortunate term "weak convergence" for this instance of weak-star convergence.

Theorem 4.14 Let X be a normed linear space with dual X^*.

(a) **(Alaoglu's Theorem)** The closed unit ball of X^* is compact in the weak-star topology relative to X.

(b) If X is separable, then the closed unit ball of X^* is a separable metric space in the weak-star topology.

REMARKS. By (a), any net $\{x^*_\alpha\}$ in X^* with $\|x^*_\alpha\|$ bounded has a subnet $\{x^*_{\alpha_\mu}\}$ and an element x^*_0 in X^* such that $x^*_{\alpha_\mu}(x) \to x^*_0(x)$ for every x in X. By (b), this conclusion about nets can be replaced by a conclusion about sequences if X is separable. Thus we recover the "preliminary form" of Alaoglu's Theorem. The results of Section III.4 give an example of the utility of the two parts of this theorem; together they lead to a proof that harmonic functions in $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ are automatically Poisson integrals of functions if $p > 1$ or of complex measures if $p = 1$.

PROOF. Let B be the closed unit ball in X^*, let $D(r)$ be the closed disk in $\mathbb{C}$ with radius r and center 0, and let $C = \bigtimes_{x \in X} D(\|x\|)$. Define $F : B \to C$ by $F(x^*) = \bigtimes_{x \in X} x^*(x)$. The function F is well defined since $|x^*(x))| \le \|x\|$ for all x^* in B and all x in X. It is continuous as a map into the product space since $x^* \mapsto x^*(x)$ is continuous for each x, it is one-one since x^* is determined by its values on each x, and it is a homeomorphism with its image by definition of weak topology. Since C is compact by the Tychonoff Product Theorem, (a) will follow if it is shown that $F(B)$ is closed in C. Let p_x denote the projection of C to its x^{th} coordinate. If x and x' are in X and if $\{f_\alpha\}$ is a net in C convergent to f_0 in C, then an equality $p_{x+x'}(f_\alpha) = p_x(f_\alpha) + p_{x'}(f_\alpha)$ for all α implies that $p_{x+x'}(f_0) = p_x(f_0) + p_{x'}(f_0)$ by continuity of $p_{x+x'}$, p_x, and $p_{x'}$. Thus the set

$$S(x, x') = \{f \in C \mid p_{x+x'}(f) = p_x(f) + p_{x'}(f)\}$$

is closed, and similarly the set

$$T(x, c) = \{f \in C \mid c p_x(f) = p_x(cf)\}$$

is closed. The intersection of all $S(x, x')$'s and all $T(x, c)$'s is the set of linear members of C, hence is exactly $F(B)$. Thus $F(B)$ is closed.

For (b), we continue with B and $D(r)$ as above, but we change C and F slightly. Let $\{x_n\}$ be a countable dense set in the norm topology of X, let $C = \bigtimes_{x_n} D(\|x_n\|)$, and define $F : B \to C$ by $F(x^*) = \bigtimes_{x_n} x^*(x_n)$. As in the proof of (a), F is continuous. It is one-one since any x^*, being continuous, is determined by its values on the dense set $\{x_n\}$. The domain is compact by (a). The range space C is a separable metric space and is in particular Hausdorff. Hence B is exhibited as homeomorphic to $F(B)$, which is a subspace of the separable metric space C and is therefore separable. $\square$

4. Stone Representation Theorem

In this section we begin to follow Alaoglu's Theorem along paths different from its use for creating limit functions and measures out of sequences that are bounded in a weak-star topology. We shall work in this section with what amounts to an example—one of the motivating examples behind a stunning idea of I. M. Gelfand around 1940 that brings algebra, real analysis, and complex analysis together in a profound way. The example gives a view of subalgebras of the algebra $B(S)$ of all bounded functions on a set S in terms of compactness. The stunning idea that came out, on which we shall elaborate shortly, is that the mechanism in the proof is the same mechanism that lies behind the Fourier transform on $\mathbb{R}^N$, that this mechanism can be cast in abstract form as a theory of commutative Banach algebras, and that the theory gives a new perspective about spectra. In particular, it leads directly to the full Spectral Theorem for bounded and unbounded self-adjoint operators, extending the theorem for compact self-adjoint operators that was proved as Theorem 2.3. In turn, the Spectral Theorem has many applications to the study of particular operators.

Let us first state the theorem about $B(S)$, then discuss Gelfand's stunning idea about the mechanism, and finally give the proof of the theorem. We shall pursue the Gelfand idea in Sections 10–11 later in this chapter.

We have discussed $B(S)$ as the Banach space of bounded complex-valued functions on a nonempty set S, the norm being the supremum norm. In this Banach space pointwise multiplication makes $B(S)$ into a complex associative algebra[11] with identity (namely the function 1), there is an operation of complex conjugation, and there is a notion of positivity (namely pointwise positivity of a function). The theorem concerns subalgebras of $B(S)$ containing 1, closed under conjugation, and closed under uniform limits.

Theorem 4.15 (Stone Representation Theorem). Let S be a nonempty set, and let $\mathcal{A}$ be a uniformly closed subalgebra of $B(S)$ with the properties that $\mathcal{A}$ is stable under complex conjugation and contains 1. Then there exist a compact Hausdorff space S_1, a function $p : S \to S_1$ with dense image, and a norm-preserving algebra isomorphism U of $\mathcal{A}$ onto $C(S_1)$ preserving conjugation and positivity, mapping 1 to 1, and having the property that $U(f)(p(s)) = f(s)$ for all s in S. If S is a Hausdorff topological space and $\mathcal{A}$ consists of continuous functions, then p is continuous.

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[11] An **associative algebra** $\mathcal{A}$ over $\mathbb{C}$ is a vector space with a $\mathbb{C}$ bilinear associative multiplication, i.e., with an operation $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ satisfying $(ab)c = a(bc)$, $a(b+c) = ab+ac$, $(a+b)c = ac+bc$, and $a(\lambda c) = (\lambda a)c = \lambda(ac)$ if λ is in $\mathbb{C}$ and a, b, c are in $\mathcal{A}$. This definition does not assume the existence of an identity element.

The idea of the proof is to consider the Banach-space dual $\mathcal{A}^*$ and focus on those members of $\mathcal{A}^*$ that are nonzero and respect multiplication—the nonzero continuous multiplicative linear functionals on $\mathcal{A}$. The ones that come immediately to mind are the evaluations at each point: for a point s of S, the evaluation at s is given by $e_s(f) = f(s)$, and it is a multiplicative linear functional, certainly of norm 1. The set S_1 in the theorem will be the set of all such continuous multiplicative linear functionals, the function p will be given by $p(s) = e_s$ for $s \in S$, and the mapping U will be given by $U(f)(\ell) = \ell(f)$ for each multiplicative linear functional ℓ.

The Banach space $\mathcal{A} \subseteq B(S)$, with its multiplication, is a **Banach algebra** in the sense that it is an associative algebra over $\mathbb{C}$, with or without identity, such that $\|fg\| \le \|f\|\|g\|$ for all f and g in $\mathcal{A}$. Another well-known Banach algebra is $L^1(\mathbb{R}^N)$. The norm in this case is the usual L^1 norm, and the multiplication is convolution, which satisfies $\|f * g\|_1 \le \|f\|_1\|g\|_1$ for all f and g in $L^1(\mathbb{R}^N)$.

The stunning idea of Gelfand's is that the formula that defines U in the Stone theorem is the same formula that gives the Fourier transform in the case of $L^1(\mathbb{R}^N)$. Specifically the nonzero multiplicative linear functionals in the case of $L^1(\mathbb{R}^N)$ are the evaluations at points of the Fourier transform, i.e., the mappings $f \mapsto \widehat{f}(y) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot y}\,dx$. These linear functionals are multiplicative because convolution goes into pointwise product under the Fourier transform.

What $\mathcal{A} \subseteq B(S)$ and $L^1(\mathbb{R}^N)$ have in common is, in the first place, that they are commutative Banach algebras. In addition, each has a conjugate-linear mapping $f \mapsto f^*$ that respects multiplication: complex conjugation in the case of $\mathcal{A}$ and the map $f \mapsto f^*$ with $f^*(x) = \overline{f(-x)}$ in the case of $L^1(\mathbb{R}^N)$. These conjugate-linear mappings interact well with the norm. The subalgebra $\mathcal{A}$ of $B(S)$ satisfies

(i) $\|ff^*\| = \|f\|\|f^*\|$ for all f,
(ii) $\|f^*\| = \|f\|$ for all f,

while $L^1(\mathbb{R}^N)$ satisfies just (ii). The theory that Gelfand developed applies best when both (i) and (ii) are satisfied, as is the case with $\mathcal{A}$ and also any L^∞ space, and it works somewhat when just (ii) holds, as with $L^1(\mathbb{R}^N)$.

Another example of a Banach algebra is the algebra $\mathcal{B}(H, H)$ of bounded linear operators from a Hilbert space H to itself, with the operator norm. The conjugate-linear mapping on $\mathcal{B}(H, H)$ is passage to the adjoint, and (i) and (ii) both hold. The thing that is missing is commutativity for $\mathcal{B}(H, H)$. However, if we take a single operator A and its adjoint A^*, assume that A commutes with A^*, and take the Banach algebra generated by A and A^*, then we have another example to which the Gelfand theory applies well. The Spectral Theorem for bounded self-adjoint operators is the eventual consequence.

The idea of considering the Banach subalgebra generated by A is a natural one because of one's experience in the subject of modern algebra: the study of all complex polynomials in a square matrix A is a useful tool in understanding a single linear transformation, including obtaining canonical forms for it like the Jordan form. Thus the use of an analogy with a topic in algebra leads one to a better understanding of a topic in analysis.

In this case ideas flowed in the reverse direction as well. The multiplicative linear functionals correspond, by passage to their kernels, to those ideals in the algebra that are maximal.[12] In effect the Banach algebra was being studied through its space of maximal ideals. About 1960, no doubt partly because of the success of the idea of considering the maximal ideals of a Banach algebra, the consideration of the totality of prime ideals of a commutative ring as a space began to play an important role in algebraic number theory and algebraic geometry.

PROOF OF THEOREM 4.15. Let S_1 be the set of all nonzero continuous multiplicative linear functionals ℓ on $\mathcal{A}$ with $\ell(\bar{f}) = \overline{\ell(f)}$. Let us see that each such has norm 1. In fact, choose f with $\ell(f) \neq 0$. Then $\ell(f) = \ell(f1) = \ell(f)\ell(1)$ shows that $\ell(1) = 1$, and hence $\|\ell\| \geq 1$. For any f with $\|f\|_{\text{sup}} \leq 1$, if we had $|\ell(f)| > 1$, then $|\ell(f)|^n = |\ell(f^n)| \leq \|\ell\|$ for all n would give a contradiction as soon as n is sufficiently large. We conclude that $\|\ell\| \leq 1$.

Therefore S_1 is a subset of the unit ball of the Banach-space dual $\mathcal{A}^*$. We give S_1 the relative topology from the weak-star topology on $\mathcal{A}^*$. Let us define the function $p : S \to S_1$, and in the process we shall have proved that S_1 is not empty. Every s in S defines an evaluation linear functional e_s in S_1 by $e_s(f) = f(s)$, and the function p is defined by $p(s) = e_s$ for s in S. To see that S_1 is a closed subset of the unit ball of $\mathcal{A}^*$ in the weak-star topology, let $\{\ell_\alpha\}$ be a net in S_1 converging to some $\ell \in \mathcal{A}^*$, the convergence being in the weak-star topology. Then we have $\ell_\alpha(fg) = \ell_\alpha(f)\ell_\alpha(g)$ and $\ell_\alpha(\bar{f}) = \overline{\ell_\alpha(f)}$ for all f and g in $\mathcal{A}$. Passing to the limit, we obtain $\ell(fg) = \ell(f)\ell(g)$ and $\ell(\bar{f}) = \overline{\ell(f)}$. Hence S_1 is closed. By Alaoglu's Theorem (Theorem 4.14a), S_1 is compact. It is Hausdorff since $\mathcal{A}^*$ is Hausdorff in the weak-star topology.

Certainly we have $\sup_{s \in S} |e_s(f)| = \|f\|_{\text{sup}}$. Since any ℓ in S_1 has $\|\ell\| \leq 1$, we obtain

$$\sup_{\ell \in S_1} |\ell(f)| = \|f\|_{\text{sup}}. \tag{$*$}$$

The definition of $U : \mathcal{A} \to C(S_1)$ is $U(f)(\ell) = \ell(f)$, and this makes $U(f)(p(s)) = U(f)(e_s) = e_s(f) = f(s)$. The function $U(f)$ on S_1 is continuous by definition of the weak-star topology. Because of the definition of S_1, U is an algebra homomorphism respecting complex conjugation and mapping 1 to 1.

[12] Checking that there are no other maximal ideals than the kernels of multiplicative linear functionals requires proving that every complex "Banach field" is 1-dimensional, an early result in the subject of Banach algebras and one that uses complex analysis in its proof. Details appear in Section 10.

Also, $(*)$ shows that U is an isometry. Since $\mathcal{A}$ is Cauchy complete, so is $U(\mathcal{A})$. Therefore $U(\mathcal{A})$ is a uniformly closed subalgebra of $C(S_1)$ stable under complex conjugation and containing the constants. It separates points of S_1 by the definition of equality of linear functionals. By the Stone–Weierstrass Theorem, $U(\mathcal{A}) = C(S_1)$. Since U is an isometry, U is one-one. Thus U is an algebra isomorphism of $\mathcal{A}$ onto $C(S_1)$.

If $p(S)$ were not dense in S_1, then Urysohn's Lemma would allow us to find a nonzero continuous function F on $C(S_1)$ with values in $[0, 1]$ such that F is 0 everywhere on $p(S)$. Since U is onto $C(S_1)$, choose $f \in \mathcal{A}$ with $U(f) = F$. If s is in S, then $0 = F(p(s)) = U(f)(p(s)) = f(s)$. Hence $\|f\|_{\text{sup}} = 0$. By $(*)$, $\ell(f) = 0$ for all $\ell \in S_1$. Then every $\ell \in S_1$ has $0 = \ell(f) = U(f)(\ell) = F(\ell)$, and $F = 0$, contradiction. We conclude that $p(S)$ is dense.

To see that U carries functions ≥ 0 to functions ≥ 0, we observe first that the identity $\ell(\bar{f}) = \overline{\ell(f)}$ for $\ell \in S_1$ and the equality $\bar{f} = f$ for f real together imply that $\ell(f) = \ell(\bar{f}) = \overline{\ell(f)}$ for f real. Hence f real implies $\ell(f)$ real. If $f \geq 0$, then $\big\| \|f\|_{\text{sup}} - f \big\|_{\text{sup}} \leq \|f\|_{\text{sup}}$. Since $\|\ell\| \leq 1$, we therefore have $\ell(\|f\|_{\text{sup}} - f) \leq \big\| \|f\|_{\text{sup}} - f \big\|_{\text{sup}} \leq \|f\|_{\text{sup}}$. Since $\ell(1) = 1$, this says that $\ell(f) \geq 0$. This inequality for all ℓ implies that $U(f) \geq 0$.

Finally suppose that S is a Hausdorff topological space and that $\mathcal{A} \subseteq C(S)$. We are to show that $p : S \to S_1$ is continuous. If $s_\alpha \to s_0$ for a net in S, we want $p(s_\alpha) \to p(s_0)$, i.e., $e_{s_\alpha} \to e_{s_0}$. According to the definition of the weak-star topology, we are thus to show that $f(s_\alpha) \to f(s_0)$ for every f in $\mathcal{A}$. But this is immediate from the continuity of f on S. □

We give three examples. A fourth example, concerning "almost periodic functions," will be considered in the problems at the end of Chapter VI. For this fourth example the compact Hausdorff space of Theorem 4.15 admits the structure of a compact group, and the representation theory of Chapter VI is applicable to describe the structure of the space of almost periodic functions.

Problems 21–25 at the end of the chapter develop the theory of Theorem 4.15 further.

EXAMPLES.

(1) $\mathcal{A} = C(S)$ with S compact Hausdorff. Then p is a homeomorphism of S onto S_1. In fact, $p(S)$ is always dense in S_1. Here p is continuous and S is compact. Thus $p(S)$ is closed and must equal S_1. The map p is one-one because Urysohn's Lemma produces functions taking different values at two distinct points s and s' of S and thus exhibiting $e_{s'}$ and e_s as distinct linear functionals. Since p is continuous and one-one from a compact space onto a Hausdorff space, it is a homeomorphism.

(2) One-point compactification. Let S be a locally compact Hausdorff space, and let $\mathcal{A}$ be the subalgebra of $C(S)$ consisting of all continuous functions having

limits at infinity. For a function f, this condition means that there is some number c such that for each $\epsilon > 0$, some compact subset K of S has the property that $|f(s) - c| \le \epsilon$ for all s not in K. Then S_1 may be identified with the one-point compactification of S.

(3) Stone–Čech compactification. Let S be a topological space, and let $\mathcal{A} = C(S)$. The resulting compact Hausdorff space S_1 is called the **Stone–Čech compactification** of S. This space tends to be huge. For example, if $S = [0, +\infty)$, the corresponding S_1 has cardinality greater than the cardinality of $\mathbb{R}$.

5. Linear Functionals and Convex Sets

For this section and the next we discuss aspects of functional analysis that lead toward the theory of distributions and toward the use of fixed-point theorems. The topic is the role of convex sets in real and complex vector spaces—first without any topology and then with an overlay of topology consistent with convex sets. Sections 7–9 will then explore the consequences of this development, first in connection with smooth functions and then in connection with fixed-point theorems.

Let X be a real or complex vector space. A subset E of X is **convex** if for each x and y in E, all points $(1-t)x + ty$ are in E for $0 \le t \le 1$.

Proposition 4.16. Convex sets in a real or complex vector space have the following elementary properties:

(a) the arbitrary intersection of convex sets is convex,
(b) if E is convex and $x_1, \dots, x_n$ are in E and $t_1, \dots, t_n$ are nonnegative reals with $t_1 + \cdots + t_n = 1$, then $t_1x_1 + \cdots + t_nx_n$ is in E,
(c) if E_1 and E_2 are convex, then so are $E_1 + E_2$, $E_1 - E_2$, and cE for any scalar c,
(d) if $L : X \to Y$ is linear between two vector spaces with the same scalars and if E is a convex subset of X, then $L(E)$ is convex in Y,
(e) if $L : X \to Y$ is linear between two vector spaces with the same scalars and if E is a convex subset of Y, then $L^{-1}(E)$ is convex in X.

PROOF. Conclusions (a), (c), (d), and (e) are completely straightforward. For (b), we induct on n, the case $n = 2$ being the definition of "convex." Suppose that the result is known for n and that members $x_1, \dots, x_{n+1}$ of X and nonnegative reals $t_1, \dots, t_{n+1}$ with sum 1 are given. We may assume that $t_1 \ne 1$. Put $s = t_2 + \cdots + t_{n+1}$ and $y = (1-t_1)^{-1}(t_2x_2 + \cdots + a_{n+1}x_{n+1})$. Since the reals $(1-t_1)^{-1}t_2, \dots, (1-t_1)^{-1}t_{n+1}$ are nonnegative and have sum 1, the inductive hypothesis shows that y is in E. Since t_1 and s are nonnegative and have sum 1, $t_1x_1 + sy = t_1x_1 + \cdots + t_{n+1}x_{n+1}$ is in E. This completes the induction. $\square$

Let E be a subset of our vector space X. We say that a point p in E is an **internal point** of E if for each x in X, there is an $\epsilon > 0$ such that $p + \delta x$ is in E for all scalars[13] δ with $|\delta| \leq \epsilon$. If p in X is neither an internal point of E nor an internal point of E^c, we say that p is a **bounding point** of E. These notions make no use of any topology on X.

Let K be a convex subset of X, and suppose that 0 is an internal point of K. For each x in X, let

$$\rho(x) = \inf\{a > 0 \mid a^{-1}x \in K\}.$$

The function $\rho(x)$ is called the **support function** of K. For an example let X be a normed linear space, and let K be the unit ball; then $\rho(x) = \|x\|$.

We are going to see that $\rho(x)$ has some bearing on controlling the linear functionals on X, as a consequence of the Hahn–Banach Theorem. By the "Hahn–Banach Theorem" here, we mean not the usual theorem for normed linear spaces[14] but the more primitive statement[15] from which that is derived:

HAHN–BANACH THEOREM. Let X be a *real* vector space, and let p be a real-valued function on X with

$$p(x + x') \leq p(x) + p(x') \qquad \text{and} \qquad p(tx) = tp(x)$$

for all x and x' in X and all real $t \geq 0$. If f is a linear functional on a vector subspace Y of X with $f(y) \leq p(y)$ for all y in Y, then there exists a linear functional F on X with $F(y) = f(y)$ for all $y \in Y$ and $F(x) \leq p(x)$ for all $x \in X$.

Before discussing linear functionals in our present context, let us observe some properties of the support function $\rho(x)$. Properties (b), (c), and (e) in the next lemma are the properties of the dominating function p in the Hahn–Banach Theorem as stated above.

Lemma 4.17. Let K be a convex subset of a vector space X, and suppose that 0 is an internal point. Then the support function $\rho(x)$ of K satisfies

(a) $\rho(x) \geq 0$,
(b) $\rho(x) < \infty$,
(c) $\rho(ax) = a\rho(x)$ for $a \geq 0$,
(d) $\rho(x) \leq 1$ for all x in K,
(e) $\rho(x + y) \leq \rho(x) + \rho(y)$,
(f) $\rho(x) < 1$ if and only if x is an internal point of K,
(g) $\rho(x) = 1$ characterizes the bounding points of K.

[13] The scalars are complex numbers if X is complex, real numbers if X is real.
[14] As in Theorem 12.13 of *Basic*.
[15] As in Lemma 12.14 of *Basic*.

PROOF. Conclusions (a), (c), and (d) are immediate, and (b) follows since 0 is an internal point of K.

For (e), let c be arbitrary with $c > \rho(x) + \rho(y)$. We show that $c^{-1}(x+y)$ is in K. Since c is arbitrary, it follows that the infimum of all numbers d with $d^{-1}(x+y)$ in K is $\leq \rho(x) + \rho(y)$; consequently $\rho(x+y)$ will have to be $\leq \rho(x) + \rho(y)$, and (e) will be proved. Thus write $c = a + b$ with $a > \rho(x)$ and $b > \rho(y)$. Since K is convex,

$$c^{-1}(x+y) = (a+b)^{-1}(x+y) = \tfrac{a}{a+b}\, a^{-1}x + \tfrac{b}{a+b}\, b^{-1}y$$

is in K, as required.

For (f), let x be an internal point of K. Then $x + \epsilon x = (1+\epsilon)x$ is in K for some $\epsilon > 0$, and hence $\rho(x) \leq (1+\epsilon)^{-1} < 1$.

Conversely suppose that $\rho(x) < 1$, and put $\epsilon = 1 - \rho(x)$. Fix y. Since 0 is an internal point of K, we can find $\mu > 0$ such that δy is in K for $|\delta| \leq \mu$. If c is any scalar of absolute value 1, then $c\mu y$ is in K, and hence $\rho(cy) \leq \mu^{-1}$. If δ is a scalar with $|\delta| < \epsilon\mu$, write $\delta = c'|\delta|$ with $|c'| = 1$. Then $\rho(\delta y) = |\delta|\rho(c'y) \leq |\delta|\mu^{-1} < \epsilon$. Applying (e) gives

$$\rho(x + \delta y) \leq \rho(x) + \rho(\delta y) = (1-\epsilon) + \rho(\delta y) < (1-\epsilon) + \epsilon = 1.$$

By definition of ρ, $1^{-1}(x + \delta y)$ is in K, i.e., $x + \delta y$ is in K. Thus x is an internal point of K.

For (g), we can argue in the same way as with (f) to see that $\rho(x) > 1$ characterizes the internal points of K^c. Therefore $\rho(x) = 1$ characterizes the bounding points of K. □

We shall now apply the Hahn–Banach Theorem to prove the basic separation theorem.

Theorem 4.18. Let M and N be disjoint nonempty convex subsets of a real or complex vector space X, and suppose that M has an internal point. Then there exists a nonzero linear functional F on X such that for some real c, $\operatorname{Re} F \leq c$ on M and $\operatorname{Re} F \geq c$ on N.

PROOF. First suppose that X is real. If m is an internal point of M, then 0 is an internal point of $M - m$, and we can replace M and N by $M - m$ and $N - m$. Changing notation, we may assume from the outset that 0 is an internal point of M.

If x_0 is in N, then $-x_0$ is an internal point of $M - N$, and 0 is an internal point of $K = M - N + x_0$. Since M and N are assumed disjoint, $M - N$ does not contain 0; thus K does not contain x_0. Let ρ be the support function of K; this function satisfies the properties of the function p in the Hahn–Banach Theorem, according to Lemma 4.17. Moreover, $\rho(x_0) \ge 1$ by Lemma 4.17f. Define $f(ax_0) = a\rho(x_0)$ for all (real) scalars a. Then f is a nonzero linear functional on the 1-dimensional space of real multiples of x_0, and it satisfies

$$\begin{aligned} a \ge 0 \qquad &\text{implies} \qquad f(ax_0) = a\rho(x_0) = \rho(ax_0), \\ a < 0 \qquad &\text{implies} \qquad f(ax_0) = af(x_0) < 0 \le \rho(ax_0). \end{aligned}$$

The Hahn–Banach Theorem shows that f extends to a linear functional F on X with $F(x) \le \rho(x)$ for all x. Then $F(x_0) \ge 1$, and Lemma 4.17 shows that $\rho(K) \le 1$. Hence

$$F(x_0) \ge 1 \qquad \text{and} \qquad F(M - N + x_0) \le 1.$$

Thus we have $F(M - N + x_0) \le F(x_0)$, $F(M - N) \le 0$, $F(m - n) \le 0$ for all m in M and n in N, and $F(m) \le F(n)$ for all m and n. Taking the supremum over m in M and the infimum over n in N gives the conclusion of the theorem for X real.

Now suppose that the vector space X is complex. We can initially regard X as a real vector space by forgetting about complex scalars, and then the previous case allows us to construct a real-linear F such that $F(M) \le c \le F(N)$. Put $G(x) = F(x) - iF(ix)$. Since $G(ix) = F(ix) - iF(i^2x) = F(ix) - iF(-x) = F(ix) + iF(x) = i(F(x) - iF(ix)) = iG(x)$, G is complex linear. The real part of G equals F, and therefore G satisfies the conclusion of the theorem. $\square$

6. Locally Convex Spaces

In this section we shall apply the discussion of convex sets and linear functionals in the context of topological vector spaces. A topological vector space X is said to be **locally convex** if there is a base for its topology that consists of convex sets.

Let us see that any topological vector space X whose topology is given by a family of seminorms $\|\cdot\|_s$ is locally convex. A base for the open sets consists of all finite intersections of sets $U(y, s, r) = \{x \mid \|x - y\|_s < r\}$ with y in X, s equal to one of the seminorm indices, and $r > 0$. If x and x' are in $U(y, s, r)$ and if $0 \le t \le 1$, then

$$\begin{aligned} \|((1-t)x + tx') - y\|_s &= \|(1-t)(x-y) + t(x'-y)\|_s \\ &\le \|(1-t)(x-y)\|_s + \|t(x'-y)\|_s \\ &= (1-t)\|x-y\|_s + t\|x'-y\|_s \\ &< (1-t)r + tr = r. \end{aligned}$$

Hence $((1-t)x+tx'$ is in $U(y,s,r)$, and $U(y,s,r)$ is convex. Since the arbitrary intersection of convex sets is convex by Proposition 4.16a, every member of the base for the topology is convex. Thus X is locally convex.

We are going to show that every locally convex topological vector space has many continuous linear functionals, enough to distinguish any two disjoint closed convex sets when one of them is compact. This result will in particular be applicable to the spaces $\mathcal{S}(\mathbb{R}^N)$ and $C^\infty(U)$ since their topologies are given by seminorms.

We begin with two lemmas that do not need an assumption of local convexity on the topological vector space.

Lemma 4.19. In any topological vector space if K_1 and K_2 are closed sets with K_1 compact, then the set $K_1 - K_2$ of differences is closed.

PROOF. It is simplest to use nets. Thus let x be a limit point of $K_1 - K_2$, and let $\{x_n\}$ be any net in $K_1 - K_2$ converging to x. Since each x_n is in $K_1 - K_2$, we can write it as $x_n = k_n^{(1)} - k_n^{(2)}$ with $k_n^{(1)}$ in K_1 and $k_n^{(2)}$ in K_2. Since K_1 is compact, $\{k_n^{(1)}\}$ has a convergent subnet, say $\{k_{n_j}^{(1)}\}$. Let $k^{(1)}$ be the limit of $\{k_{n_j}^{(1)}\}$ in K_1. Both $\{x_{n_j}\}$ and $\{k_{n_j}^{(1)}\}$ are convergent, and $\{k_{n_j}^{(2)}\}$ must be convergent because $k_{n_j}^{(2)} = k_{n_j}^{(1)} - x_{n_j}$ and subtraction is continuous. Let k_2 be its limit. This limit has to be in K_2 since K_2 is closed, and then the equation $x = k^{(1)} - k^{(2)}$ exhibits x as in $K_1 - K_2$. Hence $K_1 - K_2$ is closed. $\square$

Lemma 4.20. Let X be any topological vector space, let K_1 and K_2 be disjoint convex sets, and suppose that K_1 has nonempty interior. Then there exists a nonzero continuous linear functional F on X with $\operatorname{Re} F(K_1) \le c$ and $c \le \operatorname{Re} F(K_2)$ for some real number c.

PROOF. The key observation is that any interior point of a subset E of X is internal. In fact, if p is in E^o and x is in X, then $p + \delta x$ is in E^o for $\delta = 0$. By continuity of the vector-space operations and openness of E^o, $p + \delta x$ is in E^o for $|\delta|$ sufficiently small. Therefore p is an internal point.

Since K_1 consequently has an internal point, Theorem 4.18 produces a nonzero linear functional F such that

$$\operatorname{Re} F(K_1) \le c \quad \text{and} \quad c \le \operatorname{Re} F(K_2) \tag{$*$}$$

for some real number c. We complete the proof of the lemma by showing that F is continuous. Let f and g be the real and imaginary parts of F. Then $g(x) = -if(ix)$, and it is enough to show that f is continuous. Fix an interior point p of K_1, and choose an open neighborhood U of 0 such that $p + U \subseteq K_1$. Then

$f(U) \subseteq f(K_1) - f(p)$ since f is real linear, and $(*)$ shows that $f(U) \leq c - f(p)$. So $f(U) \leq a$ for some $a > 0$. If $V = U \cap (-U)$, then

$$f(V) = f(U \cap (-U)) \subseteq f(U) \cap f(-U) = f(U) \cap (-f(U)) \subseteq [-a, a],$$

and therefore $f(\epsilon a^{-1} V) \subseteq [-\epsilon, \epsilon]$. In other words, f is continuous at 0. Then $f(x + \epsilon a^{-1} V) \subseteq f(x) + [-\epsilon, \epsilon]$, and f is continuous everywhere. □

Theorem 4.21. Let X be a locally convex topological vector space, let K_1 and K_2 be disjoint closed convex subsets of X, and suppose that K_1 is compact. Then there exist $\epsilon > 0$, a real constant c, and a continuous linear functional F on X such that

$$\operatorname{Re} F(K_2) \leq c - \epsilon \quad \text{and} \quad c \leq \operatorname{Re} F(K_1).$$

PROOF. Lemma 4.19 shows that $K_1 - K_2$ is closed, and $K_1 - K_2$ does not contain 0 because K_1 and K_2 are disjoint. Since X is locally convex, we can choose a convex open neighborhood U of 0 disjoint from $K_1 - K_2$. Proposition 4.16c shows that $K_1 - K_2$ is convex, and Lemma 4.20 therefore applies to the sets U and $K_1 - K_2$ and yields a nonzero continuous linear functional F such that

$$\operatorname{Re} F(U) \leq d \quad \text{and} \quad d \leq \operatorname{Re} F(K_1 - K_2)$$

for some real d. Since F is not zero, we can find x_0 in X with $F(x_0) = 1$. Choose $\epsilon > 0$ such that $|a| < \epsilon$ implies ax_0 is in U. Then

$$d \geq \operatorname{Re} F(U) \supseteq \operatorname{Re} F(\{ax_0 \mid |a| < \epsilon\}) = (-\epsilon, \epsilon),$$

and hence $d \geq \epsilon$. Therefore all k_1 in K_1 and k_2 in K_2 have

$$\operatorname{Re} F(k_1) - \operatorname{Re} F(k_2) = \operatorname{Re} F(k_1 - k_2) \geq d \geq \epsilon,$$

so that $\operatorname{Re} F(k_1) \geq \epsilon + \operatorname{Re} F(k_2)$. Taking $c = \inf_{k_1 \in K_1} \operatorname{Re} F(k_1)$ now yields the conclusion of the theorem. □

Corollary 4.22. Let X be a locally convex topological vector space, let K be a closed convex subset of X, and let p be a point of X not in K. Then there exists a continuous linear functional F on X such that

$$\sup_{k \in K} \operatorname{Re} F(k) < \operatorname{Re} F(p).$$

PROOF. This is the special case of Theorem 4.21 in which the given compact set is a singleton set. □

Corollary 4.23. If X is a locally convex topological vector space and if p and q are distinct points of X, then there exists a continuous linear functional F on X such that $F(p) \neq F(q)$.

PROOF. This is the special case of Corollary 4.22 in which the given closed convex set is a singleton set. □

We conclude this section with a simple result about locally convex topological vector spaces that we shall need in the next section.

Proposition 4.24. If X is a locally convex topological vector space and Y is a closed vector subspace, then the topological vector space X/Y is locally convex.

REMARK. X/Y is a topological vector space by Proposition 4.4.

PROOF. Let E be an open neighborhood of a given point of X/Y. Without loss of generality, we may take the given point to be the 0 coset. If $q : X \to X/Y$ is the quotient map, $q^{-1}(E)$ is an open neighborhood of 0 in X. Since X is locally convex, there is a convex open neighborhood U of 0 in X with $U \subseteq q^{-1}(E)$. The map q carries open sets to open sets by Proposition 4.4 and carries convex sets to convex sets by Proposition 4.16d, and thus $q(U)$ is an open convex neighborhood of the 0 coset in X/Y contained in E. □

7. Topology on $C^\infty_{\text{com}}(U)$

In this section we carry the discussion of local convexity in Sections 5–6 along the path toward applications to smooth functions. Our objective will be to topologize the space $C^\infty_{\text{com}}(U)$ of smooth functions of compact support on the open set U of $\mathbb{R}^N$. The members of $C^\infty_{\text{com}}(U)$ extend to functions in $C^\infty_{\text{com}}(\mathbb{R}^N)$ by defining them to be 0 outside U, and we often make this identification without special comment.

The important thing about the topology will be what it accomplishes, rather than what the open sets are, and we shall therefore work toward a characterization of the topology, together with an existence proof. The characterization will be in terms of a universal mapping property, and local convexity will be part of that property. Ultimately it is possible to give an explicit description of the open sets, but we leave such a description for Problem 9 at the end of the chapter. The explicit description will show in particular that the topology is given by an uncountable family of seminorms that cannot be reduced to a countable family except when U is empty.

Let us state the universal mapping property informally now, so that the ingredients become clear. Let K be any compact subset of the given open set U of $\mathbb{R}^N$, and define C_K^∞ to be the vector space of all smooth functions of compact support on $\mathbb{R}^N$ with support contained in K. The space C_K^∞ becomes a locally convex topological vector space when we impose the countable family of seminorms $\|f\|_\alpha = \sup_{x\in K} |D^\alpha f(x)|$, with α running over all differentiation multi-indices. Set-theoretically, $C_{\text{com}}^\infty(U)$ is the union of all C_K^∞ as K runs through the compact subsets of U. The topology on $C_{\text{com}}^\infty(U)$ will be arranged so that

(i) every inclusion $C_K^\infty \subseteq C_{\text{com}}^\infty(U)$ is continuous,
(ii) whenever a linear mapping $C_{\text{com}}^\infty(U) \to X$ is given into a locally convex linear topological space X and the composition $C_K^\infty \to C_{\text{com}}^\infty(U) \to X$ is continuous for every K, then the given mapping $C_{\text{com}}^\infty(U) \to X$ is continuous.

It will automatically have the additional property

(iii) every inclusion $C_K^\infty \subseteq C_{\text{com}}^\infty(U)$ is a homeomorphism with its image.

We shall proceed somewhat abstractly, so as to be able to construct the topology of a locally convex topological vector space out of simpler data. If $(X, \mathcal{T})$ is a topological space and p is in X, we define a **local neighborhood base** for $\mathcal{T}$ at p to be a collection $\mathcal{N}_p$ of neighborhoods of p, not necessarily open, such that if V is any open set containing p, then there exists N in $\mathcal{N}_p$ with $N \subseteq V$. If X is a topological vector space with topology $\mathcal{T}$ and if $\mathcal{N}_0$ is a local neighborhood base at 0, then $\{p+N \mid N \in \mathcal{N}_0\}$ is a local neighborhood base at p because translation by x is a homeomorphism. A set is open if and only if it is a neighborhood of each of its points. Consequently we can recover $\mathcal{T}$ from a local neighborhood base $\mathcal{N}_0$ at 0 by this description: a subset V of X is open if and only if for each p in V, there exists N_p in $\mathcal{N}_0$ such that $p + N_p \subseteq V$.

Let us observe two properties of a local neighborhood base $\mathcal{N}_0$ at 0 for a topological vector space X. The first follows from the fact that X is Hausdorff, more particularly that each one-point subset of X is closed. The property is that for each $x \neq 0$ in X, there is some M_x in $\mathcal{N}_0$ with x not in M_x.

The second follows from the fact that 0 is an interior point of each member N of $\mathcal{N}_0$. The property is that 0 is an internal point of N in the sense of Section 5. The fact that interior implies internal was proved in the first paragraph of the proof of Lemma 4.20.

We shall show in Lemma 4.25 that we can arrange in the locally convex case for each member N of a local neighborhood base $\mathcal{N}_0$ at 0 to have the additional property of being **circled** in the sense that $zN \subseteq N$ for all scalars z with $|z| \leq 1$.

Then we shall see in Proposition 4.26 that we can formulate a tidy necessary and sufficient condition for a system of sets containing 0 in a real or complex vector space X to be a local neighborhood base for a topology on X that makes X into a locally convex topological vector space.

Lemma 4.25. Any locally convex topological vector space has a local neighborhood base at 0 consisting of convex circled sets.

PROOF. It is enough to show that if V is an open neighborhood of 0, then there is an open subneighborhood U of 0 that is convex and circled. Since the underlying topological vector space is locally convex, we may assume that V is convex. Replacing V by $V \cap (-V)$, we may assume by parts (a) and (c) of Proposition 4.16 that V is stable under multiplication by -1. Since V is convex, it follows that $cV \subseteq V$ for any real c with $|c| \le 1$. If the field of scalars is $\mathbb{R}$, then the proof of the lemma is complete at this point.

Thus suppose that the field of scalars is $\mathbb{C}$. If V is a convex open neighborhood of 0, put

$$W = \{u \in V \mid zu \in V \text{ for all } z \in \mathbb{C} \text{ with } |z| \le 1\}.$$

Then W is convex by Proposition 4.16a, and it is circled. Let us show that $W \supseteq \frac{1}{2}V \cap \frac{1}{2}iV$. Thus let u be an element of $\frac{1}{2}V \cap \frac{1}{2}iV$, and write it as $u = \frac{1}{2}v_1 = \frac{1}{2}iv_2$ with v_1 and v_2 in V. Let $z \in \mathbb{C}$ be given with $|z| \le 1$, and let x and y be the real and imaginary parts of z. The vectors $\pm v_1$ and 0 are in V, and V is convex; since $|x| \le 1$, xv_1 is in V. Similarly $-yv_2$ is in V. We can write $zu = \frac{1}{2}(x+iy)v_1 = \frac{1}{2}(xv_1) + \frac{1}{2}(-yv_2)$, and this is in V since V is convex. Therefore zu is in V, and u is in U. Hence $W \supseteq \frac{1}{2}V \cap \frac{1}{2}iV$, as asserted.

Let U be the interior W^o of W. Then U is an open neighborhood of 0, and we show that it is convex and circled; this will complete the proof. Let u and v be in U. Since U is open, we can find an open neighborhood N of 0 such that $u + N \subseteq U$ and $v + N \subseteq U$. If n is in N and if t satisfies $0 \le t \le 1$, then $(1-t)u + tv + n = (1-t)(u+n) + t(v+n)$ exhibits $(1-t)u + tv + n$ as a convex combination of a member of $u + N \subseteq W$ and a member of $v + N \subseteq W$, hence as a member of W. Therefore every member of $(1-t)u + tv + N$ lies in W, and U is convex.

To see that U is circled, let u and N be as in the previous paragraph with $u + N \subseteq U$. If $|z| \le 1$, then $u + N \subseteq W$ implies $z(u+N) \subseteq W$ since W is circled. Hence $zu + zN \subseteq W$. Since zN is open, $zu + zN$ is an open neighborhood of zu contained in W, and we must have $zu + zN \subseteq W^o = U$. Therefore U is circled. $\square$

Proposition 4.26. Let X be a real or complex vector space. If X has a topology making it into a locally convex topological vector space, then X has a local neighborhood base $\mathcal{N}_0$ at 0 for that topology such that

(a) each N in $\mathcal{N}_0$ is convex and circled with 0 as an internal point,
(b) whenever M and N are in $\mathcal{N}_0$, there is some P in $\mathcal{N}_0$ with $P \subseteq M \cap N$,
(c) whenever N is in $\mathcal{N}_0$ and a is a nonzero scalar, then aN is in $\mathcal{N}_0$,
(d) each $x \neq 0$ in X has some associated M_x in $\mathcal{N}_0$ such that x is not in M_x.

Conversely if $\mathcal{N}_0$ is any family of subsets of the vector space X such that (a), (b), (c), and (d) hold, then there exists one and only one topology on X making X into a locally convex topological vector space in such a way that $\mathcal{N}_0$ is a local neighborhood base at 0.

PROOF. For the direct part of the proof, Lemma 4.25 shows that there is some local neighborhood base at 0 consisting of convex circled sets. To such a local neighborhood base we are free to add any additional neighborhoods of 0. Thus we may take $\mathcal{N}_0$ to consist of *all* convex circled neighborhoods of 0. Then (b) and (c) hold, and (d) holds since the topology is Hausdorff. Since 0 is an internal point of any neighborhood of 0, (a) holds. This proves existence.

For the converse there is only one possibility for the topology $\mathcal{T}$: V is open if for each x in V, there is some N_x in $\mathcal{N}_0$ with $x + N_x \subseteq V$. This proves the uniqueness of $\mathcal{T}$, and we are to prove existence. For existence we define open sets in this way and define $\mathcal{T}$ to be the collection of all open sets. The definition makes $\varnothing$ open and the arbitrary union of open sets open, and (b) makes the intersection of two open sets open.

We shall show that the complement of any $\{x_0\}$ is open. Then it follows by taking unions that X is open, so that $\mathcal{T}$ is a topology; also we will have proved that every one-point set is closed. If $x_1 \neq x_0$, we use (d) to choose $M_{x_0-x_1}$ in $\mathcal{N}_0$ with $x_0 - x_1$ not in $M_{x_0-x_1}$. Then $x_1 + M_{x_0-x_1} \subseteq X - \{x_0\}$. Since x_1 is arbitrary, $X - \{x_0\}$ is open.

With $\mathcal{T}$ established as a topology, let us see that every member of $\mathcal{N}_0$ is a neighborhood of 0. This step involves considering the family of sets aN for fixed N in $\mathcal{N}_0$ and for arbitrary positive a. If $0 < t < 1$ and if n_1 and n_2 are in N, then $(1-t)n_1 + tn_2$ is in N since (a) says that N is convex. Hence $(1-t)N + tN \subseteq N$. If $a > 0$ and $b > 0$, then we can take $t = b(a+b)^{-1}$ and obtain $a(a+b)^{-1}N + b(a+b)^{-1}N \subseteq N$. Multiplying by $a + b$ gives

$$aN + bN \subseteq (a+b)N \qquad \text{for all positive } a \text{ and } b. \tag{$*$}$$

In particular the sets aN are nested for $a > 0$, i.e., $0 < a < a'$ implies $aN \subseteq a'N$.

From these facts we can show that each N in $\mathcal{N}_0$ is a neighborhood of 0. Given N, define $U = \bigcup_{0<a<1} aN$. This is a subset of N by the nesting property, and we shall prove that it is open. If x is in U, then x is in aN for some a with $0 < a < 1$, and $(*)$ shows that $x + \frac{1}{2}(1-a)N \subseteq U$. By (c), $\frac{1}{2}(1-a)N$ is in $\mathcal{N}_0$, and therefore $\frac{1}{2}(1-a)N$ can serve as a member N_x of $\mathcal{N}_0$ such that $x + N_x \subseteq U$. We conclude that U is open. Therefore N is a neighborhood of 0.

Next let us see that translations are homeomorphisms. If V is open and if x_0 is given, we know that each x in V has an associated N_x such that $x + N_x \subseteq V$. If y is in $x_0 + V$, then $x = y - x_0$ is in V and we see that $(y - x_0) + N_{y-x_0} \subseteq V$ and $y + N_{y-x_0} \subseteq x_0 + V$. Hence $x_0 + V$ is open, and every translation is a homeomorphism.

Let us see that addition is continuous at $(0,0)$, and then the fact that translations are homeomorphisms implies that addition is continuous everywhere. If V is an open neighborhood of 0, then the definition of open set says that there is some N in $\mathcal{N}_0$ with $0+N \subseteq V$. By (c), $\frac{1}{2}N$ is in $\mathcal{N}_0$. It is enough to prove that $(\frac{1}{2}N, \frac{1}{2}N)$ maps into V under addition. But this is immediate from $(*)$ since $\frac{1}{2}N + \frac{1}{2}N \subseteq N \subseteq V$.

Next we investigate continuity of the mapping $x \mapsto ax$ for $a \neq 0$. It is enough to show that if V is open, then so is $a^{-1}V$. Since V is open, every x in V has an associated N_x in $\mathcal{N}_0$ such that $x + N_x \subseteq V$. The most general element of $a^{-1}V$ is of the form $a^{-1}x$ with x in V, and we have $a^{-1}x + a^{-1}N_x \subseteq a^{-1}V$. Since (c) shows $a^{-1}N_x$ to be in $\mathcal{N}_0$, we conclude that $a^{-1}V$ is open.

Let us see that scalar multiplication is continuous at $(1,x)$, and then the fact that $x \mapsto ax$ is continuous for $a \neq 0$ implies that scalar multiplication is continuous everywhere except possibly at $(0,x)$. Let V be an open neighborhood of x, and choose N in $\mathcal{N}_0$ with $x+N \subseteq V$. Since N is in $\mathcal{N}_0$, (c) shows that $\frac{1}{3}N$ is in $\mathcal{N}_0$. Then 0 is an internal point of $\frac{1}{3}N$ by (a), and there exists $\epsilon > 0$ such that $-\epsilon \le c \le \epsilon$ implies that cx is in $\frac{1}{3}N$. There is no loss of generality in taking $\epsilon < 1$. Since $\frac{1}{3}N$ is circled by (a), cx is in $\frac{1}{3}N$ for $|c| \le \epsilon$. Let A be the set of scalars with $|a-1| < \epsilon$. We show that scalar multiplication carries $A \times (x + \frac{1}{3}N)$ into V. In fact, if a is in A and $\frac{1}{3}n_1$ is in $\frac{1}{3}N$, then $|a| < 2$, $\frac{1}{3}an_1$ is in $\frac{2}{3}N$, and $(*)$ gives

$$a(x + \tfrac{1}{3}n_1) = (ax - x) + (x + \tfrac{1}{3}an_1) \in \tfrac{1}{3}N + (x + \tfrac{2}{3}N) \subseteq x + N \subseteq V.$$

To complete the proof of continuity of scalar multiplication, we show continuity at all points $(0,x)$. Let V be an open neighborhood of 0 in X, and choose N in $\mathcal{N}_0$ with $0+N \subseteq V$. Since 0 is an internal point of N, there is some $\epsilon > 0$ such that cx is in N for real c with $|c| \le \epsilon$. For this ϵ, $\frac{1}{2}\epsilon x$ is in $\frac{1}{2}N$. If $|z| < 1$ and y is in $\frac{1}{2}N$, then $(z, \frac{1}{2}\epsilon x + y)$ maps to $\frac{1}{2}z\epsilon x + zy$, which lies in $\frac{1}{2}N + \frac{1}{2}N$ since N is circled. In turn, this is contained in N by $(*)$ and therefore is contained in V. So $(\frac{1}{2}\epsilon z, x + 2\epsilon^{-1}y)$ maps into V if $|z| < 1$ and y is in $\frac{1}{2}N$. Altering the definitions of z and y, we conclude that $(z, x+y)$ maps into V if $|z| < \frac{1}{2}\epsilon$ and y is in $\epsilon^{-1}N$. This proves the continuity.

Since $\{0\}$ is a closed set, Lemma 4.3 is applicable and shows that X is Hausdorff, hence is a topological vector space. Inside any open neighborhood V of 0 lies some set N in $\mathcal{U}_0$, and $\bigcup_{0<a<1} aN$ is a convex open subneighborhood of V. Therefore the topology is locally convex. $\square$

We are almost in a position to topologize $C^\infty_{\mathrm{com}}(U)$. If i_K denotes the inclusion of C^∞_K into $C^\infty_{\mathrm{com}}(U)$, we shall define a convex circled subset N in $C^\infty_{\mathrm{com}}(U)$

having 0 as an internal point to be in a local neighborhood base at 0 if $i_K^{-1}(N)$ is a neighborhood of 0 in C_K^∞ for every compact subset K of U. Then conditions (a), (b), and (c) in Proposition 4.26 will be met, and an examination of the proof of that proposition shows that we obtain a topology for $C^\infty_{\text{com}}(U)$ in which addition and scalar multiplication are continuous. What is lacking is the Hausdorff property, which follows once (d) holds in Proposition 4.26. Verifying (d) requires a construction, whose main step is given in the following lemma.

Lemma 4.27. Let X be a locally convex topological vector space, let Y be a closed vector subspace, and let Y be given the relative topology, which is locally convex. If N is a convex circled neighborhood of 0 in Y and x_0 is a point in X not in N, then there exists a convex circled neighborhood M of 0 in X such that $M \cap Y = N$ and such that x_0 is not in M.

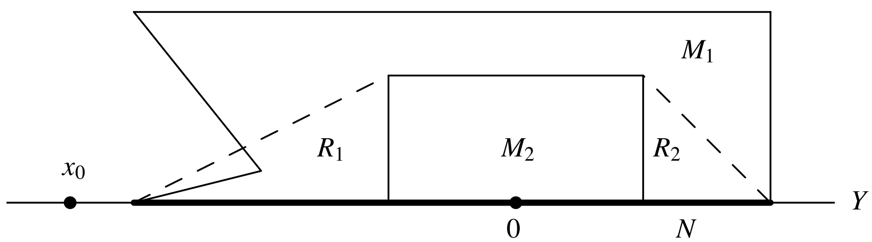

FIGURE 4.1. Extension of convex circled neighborhood of 0. The lemma extends N to the set given in the figure by $M_3 = R_1 \cup M_2 \cup R_2$.

PROOF. Since N is a neighborhood of 0 in Y and since Y has the relative topology, there exists a neighborhood M_1 of 0 in X such that $M_1 \cap Y = U$. We shall adjust M_1 to make it convex circled and to arrange that x_0 is not in it. Since X is locally convex, we can find a convex circled neighborhood M_2 of 0 contained in M_1. Taking a cue from Figure 4.1, define

$$M_3 = \{(1-t)n + tm_2 \mid n \in N,\ m_2 \in M_2,\ 0 \le t \le 1\}.$$

This is a neighborhood of 0 since it contains M_2, and it is convex circled since N and M_2 are convex circled.

We shall prove that

$$M_3 \cap Y = N.$$

Certainly $M_3 \cap Y \supseteq N$. For the reverse inclusion let m_3 be in $M_3 \cap Y$, and write $m_3 = (1-t)n + tm_2$ with $n \in N$, $m_2 \in M_2$, and $0 \le t \le 1$. If $t = 0$, then $m_3 = n$ is already in N. If $t > 0$, then $m_2 = t^{-1}(m_3 - (1-t)n)$ exhibits m_2 as a linear combination of members of Y, hence as a member of Y. Since $M_2 \subseteq M_1$, m_2 is in $M_1 \cap Y = N$. Therefore m_3 is a convex combination of the members n and m_2 of N and must lie in N since N is convex. Consequently $M_3 \cap Y = N$.

If x_0 lies in Y, then we can take $M = M_3$ since x_0 is by assumption not in N and cannot therefore be in the larger set M_3. If x_0 is not in Y, then Proposition 4.24 says that X/Y is a locally convex topological vector space. Since $x_0 + Y$ is not the 0 coset, we can find a convex circled neighborhood P of the 0 coset that does not contain $x_0 + Y$. If $q : X \to X/Y$ is the quotient map, then $q^{-1}(P)$ by Proposition 4.16e is a convex circled neighborhood of 0 in X that does not contain x_0 and satisfies $q^{-1}(P) \cap Y = Y$. Therefore $M = M_3 \cap q^{-1}(P)$ is a convex circled neighborhood of 0 in X that does not contain x_0 and satisfies $M \cap Y = N$. $\square$

Proposition 4.28. Let X be a real or complex vector space, and suppose that X is the increasing union $X = \bigcup_{p=1}^\infty X_p$ of a sequence of locally convex topological vector spaces such that for each p, X_p is a closed vector subspace of X_{p+1} and has the relative topology. Then there exists a unique topology on X making it into a locally convex topological vector space in such a way that

(a) each inclusion $i_p : X_p \to X$ is continuous,
(b) whenever $L : X \to Y$ is a linear function from X into a locally convex topological vector space Y such that $L \circ i_p : X_p \to X$ is continuous for all p, then L is continuous.

This unique topology has the property that

(c) each inclusion $i_p : X_p \to X$ is a homeomorphism with its image.

PROOF. Let $\mathcal{N}_0$ be the family of all convex circled subsets N of X having 0 as an internal point such that $i_p^{-1}(N)$ is a neighborhood of 0 in X_p for all p. We shall prove that $\mathcal{N}_0$ satisfies the four conditions (a) through (d) of Proposition 4.26, so that X has a unique topology making it into a locally convex topological vector space in such a way that $\mathcal{N}_0$ is a local neighborhood base at 0. Condition (a) holds by definition. Condition (b) holds because the intersection of two convex circled subsets with 0 as an internal point is again a convex circled set with 0 as an internal point and because the intersection of two neighborhoods is a neighborhood. Condition (c) holds because multiplication by a nonzero scalar sends convex circled sets with 0 as an internal point into convex circled sets with 0 as an internal point and because multiplication by a nonzero scalar sends neighborhoods of 0 to neighborhoods of 0.

We have to prove (d) in Proposition 4.26, namely that each $x_0 \neq 0$ in X has some associated M in $\mathcal{N}_0$ such that x_0 is not in M. Since $X = \bigcup_{p=1}^\infty X_p$, choose p_0 as small as possible so that x_0 is in X_{p_0}. Since X_{p_0} satisfies (a) through (d) and since $x_0 \neq 0$, we can find some convex circled neighborhood M_{p_0} of 0 in X_{p_0} that does not contain x_0. Proceeding inductively by means of Lemma 4.27, we can find, for each $p > p_0$, a convex circled neighborhood M_p of 0 in X_p that does not contain x_0 such that $M_p \cap X_{p-1} = M_{p-1}$. Define $M = \bigcup_{p \geq p_0} M_p$. Then M is convex circled since each M_p has this property. To see that 0 is an internal point of M, we argue as follows: for each x in X, x lies in some X_p, the set M_p has 0 as an internal point since M_p is a neighborhood of 0, M_p contains all cx for c real and small, and the larger set M contains all cx for c real and small. For each $p \geq p_0$, the set $i_p^{-1}(M)$ equals M_p, which was constructed as a neighborhood of 0 in X_p. The intersection $i_k^{-1}(M) = M_p \cap X_k$ has to be a neighborhood of 0 in X_k for $k < p$ since M_p is a neighborhood of 0 in X_p, and the set M is therefore in $\mathcal{N}_0$. Thus M meets the requirement of being a member of $\mathcal{N}_0$ that does not contain x_0, and (d) holds in Proposition 4.26.

We are left with proving (a) through (c) in the present proposition and with proving that no other topology meets these conditions. For (a), since i_p is linear, it is enough to prove continuity at 0. Hence we are to see that if N is in $\mathcal{N}_0$, then $i_p^{-1}(N)$ is a neighborhood of 0 in X_p. But this is just one of the defining conditions for the set N to be in $\mathcal{N}_0$.

For (b), since L is linear, it is enough to prove continuity at 0. Since Y is locally convex, the convex circled neighborhoods of 0 in Y form a local neighborhood base. If E is such a neighborhood, we are to show that $N = L^{-1}(E)$ is a neighborhood of 0 in X. The set E is convex and circled with 0 as an internal point, and hence the same thing is true of N. Also, $i_p^{-1}(N) = i_p^{-1}L^{-1}(E) = (L \circ i_p)^{-1}(E)$ is a neighborhood of 0 in X_p since $L \circ i_p$ is by assumption continuous. Therefore $N = L^{-1}(E)$ is in $\mathcal{N}_0$, and then $L^{-1}(E)$ is a neighborhood of 0 in the topology imposed on X. Hence L is continuous at 0 and is continuous.

For (c), we again use Lemma 4.27, except that this time we do not need a point x_0. We are to show that if N_{p_0} is a neighborhood of 0 in X_{p_0}, then $i(N_{p_0})$ is a neighborhood of 0 in the relative topology that X defines on X_{p_0}. Since X_{p_0} is locally convex, there is no loss of generality in assuming that N_{p_0} is convex circled. Proceeding inductively for $p > p_0$, we use the lemma to construct a convex circled neighborhood N_p of 0 in X_p such that $N_p \cap X_{p-1} = N_{p-1}$. Put $N = \bigcup_{p \geq p_0} N_p$. Arguing in the same way as earlier in the proof, we see that N is in $\mathcal{N}_0$. Then $i(N_{p_0}) = X_{p_0} \cap N$, and $i(N_{p_0})$ is exhibited as the intersection of X_{p_0} with a neighborhood of 0 in X. This proves (c).

Finally suppose that the constructed topology on X is $\mathcal{T}$ and that $\mathcal{T}'$ is a second topology making X into a locally convex topological vector space in such a way that (a) and (b) hold. Let $1_{\mathcal{T}}$ be the identity map from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$. By (a) for $\mathcal{T}'$, the composition $1_{\mathcal{T}} \circ i_p : X_p \to X$ is continuous. By (b) for $\mathcal{T}$, $1_{\mathcal{T}}$ is continuous from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$. Reversing the roles of $\mathcal{T}$ and $\mathcal{T}'$, we see that the identity map is continuous from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$. Therefore $1_{\mathcal{T}}$ is a homeomorphism. $\square$

In the terminology of abstract functional analysis, one says that X in Proposition 4.28 is a **strict inductive limit**[16] of the spaces X_p. With extra hypotheses that are satisfied in our case of interest, one says that X acquires the LF **topology**[17] from the X_p's.

Now let us apply the abstract theory to $C^\infty_{\text{com}}(U)$. If $\{K_p\}$ is any exhausting sequence of compact subsets of U, then we apply Proposition 4.28 with $X = C^\infty_{\text{com}}(U)$ and $X_p = C^\infty_{K_p}$. For the inclusion $X_p \subseteq X_{p+1}$, the restriction to $C^\infty_{K_p}$ of the seminorms on $C^\infty_{K_{p+1}}$ yields the seminorms for $C^\infty_{K_p}$, and therefore X_p has the relative topology as a vector subspace of X_{p+1}. The space X_p is a closed subspace because $C^\infty_{K_p}$ is Cauchy complete and because complete subsets of a metric space are closed. Thus the hypotheses are satisfied, and $C^\infty_{\text{com}}(U)$ acquires a unique topology as a locally convex topological vector space such that

(i) each inclusion $C^\infty_{K_p} \subseteq C^\infty_{\text{com}}(U)$ is continuous,

(ii) whenever a linear mapping $C^\infty_{\text{com}}(U) \to X$ is given into a locally convex linear topological space X and the composition $C^\infty_{K_p} \to C^\infty_{\text{com}}(U) \to X$ is continuous for every p, then the given mapping $C^\infty_{\text{com}}(U) \to X$ is continuous.

Furthermore

(iii) each inclusion $C^\infty_{K_p} \subseteq C^\infty_{\text{com}}(U)$ is a homeomorphism with its image.

To complete our construction, all we have to do is show that the resulting topology on $C^\infty_{\text{com}}(U)$ does not depend on the choice of exhausting sequence.

Proposition 4.29. The inductive limit topology on $C^\infty_{\text{com}}(U)$ is independent of the choice of exhausting sequence. Consequently

(a) each inclusion $C^\infty_K \subseteq C^\infty_{\text{com}}(U)$ is a homeomorphism with its image,

(b) whenever a linear mapping $C^\infty_{\text{com}}(U) \to X$ is given into a locally convex linear topological space X and the composition $C^\infty_K \to C^\infty_{\text{com}}(U) \to X$ is continuous for every compact subset K of U, then the given mapping $C^\infty_{\text{com}}(U) \to X$ is continuous.

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[16]The words "direct limit" mean the same thing as "inductive limit," but "inductive" is more common in this situation. The term "strict" refers to the fact that the successive inclusions $i_{p+1,p} : X_p \to X_{p+1}$ are one-one with $i_{p+1,p}(X_p)$ homeomorphic to X_p. The notion of "direct limit" is a construction in category theory that is useful within several different categories. Uniqueness of the direct limit up to canonical isomorphism is a formality built into the definition; existence depends on the particular category. For this situation the construction is taking place within the category of locally convex topological vector spaces (and continuous linear maps). A direct-limit construction within a different category plays a role in Problems 26–30 at the end of the chapter, and those problems are continued at the end of Chapter VI.

[17]"LF" refers to "Fréchet limit." In the usual situation the spaces X_p are assumed to be locally convex complete metric topological vector spaces, i.e., "Fréchet spaces." The X_p's have this property in the application to $C^\infty_{\text{com}}(U)$.

PROOF. Write X for $C^\infty_{\text{com}}(U)$ with its topology defined relative to an exhausting sequence $\{K_p\}$ of compact subsets of U, and write Y for $C^\infty_{\text{com}}(U)$ with its topology defined relative to an exhausting sequence $\{K'_p\}$. If K_k is a member of the sequence $\{K_p\}$, then $K_k \subseteq K'_p$ for $p \geq$ some index p_0 depending on k since the interiors of the sets K'_p cover the compact set K_k. The inclusion $K_k \subseteq K'_p$ is continuous for $p \geq p_0$, and therefore the composition $K_k \to K'_{p_0} \to Y$ is continuous. This continuity for all k implies that the identity map from X into Y is continuous. Reversing the roles of X and Y, we see that the identity map is a homeomorphism. $\square$

8. Krein–Milman Theorem

In this section we carry the discussion of local convexity in Sections 5–6 along the path toward fixed-point theorems. Our objective will be to prove a fundamental existence theorem about "extreme points."

If K is a convex set in a real or complex vector space and if x_0 is in K, we say that x_0 is an **extreme point** of K if x_0 is not in the interior of any **line segment** belonging to K, i.e., if

$$x_0 = (1-t)x + ty \text{ with } 0 < t < 1 \text{ and } x, y \in K \qquad \text{implies} \qquad x_0 = x = y.$$

Let X be a topological vector space, and let K be a closed convex subset of X. A nonempty closed convex subset S of K is called a **face** if whenever ℓ is a line segment belonging to K, in the above sense, and ℓ has an interior point in S, then the whole line segment belongs to S. With this definition, x_0 is an extreme point of K if and only if the singleton set $\{x_0\}$ is a face.

If E is a subset of X, then the **closed convex hull** of E is defined to be the intersection of all closed convex subsets of X that contain E. It may be described explicitly as the closure of the set of all convex combinations of members of E.

Theorem 4.30 (Krein–Milman Theorem). If K is a compact convex set in a locally convex topological vector space, then K is the closed convex hull of the set of extreme points of K. In particular, if K is nonempty, then K has an extreme point.

PROOF. Let X be the underlying topological vector space. We may assume, without loss of generality, that K is nonempty. Let us see that if f is any continuous linear functional on X, then the subset of K on which $\operatorname{Re} f$ assumes its maximum value is a face. In fact, let S be the subset where $g = \operatorname{Re} f$ assumes its maximum value m. Then S is nonempty since K is compact and g is continuous, and the continuity and real linearity of g imply that S is closed and convex. To check that S is a face, let x_0 be in S, and suppose that $x_0 = (1-t)x + ty$ with $0 < t < 1$ and x, y in K. Then

$$m = g(x_0) = (1-t)g(x) + tg(y) \le m(1-t) + tm = m.$$

Equality must hold throughout, and therefore $g(x) = m = g(y)$. Hence x and y are in S, and S is a face.

Next let us see that any face of K contains an extreme point. In fact, order the faces by inclusion downward. The intersection of a chain of faces is nonempty by compactness and hence is a face that provides a lower bound for the chain. By Zorn's Lemma there exists a minimal face S_1. Arguing by contradiction, suppose that S_1 contains at least two points. Then Corollary 4.23 and the local convexity of X yield a continuous linear functional whose real part takes distinct values at the two points. From the previous paragraph we find that S_1 contains a proper face S. A face of a face is a face. Thus S is a face of K strictly smaller than the minimal face S_1, and we arrive at a contradiction.

Now we can complete the proof. If E denotes the closed convex hull of the set of extreme points of K, then certainly $E \subseteq K$. Arguing by contradiction, suppose that equality fails: Let x_0 be in K but not in E. Then Corollary 4.22 and the local convexity of X produce a continuous linear functional whose real part has supremum on E strictly less than the value at x_0. The first paragraph of the proof shows that the subset of K where the real part of this linear functional takes the value at x_0 is a face of K, and the second paragraph shows that this face has an extreme point. This extreme point is not in E, and we arrive at a contradiction. □

Compact convex subsets of $\mathbb{R}^N$ arise in practical maximum-minimum problems involving several variables, typically economic variables. Often the compact convex set is a polyhedron, and the function to be maximized is the sum of a constant and a linear function. The Krein–Milman Theorem produces extreme points, and the basic techniques of the subject of linear programming show that the maximum is attained at an extreme point and show how to find this extreme point.

A natural place where infinite-dimensional compact convex sets arise is in the weak-star topology on the closed unit ball of the dual of a normed linear space. Alaoglu's Theorem says that this set is compact, and it is certainly convex. The Hahn–Banach Theorem is what shows that this compact convex set contains a nonzero element when the normed linear space is nonzero.

When the whole closed unit ball is the set of interest, let us see what the extreme points are like in certain situations. If the underlying normed linear space is a Hilbert space, then the real part of a continuous linear functional takes its maximum value at a single point of the closed unit ball. The upshot of this fact is that the proof of the Krein–Milman Theorem above degenerates; Zorn's

Lemma is not needed, for example, to produce an extreme point. The proof degenerates in the same way, in fact, whenever one considers some L^p space with $1 < p < \infty$.

The case of L^∞ is more interesting. Let us work with real-valued functions in the context of a σ-finite measure space, regarding L^∞ as the dual of L^1. The extreme points of the closed unit ball are all the L^∞ functions that take only the values -1 and $+1$.

Similarly we can consider the space $C([0,1])$ of continuous functions on $[0,1]$. Again let us work with real-valued functions. Suppose that this Banach space is the dual of some normed linear space. Then the closed unit ball of $C([0,1])$ forms a compact convex set in the weak-star topology. As with L^∞, the extreme points are the functions that take only the values -1 and $+1$. The functions have to be continuous, however, and they are therefore constant. So we get only two extreme points, the constant functions -1 and $+1$, and their closed convex hull contains only constant functions. The conclusion is that $C([0,1])$ is not the dual of any normed linear space.

We can argue similarly with measures and L^1 functions. Suppose that X is a compact Hausdorff space. The Banach space $M(X)$ of regular complex Borel measures on X is the dual of $C(X)$, and the set of nonnegative Borel measures of total mass ≤ 1 is a closed compact subset of the unit ball in the weak-star topology. This set has to be the closed convex hull of its extreme points. Indeed, as is pointed out in Problem 17 at the end of the chapter, the extreme points of this set are 0 and the point masses of mass 1 at the points of X; the statement of the theorem is reflected in the fact that any regular Borel measure on X with total mass ≤ 1 is a weak-star limit of linear combinations of point masses.

We can consider similarly the space $L^1([0,1])$ of Borel functions on $[0,1]$ integrable with respect to Lebesgue measure. Suppose that this Banach space is the dual of some normed linear space. Then the closed unit ball of $L^1([0,1])$ forms a compact convex set in the weak-star topology. Problem 18 at the end of the chapter shows that the extreme points are trying to be the functions whose mass is concentrated at a single point, and there are none. The conclusion is that $L^1([0,1])$ is not the dual of any normed linear space.

The Krein–Milman Theorem begins to show its power when applied to more subtle closed convex subsets of a unit ball in the weak-star topology. Here is an example that lies behind the foundations of the theory of locally compact abelian groups.[18] For concreteness we work with complex-valued functions on the integers, i.e., doubly infinite sequences. Such a function $f(n)$ is said to be **positive definite** if $\sum_{j,k} c(j)f(j-k)\overline{c(k)} \ge 0$ for all functions $c(n)$ on the integers with finite support. Positive definite functions are easily checked to

[18] Such groups are defined in Chapter VI.

have $f(0) \geq 0$ and $|f(n)| \leq f(0)$. In particular, the set K of positive definite functions f with $f(0) = 1$ may be regarded as a subset of the closed unit ball of L^∞ of the integers with the counting measure, a space sometimes called ℓ^∞. Weak-star convergence for such functions is the same as pointwise convergence, and it follows that K is closed, hence compact. Checking the definition, we see that K is convex. The Krein–Milman Theorem tells us that K is the closed convex hull of its extreme points. It is shown in Problem 20 at the end of the chapter that the extreme points are the functions $f_\theta(n) = e^{in\theta}$ for real θ.

By way of introduction to the next section, let us consider one more example. Let S be a compact Hausdorff space, and let F be any homeomorphism of S. Put $X = C(S)$. In the weak-star topology on $M(S)$, the nonnegative regular Borel measures μ with $\mu(S) = 1$ form a compact convex subset K_1 of $M(S)$. The Markov–Kakutani Theorem in the next section shows that there exist elements of K_1 invariant under F. The invariant such measures therefore form a nonempty compact convex subset K of K_1. According to the Krein–Milman Theorem, K is the closed convex hull of its set of extreme points. As shown in Problem 19 at the end of the chapter, the μ's that are extreme points have the interesting property that all Borel subsets that are carried onto themselves by the homeomorphism F have measure 0 or 1; the usual name for this phenomenon is that μ is **ergodic** with respect to F. Since the Krein–Milman Theorem is saying that extreme points exist, we obtain the consequence that for each homeomorphism F of S, there is some regular Borel measure μ with $\mu(S) = 1$ that is ergodic with respect to F.

9. Fixed-Point Theorems

In this section we continue the discussion of convexity and local convexity. We shall give two fixed-point theorems.

Theorem 4.31 (Markov–Kakutani Theorem). Let K be a compact convex set in a topological vector space X, and let $\mathcal{F}$ be a commuting family of continuous linear mappings carrying K into itself. Then there exists a point p in K such that $T(p) = p$ for all T in $\mathcal{F}$.

PROOF. For each integer $n \geq 1$ and member T of $\mathcal{F}$, let

$$T_n = \tfrac{1}{n}(I + T + T^2 + \cdots + T^{n-1}).$$

Let $\mathcal{K}$ be the family of all subsets of X that arise as $T_n(K)$ for some $n \geq 1$ and some T in $\mathcal{F}$. Each such set is a compact convex subset of K, being the image of a compact convex set under a continuous linear mapping that carries K into itself. If $\{T^{(i)}_{n_i}\}_{i=1}^r$ is a finite subset of $\mathcal{F}$ and each n_i is ≥ 1, then

$$T^{(1)}_{n_1} T^{(2)}_{n_2} \cdots T^{(r)}_{n_r}(K) \subseteq T^{(1)}_{n_1} T^{(2)}_{n_2} \cdots T^{(r-1)}_{n_{r-1}}(K) \subseteq \cdots \subseteq T^{(1)}_{n_1}(K).$$

By symmetry and commutativity of the operators,

$$T_{n_1}^{(1)} T_{n_2}^{(2)} \cdots T_{n_r}^{(r)}(K) \subseteq \bigcap_{j=1}^{r} T_{n_j}^{(j)}(K).$$

Thus the members of $\mathcal{K}$ have the finite-intersection property. By compactness their intersection is nonempty. Let p be in the intersection. We shall show that $T(p) = p$ for all T in $\mathcal{F}$.

Arguing by contradiction, suppose that T is given in $\mathcal{F}$ with $T(p) \neq p$. Choose a neighborhood U of 0 in X such that $T(p) - p$ is not in U. The fact that p is in the intersection of all the sets in $\mathcal{K}$ implies that p is in $T_n(K)$ for $n \geq 1$ and thus

$$p = n^{-1}(I + T + T^2 + \cdots + T^{n-1})(q_n)$$

for some q_n in K. Applying $T - I$ to this equality, we obtain

$$T(p) - p = n^{-1}(T^n - I)(q_n).$$

Since the left side is not in U, the right side is not in U. Since $T^n(q_n)$ and q_n are in K, it follows that $\frac{1}{n}(K - K)$ is not contained in U for any n. But $K - K$ is a compact set, being the image under the subtraction mapping of the compact set $K \times K$, and this conclusion contradicts Lemma 4.7. $\square$

Let us return to the example at the end of the previous section. As in that example, let S be a compact Hausdorff space, and let F be any homeomorphism of S. Put $X = C(S)$. In the weak-star topology on $M(S)$, the nonnegative regular Borel measures μ with $\mu(S) = 1$ form a compact convex subset K_1 of $M(S)$. The homeomorphism F acts on $M(S)$ by the formula $T_F(\rho)(E) = \rho(F^{-1}(E))$. The mapping T_F is linear, and it follows from the definitions that T_F satisfies $\|T_F(\rho)\|_{M(S)} = \|\rho\|_{M(S)}$. Thus T_F has norm 1 and is continuous. It maps K_1 into itself. Putting $\mathcal{F} = \{T_F\}$ and applying Theorem 4.31, we obtain the existence of a nonzero F invariant measure on S. The discussion in the previous section went on to observe that the subset K of F invariant measures in K_1, which we now know to be nonempty, is compact convex in a locally convex topological vector space. Thus K is a set to which we can apply the Krein–Milman Theorem, and the extreme points turn out to be the ergodic invariant measures.

Theorem 4.32 (Schauder–Tychonoff Theorem). Let K be a compact convex set in a locally convex topological vector space, and let F be a continuous function from K into itself. Then there exists p in K with $F(p) = p$.

The proof of Theorem 4.32 is long and will be omitted.[19] The power in the result comes from its applicability to nonlinear mappings. In the special case in which K is the closed unit ball in $\mathbb{R}^N$, it reduces to the celebrated Brouwer Fixed-Point Theorem.

This kind of theorem has applications to economics, where fixed-point theorems prove the existence of equilibrium points for certain systems. The theorem does not by itself address stability of such an equilibrium point, however.

By way of illustration, let us return to a comparatively simple situation that was studied in Chapter IV of *Basic*. The usual Picard–Lindelöf Existence Theorem[20] for the initial-value problem with a system $y' = f(t, y)$ of ordinary differential equations assumes continuity of f and also a Lipschitz condition for f in the y variable. A variant, the Cauchy–Peano Existence Theorem, is the subject of problems at the end of Chapter IV of *Basic*. It assumes only continuity for f and obtains existence of solutions, with uniqueness being lost. The Cauchy–Peano result is proved using Ascoli's Theorem and a nonobvious construction.

Ascoli's Theorem, as we know from Section X.9 of *Basic*, is intimately connected with compactness. Let us see how to combine Ascoli's Theorem and the Schauder–Tychonoff Theorem to obtain a more transparent proof of the Cauchy–Peano result than was suggested in the problems at the end of Chapter IV of *Basic*. To keep the notation simple, we stick with the case of a single equation, rather than a system. We suppose that $f(t, y)$ is continuous on an open subset D of $\mathbb{R}^2$. Let (t_0, y_0) be in D, and let R be a closed rectangle in D centered at (t_0, y_0) and having the form

$$R = \big\{(t, y) \bigm| |t - t_0| \le a \text{ and } |y - y_0| \le b\big\}.$$

Suppose that $|f(t, y)| \le M$ on R. Put $a' = \min\{a, \frac{b}{M}\}$. The theorem is that there exists a continuously differentiable solution $y(t)$ to the initial-value problem $y' = f(t, y)$, $y(t_0) = y_0$, $|t - t_0| < a'$.

For the proof let X be the Banach space $C(\{t \bigm| |t - t_0| \le a'\})$, and let K be the closure of the set

$$E = \left\{ y \in X \;\middle|\; \begin{array}{c} \text{(i) } y(t_0) = y_0, \\ \text{(ii) } y' \text{ is continuous for } |t - t_0| \le a', \\ \text{(iii) } |y'(t)| \le M \text{ for } |t - t_0| \le a' \end{array} \right\}$$

in the Banach space X. Condition (iii) makes E an equicontinuous family, and (i) and (iii) together make E pointwise bounded. Lemma 10.47 of *Basic* shows that the closure K is equicontinuous and pointwise bounded. Ascoli's Theorem

[19] A proof may be found in Dunford–Schwartz's *Linear Operators*, Part I, pp. 453–456 and 467–469.

[20] Theorem 4.1 of *Basic*.

therefore shows that K is compact. Define a function F carrying the space K of functions to another space of functions by

$$F(y)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

For y in E, we have $|y(s) - y_0| \leq M|s - t_0| \leq Ma' \leq b$, and thus $(s, y(s))$ is in the rectangle R. Hence $F(y)$ satisfies (i), (ii), and (iii) and is in E. So F carries E to itself. The formula for F makes clear that F extends to a continuous mapping on K in the supremum-norm topology. Since $F(E) \subseteq E$, we obtain $F(K) \subseteq K$. The set K is compact convex in a Banach space, which is locally convex. The Schauder–Tychonoff Theorem applies to F, and the fixed point it produces is the desired solution.

10. Gelfand Transform for Commutative C^* Algebras

Alaoglu's Theorem, obtained in Section 3, leads in several directions in functional analysis, and we now return to its ramifications for spectral theory. The Stone Representation Theorem in Section 4 gave a concrete example of what we shall be investigating, showing that certain subalgebras of the algebra $B(S)$ of all complex-valued bounded functions on a set S can be realized as the algebra of all complex-valued continuous functions on a suitable compact Hausdorff space. The present section is devoted to a generalization due to I. M. Gelfand of this result to certain algebras besides $B(S)$; a different special case of this generalization will yield in the next section the Spectral Theorem for bounded self-adjoint operators on a Hilbert space.

Recall from Section 4 that a complex **Banach algebra** $\mathcal{A}$ is a complex associative algebra having a norm that makes it into a Banach space such that $\|ab\| \leq \|a\|\|b\|$ for all a and b in $\mathcal{A}$. We shall not consider $\mathcal{A} = 0$ as a Banach algebra. Nor shall we have any occasion to consider real Banach algebras. The inequality concerning the norm under multiplication implies that multiplication is continuous. If the Banach algebra has an identity, the same inequality implies that $\|1\| \geq 1$.

EXAMPLES.

(1) If S is a nonempty set, then the algebra $B(S)$ of all bounded complex-valued functions on S is a commutative Banach algebra. The function 1 is an identity. If S has a topology, then the subalgebra $C(S)$ of bounded continuous functions gives another example of a commutative Banach algebra with identity.

(2) If (S, μ) is a σ-finite measure space, then pointwise multiplication and the essential-supremum norm make $L^\infty(S, \mu)$ into a commutative Banach algebra with identity.

(3) In Euclidean space $\mathbb{R}^N$, the Banach space $L^1(\mathbb{R}^N)$ with Lebesgue measure becomes a commutative Banach algebra with convolution as multiplication: $(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,dy = \int_{\mathbb{R}^N} f(y)g(x-y)\,dy$. This Banach algebra does not have an identity. A variant of this Banach algebra may be defined using functions on $\mathbb{R}^N$ periodic in each variable with period 2π, the measure being $(2\pi)^{-N}\,dx$, and convolution being the multiplication. Still another variant uses functions on $\mathbb{Z}^N$ integrable with respect to the counting measure, and convolution is again the multiplication.

4) If H is a complex Hilbert space, then the algebra $\mathcal{B}(H, H)$ of all bounded linear operators from H to itself is a Banach algebra with identity when the norm is the operator norm and the multiplication is composition of operators.

The example of L^1 is so important that one does not want automatically to impose a condition on a Banach algebra that it contain an identity. Nevertheless, it is always possible to adjoin an identity to a Banach algebra if one wants, as the following proposition shows.

Proposition 4.33. Let $\mathcal{A}$ be a complex Banach algebra, and let

$$\mathcal{B} = \{(a, \lambda) \mid a \text{ is in } \mathcal{A} \text{ and } \lambda \text{ is in } \mathbb{C}\} = \mathcal{A} \oplus \mathbb{C}$$

as a vector space. Define

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

and

$$\|(a, \lambda)\| = \|a\| + |\lambda|.$$

Then $\mathcal{B}$ is a complex Banach algebra with identity $(0, 1)$, and the map $a \mapsto (a, 0)$ is a norm-preserving algebra homomorphism of $\mathcal{A}$ onto a closed ideal in $\mathcal{B}$.

REMARKS. The formula for the multiplication is motivated by expansion of the product $(a + \lambda)(b + \mu)$, and the formula for the norm is motivated by the norm of the element $f\,dx + \delta_0$ in $M(\mathbb{R}^N)$, where δ_0 is a point mass of weight 1 at the origin. We omit the proof of the proposition, since we shall not pursue L^1 very far from this point of view.

To proceed further, let us go back to our examples and see what can be said about them. For $B(S)$ in Example 1, the Stone Representation Theorem realized certain subalgebras as $C(X)$ for some compact Hausdorff space X. The space X is the space of all nonzero continuous multiplicative linear functionals respecting complex conjugation, regarded as a closed subset of the set of all continuous linear functionals of norm ≤ 1 with the weak-star topology. Evaluations at points of S provide examples of members of X, and X is just the closure of those evaluations.

To what extent might multiplicative linear functionals help us understand the other examples? For L^∞ in Example 2, the notion of multiplicative linear functional is meaningful, but it is not clear that any nonzero ones exist. At points of the measure space of positive measure, evaluations are well defined and yield multiplicative linear functionals. But if every one-point set of the measure space has measure 0, then it is not clear how to proceed.

For L^1 in Example 3, the answer is more decisive. The most general continuous linear functional is integration with an L^∞ function, and the nonzero continuous multiplicative linear functionals are the ones where the L^∞ function is an exponential $x \mapsto e^{ix\cdot y}$ for some y in $\mathbb{R}^N$. Let us sketch the argument. If a multiplicative linear functional ℓ is given by the nonzero L^∞ function φ, then the condition $\ell(f * g) = \ell(f)\ell(g)$ translates into the condition

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x)g(y)\varphi(x+y)\,dx\,dy = \int_{\mathbb{R}^N\times\mathbb{R}^N} f(x)g(y)\varphi(x)\varphi(y)\,dx\,dy.$$

Since f and g are arbitrary, $\varphi(x+y) = \varphi(x)\varphi(y)$ a.e. $[dx\,dy]$. Letting p be in $C_{\text{com}}(\mathbb{R}^N)$ and integrating this equation with $p(y)$ gives

$$\int_{\mathbb{R}^N} p(y)\varphi(x+y)\,dy = \varphi(x)\int_{\mathbb{R}^N} p(y)\varphi(y)\,dy \quad \text{a.e. } [dx].$$

The left side, upon the change of variables $y \mapsto -y$, is the convolution of a function in $C_{\text{com}}(\mathbb{R}^N)$ and a function in $L^\infty(\mathbb{R}^N)$. It is therefore continuous as a function of x. On the right side some p has $\int_{\mathbb{R}^N} p(y)\varphi(y)\,dy \neq 0$ since φ is not the 0 function almost everywhere. Fixing such a p and dividing by $\int_{\mathbb{R}^N} p(y)\varphi(y)\,dy$, we see that $\varphi(x)$ is almost everywhere equal to a certain continuous function. We may therefore adjust φ on a set of measure 0 to be continuous. Once adjusted, φ satisfies $\varphi(x+y) = \varphi(x)\varphi(y)$ everywhere. It is then a simple matter to see that φ is an exponential, as asserted.

Example 4 is something like Example 2. Suppose that A is a bounded self-adjoint operator on the Hilbert space H. We can form the smallest subalgebra of $\mathcal{B}(H, H)$ containing A and the identity, and we can look for multiplicative linear functionals. Theorem 2.3 addresses a situation in which we can identify such functionals. If A is compact, then the theorem gives an orthonormal basis of eigenvectors, and every member of this algebra acts as a scalar on each eigenvector. So each eigenvector yields, via the corresponding set of eigenvalues, a multiplicative linear functional. If A is not compact, however, eigenvectors need not exist, and then it is unclear where to look to find nonzero multiplicative linear functionals.

A series of theoretical insights now comes into play. An associative algebra with identity need not have nonzero multiplicative linear functionals, but it always has maximal ideals. These come from Zorn's Lemma, the proper ideals being those ideals not containing the identity. Accordingly, we shall think in terms of maximal ideals. These turn out to be closed, because as we shall see, there is a neighborhood of the identity where every element is invertible with an inverse given by the sum of a geometric series. The quotient of a commutative complex Banach algebra with identity by a (closed) maximal ideal is then a complex Banach algebra in which every nonzero element is invertible. The remarkable fact is that such a quotient necessarily is 1-dimensional. Then it follows that the maximal ideals all correspond to continuous multiplicative linear functionals after all, and their existence has been established. Let us run through the steps.

Let $\mathcal{A}$ be a Banach algebra with identity, at first not necessarily commutative. If a is in $\mathcal{A}$, then a **right inverse** to a is an element b with $ab = 1$. If a has a right inverse b and if a has a **left inverse** c, then the two are equal as a consequence of the associativity of multiplication: $c = c1 = c(ab) = (ca)b = 1b = b$. So a has a two-sided inverse, which we call simply an **inverse**, and we say that a is **invertible**.

Proposition 4.34. Let $\mathcal{A}$ be a Banach algebra with identity. If $\|a\| < 1$, then $1 - a$ is invertible and $\|(1-a)^{-1}\| \le (1 - \|a\|)^{-1}$.

PROOF. Form $\sum_{n=0}^{\infty} a^n$. This series is Cauchy since $\|a^n\| \le \|a\|^n$ implies $\big\|\sum_{n=M}^{N} a^n\big\| \le \sum_{n=M}^{N} \|a\|^n \le \|a\|^M(1-\|a\|)^{-1}$. Since $\mathcal{A}$ is complete, the series $\sum_{n=0}^{\infty} a^n$ is convergent. Let b be its sum. Then we have $(1-a)\big(\sum_{n=0}^{N} a^n\big) = \big(\sum_{n=0}^{N} a^n\big)(1-a) = 1 - a^{N+1}$, and hence $(1-a)b = b(1-a) = 1$. Also, $\|b\| \le \sum_{n=0}^{\infty} \|a\|^n = (1-\|a\|)^{-1}$. □

Corollary 4.35. In a Banach algebra with identity, the invertible elements form an open set. More particularly if $\|a\|$ is invertible and $\|x - a\| < \|a^{-1}\|^{-1}$, then x is invertible.

PROOF. Let U be the set of invertible elements, and let a be in U. If $\|x - a\| < \|a^{-1}\|^{-1}$, then

$$\|a^{-1}x - 1\| = \|a^{-1}(x-a)\| \le \|a^{-1}\|\|x-a\| < 1,$$

and Proposition 4.34 shows that $1 - (1 - a^{-1}x) = a^{-1}x$ is invertible. Hence x is invertible. □

Proposition 4.36. If $\mathcal{A}$ is a Banach algebra with identity and U is the open set of invertible elements, then inversion is a continuous map of U into itself.

PROOF. Let a be in U, and let $\|x-a\| < \|a^{-1}\|^{-1}$, so that x is in U by Corollary 4.35. Then

$$\|x^{-1}-a^{-1}\| = \|x^{-1}(x-a)a^{-1}\| \le \|a^{-1}\|\|x^{-1}\|\|x-a\|,$$

and continuity will follow if we show that $\|x^{-1}\| \le M < \infty$ for x near a. Computation and Proposition 4.34 give

$$\|x^{-1}\| = \|(a-(a-x))^{-1}\| = \|a^{-1}(1-(1-xa^{-1}))^{-1}\| \le \frac{\|a^{-1}\|}{1-\|1-xa^{-1}\|},$$

and the desired boundedness follows from continuity of multiplication. □

Let $\mathcal{A}$ be a complex Banach algebra with identity. If a is in $\mathcal{A}$, the **spectrum** of a is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a-\lambda \text{ is not invertible}\}.$$

It will be proved in Corollary 4.39 below that $\sigma(a)$ is always nonempty. The **resolvent set** $P(a)$ of a is the complement of $\sigma(a)$ in $\mathbb{C}$. The **resolvent** of a is the function

$$R(\lambda) = (a-\lambda)^{-1} \qquad \text{from } P(a) \text{ into } \mathcal{A}.$$

The **spectral radius** of a, denoted by $r(a)$, is

$$r(a) = \sup\bigl\{|\lambda| \bigm| \lambda \text{ is in } \sigma(a)\bigr\}.$$

Proposition 4.37. For a in a complex Banach algebra $\mathcal{A}$ with identity, $\sigma(a)$ is compact and $r(a)$ is $\le \|a\|$.

PROOF. The function $\lambda \mapsto a-\lambda$ is continuous, and the set U of invertible elements is open, the latter by Corollary 4.35. Thus $P(a) = \{\lambda \mid a-\lambda \text{ is in } U\}$ is open. Hence the complement $\sigma(a)$ is closed. Fix λ with $\lambda > \|a\|$. Then $\|\lambda^{-1}a\| < 1$, and therefore $\lambda^{-1}a - 1$ is in U. Since $\lambda \ne 0$, $a-\lambda$ is in U. Thus λ is in $P(a)$. It follows that $\sigma(a)$ is contained in $\bigl\{\lambda \bigm| |\lambda| \le \|a\|\bigr\}$ and that $r(a) \le \|a\|$. Since $\sigma(a)$ is then bounded, as well as closed, $\sigma(a)$ is compact. □

We say that a function φ from an open subset V of $\mathbb{C}$ into the complex Banach algebra $\mathcal{A}$ is **weakly analytic** on V if $\ell \circ \varphi$ is an analytic function on V for every ℓ in the dual space $\mathcal{A}^*$.

Theorem 4.38. If $\mathcal{A}$ is a complex Banach algebra with identity and if a is in $\mathcal{A}$, then $R(\lambda) = (a-\lambda)^{-1}$ is weakly analytic on $P(a)$ with $\lim_{\lambda\to\infty} \|R(\lambda)\| = 0$.

PROOF. Let λ_0 be in $P(a)$, and let ℓ be in $\mathcal{A}^*$. Writing

$$a - \lambda = (a - \lambda_0)\big(1 - (a - \lambda_0)^{-1}(\lambda - \lambda_0)\big)$$

and applying Proposition 4.34, we see that $a - \lambda$ is invertible if the condition $\|(a - \lambda_0)^{-1}(\lambda - \lambda_0)\| < 1$ is satisfied. In this case,

$$(a - \lambda)^{-1} = (a - \lambda_0)^{-1} \textstyle\sum_{n=0}^{\infty} (a - \lambda_0)^{-n}(\lambda - \lambda_0)^n,$$

and the continuity of ℓ yields

$$\ell\big((a - \lambda)^{-1}\big) = \sum_{n=0}^{\infty} \ell\big((a - \lambda_0)^{-n-1}\big)(\lambda - \lambda_0)^n,$$

with the series convergent. Therefore $R(\lambda)$ is weakly analytic.

To establish that $\lim_{\lambda\to\infty} \|(a - \lambda)^{-1}\| = 0$, we write

$$(a - \lambda)^{-1} = \big(\lambda(\lambda^{-1}a - 1)\big)^{-1} = \lambda^{-1}(\lambda^{-1}a - 1)^{-1}.$$

Proposition 4.34 gives

$$\|(\lambda^{-1}a - 1)^{-1}\| \le (1 - |\lambda|^{-1}\|a\|)^{-1},$$

and the right side tends to 1 as λ tends to infinity. Hence $\lim_{\lambda\to\infty} \|(a-\lambda)^{-1}\| = 0$. $\square$

Corollary 4.39. If $\mathcal{A}$ is a complex Banach algebra with identity, then $\sigma(a)$ is nonempty for each a in $\mathcal{A}$.

PROOF. If $\sigma(a)$ were to be empty, then every ℓ in $\mathcal{A}^*$ would have $\lambda \mapsto \ell((a - \lambda)^{-1})$ entire and vanishing at infinity, by Theorem 4.38. By Liouville's Theorem, we would have $\ell((a-\lambda)^{-1}) = 0$ for every a and λ. Since ℓ is arbitrary, the Hahn–Banach Theorem would give $(a - \lambda)^{-1} = 0$, contradiction. $\square$

Corollary 4.40 (Gelfand–Mazur Theorem). The only complex Banach algebra with identity in which every nonzero element is invertible is $\mathbb{C}$ itself.

PROOF. Suppose that $\mathcal{A}$ is a complex Banach algebra with identity with every nonzero element invertible. If a is given in $\mathcal{A}$, $\sigma(a)$ is not empty, according to Corollary 4.39. Choose λ in $\sigma(a)$. Then $a - \lambda$ is not invertible. Since every nonzero element of $\mathcal{A}$ is by assumption invertible, $a - \lambda = 0$. Hence $a = \lambda$. Thus $\mathcal{A}$ consists of the scalar multiples of the identity. $\square$

Corollary 4.41. If $\mathcal{A}$ is a commutative complex Banach algebra with identity, then the nonzero multiplicative linear functionals on $\mathcal{A}$ stand in one-one correspondence with the maximal ideals of $\mathcal{A}$, the correspondence being

$$\ell = \left\{\begin{matrix}\text{multiplicative} \\ \text{linear functional}\end{matrix}\right\} \quad \longrightarrow \quad \ker \ell = \text{maximal ideal}$$

with inverse

$$I = \left\{\begin{matrix}\text{maximal ideal,} \\ \text{necessarily with} \\ \mathcal{A} = I \oplus \mathbb{C}1\end{matrix}\right\} \quad \longrightarrow \quad \ell \ \text{ defined by } \ell(x, \lambda) = \lambda.$$

Every nonzero multiplicative linear functional is continuous with norm ≤ 1, and every maximal ideal is closed. Every nonzero multiplicative linear functional carries 1 into 1.

REMARKS. The proof will make use of Problem 4 in Chapter XII of *Basic*: if X is a Banach space and Y is a closed subspace, then the vector space X/Y becomes a normed linear space under the definition $\|x + Y\| = \inf_{y \in Y} \|x + y\|$, and the resulting metric on X/Y is complete. Problem 1 at the end of the present chapter points out that the Banach space X/Y obtained this way has the same topology as the quotient topological vector space X/Y defined in Section 1.

PROOF. We may assume $\mathcal{A} \neq 0$. If ℓ is a nonzero multiplicative linear functional, then its kernel is an ideal of codimension 1, hence is a maximal ideal. Conversely if I is a maximal ideal, then no element of I can be invertible. Since the set U of invertible elements is open, according to Corollary 4.35, the set I is at positive distance from 1. Thus the closure I^{cl}, which is an ideal, does not contain 1. Since I is maximal, $I^{\text{cl}} = I$. Thus I is closed. By the above remarks, $\mathcal{A}/I$ is a complex Banach space. Its multiplication makes it into a complex Banach algebra because if we take the infimum over $y_1 \in I$ and $y_2 \in I$ of the right side of the inequality

$$\begin{aligned}
\|a_1 a_2 + I\| &\leq \|a_1 a_2 + (y_1 a_2 + a_1 y_2 + y_1 y_2)\| \\
&= \|(a_1 + y_1)(a_2 + y_2)\| \\
&\leq \|a_1 + y_1\| \|a_2 + y_2\|,
\end{aligned}$$

we obtain $\|a_1 a_2 + I\| \leq \|a_1 + I\| \|a_2 + I\|$. The quotient $\mathcal{A}/I$ is also a field, being the quotient of a nonzero commutative ring with identity by a maximal ideal. By Corollary 4.40, $\mathcal{A}/I \cong \mathbb{C}$. Hence I has codimension 1, and $\mathcal{A} = I \oplus \mathbb{C}1$ as vector spaces. If we define a linear functional ℓ by $\ell(x, \lambda) = \lambda$, then we readily check that ℓ is multiplicative and has kernel I. To see that ℓ is continuous, one way to proceed is to use the Hahn–Banach Theorem: Since I is closed and 1 is not in I, there exists a continuous linear functional ℓ' with $\ell'(1) \neq 0$ and $\ell'(I) = 0$. Then $\ell = \ell'(1)^{-1}\ell(1)\ell'$, and therefore ℓ is continuous.

This establishes the correspondence. To check that it is one-one, it is enough to see that any nonzero multiplicative linear functional carries 1 into 1. If ℓ is a nonzero multiplicative linear functional, then $\ell(a) = \ell(a)\ell(1) = \ell(a)\ell(1)$. If we choose a with $\ell(a) \neq 0$, then we can divide and conclude that $\ell(1) = 1$.

Finally we check the norm of the nonzero multiplicative linear functional ℓ. If a in $\mathcal{A}$ has $\|a\| \leq 1$, then $|\ell(a)|^n = |\ell(a^n)| \leq \|\ell\|\|a^n\| \leq \|\ell\|\|a\|^n \leq \|\ell\|$. Since $n \geq 1$ is arbitrary, we must have $|\ell(a)| \leq 1$. Taking the supremum over a, we obtain $\|\ell\| \leq 1$. $\square$

If $\mathcal{A}$ is a commutative complex Banach algebra with identity, we denote its space of maximal ideals by $\mathcal{A}^*_{\mathrm{m}}$. For $\mathcal{A} \neq 0$, this space is nonempty by an application of Zorn's Lemma to the set of all proper ideals of $\mathcal{A}$. Using the identification via Corollary 4.41 of $\mathcal{A}^*_{\mathrm{m}}$ as a set of linear functionals of norm ≤ 1, we can regard $\mathcal{A}^*_{\mathrm{m}}$ as a subset of the unit ball of the dual $\mathcal{A}^*$. We give $\mathcal{A}^*_{\mathrm{m}}$ the relative topology from the weak-star topology on $\mathcal{A}^*$.

Proposition 4.42. If $\mathcal{A}$ is a commutative complex Banach algebra with identity, then the weak-star topology makes the maximal ideal space $\mathcal{A}^*_{\mathrm{m}}$ into a compact Hausdorff space.

PROOF. Corollary 4.41 identifies $\mathcal{A}^*_{\mathrm{m}}$ with a subset of the unit ball of $\mathcal{A}^*$, which is compact in the weak-star topology by Alaoglu's Theorem (Theorem 4.14) and is also Hausdorff. All we have to do is show that $\mathcal{A}^*_{\mathrm{m}}$ is a closed subset. For each a and b in $\mathcal{A}$, the set $\{\ell \in \mathcal{A}^* \mid \ell(ab) = \ell(a)\ell(b)\}$ is closed since the functions $\ell \mapsto \ell(ab)$ and $\ell \mapsto \ell(a)\ell(b)$ are continuous from the weak-star topology into $\mathbb{C}$. Hence the intersection over all a and b is closed. The set $\mathcal{A}^*_{\mathrm{m}}$ is the intersection of this set with the closed set $\{\ell \in \mathcal{A}^* \mid \ell(1) = 1\}$ and is therefore closed. $\square$

For L^1 or any other complex Banach algebra $\mathcal{A}$ not containing an identity, the prescription for applying the above theory to $\mathcal{A}$ is to adjoin an identity and form $\mathcal{A} \oplus \mathbb{C}$, apply the results to $\mathcal{A} \oplus \mathbb{C}$, and then see what happens when the identity is removed. For Proposition 4.42, $\mathcal{A}$ is one of the maximal ideals in $\mathcal{A} \oplus \mathbb{C}$. Removing it from $(\mathcal{A} \oplus \mathbb{C})^*_{\mathrm{m}}$ yields a locally compact Hausdorff space whose one-point compactification is $(\mathcal{A} \oplus \mathbb{C})^*_{\mathrm{m}}$.

It is now just a formality to obtain a mapping of any commutative complex Banach algebra $\mathcal{A}$ with identity into $C(\mathcal{A}^*_{\mathrm{m}})$. The **Gelfand transform** $a \mapsto \widehat{a}$ is the mapping of $\mathcal{A}$ into $C(\mathcal{A}^*_{\mathrm{m}})$ given by $\widehat{a}(\ell) = \ell(a)$ for each nonzero multiplicative linear functional ℓ on $\mathcal{A}$.

In the context of a suitable subalgebra of $B(S)$, the Gelfand transform is just the evaluation of all nonzero multiplicative linear functionals on the members of the subalgebra. Such linear functionals turn out automatically to respect complex conjugation.[21] The evaluations at the points of S are a dense subset of these. The Stone Representation Theorem says that the Gelfand transform is a norm-preserving algebra isomorphism.

In the context of $L^1(\mathbb{R}^N)$, the Gelfand transform is just the Fourier transform. The nonzero multiplicative linear functionals are the functions $\ell_y(f) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot y}\,dx$ for $y \in \mathbb{R}^N$, i.e., $\ell_y(f) = \widehat{f}(y)$. The Gelfand transform is the mapping of f to the resulting function of ℓ_y or of y. It is therefore exactly the Fourier transform $f \mapsto \widehat{f}$ if we parametrize $L^1(\mathbb{R}^N)^*_{\mathrm{m}}$ by the variable y.

The Gelfand transform makes sense for our other two examples as well, for L^∞ and for the complex Banach algebra generated by the identity and a single self-adjoint bounded linear operator on a Hilbert space. But we do not so far get much insight into what the Gelfand transform does for these cases. We can summarize all the formalism as follows.

Proposition 4.43. If $\mathcal{A}$ is a commutative complex Banach algebra with identity, then the Gelfand transform is an algebra homomorphism of norm ≤ 1 of $\mathcal{A}$ into $C(\mathcal{A}^*_{\mathrm{m}})$ carrying 1 to 1, and its kernel is the intersection of all maximal ideals of $\mathcal{A}$. Moreover, for each a and b in $\mathcal{A}$,

(a) $\sigma(a)$ is the image of the function $\widehat{a}$ in $\mathbb{C}$,
(b) $r(a) = \|\widehat{a}\|_{\sup}$,
(c) $r(a+b) \le r(a) + r(b)$ and $r(ab) \le r(a)r(b)$.

PROOF. The Gelfand transform is an algebra homomorphism because

$$\widehat{ab}\,(\ell) = \ell(ab) = \ell(a)\ell(b) = \widehat{a}(\ell)\widehat{b}(\ell)$$

for all ℓ in $\mathcal{A}^*_{\mathrm{m}}$. Corollary 4.41 shows that each ℓ in $\mathcal{A}^*_{\mathrm{m}}$ has norm ≤ 1, and therefore $|\widehat{a}(\ell)| = |\ell(a)| \le \|a\|$. Hence $\|\widehat{a}\|_{\sup} \le \|a\|$, and the Gelfand transform has norm ≤ 1. Corollary 4.41 shows that every nonzero multiplicative linear functional carries 1 into 1, and therefore the Gelfand transform carries 1 into 1.

The kernel of the Gelfand transform is the set of all a in $\mathcal{A}$ with $\widehat{a}(\ell) = 0$ for all ℓ, thus the set of all a with $\ell(a) = 0$ for all ℓ, thus the intersection of the kernels of all ℓ's.

For (a), we observe that a is invertible if and only if $a\mathcal{A} = \mathcal{A}$, if and only if a is not in any maximal ideal, if and only if $\widehat{a}$ is nowhere vanishing. Thus a complex number λ is in $\sigma(a)$ if and only if $a - \lambda$ is not invertible, if and only if $\widehat{a} - \lambda$ is somewhere vanishing, if and only if λ is in the image of $\widehat{a}$. This proves (a).

[21] The verification for an algebra as in Theorem 4.15 that the nonzero multiplicative linear functionals automatically respect complex conjugation is embedded in the proof of Theorem 4.48 below. See the paragraph of the proof containing the display (†) and the two paragraphs that follow it.

Conclusion (b) is immediate from (a) and the definition of $r(a)$, and (c) follows from (b) and the inequalities satisfied by the supremum norm. This completes the proof. $\square$

Proposition 4.43 isolates the real problem, which is to say something quantitative about the intersection of the kernels of all maximal ideals, about $\sigma(a)$, and about $r(a)$. For our purposes it will be enough to have the spectral radius formula that is proved in Corollary 4.46 below.

Theorem 4.44 (Spectral Mapping Theorem). If $\mathcal{A}$ is a complex Banach algebra with identity, if a is in $\mathcal{A}$, and if Q is any polynomial in one variable, then $Q(\sigma(a)) = \sigma(Q(a))$.

REMARKS. The left side $Q(\sigma(a))$ is understood to be the image under Q of the set $\sigma(a)$, while the right side $\sigma(Q(a))$ is the spectrum of $Q(a)$, i.e., the spectrum of the member of $\mathcal{A}$ obtained by substituting a for the variable in Q.

PROOF. First we show that $Q(\sigma(a)) \subseteq \sigma(Q(a))$. Let λ_0 be in $\sigma(a)$, so that $a - \lambda_0$ is not invertible. Arguing by contradiction, suppose that $Q(a) - Q(\lambda_0)$ is invertible, say with b as two-sided inverse. Let S be the polynomial defined by $Q(\lambda) - Q(\lambda_0) = (\lambda - \lambda_0)S(\lambda)$. Since b is a two-sided inverse of $Q(a) - Q(\lambda_0) = (a - \lambda_0)S(a)$, we have $1 = b(a - \lambda_0)S(a) = (bS(a))(a - \lambda_0)$ and $1 = (a - \lambda_0)(S(a)b)$. Thus $a - \lambda_0$ has a left inverse $bS(a)$ and a right inverse $S(a)b$, and $a - \lambda_0$ must be invertible, contradiction.

For the reverse inclusion $\sigma(Q(a)) \subseteq Q(\sigma(a))$, suppose that λ_0 is in $\sigma(Q(a))$. Let $\lambda_1, \dots, \lambda_n$ be the roots of $Q(\lambda) - \lambda_0$ repeated according to their multiplicities. Then we have $Q(\lambda) - \lambda_0 = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ for some nonzero constant c. Substitution of a for λ gives

$$Q(a) - \lambda_0 = c(a - \lambda_1) \cdots (a - \lambda_n).$$

Since $Q(a) - \lambda_0$ is by assumption not invertible, some $a - \lambda_j$ is not invertible. For this j, λ_j is in $\sigma(a)$. Since λ_j is a root of $Q(\lambda) - \lambda_0$, we have $Q(\lambda_j) - \lambda_0 = 0$, i.e., $Q(\lambda_j) = \lambda_0$. Hence λ_0 is exhibited as Q of the member λ_j of $\sigma(a)$. $\square$

Corollary 4.45. If $\mathcal{A}$ is a complex Banach algebra with identity and if a is in $\mathcal{A}$, then $r(a^n) = r(a)^n$ for every integer $n \geq 1$.

PROOF. This follows by taking $Q(\lambda) = \lambda^n$ in Theorem 4.44 and then using the definition of the function r. $\square$

Corollary 4.46 (spectral radius formula). If $\mathcal{A}$ is a complex Banach algebra with identity and if a is in $\mathcal{A}$, then

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n},$$

the limit existing.

PROOF. For every n, Corollary 4.45 and Proposition 4.37 give $r(a)^n = r(a^n) \le \|a^n\|$ and thus $r(a) \le \|a^n\|^{1/n}$. Hence

$$r(a) \le \liminf_n \|a^n\|^{1/n}. \qquad (*)$$

If $|\lambda| < \|a\|^{-1}$ and ℓ is in the dual space $\mathcal{A}^*$, then Proposition 4.34 yields

$$(1-\lambda a)^{-1} = \textstyle\sum_{n=0}^\infty a^n\lambda^n \qquad \text{and therefore} \qquad \ell((1-\lambda a)^{-1}) = \textstyle\sum_{n=0}^\infty \ell(a^n)\lambda^n.$$

Theorem 4.38 shows that $\lambda \mapsto \ell((1-\lambda a)^{-1})$ is analytic for λ^{-1} in $P(a)$, and Proposition 4.37 shows that this analyticity occurs for $|\lambda|^{-1} > r(a)$, hence for $|\lambda| < r(a)^{-1}$. Therefore the power series $\sum_{n=0}^\infty \ell(a^n)\lambda^n$ is convergent for $|\lambda| < r(a)^{-1}$. Since the terms of a convergent series are bounded, each fixed λ within the disk of convergence must have $|\ell(a^n)||\lambda^n| \le M_\ell$ for some constant M_ℓ. That is,

$$|\ell(\lambda^n a^n)| \le M_\ell \qquad (**)$$

for all n. Each linear functional on $\mathcal{A}^*$ given by $\ell \mapsto \ell(\lambda^n a^n)$ is bounded, and therefore the system of such linear functionals as n varies, which has been shown in $(**)$ to be pointwise bounded, satisfies $\|\lambda^n a^n\| \le M$ by the Uniform Boundedness Theorem. Consequently $|\lambda|\|a^n\|^{1/n} \le M^{1/n}$. Taking the limsup of both sides gives $|\lambda| \limsup_n \|a^n\|^{1/n} \le 1$, and hence $\limsup_n \|a^n\|^{1/n} \le |\lambda|^{-1}$. Since λ is an arbitrary complex number with $|\lambda|^{-1} > r(a)$, we obtain $\limsup_n \|a^n\|^{1/n} \le r(a)$. In combination with $(*)$, this inequality completes the proof. $\square$

The spectral radius formula gives us the following quantitative conclusion about the Gelfand transform.

Corollary 4.47. The Gelfand transform for a commutative complex Banach algebra $\mathcal{A}$ with identity is norm preserving from $\mathcal{A}$ to $C(\mathcal{A}^*_{\mathrm{m}})$ if and only if $\|a^2\| = \|a\|^2$ for all a in $\mathcal{A}$.

PROOF. If $\|a^2\| = \|a\|^2$ for all a, then induction gives $\|a^{2^n}\| = \|a\|^{2^n}$ and thus $\|a\| = \|a^{2^n}\|^{2^{-n}}$. Hence $\|a\| = \lim_n \|a^{2^n}\|^{2^{-n}}$. This limit equals $r(a)$ by the spectral radius formula (Corollary 4.46), and $r(a)$ equals $\|\widehat{a}\|_{\mathrm{sup}}$ by Proposition 4.43b. Therefore $\|a\| = \|\widehat{a}\|_{\mathrm{sup}}$.

Conversely if $\|\widehat{a}\|_{\mathrm{sup}} = \|a\|$ for all a, then $r(a) = \|a\|$ by Proposition 4.43b, and $\|a^2\| = r(a^2) = r(a)^2 = \|a\|^2$ by Corollary 4.45. $\square$

This represents some progress. The condition $\|a^2\| = \|a\|^2$ is satisfied in L^∞, and hence the Gelfand transform is a norm-preserving algebra homomorphism of L^∞ into $C(\mathcal{A}^*_{\mathrm{m}})$. In L^1 after an identity is adjoined, the condition $\|a^2\| = \|a\|^2$

is not universally satisfied, and the corollary says that the Gelfand transform, i.e., the Fourier transform, is not norm preserving; this conclusion has content, but it is not a surprise. In the case of the complex Banach algebra generated by the identity and a bounded self-adjoint operator A, the condition $\|a^2\| = \|a\|^2$ is satisfied for $a = A$ as a consequence of Proposition 2.2 with $L = A^*A$, but it is less transparent what happens with other operators in the Banach algebra that are not self adjoint.

The final step is to bring the operation $(\,\cdot\,)^*$ into play. An **involution** of a complex Banach algebra $\mathcal{A}$ is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself with the properties that the following hold for all a and b in $\mathcal{A}$:

(i) $a^{**} = a$,
(ii) $(a+b)^* = a^* + b^*$,
(iii) $(\lambda a)^* = \bar{\lambda} a^*$ for all λ in $\mathbb{C}$,
(iv) $(ab)^* = b^* a^*$.

A complex Banach algebra $\mathcal{A}$ with involution $(\,\cdot\,)^*$ is called a C^* **algebra** if

(v) $\|a^*a\| = \|a\|^2$ for all a in $\mathcal{A}$.

Our examples—$B(S)$ and certain subalgebras, L^∞, L^1, and $\mathcal{B}(H, H)$ are all complex Banach algebras with involution. For $B(S)$ and L^∞, the involution is complex conjugation. For L^1, it is $f \mapsto g$ with $g(x) = \overline{f(-x)}$, and for $\mathcal{B}(H, H)$ it is adjoint. Of these examples all but L^1 are C^* algebras.

Observe that (i) and (iv) imply that the element 1, if it is present, has to satisfy $1^* = 1$ because $1 = (1^*)^* = (11^*)^* = 1^{**}1^* = 11^* = 1^*$. If (v) holds also, then (v) with $a = 1$ shows that $\|1\| = 1$.

Theorem 4.48. If $\mathcal{A}$ is a commutative C^* algebra with identity, then the Gelfand transform is a norm-preserving algebra isomorphism of $\mathcal{A}$ *onto* $C(\mathcal{A}^*_{\mathrm{m}})$, and it carries $(\,\cdot\,)^*$ into complex conjugation.

PROOF. For any a in $\mathcal{A}$, (v) gives $\|a\|^2 = \|a^*a\| \le \|a^*\|\|a\|$. If $a = 0$, then $a^* = 0$; otherwise division by $\|a\|$ gives $\|a\| \le \|a^*\|$. Applying this inequality to a^* and using (i), we obtain

$$\|a^*\| = \|a\|. \tag{$*$}$$

Next suppose that b is an element of $\mathcal{A}$ with $b^* = b$. Raising to powers gives $(b^{2^n})^* = (b^{2^n})^*$ for $n \ge 0$. Then (v) gives $\|b^{2^n}\| = \|(b^{2^{n-1}})^* b^{2^{n-1}}\| = \|b^{2^{n-1}}\|^2$, and induction shows that $\|b^{2^n}\| = \|b\|^{2^n}$. Hence $\|b\| = \|b^{2^n}\|^{2^{-n}}$. Taking the limit and applying the spectral radius formula and Proposition 4.43b, we obtain

$$\|b\| = \lim_n \|b^{2^n}\|^{2^{-n}} = r(b) = \|\widehat{b}\|_{\sup}. \tag{$**$}$$

The Gelfand transform is an algebra homomorphism by Proposition 4.43. If a general a is given in $\mathcal{A}$, then we can apply $(*)$ to a and $(**)$ to $b = a^*a$ to obtain

$$\begin{aligned}\|a^*\|\|a\| = \|a\|^2 = \|a^*a\| = \|b\| = \|\widehat{b}\|_{\sup} = \|\widehat{a^*a}\|_{\sup} \\ = \|\widehat{a^*}\,\widehat{a}\|_{\sup} \leq \|\widehat{a^*}\|_{\sup}\|\widehat{a}\|_{\sup} \leq \|a^*\|\|a\|,\end{aligned}$$

the last inequality holding since the Gelfand transform has norm ≤ 1 according to Proposition 4.43. The end expressions are equal, and equality must hold throughout. Therefore $\|\widehat{a}\|_{\sup} = \|a\|$, and the Gelfand transform is norm preserving.

In working toward proving that the Gelfand transform carries $(\,\cdot\,)^*$ into complex conjugation, we first show that

$$b^* = b \qquad \text{implies} \qquad i \text{ is not in } \sigma(b). \tag{$\dagger$}$$

Assuming the contrary, we find that 1 is in $\sigma(-ib)$. By the Spectral Mapping Theorem (Theorem 4.44), $\lambda + 1$ is in $\sigma(\lambda - ib)$ for all real λ. Hence

$$\begin{aligned}(\lambda+1)^2 \leq (r(\lambda - ib))^2 \leq \|\lambda - ib\|^2 = \|(\lambda - ib)^*(\lambda - ib)\| \\ = \|(\lambda + ib)(\lambda - ib)\| = \|\lambda^2 + b^2\| \leq \lambda^2\|1\| + \|b^2\| = \lambda^2 + \|b^2\|,\end{aligned}$$

and $2\lambda + 1 \leq \|b\|^2$ for all real λ. This is a contradiction, and $(\dagger)$ is proved.

Next let us deduce from $(\dagger)$ that

$$b^* = b \qquad \text{implies} \qquad \sigma(b) \subseteq \mathbb{R}. \tag{$\dagger\dagger$}$$

Suppose that $\lambda = \alpha + i\beta$ has α and β real and $\beta \neq 0$. Then $\beta^{-1}(b - \lambda) = \beta^{-1}(b - \alpha) - i$. The element $\beta^{-1}(b - \lambda)$ has $(\beta^{-1}(b-\alpha))^* = \beta^{-1}(b - \alpha)$, and $(\dagger)$ shows that i is not in its spectrum. Therefore $\beta^{-1}(b - \lambda) = \beta^{-1}(b - \alpha) - i$ is invertible. Since $\beta \neq 0$, $b - \lambda$ is invertible. Therefore λ is not in $\sigma(b)$. This proves $(\dagger\dagger)$.

Now we shall show that the Gelfand transform carries $(\,\cdot\,)^*$ into complex conjugation. Let a be in $\mathcal{A}$, and write $a = \frac{1}{2}(a + a^*) + \frac{1}{2i}((ia) + (ia)^*) = b + ic$ with $b^* = b$ and $c^* = c$. Then $a^* = b - ic$. From $(\dagger\dagger)$ we know that $\widehat{b}$ and $\widehat{c}$ are real-valued. Therefore $\widehat{a^*}(\ell) = \widehat{b}(\ell) - i\widehat{c}(\ell) = \overline{\widehat{b}(\ell) + i\widehat{c}(\ell)} = \overline{\widehat{a}(\ell)}$, as asserted.

Since the Gelfand transform is norm preserving, respects products, and carries 1 into 1, its image is a uniformly closed subalgebra of $C(\mathcal{A}^*_{\mathrm{m}})$. The fact that $(\,\cdot\,)^*$ is carried into complex conjugation implies that the image is closed under complex conjugation. The image separates points of $\mathcal{A}$ by definition of equality of linear functionals. By the Stone–Weierstrass Theorem the image is all of $C(\mathcal{A}^*_{\mathrm{m}})$. This completes the proof. $\square$

Among our examples, if $\mathcal{A}$ is a conjugate-closed Banach subalgebra of $B(S)$ with identity, then Theorem 4.48 reproduces the Stone Representation Theorem (Theorem 4.15).

Second if $\mathcal{A}$ is L^∞, Theorem 4.48 gives us something new, identifying L^∞ with $C((L^\infty)^*_{\mathrm{m}})$. We do not get a total understanding of $(L^\infty)^*_{\mathrm{m}}$, but we do get some understanding from the fact that every ideal is contained in a maximal ideal. We can produce an ideal in L^∞ merely by specifying a measurable subset; the ideal consists of all essentially bounded functions, modulo null functions, that vanish on that set. As the set gets smaller, we get closer to the situation of a maximal ideal.

Third if $\mathcal{A}$ is L^1, Theorem 4.48 gives us no information since L^1 is not a C^* algebra. The theory of complex Banach algebras can be pursued in a direction that specializes to more information about L^1, but we shall not follow such a route.

Fourth if $\mathcal{A}$ is the complex Banach algebra generated by the identity and a bounded self-adjoint operator A on a Hilbert space H, then Theorem 4.48 is applicable and realizes the algebra as $C(\mathcal{A}^*_{\mathrm{m}})$. We shall see in the next section that $\mathcal{A}^*_{\mathrm{m}}$ can be viewed as the spectrum $\sigma(A)$. However, the Hilbert space H plays no role in this realization, and we therefore cannot expect to learn much about our original operator from $C(\mathcal{A}^*_{\mathrm{m}})$. For example we cannot distinguish between the two operators on $\mathbb{C}^3$ given by diagonal matrices diag$(1, 1, 2)$ and diag$(1, 2, 2)$ on the basis of the spectrum of each. The goal of the next section is to remedy this defect.

Since we shall want to consider operators in $\mathcal{B}(H, H)$ as belonging to more than one C^* algebra, let us take another look at the definition of the spectrum of an element. The spectrum of a, as a member of $\mathcal{A}$, is the set of complex λ for which $(a-\lambda)^{-1}$ fails to exist as a member of $\mathcal{A}$. Certainly if we have $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and a is in $\mathcal{A}_1$, then the failure of $(a-\lambda)^{-1}$ to exist in $\mathcal{A}_2$ implies the failure of $(a-\lambda)^{-1}$ to exist in $\mathcal{A}_1$. Hence the spectrum relative to $\mathcal{A}_1$ contains the spectrum relative to $\mathcal{A}_2$. The spectrum is the smallest for $\mathcal{A} = \mathcal{B}(H, H)$. The following corollary implies that for operators A with $AA^* = A^*A$, the smallest possible spectrum is already achieved for the C^* algebra generated by 1, A, and A^*.

Corollary 4.49. If $\mathcal{A}$ is a C^* algebra with identity and if a is an invertible element of $\mathcal{A}$ such that $aa^* = a^*a$, then a is invertible already in the smallest closed subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ containing 1, a, and a^*.

PROOF. Since $a^{-1}a^* = a^{-1}(a^*a)a^{-1} = a^{-1}(aa^*)a^{-1} = a^*a^{-1}$, the smallest closed subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ containing 1, a, a^*, a^{-1}, and a^{-1*} is commutative, hence is a commutative C^* algebra with identity. Form the Gelfand transform $b \mapsto \widehat{b}$ for $\mathcal{A}_1$. Then $\widehat{a}$ and $\widehat{a^{-1}}$ are reciprocals, and the image of $\widehat{a}$ is therefore bounded away from 0. By the Stone–Weierstrass Theorem we can find a sequence

$\{p_n(z, \bar{z})\}$ of polynomial functions that converge uniformly on the compact image of $\widehat{a}$ to $1/z$. Since by Theorem 4.48, the Gelfand transform is isometric for $\mathcal{A}_1$, we have $a^{-1} = \lim p_n(a, a^*)$ in $\mathcal{A}_1$, and a^{-1} is therefore exhibited as a member of $\mathcal{A}_0$. $\square$

11. Spectral Theorem for Bounded Self-Adjoint Operators

The goal of this section is to expand upon Theorem 4.48 in the case of a commutative C^* algebra of bounded linear operators on a Hilbert space in such a way that the Hilbert space plays a decisive role. The result will be the Spectral Theorem, and we shall see how the Spectral Theorem enables one to compute with the operators in question. The theorem to be given here is limited to the case of a separable Hilbert space, and the assumption of separability will be included in all our results about general spaces $\mathcal{B}(H, H)$. The Spectral Theorem will enable us to view the operators in question as multiplications by L^∞ functions on an L^2 space, and we therefore begin with that example.

EXAMPLE. Let (S, μ) be a finite measure space, and let H be the Hilbert space $H = L^2(S, \mu)$. For f in $L^\infty(X, \mu)$, define $M_f : L^2 \to L^2$ by $M_f(g) = fg$. The computation

$$\|M_f(g)\|_2^2 = \int_X |fg|^2\,d\mu \le \|f\|_\infty^2 \int_X |g|^2\,d\mu = \|f\|_\infty^2 \|g\|_2^2$$

shows that M_f is a bounded operator on H with $\|M_f\| \le \|f\|_\infty$. Shortly we shall check that equality holds:

$$\|M_f\| = \|f\|_\infty. \qquad (*)$$

But first, let us observe that

$$M_{fg} = M_f M_g, \quad M_{\alpha f + \beta g} = \alpha M_f + \beta M_g, \quad M_f^* = M_{\bar{f}}, \quad M_1 = I.$$

These facts, in combination with $(*)$, say that $f \mapsto M_f$ is a norm-preserving C^* algebra isomorphism of the commutative C^* algebra $L^\infty(S, \mu)$ onto the subalgebra

$$\mathcal{M}(L^2(S, \mu)) = \{M_f \in \mathcal{B}(L^2(S, \mu), L^2(S, \mu)) \mid f \in L^\infty(S, \mu)\}$$

of the C^* algebra $\mathcal{B}(L^2(S, \mu), L^2(S, \mu))$. The algebra $\mathcal{M}(L^2(S, \mu))$ is called the **multiplication algebra** on $L^2(S, \mu)$. Returning to the verification of $(*)$, let $\epsilon > 0$ be given with $\epsilon \le \|f\|_\infty$, and let

$$E = \big\{x \bigm| |f(x)| \ge \|f\|_\infty - \epsilon\big\}.$$

Then $0 < \mu(E) < \infty$, and we take g to be the function that is 1 on E and is 0 on E^c. Then $\|g\|_2 = \mu(E)^{1/2}$, and

$$\|fg\|_2^2 = \int_X |fg|^2\,d\mu = \int_E |f|^2\,d\mu \geq (\|f\|_\infty - \epsilon)^2 \mu(E).$$

Therefore

$$(\|f\|_\infty - \epsilon)\mu(E)^{1/2} \leq \|M_f g\|_2 \leq \|M_f\|\|g\|_2 = \|M_f\|\mu(E)^{1/2},$$

and $\|f\|_\infty - \epsilon \leq \|M_f\|$. Since we already know that $\|M_f\| \leq \|f\|_\infty$ and since ϵ is arbitrary, we conclude that $(*)$ holds.

Now let us consider an arbitrary bounded self-adjoint linear operator on a separable Hilbert space. We mentioned at the end of Section 10 the two operators on $\mathbb{C}^3$ given by diagonal matrices $\operatorname{diag}(1, 1, 2)$ and $\operatorname{diag}(1, 2, 2)$. The C^* algebras generated by these operators are isomorphic 2-dimensional algebras, and hence there is no way to superimpose on the setting of Theorem 4.48 the action of the operators on the Hilbert space $\mathbb{C}^3$ if we consider these operators by themselves. The operators do get distinguished, however, if we enlarge the C^* algebra under consideration, working instead with the 3-dimensional commutative C^* algebra of all diagonal matrices. In the general situation, as long as we are going to enlarge the algebra of operators under consideration, we may as well enlarge it as much as possible while keeping it commutative.

If H is a Hilbert space, a **maximal abelian self-adjoint subalgebra** in $\mathcal{B}(H, H)$ is a commutative C^* subalgebra of $\mathcal{B}(H, H)$ that is not contained in any larger commutative subalgebra of $\mathcal{B}(H, H)$ that is closed under $(\,\cdot\,)^*$. In the example with $H = \mathbb{C}^3$ in the previous paragraph, the 3-dimensional algebra of diagonal matrices is a maximal abelian self-adjoint subalgebra.

For general H, we shall obtain a simple criterion for a subalgebra to be maximal abelian self-adjoint, we shall show that the multiplication algebra for an L^2 space with respect to a finite measure meets this criterion, and then we shall see that maximal abelian self-adjoint subalgebras have a special property that will allow us to incorporate the Hilbert space into an application of Theorem 4.48.

If $\mathcal{T}$ is a subset of $\mathcal{B}(H, H)$, let

$$\mathcal{T}' = \{A \in \mathcal{B}(H, H) \mid AB = BA \text{ for all } B \in \mathcal{T}\}.$$

The set $\mathcal{T}'$ is a subalgebra of $\mathcal{B}(H, H)$ containing the identity and called the **commuting algebra** of $\mathcal{T}$. It has the following properties:

(i) $\mathcal{T}'$ is closed in the operator-norm topology,
(ii) $\mathcal{T}' \supseteq \mathcal{T}$ if and only if $\mathcal{T}$ is commutative,

(iii) if $\mathcal{T}$ is stable under $(\,\cdot\,)^*$, then $\mathcal{T}'$ is stable under $(\,\cdot\,)^*$ and hence is a C^* subalgebra of $\mathcal{B}(H, H)$,

(iv) a subalgebra $\mathcal{A}$ of $B(H, H)$ stable under $(\,\cdot\,)^*$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$ if and only if $\mathcal{A}' = \mathcal{A}$.

All of these properties are verified by inspection except possibly the assertion in (iv) that $\mathcal{A}$ maximal implies that $\mathcal{A}'$ does not strictly contain $\mathcal{A}$. For this assertion let $\mathcal{A}$ be maximal, and suppose that B lies in $\mathcal{A}'$ but not $\mathcal{A}$. Since $\mathcal{A}$ is stable under $(\,\cdot\,)^*$, B^* lies in $\mathcal{A}'$, and so does $B + B^*$. Then $B + B^*$ and $\mathcal{A}$ together generate a C^* subalgebra that is commutative and strictly contains $\mathcal{A}$, in contradiction to the maximality of $\mathcal{A}$. This proves (iv).

Proposition 4.50. If (S, μ) is a finite measure space, then the multiplication algebra on $L^2(S, \mu)$ is a maximal abelian self-adjoint subalgebra of the algebra $\mathcal{B}(L^2(S, \mu), L^2(S, \mu))$.

PROOF. Write $\mathcal{M}$ for $\mathcal{M}(L^2(S, \mu))$. Since $\mathcal{M}$ is commutative, (ii) shows that $\mathcal{M}' \supseteq \mathcal{M}$. Since $\mathcal{M}$ is stable under $(\,\cdot\,)^*$, (iv) shows that it is enough to prove that $\mathcal{M}' \subseteq \mathcal{M}$. Thus let T be in $\mathcal{M}'$, and define an L^2 function g by $g = T(1)$. If f is in L^∞, then the fact that T is in $\mathcal{M}'$ implies that

$$Tf = TM_f(1) = M_f T(1) = M_f g = fg.$$

If the set where $N \le |g(x)| \le N + 1$ has positive measure, then an argument in the example with $L^2(S, \mu)$ shows that $\|T\| \ge N$. Since T is assumed bounded, we conclude that g is in L^∞. Therefore $Tf = M_g f$ for all f in L^∞. Since L^∞ is dense in L^2 for a finite measure space and since T and M_g are both bounded, $T = M_g$. Therefore T is exhibited as in $\mathcal{M}$, and the proof that $\mathcal{M}' \subseteq \mathcal{M}$ is complete. □

We come now to the special property of maximal abelian self-adjoint subalgebras that will allow us to bring the Hilbert space into play when applying Theorem 4.48 to these subalgebras. If $\mathcal{A}$ is any subalgebra of $\mathcal{B}(H, H)$, a vector x in H is called a **cyclic vector** for $\mathcal{A}$ if the vector subspace $\mathcal{A}x$ of H is dense in H.

Lemma 4.51. Let H be a complex Hilbert space, let $K \subseteq H$ be a closed vector subspace, and let E be the orthogonal projection of H on K. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(H, H)$ that is stable under $(\,\cdot\,)^*$ and has the property that $A(K) \subseteq K$ for all A in $\mathcal{A}$, then E is in $\mathcal{A}'$.

PROOF. Since $A(K) \subseteq K$, $AE(x)$ is in K for all x in H. Therefore $AE(x) = EAE(x)$ for all x in H, and $AE = EAE$. Since $E^* = E$ and since $\mathcal{A}$ is stable under $(\,\cdot\,)^*$, $A^*E = EA^*E$. Consequently $EA = E^*A = (A^*E)^* = (EA^*E)^* = EAE = AE$, and E is in $\mathcal{A}'$. □

Proposition 4.52. If H is a complex separable Hilbert space and $\mathcal{A}$ is a maximal self-adjoint subalgebra of $\mathcal{B}(H, H)$, then $\mathcal{A}$ has a cyclic vector.

REMARKS. The 2-dimensional subalgebras that we introduced in connection with $\mathbb{C}^3$ have no cyclic vectors, as we see by a count of dimensions; however, the full 3-dimensional diagonal subalgebra has $\begin{pmatrix}1\\1\\1\end{pmatrix}$ as a cyclic vector since

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}.$$

PROOF. For each x in H, form the closed vector subspace $(\mathcal{A}x)^{\text{cl}}$. Since the identity is in $\mathcal{A}$, x is in $\mathcal{A}x$. Since $\mathcal{A}x$ is stable under $\mathcal{A}$ and since the members of $\mathcal{A}$ are bounded operators, $(\mathcal{A}x)^{\text{cl}}$ is stable under $\mathcal{A}$. The vector subspace $\mathcal{A}x$ has the property that

$$y \perp \mathcal{A}x \qquad \text{implies} \qquad \mathcal{A}y \perp \mathcal{A}x \tag{$*$}$$

because $(Ax, By) = (y, A^*Bx) = 0$ if A and B are in $\mathcal{A}$. Consider orthonormal subsets $\{x_\alpha\}$ in H such that $\mathcal{A}x_\alpha \perp \mathcal{A}x_\beta$ for $\alpha \neq \beta$. Such sets exist, the empty set being one. By Zorn's Lemma let $S = \{x_\alpha\}$ be a maximal such set. This maximal S has the property that

$$H = \Big(\sum_{x_\alpha \in S} \mathcal{A}x_\alpha\Big)^{\text{cl}},$$

since otherwise we could obtain a contradiction by adjoining any unit vector in $\big(\big(\sum_{x_\alpha \in S} \mathcal{A}x_\alpha\big)^{\text{cl}}\big)^\perp$ to S and applying $(*)$. Since H is separable, S is countable. Let us enumerate its members as $x_1, x_2, \dots$. Put $z = \sum_{n=1}^\infty 2^{-n} x_n$. This series converges in H since H is complete, and we shall prove that the sum z is a cyclic vector for $\mathcal{A}$.

Lemma 4.51 implies that the orthogonal projection E_n of H onto $(\mathcal{A}x_n)^{\text{cl}}$ is in $\mathcal{A}'$. Since $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$, $\mathcal{A}' = \mathcal{A}$. Hence E_n is in $\mathcal{A}$. Therefore $\mathcal{A}z \supseteq \mathcal{A}E_n z = \mathcal{A}2^{-n}x_n = \mathcal{A}x_n$ for all n, and we obtain $(\mathcal{A}z)^{\text{cl}} \supseteq \big(\sum_n \mathcal{A}x_n\big)^{\text{cl}} = H$. This completes the proof. $\square$

If H_1 and H_2 are complex Hilbert spaces, a **unitary operator** U from H_1 to H_2 is a linear operator from H_1 *onto* H_2 with $\|Ux\|_{H_2} = \|x\|_{H_1}$ for all x in H_1. Such an operator is invertible, and its inverse is unitary. By means of polarization, one sees that a unitary operator satisfies also the identity $(Ux, Uy)_{H_2} = (x, y)_{H_1}$, i.e., that the inner product is preserved. Therefore a unitary operator provides the natural notion of isomorphism between two Hilbert spaces.

Theorem 4.53. If H is a nonzero complex separable Hilbert space and $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$, then there exists a measure space (S, μ) with $\mu(S) = 1$ and a unitary operator $U : H \to L^2(S, \mu)$ such that

$$U\mathcal{A}U^{-1} = \mathcal{M}(L^2(S, \mu)).$$

REMARK. In other words, under the assumption that H is separable, any maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$ is isomorphic to the multiplication algebra for the L^2 space relative to some finite measure.

PROOF. Applying Proposition 4.52, let z be a unit cyclic vector for $\mathcal{A}$. Let us see that the linear map of $\mathcal{A}$ into H given by $A \mapsto Az$ is one-one. In fact, if $Az = 0$, then every B in $\mathcal{A}$ has $A(Bz) = BAz = B0 = 0$. Since $\mathcal{A}z$ is dense in H and A is bounded, A is 0.

We saw before Proposition 4.50 that $\mathcal{A}$ is a commutative C^* algebra with identity. By Theorem 4.48 the Gelfand transform $A \mapsto \widehat{A}$ is a norm-preserving algebra isomorphism of $\mathcal{A}$ onto $C(\mathcal{A}_{\mathrm{m}}^*)$ carrying $(\,\cdot\,)^*$ to complex conjugation. Define a linear functional ℓ on $C(\mathcal{A}_{\mathrm{m}}^*)$ by

$$\ell(\widehat{A}) = (Az, z)_H,$$

the inner product being the inner product in H. Let us see that the linear functional ℓ is positive. In fact, any function ≥ 0 in $C(\mathcal{A}_{\mathrm{m}}^*)$ is the absolute value squared of some element of $C(\mathcal{A}_{\mathrm{m}}^*)$, hence is of the form $|\widehat{A}|^2$. Then

$$\ell(|\widehat{A}|^2) = \ell(\overline{\widehat{A}}\,\widehat{A}) = \ell(\widehat{A^*A}) = (A^*Az, z)_H = (Az, Az)_H \geq 0.$$

By the Riesz Representation Theorem, there exists a unique regular Borel measure μ on $\mathcal{A}_{\mathrm{m}}^*$ such that

$$\ell(\widehat{A}) = \int_{\mathcal{A}_{\mathrm{m}}^*} \widehat{A}\, d\mu$$

for all $\widehat{A}$ in $C(\mathcal{A}_{\mathrm{m}}^*)$. The measure μ has total mass equal to $\ell(1) = \ell(\widehat{I}) = (Iz, z)_H = \|z\|_H^2 = 1$.

We shall now construct the unitary operator U carrying H to $L^2(\mathcal{A}_{\mathrm{m}}^*, \mu)$. On the dense vector subspace $\mathcal{A}z$ of H, define a linear mapping U_0 by

$$U_0 Az = \widehat{A} \in C(\mathcal{A}_{\mathrm{m}}^*) \subseteq L^2(\mathcal{A}_{\mathrm{m}}^*, \mu).$$

This is well defined since, as we have seen, $Az = 0$ implies $A = 0$. On the vector subspace $\mathcal{A}z$, we have

$$\|U_0 Az\|^2_{L^2(\mathcal{A}_{\mathrm{m}}^*)} = \int_{\mathcal{A}_{\mathrm{m}}^*} |\widehat{A}|^2\, d\mu = \int_{\mathcal{A}_{\mathrm{m}}^*} \widehat{A^*A}\, d\mu = \ell(A^*A) = (A^*Az, z)_H = \|Az\|_H^2.$$

Hence U_0 is an isometry from the dense subset $\mathcal{A}z$ of H into $L^2(\mathcal{A}^*_{\mathrm{m}})$. By uniform continuity, U_0 extends to an isometry U from H into $L^2(\mathcal{A}^*_{\mathrm{m}})$. As the continuous extension of the linear function U_0, U is linear. The image of U contains $C(\mathcal{A}^*_{\mathrm{m}})$, which is dense in $L^2(\mathcal{A}^*_{\mathrm{m}}, \mu)$, and the image is complete, being isometric with H. Therefore the image of U is closed. Consequently U carries H onto $L^2(\mathcal{A}^*_{\mathrm{m}}, \mu)$ and is unitary.

We still have to check that $U\mathcal{A}U^{-1} = \mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu))$. If A and B are in $\mathcal{A}$, then

$$UAU^{-1}(\widehat{B}) = UA(Bz) = U(ABz) = \widehat{AB} = \widehat{A}\,\widehat{B} = M_{\widehat{A}}\widehat{B}.$$

Since UAU^{-1} and $M_{\widehat{A}}$ are bounded and since the $\widehat{B}$'s are dense in $L^2(\mathcal{A}^*_{\mathrm{m}}, \mu)$, $UAU^{-1} = M_{\widehat{A}}$. Therefore $U\mathcal{A}U^{-1} \subseteq \mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu))$. Next let T be in $\mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu))$. Then T commutes with every member of $\mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu))$ and in particular with every UAU^{-1}. Thus $TUAU^{-1} = UAU^{-1}T$ for all A in $\mathcal{A}$, and $U^{-1}TUA = AU^{-1}TU$. Since A is arbitrary in $\mathcal{A}$, $U^{-1}TU$ is in $\mathcal{A}'$. But $\mathcal{A}$ is assumed to be a maximal abelian self-adjoint subalgebra, and therefore $\mathcal{A}' = \mathcal{A}$. Consequently $U^{-1}TU$ is in $\mathcal{A}$. Say that $U^{-1}TU = A_0$. Then $T = UA_0U^{-1}$, and T is in $U\mathcal{A}U^{-1}$. Therefore $U\mathcal{A}U^{-1} = \mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu))$. □

The Spectral Theorem for a single bounded self-adjoint operator will be an immediate consequence of Theorem 4.53 and an application of Zorn's Lemma. But let us state the result (Theorem 4.54) so that it applies to a wider class of operators—and to a commuting family of such operators rather than just one.

The first step is to define the kinds of bounded linear operators of interest. Let H be a complex Hilbert space. A bounded linear operator $A : H \to H$ is said to be

- **normal** if $A^*A = AA^*$,
- **positive semidefinite** if it is self adjoint[22] and $(Ax, x) \geq 0$ for all $x \in H$,
- **unitary** if A is onto H and has $\|Ax\| = \|x\|$ for all $x \in H$.

Self-adjoint operators, having $A^* = A$, are certainly normal. Every operator of the form A^*A for some bounded linear A is positive semidefinite. The definition of "unitary" merely specializes the definition before Theorem 4.53 to the case that $H_1 = H_2$. Unitary operators A in the present setting, according to Proposition 2.6, are characterized by the condition that A is invertible with $A^{-1} = A^*$, and unitary operators are therefore normal.

In the case of multiplication operators M_f by L^∞ functions on L^2 of a finite measure space, the adjoint of M_f is $M_{\bar{f}}$. Then every M_f is normal, M_f is self adjoint if and only if f is real-valued a.e., M_f is positive semidefinite if and only if f is ≥ 0 a.e., and M_f is unitary if and only if $|f| = 1$ a.e.

[22] The condition "self adjoint" can be shown to be automatic in the presence of the inequality $(Ax, x) \geq 0$ for all x, but we shall not need to make use of this fact.

Theorem 4.54 (Spectral Theorem for bounded normal operators). Let $\{A_\alpha\}_{\alpha\in E}$ be a family of bounded normal operators on a complex separable Hilbert space H, and suppose that $A_\alpha A_\beta = A_\beta A_\alpha$ and $A_\alpha A_\beta^* = A_\beta^* A_\alpha$ for all α and β. Then there exist a finite measure space (S, μ), a unitary operator $U : H \to L^2(S, \mu)$, and a set $\{f_\alpha\}_{\alpha\in E}$ of functions in $L^\infty(S, \mu)$ such that $UA_\alpha U^{-1} = M_{f_\alpha}$ for all α in E.

PROOF. Let $\mathcal{A}_0$ be the complex subalgebra of $\mathcal{B}(H, H)$ generated by I and all A_α and A_α^* for α in E. This algebra is commutative and is stable under $(\,\cdot\,)^*$. Let $\mathcal{S}$ be the set of all commutative subalgebras of $\mathcal{B}(H, H)$ containing $\mathcal{A}_0$ and stable under $(\,\cdot\,)^*$, and partially order $\mathcal{S}$ by inclusion upward. The union of the members of a chain in $\mathcal{S}$ is an upper bound for the chain, and Zorn's Lemma therefore produces a maximal element $\mathcal{A}$ in $\mathcal{S}$. Since $\mathcal{A}$ is maximal, it is necessarily closed in the operator-norm topology. Then $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$, and Theorem 4.53 is applicable. The theorem yields a finite measure space (S, μ) and a unitary operator $U : H \to L^2(S, \mu)$ such that $U\mathcal{A}U^{-1} = \mathcal{M}(L^2(S, \mu))$. For each α in E, we then have $UA_\alpha U^{-1} = M_{f_\alpha}$ for some f_α in $L^\infty(S, \mu)$, as required. $\square$

In a corollary we shall characterize the spectra of operators that are self adjoint, or positive definite, or unitary. Implicitly in the statement and proof, we make use of Corollary 4.49 when referring to $\sigma(A)$: the set $\sigma(A)$ is independent of the Banach subalgebra of $\mathcal{B}(H, H)$ from which it is computed as long as that subalgebra contains I, A, and A^*. The corollary needs one further thing beyond Theorem 4.54, and we give that in the lemma below.

Lemma 4.55. Let (S, μ) be a finite measure space, and form the Hilbert space $L^2(S, \mu)$. For f in $L^\infty(S, \mu)$, let M_f be the operation of multiplication by f. Define the **essential image** of f to be

$$\big\{\lambda_0 \in \mathbb{C} \bigm| \mu\big(f^{-1}(\{\lambda \in \mathbb{C} \bigm| |\lambda - \lambda_0| < \epsilon\})\big) > 0 \text{ for every } \epsilon > 0\big\}.$$

Then

$$\sigma(M_f) = \text{essential image of } f.$$

PROOF. To prove $\subseteq$ in the asserted equality, let λ_0 be outside the essential image, and choose $\epsilon > 0$ such that $f^{-1}(\{|\lambda - \lambda_0| < \epsilon\})$ has measure 0. Then $|f(x) - \lambda_0| \ge \epsilon$ a.e. Hence $1/(f - \lambda_0)$ is in L^∞, and $M_{1/(f-\lambda_0)}$ exhibits $M_{f-\lambda_0}$ as invertible. Thus λ_0 is not in $\sigma(M_f)$.

For the inclusion $\supseteq$, suppose that $M_{f-\lambda_0}$ is invertible, with inverse T. For every g in L^∞, we have $M_{f-\lambda_0}M_g = M_g M_{f-\lambda_0}$. Multiplying this equality by T twice, we obtain $M_g T = T M_g$. By Proposition 4.50, T is of the form $T = M_h$ for some h in L^∞. Then we must have $(f - \lambda_0)h = 1$ a.e. Hence $|f(x) - \lambda_0| \ge \|h\|_\infty^{-1}$ a.e., and λ_0 is outside the essential image. This proves the lemma. $\square$

Corollary 4.56. Let H be a complex separable Hilbert space, let A be a normal operator in $\mathcal{B}(H, H)$, and let $\sigma(A)$ be the spectrum of A. Then

(a) A is self adjoint if and only if $\sigma(A) \subseteq \mathbb{R}$,
(b) A is positive semidefinite if and only if $\sigma(A) \subseteq [0, +\infty)$,
(c) A is unitary if and only if $\sigma(A) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$.

PROOF. The corollary is immediate from Theorem 4.54 as long as the corollary is proved for any multiplication operator $A = M_f$ by an L^∞ function f on the Hilbert space $L^2(S, \mu)$. For this purpose we shall use Lemma 4.55.

In the case of (a), the operator M_f is self adjoint if and only if f is real-valued a.e. If f is real-valued, then the definition of essential image shows that λ_0 is not in the essential image if λ_0 is nonreal. Conversely if every nonreal λ_0 is outside the essential image, then to each such λ_0 we can associate a number $\epsilon_{\lambda_0} > 0$ for which $f^{-1}(\{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \epsilon_{\lambda_0}\})$ has μ measure 0. Countably many of the open sets $\{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \epsilon_{\lambda_0}\}$ cover the complement of $\mathbb{R}$ in $\mathbb{C}$, and their inverse images under f have μ measure 0. Therefore the inverse image under f of the union has μ measure 0, and $\mu(f^{-1}(\mathbb{R}^c)) = 0$. That is, f is real-valued a.e. This proves (a), and the arguments for (b) and (c) are completely analogous. $\square$

The power of the Spectral Theorem comes through the functional calculus that it implies for working with operators. We shall prove the relevant theorem and then give five illustrations of how it is used.

Theorem 4.57 (functional calculus). Fix a bounded normal operator A on a complex separable Hilbert space H. Then there exists one and only one way to define a system of operators $\varphi(A)$ for every bounded Borel function φ on $\sigma(A)$ such that

(a) $z(A) = A$ for the function $\varphi(z) = z$, and $1(A) = I$ for the constant function 1,
(b) $\varphi \mapsto \varphi(A)$ is an algebra homomorphism into $\mathcal{B}(H, H)$,
(c) $\varphi(A)^* = \overline{\varphi}(A)$,
(d) $\lim_n \varphi_n(A)x = \varphi(A)x$ for all $x \in H$ whenever $\varphi_n \to \varphi$ pointwise with $\{\varphi_n\}$ uniformly bounded.

The operators $\varphi(A)$ have the additional properties that

(e) $\varphi(A)$ is normal, and all the operators $\varphi(A)$ commute,
(f) $\|\varphi(A)\| \le \|\varphi\|_{\sup}$,
(g) $\lim_n \varphi_n(A) = \varphi(A)$ in the operator-norm topology whenever $\varphi_n \to \varphi$ uniformly,
(h) $\sigma(\varphi(A)) \subseteq (\varphi(\sigma(A))^{\text{cl}}$,
(i) **(spectral mapping property)** $\sigma(\varphi(A)) = \varphi(\sigma(A))$ if φ is continuous.

PROOF OF EXISTENCE. Apply Theorem 4.54 to the singleton set $\{A\}$, obtaining a finite measure space (S, μ), a unitary operator $U : H \to L^2(S, \mu)$, and an L^∞ function f_A on S such that $UAU^{-1} = M_{f_A}$. Examining the proofs of Theorems 4.53 and 4.54, we see that we can take S to be a certain compact Hausdorff space $\mathcal{A}^*_{\mathrm{m}}$, μ to be a regular Borel measure on S, and the function f_A to be the Gelfand transform $\widehat{A}$, therefore continuous. In the construction of Theorem 4.53, the measure μ has the property that $\int_S |\widehat{B}|^2\, d\mu = \|Bz\|_H$ for every B in $\mathcal{A}$, where z is a cyclic vector. Therefore $B \neq 0$ implies $\int_S |\widehat{B}|^2\, d\mu > 0$. Since $|\widehat{B}|^2$ is the most general continuous function ≥ 0 on S, μ assigns positive measure to every nonempty open set.

For any bounded Borel function φ on $\sigma(A)$, the function $\varphi \circ f_A$ is a well-defined function on S since Proposition 4.43a shows that the image of $\widehat{A} = f_A$ is $\sigma(A)$. The function $\varphi \circ f_A$ is a bounded Borel function since φ^{-1} of an open set in $\mathbb{C}$ is a Borel set of $\mathbb{C}$ and since f_A^{-1} of a Borel set of $\mathbb{C}$ is a Borel set of S. Thus it makes sense to define

$$\varphi(A) = U^{-1} M_{\varphi \circ f_A} U.$$

Then we see that properties (a) through (i) are satisfied for any given normal A on H if they are valid in the special case of any M_f on $L^2(S, \mu)$ with f continuous, S compact Hausdorff, μ a regular Borel measure assigning positive measure to every nonempty open set, and $\varphi(M_f)$ defined for arbitrary bounded Borel functions φ on the image of f by

$$\varphi(M_f) = M_{\varphi \circ f}.$$

Properties (a) through (c) for multiplication operators are immediate, (d) follows by dominated convergence, (e) and (f) are immediate, and (g) follows directly from (f). We are left with properties (h) and (i).

Lemma 4.55 identifies the spectrum of a multiplication operator by an L^∞ function with the essential image of the function. Using this identification, we see that (h) and (i) follow in our special case if it is proved for f continuous that

$$\text{essential image of } \varphi \circ f \subseteq (\varphi(\text{essential image of } f))^{\mathrm{cl}}, \quad \varphi \text{ bounded Borel}, \quad (*)$$

$$\text{essential image of } \varphi \circ f = \varphi(\text{essential image of } f), \quad \varphi \text{ continuous}. \quad (**)$$

Let us see that these follow if we prove that

$$\text{essential image of } \psi \subseteq (\text{image } \psi)^{\mathrm{cl}} \quad \text{for } \psi : S \to \mathbb{C} \text{ bounded Borel}, \quad (\dagger)$$

$$\text{essential image of } \psi = \text{image } \psi \quad \text{for } \psi : S \to \mathbb{C} \text{ continuous}. \quad (\dagger\dagger)$$

In fact, if (†) and (††) hold, then for ($*$) we have

$$\begin{aligned} \text{essential image}(\varphi\circ f) &\subseteq (\text{image}(\varphi\circ f))^{\text{cl}} && \text{by } (\dagger) \text{ for } \varphi\circ f \\ &= (\varphi(\text{image } f))^{\text{cl}} \\ &= (\varphi(\text{essential image } f))^{\text{cl}} && \text{by } (\dagger\dagger) \text{ for } f. \end{aligned}$$

For ($**$) we have

$$\begin{aligned} \text{essential image}(\varphi\circ f) &= \text{image}(\varphi\circ f) && \text{by } (\dagger\dagger) \text{ for } \varphi\circ f \\ &= \varphi(\text{image } f) \\ &= \varphi(\text{essential image } f) && \text{by } (\dagger\dagger) \text{ for } f. \end{aligned}$$

Thus it is enough to prove (†) and (††). For (†) let λ_0 be in the essential image of ψ. Then for each $n \geq 1$, $\mu\big(\psi^{-1}\{\lambda \mid |\lambda-\lambda_0| < \frac{1}{n}\}\big) > 0$, and hence $\psi^{-1}\{\lambda \mid |\lambda-\lambda_0| < \frac{1}{n}\} \neq \varnothing$. Thus there exists $\lambda = \lambda_n$ with λ_n in the image of ψ such that $|\lambda-\lambda_0| < \frac{1}{n}$, and λ_0 is exhibited as a member of (image $\psi)^{\text{cl}}$.

For (††) we first show that the image of ψ lies in the essential image of ψ if ψ is continuous. Thus let λ_0 be in the image of ψ. Then $\psi^{-1}\{\lambda \mid |\lambda-\lambda_0| < \epsilon\}$ is nonempty, and it is open since ψ is continuous. Since nonempty open sets of S have positive μ measure, we conclude that λ_0 is in the essential image of ψ. Then

$$\begin{aligned} \text{image } \psi &\subseteq \text{essential image } \psi && \text{by what we have just proved} \\ &\subseteq (\text{image } \psi)^{\text{cl}} && \text{by } (\dagger) \\ &= \text{image } \psi && \text{since } S \text{ is compact and } \psi \text{ is continuous,} \end{aligned}$$

and (††) follows. This completes the proof of existence and the list of properties in Theorem 4.57. □

PROOF OF UNIQUENESS. Properties (a) through (c) determine $\varphi(A)$ whenever φ is a polynomial function of z and $\bar{z}$. By the Stone–Weierstrass Theorem any continuous φ on a compact set such as $\sigma(A)$ is the uniform limit of such polynomials, and hence (d) implies that $\varphi(A)$ is determined whenever φ is continuous.

The indicator function of a compact subset of $\mathbb{C}$ is the decreasing pointwise limit of a sequence of continuous functions of compact support, and hence (d) implies that $\varphi(A)$ is determined whenever φ is the indicator function of a compact set. Applying (b) twice, we see that $\varphi(A)$ is determined whenever φ is the indicator function of any finite disjoint union of differences of compact sets. Such sets form[23] the smallest algebra of sets containing the compact subsets of

[23]By Lemma 11.2 of *Basic*.

$\sigma(A)$. Another application of (d), together with the Monotone Class Lemma,[24] shows that $\varphi(A)$ is determined whenever φ is the indicator function of any Borel subset of $\sigma(A)$. Any bounded Borel function on $\sigma(A)$ is the uniform limit of finite linear combinations of indicator functions of Borel sets, and hence one more application of (b) and (d) shows that $\varphi(A)$ is determined whenever φ is a bounded Borel function on $\sigma(A)$. $\square$

Corollary 4.58. If H is a complex separable Hilbert space, then every positive semidefinite operator in $\mathcal{B}(H, H)$ has a unique positive semidefinite square root.

REMARKS. This is an important application of the Spectral Theorem and the functional calculus. It is already important when applied to operators of the form A^*A with A in $\mathcal{B}(H, H)$. For example the corollary allows us in the definition of trace-class operator before Proposition 2.8 to drop the assumption that the operator is compact; it is enough to assume that it is bounded.

PROOF. If A is positive semidefinite, then $\sigma(A) \subseteq [0, \infty)$ by Corollary 4.56b. The usual square root function $\sqrt{}$ on $[0, \infty)$ is bounded on $\sigma(A)$, and we can form $\sqrt{A}$ by Theorem 4.57. Then (a) and (b) in Theorem 4.57 imply that $(\sqrt{A})^2 = A$, and (i) implies that $\sqrt{A}$ is positive semidefinite. This proves existence.

For uniqueness let B be positive semidefinite with $B^2 = A$. Because of the uniqueness assertion in Theorem 4.57, we have at our disposal the maximal abelian self-adjoint subalgebra of $\mathcal{B}(H, H)$ that is recalled from Theorem 4.53 and used to define operators $\varphi(A)$ in the proof of Theorem 4.57. Let $\mathcal{A}_0$ be the smallest C^* algebra in $\mathcal{B}(H, H)$ containing I, A, and B, and extend $\mathcal{A}_0$ to a maximal abelian self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{B}(H, H)$. We use this $\mathcal{A}$ to define $\sqrt{A}$. On the compact Hausdorff space, $\widehat{\sqrt{A}}$ and $\widehat{B}$ are both nonnegative square roots of $\widehat{A}$ and must be equal. Since the Gelfand transform for $\mathcal{A}$ is one-one, $B = \sqrt{A}$. $\square$

Corollary 4.59. Let H be a complex separable Hilbert space, and let A and B be bounded normal operators on H such that A commutes with B and B^*. Then each $\varphi(A)$, for φ a bounded Borel function on $\sigma(A)$, commutes with B and B^*.

PROOF. As in the proof of the previous corollary, we have at our disposal the maximal abelian self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{B}(H, H)$ that is used to define operators $\varphi(A)$. We choose one containing I, A, and B. Then $\varphi(A)$ is in $\mathcal{A}$ and hence commutes with B and B^*. $\square$

Corollary 4.60. Let A be a bounded normal operator on a complex separable Hilbert space, let $\varphi_2 : \sigma(A) \to \mathbb{C}$ be a continuous function,

[24] Lemma 5.43 of *Basic*.

and let $\varphi_1 : \varphi_2(\sigma(A)) \to \mathbb{C}$ be a bounded Borel function. Then $\varphi_1(\varphi_2(A)) = (\varphi_1 \circ \varphi_2)(A)$.

REMARK. If $\varphi_2(z) = \bar{z}$, this corollary recovers property (c) in Theorem 4.57.

PROOF. The uniqueness in Theorem 4.57 shows that the operators $\varphi(\varphi_2(A))$ form the unique system defined for bounded Borel functions $\varphi : \sigma(\varphi_2(A)) \to \mathbb{C}$ such that $z(\varphi_2(A)) = \varphi_2(A)$, $1(\varphi_2(A)) = 1$, $\varphi \mapsto \varphi(\varphi_2(A)$ is an algebra homomorphism, $\varphi(\varphi_2(A))^* = \overline{\varphi}(\varphi_2(A))$, and $\lim \varphi_n(\varphi_2(A))x = \varphi(\varphi_2(A))x$ for all x whenever $\varphi_n \to \varphi$ pointwise and boundedly on $\sigma(\varphi_2(A))$.

We now consider the system formed from $\psi(A)$, specialize to functions $\psi = \varphi \circ \varphi_2$, and make use of the properties of $\psi(A)$ as stated in the existence half of the theorem. Theorem 4.57i shows that $\sigma(\varphi_2(A)) = \varphi_2(\sigma(A))$. We have $(z \circ \varphi_2)(A) = \varphi_2(A)$ trivially and $(1 \circ \varphi_2)(A) = 1(A) = 1$ by (a) for the system $\psi(A)$. The map $\varphi \mapsto (\varphi \circ \varphi_2)(A)$ is an algebra homomorphism as a special case of (b) for $\psi(A)$. The formula $(\varphi \circ \varphi_2)(A)^* = \overline{\varphi \circ \varphi_2}(A) = (\overline{\varphi} \circ \varphi_2)(A)$ is a special case of (c) for $\psi(A)$. And the formula $\lim(\varphi_n \circ \varphi_2)(A)x = (\varphi \circ \varphi_2)(A)x$ is a special case of (d) for $\psi(A)$. Therefore the system $(\varphi \circ \varphi_2)(A)$ has the properties that uniquely determine the system $\varphi(\varphi_2(A))$, and we must have $\varphi(\varphi_2(A)) = (\varphi \circ \varphi_2)(A)$ for every bounded Borel function φ on $\sigma(\varphi_2(A))$. □

Corollary 4.61. If A is a bounded normal operator on a complex separable Hilbert space, then there exists a sequence $\{S_n\}$ of bounded linear operators of the form $S_n = \sum_{i=1}^{N_n} c_{i,n} E_{i,n}$ converging to A in the operator-norm topology and having the property that each $E_{i,n}$ is an orthogonal projection of the form $\varphi(A)$.

PROOF. Choose a sequence of simple Borel functions s_n on $\sigma(A)$ converging uniformly to the function z, and let $S_n = s_n(A)$. Then apply Theorem 4.57. □

Corollary 4.62. If A is a bounded normal operator on a complex separable Hilbert space H of dimension > 1, then there exists a nontrivial orthogonal projection that commutes with every bounded normal operator that commutes with A and A^*. Hence there is a nonzero proper closed vector subspace K of H such that $B(K) \subseteq K$ for every bounded normal operator B commuting with A and A^*.

PROOF. This is a special case of Corollary 4.61. □

This completes our list of illustrations of the functional calculus associated with the Spectral Theorem. We now prove a result mentioned near the end of Section 10, showing how the spectrum of an operator relates to spaces of maximal ideals.

Proposition 4.63. Let A be a bounded normal operator on a complex separable Hilbert space H, and let $\mathcal{A}$ be the smallest C^* algebra of $\mathcal{B}(H, H)$ containing I, A, and A^*. Then the maximal ideal space $\mathcal{A}_{\mathrm{m}}^*$ is canonically homeomorphic to the spectrum $\sigma(A)$.

PROOF. Let $B \mapsto \widehat{B}$ be the Gelfand transform for $\mathcal{A}$, carrying $\mathcal{A}$ to $C(\mathcal{A}_{\mathrm{m}}^*)$. Proposition 4.43a shows that the image of $\widehat{A}$ in $\mathbb{C}$ is $\sigma(A)$, and Corollary 4.49 shows that this version of $\sigma(A)$ is the same as the one obtained from $\mathcal{B}(H, H)$. Therefore we obtain a map $C(\sigma(A)) \to C(\mathcal{A}_{\mathrm{m}}^*)$ by the definition $f \mapsto f \circ \widehat{A}$. This map is an algebra homomorphism respecting conjugation, and it satisfies $\|f\|_{\sup} = \|f \circ \widehat{A}\|_{\sup}$ since the function $\widehat{A}$ is onto $\sigma(A)$. This equality of norms implies that the map $f \mapsto f \circ \widehat{A}$ is one-one.

To see that $f \mapsto f \circ \widehat{A}$ is onto $C(\mathcal{A}_{\mathrm{m}}^*)$, we observe that the operators $p(A, A^*)$, for p a polynomial in z and $\bar{z}$, are dense in $\mathcal{A}$ since I, A, and A^* generate $\mathcal{A}$. Using that $\widehat{(\cdot)}$ is a norm-preserving isomorphism of $\mathcal{A}$ onto $C(\mathcal{A}_{\mathrm{m}}^*)$, we see that the members $\widehat{p(A, A^*)}$ of $C(\mathcal{A}_{\mathrm{m}}^*)$ are dense in $C(\mathcal{A}_{\mathrm{m}}^*)$. Since $C(\sigma(A))$ is complete and $f \mapsto f \circ \widehat{A}$ is norm preserving, the image is closed. Therefore $f \mapsto f \circ \widehat{A}$ carries $C(\sigma(A))$ onto $C(\mathcal{A}_{\mathrm{m}}^*)$.

Hence we have a canonical isomorphism of commutative C^* algebras $C(\sigma(A))$ and $C(\mathcal{A}_{\mathrm{m}}^*)$. The maximal ideal spaces must be canonically homeomorphic. The maximal ideal space of $C(\sigma(A))$ contains $\sigma(A)$ because of the point evaluations but can be no larger than $\sigma(A)$ since the Stone Representation Theorem (Theorem 4.15) shows that the necessarily closed image of $\sigma(A)$ is dense in $\big(C(\sigma(A))\big)_{\mathrm{m}}^*$. $\square$

FURTHER REMARKS. A version of the Spectral Theorem is valid also for *unbounded* self-adjoint operators on a complex separable Hilbert space. Such operators are of importance since they enable one to use functional analysis directly with linear differential operators, which may be expected to be unbounded. The operator L in the Sturm–Liouville theory of Chapter I is an example of the kind of operator that one wants to handle directly. The subject has to address a large number of technical details, particularly concerning domains of operators, and the definitions have to be made just right. The prototype of an unbounded self-adjoint operator is the multiplication operator M_f on our usual $L^2(S, \mu)$ corresponding to an unbounded real-valued function f that is finite almost everywhere; the domain of M_f is the dense vector subspace of members of L^2 whose product with f is in L^2. Just as in this example, the domain of an unbounded self-adjoint operator is forced by the definitions to be a dense but proper vector subspace of the whole Hilbert space. Once one is finally able to state the Spectral Theorem for unbounded self-adjoint operators precisely, the result is proved by reducing it to Theorem 4.54. Specifically if T is an unbounded self-adjoint operator on H, then one shows that $(T + i)^{-1}$ is a globally defined bounded normal operator.

Application of Theorem 4.54 to $(T+i)^{-1}$ yields an L^∞ function g such that the unitary operator $U : H \to L^2(S,\mu)$ carries $(T+i)^{-1}$ to g. One wants T to be carried to f, and hence the definition should force $1/(f+i) = g$. In other words, f is defined by the equation $f = 1/g - i$. One checks that the unitary operator U from H to L^2 indeed carries T to M_f. For a discussion of the use of the Spectral Theorem in connection with partial differential equations, the reader can look at Parts 2 and 3 of Dunford–Schwartz's *Linear Operators*.

BIBLIOGRAPHICAL REMARKS. The exposition in Section 3–6 and Section 8–9 is based on that in Part 1 of Dunford–Schwartz's *Linear Operators*. The exposition in Section 7 is based on that in Treves's *Topological Vector Spaces, Distributions and Kernels*.

12. Problems

1. Let X be a Banach space, and let Y be a closed vector subspace. Take as known (from Problem 4 in Chapter XII of *Basic*) that X/Y becomes a normed linear space under the definition $\|x+Y\| = \inf_{y\in Y}\|x+y\|$ and that the resulting norm is complete. Prove that the topology on X/Y obtained this way coincides with the quotient topology on X/Y as the quotient of a topological vector space by a closed vector subspace.
2. Let $T : X \to Y$ be a linear function between Banach spaces such that $T(X)$ is finite-dimensional and $\ker(T)$ is closed. Prove that T is continuous.
3. Using the result of Problem 1, derive the Interior Mapping Theorem for Banach spaces from the special case in which the mapping is one-one.
4. If X is a finite-dimensional normed linear space, why must the norm topology coincide with the weak topology?
5. Let H be a separable infinite-dimensional Hilbert space. Give an example of a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ for all n and with $\{x_n\}$ tending to 0 weakly.
6. In a σ-finite measure space (S,μ), suppose that the sequence $\{f_n\}$ tends weakly to f in $L^2(S,\mu)$ and that $\lim_n \|f_n\|_2 = \|f\|_2$. Prove that $\{f_n\}$ tends to f in the norm topology of $L^2(S,\mu)$.
7. Let X be a normed linear space, let $\{x_n\}$ be a sequence in X with $\{\|x_n\|\}$ bounded, and let x_0 be in X. Prove that if $\lim_n x^*(x_n) = x^*(x_0)$ for all x^* in a dense subset of X^*, then $\{x_n\}$ tends to x_0 weakly.
8. Fix p with $0 < p < 1$. It was shown in Section 1 that the set of Borel functions f on [0, 1] with $\int_{[0,1]} |f|^p\,dx < \infty$, with two functions identified when they are equal almost everywhere, forms a topological vector space $L^p([0,1])$ under the metric $d(f,g) = \int_{[0,1]} |f-g|\,dx$. Put $D(f) = \int_{[0,1]} |f|^p\,dx$.

(a) Show for each positive integer n that any function f with $D(f) = 1$ can be written as $f = \frac{1}{n}(f_1 + \cdots + f_n)$ with $D(f_j) = n^{-(1-p)}$.
(b) Deduce from (a) that if f has $D(f) = 1$, then an arbitrarily large multiple of f can be written as a convex combination of functions f_j with $D(f_j) \le 1$.
(c) Deduce from (b) for each $\varepsilon > 0$ that the smallest convex set containing all f's with $D(f) \le \varepsilon$ is all of $L^p([0,1])$.
(d) Why must $L^p([0,1])$ fail to be locally convex?
(e) Prove that $L^p([0,1])$ has no nonzero continuous linear functionals.

9. Let U be a nonempty open set in $\mathbb{R}^N$, and let $\{K_p\}_{p\ge 0}$ be an exhausting sequence of compact subsets of U with $K_0 = \varnothing$. Let M be the set of all monotone increasing sequences of integers $m_p \ge 0$ tending to infinity, and let E be the set of all monotone decreasing sequences of real numbers $\varepsilon_p > 0$ tending to 0. For each pair $(m, \varepsilon) = \big(\{m_p\}, \{\varepsilon_p\}\big)$ with $m \in M$ and $\varepsilon \in E$, define a seminorm $\|\cdot\|_{m,\varepsilon}$ on $C^\infty_{\text{com}}(U)$ by

$$\|\varphi\|_{m,\varepsilon} = \sup_{p\ge 0} \varepsilon_p^{-1}\Big(\sup_{x\notin K_p}\ \sup_{|\alpha|\le m_p} |(D^\alpha\varphi)(x)|\Big).$$

Denote the inductive limit topology on $C^\infty_{\text{com}}(U)$ by $\mathcal{T}$ and the topology defined with the above uncountable family of seminorms by $\mathcal{T}'$.
(a) Verify for φ in $C^\infty(U)$ that $\|\varphi\|_{m,\varepsilon} < \infty$ for all pairs (m,ε) if and only if φ is in $C^\infty_{\text{com}}(U)$.
(b) Prove that the identity mapping $(C^\infty_{\text{com}}(U), \mathcal{T}) \to (C^\infty_{\text{com}}(U), \mathcal{T}')$ is continuous.
(c) For $p \ge 0$, fix $\psi_p \ge 0$ in $C^\infty_{\text{com}}(U)$ with $\sum_p \psi_p = 1$, $\psi_0 \ne 0$ on K_2, and

$$\psi_p(x)\begin{cases} \ne 0 \text{ for } x \text{ in } K_{p+2} - K^0_{p+1}, \\ = 0 \text{ for } x \text{ in } (K^0_{p+3})^c \text{ and for } x \text{ in } K_p. \end{cases}$$

A basic open neighborhood N of 0 in $(C^\infty_{\text{com}}(U), \mathcal{T})$ is a convex circled set with 0 as an internal point satisfying conditions of the following form: for each $p \ge 0$, there exist an integer n_p and a real $\delta_p > 0$ such that a member φ of $C^\infty_{K_{p+3}}$ is in $N \cap C^\infty_{K_{p+3}}$ if and only if $\sup_{x\in K_{p+3}} \sup_{|\alpha|\le n_p} |D^\alpha\varphi(x)| < \delta_p$. Prove that there exists a pair (m,ε) such that $\|\varphi\|_{m,\varepsilon} < 1$ implies that $2^{p+1}\psi_p\varphi$ is in $N \cap C^\infty_{K_{p+3}}$ for all $p \ge 0$.
(d) With notation as in (c), show that the function $\varphi = \sum_{p\ge 0} 2^{-(p+1)}(2^{p+1}\psi_p\varphi)$ is in N whenever $\|\varphi\|_{m,\varepsilon} < 1$. Conclude that the identity mapping from $(C^\infty_{\text{com}}(U), \mathcal{T}')$ to $(C^\infty_{\text{com}}(U), \mathcal{T})$ is continuous and that $\mathcal{T}$ and $\mathcal{T}'$ are therefore the same.
(e) Exhibit a sequence of closed nowhere dense subsets of $C^\infty_{\text{com}}(U)$ with union $C^\infty_{\text{com}}(U)$, thereby showing that the hypotheses of the Baire Category Theorem must not be satisfied in $C^\infty_{\text{com}}(U)$.

10. Prove or disprove: If H is an infinite-dimensional separable Hilbert space, then $\mathcal{B}(H, H)$ is separable in the operator-norm topology.

11. Let S be a compact Hausdorff space, let μ be a regular Borel measure on S, and regard $\mathcal{A} = \{\text{multiplications by } C(S)\}$ as a subalgebra of $\mathcal{M}(L^2(S, \mu))$. Prove that the commuting algebra $\mathcal{A}'$ of $\mathcal{A}$ within $\mathcal{B}(L^2(S, \mu), L^2(S, \mu))$ is $\mathcal{M}(L^2(S, \mu))$.

12. Prove that if A is a bounded normal operator on a separable complex Hilbert space H, then $\|A\| = \sup_{\|x\| \leq 1} |(Ax, x)_H|$.

13. Let H be a separable complex Hilbert space, let $\mathcal{A}$ be a commutative C^* subalgebra of $\mathcal{B}(H, H)$ with identity, and suppose that $\mathcal{A}$ has a cyclic vector. Prove that there exist a regular Borel measure μ on $\mathcal{A}^*_{\mathrm{m}}$ and a unitary operator $U : H \to L^2(\mathcal{A}^*_{\mathrm{m}}, \mu)$ such that

$$U\mathcal{A}U^{-1} = \{\text{multiplications by } C(\mathcal{A}^*_{\mathrm{m}})\} \subseteq \mathcal{M}(L^2(\mathcal{A}^*_{\mathrm{m}}, \mu)).$$

14. Let A be a bounded normal operator on a separable complex Hilbert space H, and let $\mathcal{A}$ be the smallest C^* subalgebra of $\mathcal{B}(H, H)$ containing I, A, and A^*. Suppose that $\mathcal{A}$ has a cyclic vector. Prove that there exists a Borel measure on the spectrum $\sigma(A)$ and a unitary mapping $U : H \to L^2(\sigma(A), \mu)$ such that

$$U\mathcal{A}U^{-1} = \{\text{multiplications by } C(\sigma(A))\} \subseteq \mathcal{M}(L^2(\sigma(A), \mu))$$

and such that UAU^{-1} is the multiplication operator M_z.

15. Form the multiplication operator M_x on $L^2([0, 1])$, and let $\mathcal{A}$ be the smallest C^* subalgebra of $\mathcal{B}(L^2([0, 1]), L^2([0, 1]))$ containing I and M_x.
    (a) Prove that the function 1 is a cyclic vector for $\mathcal{A}$.
    (b) Identify the spectrum $\sigma(M_x)$.
    (c) Prove in the context of the functional calculus of the Spectral Theorem that the operator $\varphi(M_x)$ is M_φ for every bounded Borel function φ on the spectrum $\sigma(M_x)$.

16. Let A and B be bounded normal operators on a separable complex Hilbert space H such that A commutes with B and B^*. Let $\mathcal{A}$ be the smallest C^* subalgebra of $\mathcal{B}(H, H)$ containing I, A, A^*, B, and B^*.
    (a) Prove that $\mathcal{A}^*_{\mathrm{m}}$ is canonically homeomorphic to the subset $\sigma(A, B)$ of $\sigma(A) \times \sigma(B) \subseteq \mathbb{C}^2$ given by $\sigma(A, B) = \{(\widehat{A}(\ell), \widehat{B}(\ell))\}_{\ell \in \mathcal{A}^*_{\mathrm{m}}}$.
    (b) Prove under the identification of (a) that $\widehat{A}$ is identified with the function z_1 and $\widehat{B}$ is identified with z_2.

Problems 17–20 concern the set of extreme points in particular closed subsets of locally convex topological vector spaces.

17. Let S be a compact Hausdorff space, and let K be the set of all regular Borel measures on S with $\mu(S) \leq 1$. Give K the weak-star topology relative to $C(S)$. Prove that the extreme points of K are 0 and the point masses of total measure 1.

18. In $L^1([0, 1])$, suppose that f has norm 1 and that E is a Borel subset such that $\int_E |f|\,dx > 0$ and $\int_{E^c} |f|\,dx > 0$. Let f_1 be f on E and be 0 on E^c, and let f_2 be f on E^c and be 0 on E.
    (a) Prove that f is a nontrivial convex combination of $\|f_1\|_1^{-1} f_1$ and $\|f_2\|_1^{-1} f_2$.
    (b) Conclude that the closed unit ball of $L^1([0, 1])$ has no extreme points.

19. Let S be a compact Hausdorff space, and let K_1 be the set of all regular Borel measures on S with $\mu(S) = 1$. Give K_1 the weak-star topology relative to $C(S)$. Let F be a homeomorphism of S. Within K_1, let K be the subset of members μ of K_1 that are F invariant in the sense that $\mu(E) = \mu(F^{-1}(E))$ for all Borel sets E.
    (a) Prove that K is a compact convex subset of $M(S)$ in the weak-star topology relative to $C(S)$.
    (b) A member μ of K is said to be **ergodic** if every Borel set E such that $F(E) = E$ has the property that $\mu(E) = 0$ or $\mu(E) = 1$. Prove that every extreme point of K is ergodic.
    (c) Is every ergodic measure in K necessarily an extreme point?

20. Regard the set $\mathbb{Z}$ of integers as a measure space with the counting measure imposed. As in Section 8, a complex-valued function $f(n)$ on $\mathbb{Z}$ is said to be **positive definite** if $\sum_{j,k} c(j) f(j-k)\overline{c(k)} \geq 0$ for all complex-valued functions $c(n)$ on the integers with finite support.
    (a) Prove that every positive definite function f has $f(0) \geq 0$, $f(-n) = \overline{f(n)}$, and $|f(n)| \leq f(0)$.
    (b) Prove that a bounded sequence in $L^\infty(\mathbb{Z})$ converges weak-star relative to $L^1(\mathbb{Z})$ if and only if it converges pointwise.
    (c) In view of (a), the set K of positive definite functions f with $f(1) = 1$ is a subset of the closed unit ball of $L^\infty(\mathbb{Z})$. Prove that the set K is convex and is compact in the weak-star topology relative to $L^1(\mathbb{Z})$.
    (d) Prove that every function $f_\theta(n) = e^{in\theta}$ with θ real is an extreme point of K.
    (e) Take for granted the fact that every positive definite function on $\mathbb{Z}$ is the sequence of Fourier coefficients of some Borel measure on the circle. (The corresponding fact for positive definite functions on $\mathbb{R}^N$ is proved in Problems 8–12 of Chapter VIII of *Basic*.) Prove that the set K has no other extreme points besides the ones in (d).

Problems 21–25 elaborate on the Stone Representation Theorem, Theorem 4.15. The first of the problems gives a direct proof, without using the Gelfand–Mazur Theorem, that every multiplicative linear functional is continuous in the context of Theorem 4.15.

21. Let S be a nonempty set, and let $\mathcal{A}$ be a uniformly closed subalgebra of $B(S)$ containing the constants and stable under complex conjugation. Let C be a complex number with $|C| > 1$, let f be a member of $\mathcal{A}$ with $\|f\|_{\sup} \leq 1$, and let ℓ be a multiplicative linear functional on $\mathcal{A}$.
    (a) Show that $\sum_{n=0}^{\infty}(f/C)^n$ converges and that its sum x provides an inverse to $1 - (f/C)$ under multiplication.
    (b) By applying ℓ to the identity $(1 - (f/C))x = 1$, prove that $\ell(f) = C$ is impossible.
    (c) Conclude from (b) that $\|\ell\| \leq 1$, hence that ℓ is automatically bounded.
22. Let S be a compact Hausdorff space, and let ℓ be a multiplicative linear functional on $C(S)$ such that $\ell(\bar{f}) = \overline{\ell(f)}$ for all f in $C(S)$. Prove that ℓ is the evaluation e_s at some point s of S.
23. Let S and T be two compact Hausdorff spaces, and let $U : C(S) \to C(T)$ be an algebra homomorphism that carries 1 to 1 and respects complex conjugation.
    (a) Prove that there exists a unique continuous map $u : T \to S$ such that $(Uf)(t) = f(u(t))$ for every $t \in T$ and $f \in C(S)$.
    (b) Prove that if U is one-one, then u is onto.
    (c) Prove that if U is an isomorphism, then u is a homeomorphism.
24. Let X be a compact Hausdorff space, and let $\mathcal{A}$ and $\mathcal{B}$ be uniformly closed subalgebras of $B(X)$ containing the constants and stable under complex conjugation. Suppose that $\mathcal{A} \subseteq \mathcal{B}$. Suppose that S, p, U and T, q, V are data such that S and T are compact Hausdorff spaces, $p : X \to S$ and $q : X \to T$ are functions with dense image, and $U : \mathcal{A} \to C(S)$ and $V : \mathcal{B} \to C(T)$ are algebra isomorphisms carrying 1 to 1 and respecting complex conjugations such that for every $x \in X$, $(Uf)(p(x)) = x$ for all $f \in \mathcal{A}$ and $(Vg)(q(x)) = x$ for all $g \in \mathcal{B}$. Prove that there exists a unique continuous map $\Phi : T \to S$ such that $p = \Phi \circ q$. Prove also that this map satisfies $(Uf)(\Phi(t)) = (Vf)(t)$ for all f in $\mathcal{A}$.
25. Formulate and prove a uniqueness statement to complement the existence statement in Theorem 4.15.

Problems 26–30 concern inductive limits. As mentioned in a footnote in the text, "direct limit" is a construction in category theory that is useful within several different settings. These problems concern the setting of topological spaces and continuous maps between them. For this setting a direct limit is something attached to a **directed system** of topological spaces and continuous maps. For the latter let $(I, \leq)$ be a directed set, and suppose that W_i is a topological space for each i in I. Suppose that a one-one continuous map $\psi_{ji} : W_i \to W_j$ is defined whenever $i \leq j$, and suppose that these maps satisfy $\psi_{ii} = 1$ and $\psi_{ki} = \psi_{kj} \circ \psi_{ji}$ whenever $i \leq j \leq k$. A **direct limit** of this directed system consists of a topological space W and continuous maps $q_i : W_i \to W$ for each i in I satisfying the following universal mapping property: whenever continuous maps $\varphi_i : W_i \to Z$ are given for each i such that $\varphi_j \circ \psi_{ji} = \varphi_i$

for $i \le j$, then there exists a unique continuous map $\Phi : W \to Z$ such that $\varphi_i = \Phi \circ q_i$ for all i.

26. Suppose that a directed system of topological spaces and continuous maps is given with notation as above. Let $\coprod_i W_i$ denote the disjoint union of the spaces W_i, topologized so that each W_i appears as an open subset of the disjoint union. Define an equivalence relation $\sim$ on $\coprod W_i$ as follows: if x_i is in W_i and x_j is in W_j, then $x_i \sim x_j$ means that there is some k with $i \le k$ and $j \le k$ such that $\psi_{ki}(x_i) = \psi_{kj}(x_j)$.
    (a) Prove that $\sim$ is an equivalence relation.
    (b) Prove that elements x_i in W_i and x_j in W_j have $x_i \sim x_j$ if and only if every l with $i \le l$ and $j \le l$ has $\psi_{li}(x_i) = \psi_{lj}(x_j)$.
27. Define W to be the quotient $\coprod_i W_i \big/ \sim$, and give W the quotient topology. Let $q : \coprod_i W_i \to W$ be the quotient map. Prove that W and the system of maps $q\big|_{W_i}$ form a direct limit of the given directed system.
28. Prove that if $(V, \{p_i\})$ and $(W, \{q_i\})$ are two direct limits of the given system, then there exists a unique homeomorphism $F : V \to W$ such that $q_i = F \circ p_i$ for all i in I.
29. Suppose that each map $\psi_i : W_i \to W_j$ is a homeomorphism onto an open subset.
    (a) Prove that the quotient map $q : \coprod_i W_i \to W$ carries open sets to open sets.
    (b) Prove that the direct limit W is Hausdorff if each given W_i is Hausdorff.
    (c) Prove that the direct limit W is locally compact Hausdorff if each W_i is locally compact Hausdorff.
    (d) Give an example in which each W_i is compact Hausdorff but the direct limit W is not compact.
30. Let I be a nonempty index set, and let S_0 be a finite subset. Suppose that a locally compact Hausdorff space X_i is given for each $i \in I$ and that a compact open subset K_i is specified for each $i \notin S_0$. For each finite subset S of I containing S_0, define
$$X(S) = \big(\textstyle\bigtimes_{i \in S} X_i\big) \times \big(\textstyle\bigtimes_{i \notin S} K_i\big),$$
giving it the product topology. If S_1 and S_2 are two finite subsets of I containing S_0 such that $S_1 \subseteq S_2$, then the inclusion $\psi_{S_2 S_1} : X(S_1) \to X(S_2)$ is a homeomorphism onto an open set, and these homeomorphisms are compatible under composition. The resulting direct limit X is called the **restricted direct product** of the X_i's with respect to the K_i's. Prove that X is locally compact Hausdorff and that elements of X may be regarded as tuples (x_i) for which x_i is in X_i for all i while x_i is in K_i for all but finitely many i.