Chapter 4

STRAIGHTNESS ON CYLINDERS AND CONES

If a cut were made through a cone parallel to its base, how should we conceive of the two opposing surfaces which the cut has produced — as equal or as unequal? If they are unequal, that would imply that a cone is composed of many breaks and protrusions like steps. On the other hand, if they are equal, that would imply that two adjacent intersection planes are equal, which would mean that the cone, being made up of equal rather than unequal circles, must have the same appearance as a cylinder; which is utterly absurd. — Democritus of Abdera (~460 – ~380 B.C.)

This quote shows that cylinders and cones were the subject of mathematical inquiry before Euclid (~365 – ~300 B.C.). In this chapter we investigate straightness on cones and cylinders. You should be comfortable with straightness as a *local intrinsic notion* — this is the bug's view. This notion of straightness is also the basis for the notion of *geodesics* in differential geometry. Chapters 4 and 5 can be covered in either order, but we think that the experience with cylinders and cones in Problem **4.1** will help the reader to understand the hyperbolic plane in Problem **5.1**. If the reader is comfortable with straightness as a local intrinsic notion, then it is also possible to skip Chapter 4 if Chapters 18 and 24 on geometric manifolds are not going to be covered. However, we suggest that you read the sections at the end of this chapter — Is "Shortest" Always "Straight"? and Relations to Differential Geometry — at least enough to find out what Euclid's Fourth Postulate has to do with cones and cylinders.

When looking at great circles on the surface of a sphere, we were able (except in the case of central symmetry) to see all the symmetries of straight lines from global extrinsic points of view. For example, a great circle extrinsically divides a sphere into two hemispheres that are mirror images of each other. Thus, on a sphere, it is a natural tendency to use the more familiar and comfortable extrinsic lens instead of taking the bug's local and intrinsic point of view. However, on a cone and cylinder you must use the local, intrinsic point of view because there is no extrinsic view that will work except in special cases.

PROBLEM 4.1 STRAIGHTNESS ON CYLINDERS AND CONES

a. *What lines are straight with respect to the surface of a cylinder or a cone? Why? Why not?*

b. *Examine:*

- *Can geodesics intersect themselves on cylinders and cones?*
- *Can there be more than one geodesic joining two points on cylinders and cones?*
- *What happens on cones with varying cone angles, including cone angles greater than 360°?* These are discussed starting in the next section.

SUGGESTIONS

Problem **4.1** is similar to Problem **2.1**, but this time the surfaces are cylinders and cones. Make paper models but consider the cone or cylinder as continuing indefinitely with no top or bottom (except, of course, at the cone point). Again, imagine yourself as a bug whose whole universe is a cone or cylinder. As the bug crawls around on one of these surfaces, what will the bug experience as straight? As before, paths that are straight with respect to a surface are often called the "geodesics" for the surface.

As you begin to explore these questions, it is likely that many other related geometric ideas will arise. Do not let seemingly irrelevant excess geometric baggage worry you. Often, you will find yourself getting lost in a tangential idea, and that's understandable. Ultimately, however, the exploration of related ideas will give you a richer understanding of the scope and depth of the problem. In order to work through possible confusion on this problem, try some of the following suggestions others have found helpful. Each suggestion involves constructing or using models of cones and cylinders.

- You may find it helpful to explore cylinders first before beginning to explore cones. This problem has many aspects but focusing at first on the cylinder will simplify some things.
- If we make a cone or cylinder by rolling up a sheet of paper, will "straight" stay the same for the bug when we unroll it? Conversely, if we have a straight

line drawn on a sheet of paper and roll it up, will it continue to be experienced as straight for the bug crawling on the paper? We are assuming here that the paper will not stretch, and its thickness is negligible.

- Lay a stiff ribbon or straight strip of paper on a cylinder or cone. Convince yourself that it will follow a straight line with respect to the surface. Also, convince yourself that straight lines on the cylinder or cone, when looked at locally and intrinsically, have the same symmetries as on the plane.
- If you intersect a cylinder by a flat plane and unroll it, what kind of curve do you get? Is it ever straight? (One way to see this curve is to dip a paper cylinder into water.)
- On a cylinder or cone, can a geodesic ever intersect itself? How many times? This question is explored in more detail in Problem **4.2**, which the interested reader may turn to now.
- Can there be more than one geodesic joining two points on a cylinder or cone? How many? Is there always at least one? Again, this question is explored in more detail in Problem **4.2**.

There are several important things to keep in mind while working on this problem. First, **you absolutely must make models**. If you attempt to visualize lines on a cone or cylinder, you are bound to make claims that you would easily see are mistaken if you investigated them on an actual cone or cylinder. Many students find it helpful to make models using transparent material.

Second, as with the sphere, you must think about lines and triangles on the cone and cylinder in an intrinsic way — always looking at things from a bug's point of view. We are not interested in what's happening in 3-space, only what you would see and experience if you were restricted to the surface of a cone or cylinder.

And last, but certainly not least, you must look at cones of different shapes, that is, cones with varying cone angles.

CONES WITH VARYING CONE ANGLES

Geodesics behave differently on differently shaped cones. So an important variable is the cone angle. The ***cone angle*** is generally defined as the angle measured around the point of the cone on the surface. Notice that this is an intrinsic description of angle. The bug could measure a cone angle (in radians) by determining the circumference of an intrinsic circle with center at the cone point and then dividing that circumference by the radius of the circle. We can determine the cone angle extrinsically in the following way: Cut the cone along a ***generator*** (a line on the cone through the cone point) and flatten the cone. The measure of the cone angle is then the angle measure of the flattened planar sector.

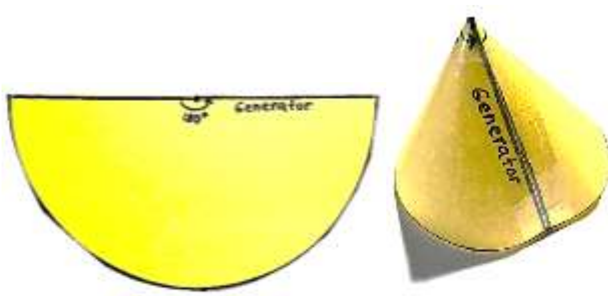

Figure 4.1 Making a 180° cone

For example, if we take a piece of paper and bend it so that half of one side meets up with the other half of the same side, we will have a 180-degree cone (Figure 4.1). A 90º cone is also easy to make — just use the corner of a paper sheet and bring one side around to meet the adjacent side.

Also be sure to look at larger cones. One convenient way to do this is to make a cone with a variable cone angle. This can be accomplished by taking a sheet of paper and cutting (or tearing) a slit from one edge to the center. (See Figure 4.2.) A rectangular sheet will work but a circular sheet is easier to picture. Note that it is not necessary that the slit be straight!

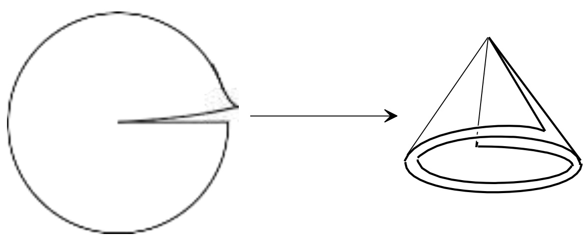

Figure 4.2 A cone with variable cone angle (0 – 360°)

You are already familiar with a 360º cone — it's just a plane. The cone angle can also be larger than 360º. A common larger cone is the 450º cone. You probably have a cone like this somewhere on the walls, floor, and ceiling of your room. You can easily make one by cutting a slit in a piece of paper and inserting a 90º slice (360º + 90º = 450º) as in Figure 4.3.

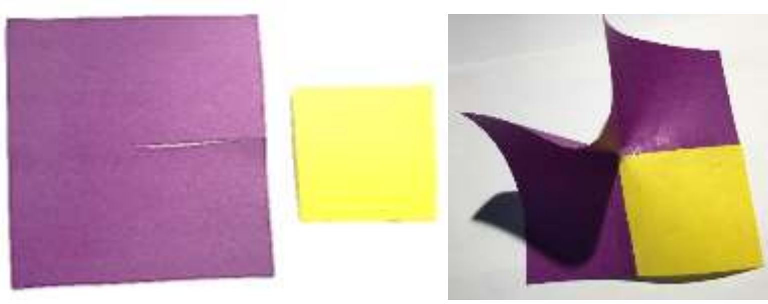

Figure 4.3 How to make a 450º cone

Two cone angles on a ceiling

You may have trouble believing that this is a cone but remember that just because it cannot hold ice cream does not mean it is not a cone. If you will look around in the room you are, perhaps you can locate a corner where five right angles meet – that is 450º cone. It is important to realize that when you change the shape of the cone like this (that is, either it is with ruffles or straight lines), you are only changing its extrinsic appearance. Intrinsically (from the bug's point of view) there is no difference.

It may be helpful for you to discuss some definitions of a cone, such as the following: *Take any simple* (*non-intersecting*) *closed curve **a** on a sphere and the center **P** of the sphere. A **cone** is the union of the rays that start at **P** and go through each point on **a***. The cone angle is then equal to (length of ***a***)/ (radius of sphere), in radians. *Do you see why*?

You can also make a cone with variable angle of more than 180°: Take two sheets of paper and slit them together to their centers as in Figure 4.4. Tape the right side of the top slit to the left side of the bottom slit as pictured. Now slide the other sides of the slits. Try it!

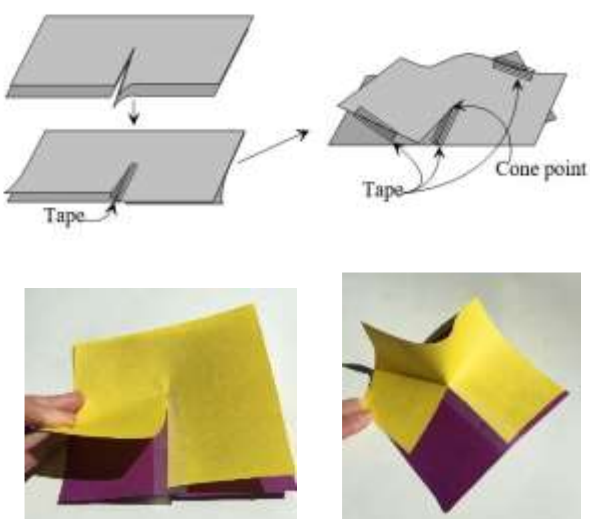

Figure 4.4 Variable cone angle larger than 360°

Experiment by making paper examples of cones like those shown in Figure 4.4. What happens to the triangles and lines on a 450º cone? Is the shortest path always straight? Does every pair of points determine a straight line?

Finally, also consider line symmetries on the cone and cylinder. Check to see if the symmetries you found on the plane will work on these surfaces and remember to think intrinsically and locally. A special class of geodesics on the cone and cylinder is the generators. These are the straight lines that go through the cone point on the cone or go parallel to the axis of the cylinder. These lines have some extrinsic symmetries (*can you see which ones*?), but in general, geodesics have only local, intrinsic symmetries. Also, on the cone, think about the region near the cone point — what is happening there that makes it different from the rest of the cone?

Few more explorations of geodesics on the cone can be found http://www.rdrop.com/~half/Creations/Puzzles/cone.geodesics/index.html

It is best if you experiment with paper models to find out what geodesics look **like on the cone and cylinder before reading further.**

GEODESICS ON CYLINDERS

Let us first look at the three classes of straight lines on a cylinder. When walking on the surface of a cylinder, a bug might walk along a vertical generator. See Figure 4.5.

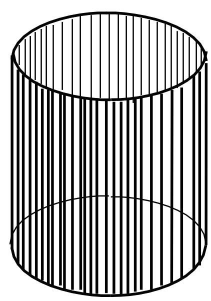

Figure 4.5 Vertical generators are straight

It might walk along an intersection of a horizontal plane with the cylinder, what we will call a *great circle*. See Figure 4.6

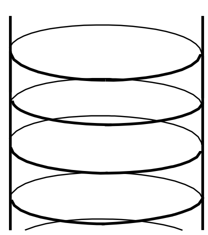

Figure 4.6 Great circles are intrinsically straight

Or, the bug might walk along a spiral or helix of constant slope around the cylinder. See Figure 4.7 and the photo at the beginning of this chapter depicting lightening damage to the tree. Watch a squirrel running up the tree!

Figure 4.7 Helixes are intrinsically straight
Helixes can be seen on outside parking garages and in sculptures

Why are these geodesics? How can you convince yourself? And why are these the only geodesics?

GEODESICS ON CONES

Now let us look at the classes of straight lines on a cone.

Walking along a generator: When looking at straight paths on a cone, you will be forced to consider straightness at the cone point. You might decide that there is no way the bug can go straight once it reaches the cone point, and thus a straight path leading up to the cone point ends there. Or you might decide that the bug can find a continuing path that has at least some of the symmetries of a straight line. Do you see which path this is? Or you might decide that the straight continuing path(s?) is the limit of geodesics that just miss the cone point. See Figure 4.8.

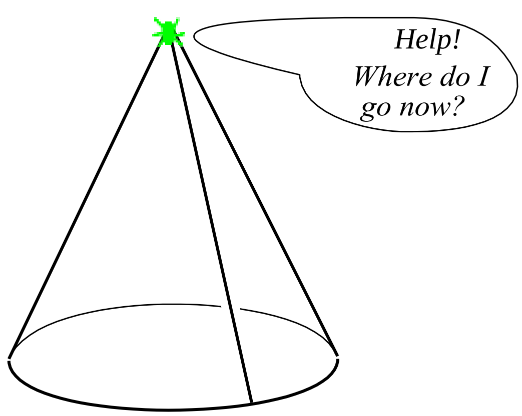

Figure 4.8 Bug walking straight over the cone point

Walking straight and around: If you use a ribbon on a 90º cone, then you can see that this cone has a geodesic like the one depicted in Figure 4.9. This particular geodesic intersects itself. However, check to see that this property depends on the cone angle. In particular, if the cone angle is more than 180°, then geodesics do not intersect themselves. And if the cone angle is less than 90°, then geodesics (except for generators) intersect at least two times. Try it out! Later, in Problem **4.2**, we will describe a tool that will help you determine how the number of self- intersections depends on the cone angle.

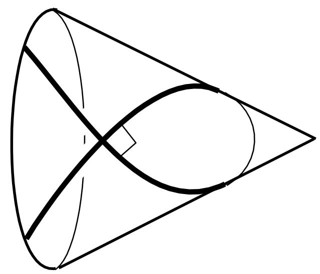

Figure 4.9 A geodesic intersecting itself on a 90° cone

PROBLEM 4.2 GLOBAL PROPERTIES OF GEODESICS

Now we will look more closely at long geodesics that wrap around on a cylinder or cone. Several questions have arisen.

a. *How do we determine the different geodesics connecting two points? How many are there? How does it depend on the cone angle? Is there always at least one geodesic joining each pair of points? How can we justify our conjectures?*

b. *How many times can a geodesic on a cylinder or cone intersect itself? How are the self-intersections related to the cone angle? At what angle does the geodesic intersect itself? How can we justify these relationships?*

SUGGESTIONS

Here we offer the tool of covering spaces, which may help you explore these questions. The method of coverings is so named because it utilizes layers (or sheets) that each cover the surface. We will first start with a cylinder because it is easier and then move on to a cone.

n -SHEETED COVERINGS OF A CYLINDER

To understand how the method of coverings works, imagine taking a paper cylinder and cutting it axially (along a vertical generator) so that it unrolls into a plane. This is probably the way you constructed cylinders to study this problem before. The unrolled sheet (a portion of the plane) is said to be a ***1-sheeted covering*** of the cylinder. See Figure

4.10. If you marked two points on the cylinder, *A* and *B*, as indicated in the figure, when the cylinder is cut and unrolled into the covering, these two points become two points on the covering (which are labeled by the same letters in the figure). The two points on the covering are said to be ***lifts*** of the points on the cylinder.

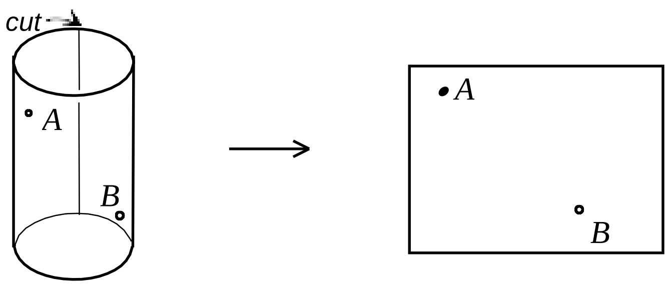

Figure 4.10 A 1-sheeted covering of a cylinder

Now imagine attaching several of these "sheets" together, end to end. When rolled up, each sheet will go around the cylinder exactly once — they will each cover the cylinder. (Rolls of toilet paper or paper towels give a rough idea of coverings of a cylinder.) Also, each sheet of the covering will have the points *A* and *B* in identical locations. You can see this (assuming the paper thickness is negligible) by rolling up the coverings and making points by sticking a sharp object through the cylinder. This means that all the *A*'s are coverings of the same point on the cylinder and all the *B*'s are coverings of the same point on the cylinder. We just have on the covering several representations, or ***lifts***, of each point on the cylinder. Figure 4.11 depicts a 3-sheeted covering space for a cylinder and six geodesics joining *A* to *B*. (One of them is the most direct path from *A* to *B* and the others spiral once, twice, or three times around the cylinder in one of two directions.)

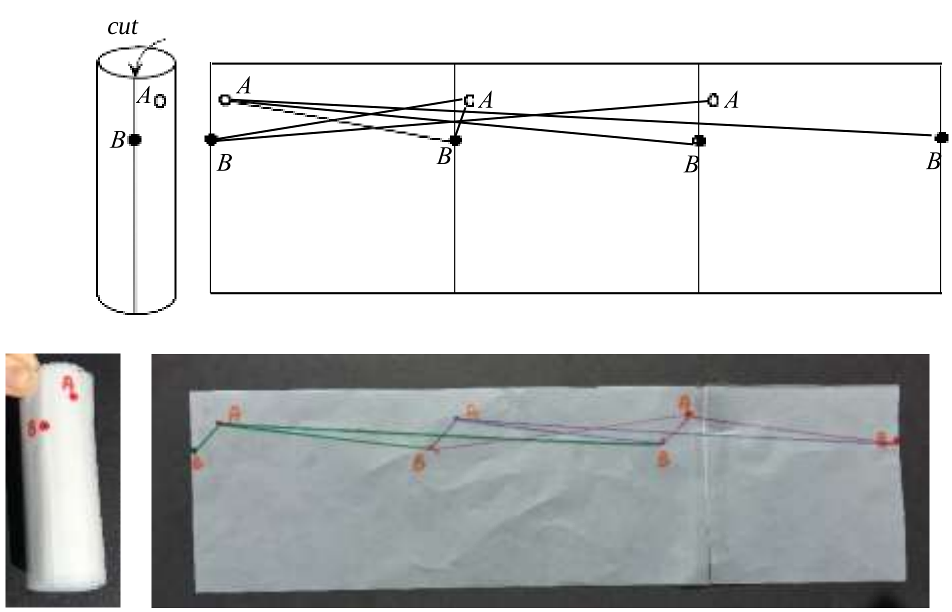

Figure 4.11 A 3-sheeted covering space for a cylinder

We could also have added more sheets to the covering on either the right or left side. You can now roll these sheets back into a cylinder and see what the geodesics look like. Remember to roll sheets up so that each sheet of the covering covers the cylinder exactly once — all of the vertical lines between the coverings should lie on the same generator of the cylinder. Note that if you do this with ordinary paper, part or all of some geodesics will be hidden, even though they are all there. It may be easier to see what's happening if you use transparencies.

This method works because straightness is a local intrinsic property. Thus, lines that are straight when the coverings are laid out in a plane will still be straight when rolled into a cylinder. Remember that bending the paper does not change the intrinsic nature of the surface. Bending only changes the curvature that we see extrinsically. It is important always to look at the geodesics from the bug's point of view. The cylinder and its covering are locally isometric.

Use coverings to investigate Problem **4.2** on the cylinder. The global behavior of straight lines may be easier to see on the covering.

n-SHEETED (BRANCHED) COVERINGS OF A CONE

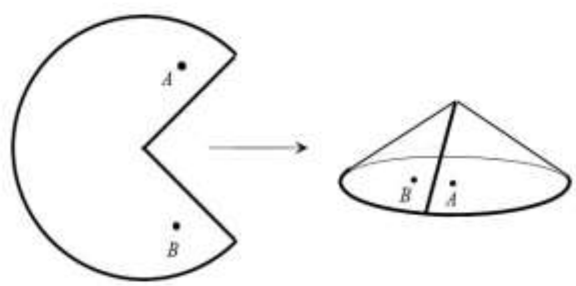

Figure 4.12 1-sheeted covering of a 270° cone

Figure 4.12 shows a 1-sheeted covering of a cone. The sheet of paper and the cone are locally isometric except at the cone point. The cone point is called a ***branch point*** of the covering. We talk about lifts of points on the cone in the same way as on the cylinder. In Figure 4.12 we depict a 1-sheeted covering of a 270° cone and label two points and their lifts.

A 4-sheeted covering space for a cone is depicted in Figure 4.13. Each of the rays drawn from the center of the covering is a lift of a single ray on the cone. Similarly, the points marked on the covering are the lifts of the points *A* and *B* on the cone. In the covering there are four segments joining a lift of *A* to different lifts of *B*. Each of these segments is the lift of a different geodesic segment joining *A* to *B*.

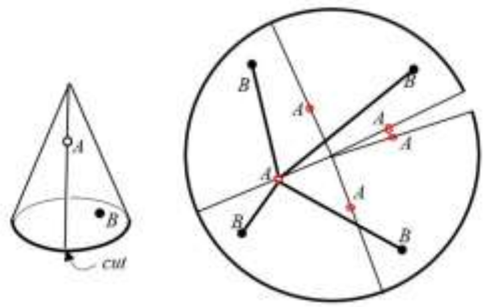

Figure 4.13 4-sheeted covering space for a 89° cone

Think about ways that the bug can use coverings as a tool to expand its exploration of surface geodesics. Also, think about ways you can use coverings to justify your observations in an intrinsic way. It is important to be precise; you don't want the bug to get lost! Count the number of ways in which you can connect two points with a straight line and relate those countings with the cone angle. Does the number of straight paths only depend on the cone angle? Look at the 450° cone and see if it is always possible to connect any two points with a straight line. **Make paper models!** It is not possible to get an equation that relates the cone angle to the number of geodesics joining every pair of points. However, it is possible to find a formula that works for most pairs. Make covering spaces for cones of different size angles and refine the guesses you have already made about the numbers of self-intersections.

In studying the self-intersections of a geodesic ***l*** on a cone, it may be helpful for you to consider the ray ***R*** that is perpendicular to the line ***l***. (See Figure 4.14.) Now study one lift of the geodesic ***l*** and its relationship to the lifts of the ray ***R***. Note that the seams between individual wedges are lifts of ***R***.

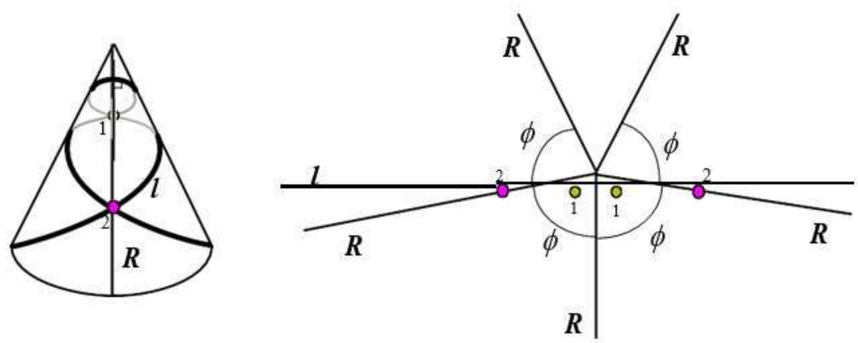

Figure 4.14 Self-intersections on a cone with angle ϕ

A recent tidbit about coverings: In 1914 Henri Lebesgue (French, 1875-1941) posed a question: What is the shape with the smallest area that can completely cover a host of other shapes (which all share a certain trait in common)? The shapes should be such that no two points are further than one unit apart. In 2014 retired software engineer Philip Gibbs ran computer simulations on 200 randomly generated shapes with diameter 1. He kept "trimming corners" of hexagon and found that to be smallest known covering. (https://www.quantamagazine.org/amateur-mathematician-finds-smallest-universal-cover-20181115/)

LOCALLY ISOMETRIC

By now you should realize that when a piece of paper is rolled or bent into a cylinder or cone, the bug's local and intrinsic experience of the surface does not change except at the cone point. Extrinsically, the piece of paper and the cone are different, but in terms of the local geometry intrinsic to the surface they differ only at the cone point.

Two geometric spaces, **G** and **H**, are said to be ***locally isometric*** at the points *G* in **G** and *H* in **H** if the local intrinsic experience at *G* is the same as the experience at *H*. That is, there are neighborhoods of *G* and *H* that are identical in terms of their intrinsic geometric properties. A cylinder and the plane are locally isometric (at each point) and the plane and a cone are locally isometric except at the cone point. Two cones are locally isometric at the cone points only if their cone angles are the same. Because cones and cylinders are locally isometric with the plane, locally they have the same geometric properties. Later, we will show that a sphere is not locally isometric with the plane — *be on the lookout for a result that will imply this*.

IS "SHORTEST" ALWAYS "STRAIGHT"?

We are often told that "a straight line is the shortest distance between two points," but is this really true? As we have already seen on a sphere, two points not opposite each other are connected by two straight paths (one going one way around a great circle and one going the other way). Only one of these paths is shortest. The other is also straight, but not the shortest straight path.

Consider a model of a cone with angle 450°. Notice that such cones appear commonly in buildings as so-called "outside corners" (see Figure 4.3). It is best, however, to have a paper model that can be flattened.

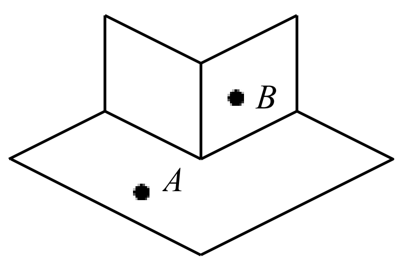

Figure 4.15 There is no straight (symmetric) path from *A* to *B*

Use your model to investigate which points on the cone can be joined by straight lines (in the sense of having reflection-in-the-line symmetry). In particular, look at points such as those labeled *A* and *B* in Figure 4.15. Convince yourself that there is no path from *A* to *B* that is straight (in the sense of having reflection-in-the-line symmetry), and for these points the shortest path goes through the cone point and thus is not straight (in the sense of having symmetry).

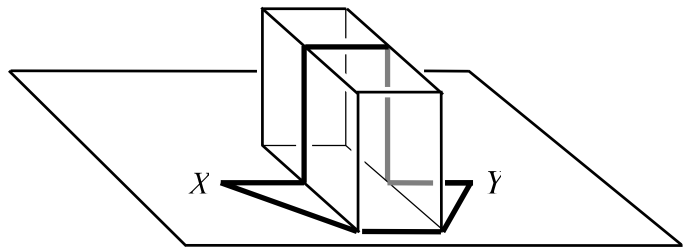

Figure 4.16 The shortest path is not straight (in the sense of symmetry)

Here is another example: Think of a bug crawling on a plane with a tall box sitting on that plane (refer to Figure 4.16). This combination surface — the plane with the box sticking out of it — has eight cone points. The four at the top of the box have 270° cone angles, and the four at the bottom of the box have 450° cone angles (180° on the box and 270° on the plane). What is the shortest path between points *X* and *Y*, points on opposite sides of the box? Is the straight path the shortest? Is the shortest path straight? To check that the shortest path is not straight, try to see that at the bottom corners of the box the two sides of the path have different angular measures. (If *X* and *Y* are close to the box, then the angle on the box side of the path measures a little more than 180° and the angle on the other side measures almost 270°.)

Relations to Differential Geometry

We see that sometimes a straight path is not shortest, and the shortest path is not straight. Does it then make sense to say (as most books do) that in Euclidean geometry a straight line is the shortest distance between two points? In differential geometry, on "smooth" surfaces, "straight" and "shortest" are more nearly the same. A *smooth* surface is essentially what it sounds like. More precisely, a surface is smooth at a point if, when you zoom in on the point, the surface becomes indistinguishable from a flat plane. (For details of this definition, see Problem **4.1** in [**DG**: Henderson,

https://projecteuclid.org/euclid.bia/1399917369].

See also the last section and especially the endnote in Chapter 1.) Note that a cone is not smooth at the cone point, but a sphere and a cylinder are both smooth at every point. The following is a theorem from differential geometry:

> **Theorem 4.1:** *If a surface is smooth, then an intrinsically straight line* (*geodesic*) *on the surface is always the shortest path between "nearby" points. If the surface is also complete* (*every geodesic on it can be extended indefinitely*), *then any two points can be joined by a geodesic that is the shortest path between them.* See [**DG**: Henderson], Problems **7.4b** and **7.4d**.

Consider a planar surface with a hole removed. Check that for points near opposite sides of the hole, the shortest path (on the planar surface with hole removed) is not straight because the shortest path must go around the hole. *We encourage the reader to discuss how each of the previous examples and problems is in harmony with this theorem.*

Note that the statement "every geodesic on the surface can be extended indefinitely" is a reasonable interpretation of Euclid's Second Postulate: *Every limited straight line can be extended indefinitely to a* (*unique*) *straight line*. Note that the Second Postulate does not hold on a cone unless you consider geodesics to continue through the cone point.

Also, Euclid defines a right angle as follows: *When a straight line intersects another straight line such that the adjacent angles are equal to one another, then the equal angles are called* ***right angles***. Note that if you consider geodesics to continue through the cone point, then right angles at a cone point are not equal to right angles at points where the cone is locally isometric to the plane.

And Euclid goes on to state as his Fourth Postulate: *All right angles are equal*. Thus, Euclid's Second Postulate or Fourth Postulate rules out cones and any surface with isolated cone points. What is further ruled out by Euclid's Fourth Postulate would depend on formulating more precisely just what it says. It is not clear (at least to the authors!) whether there is something we would want to call a surface that could be said to satisfy Euclid's Fourth Postulate and not be a smooth surface. However, we can see that Euclid's postulate at least gives part of the meaning of "smooth surface," because it rules out isolated cone points.

When we were in high school geometry class, we were confused why Euclid would have made such a postulate as his Postulate 4 — how could they possibly *not* be equal? In this chapter we have discovered that on cones right angles are not all equal.