Chapter 16

INVERSIONS IN CIRCLES

Q: How does a geometer capture a lion in the desert?

A: Build a circular cage in the desert, enter it, and lock it. Now perform an inversion with respect to the cage. Then you are outside, and the lion is locked in the cage.

— A mathematical joke from before 1938

We will now study inversions in a circle (which is the analogue of reflection in a line) and its applications. Though inversion in a circle can be defined on spheres and hyperbolic planes, it seems to have no significant applications on these surfaces. Therefore, in this chapter we will only consider the case of the Euclidean plane.

To study Chapter 16, the only results needed in Chapters 10–15 are

PROBLEM 13.4: The **AAA similarity** and **SAS similarity** criteria for triangles on the plane.

PROBLEM 15.1b: *On a plane, if two lines through a point P intersect a circle at points A, A' (possibly coincident) and B, B' (possibly coincident), then* $|PA| \times |PA'| = |PB| \times |PB'|$.

If you are willing to assume these criteria for similar triangles, then you can work through Chapter 16 without Chapters 10–15.

EARLY HISTORY OF INVERSIONS

Apollonius of Perga (c. 250–175 B.C.) was famous in his time for work on astronomy (Navigation/Stargazing Strand), before his now well-known work on conic sections.

Unfortunately, Apollonius' original work on astronomy and most of his mathematical work (except for *Conics*, **[AT**: Apollonius]**)** has been lost and we only know about it from a commentary by Pappus of Alexandria (290–350 A.D.). According to Pappus, Apollonius investigated one particular family of circles and straight lines. Apollonius defined the curve:

> $c_k(A, B)$ *is the locus of points P such that* $PA = k \times PB$, *where A and B are two points in the Euclidean plane, and k is a positive constant.*

This curve is a straight line if $k = 1$ and a circle otherwise and is usually called an ***Apollonian circle***. Apollonius proved (see Problem **16.1b**) that a circle *c* (with center *C* and radius *r*) belongs to the family $\{c_k(A,B)\}$ if and only if $BC \times AC = r^2$ and *A* and *B* are on the same ray from *C*. In modern terms, we use "$BC \times AC = r^2$ and *A* and *B* are on the same ray from *C* " as the definition of *A* and *B* being inversions of each other with respect to the circle *c*. There is indirect evidence (see [**HI**: Calinger], page 181) that Apollonius used inversion in a circle to solve astronomical problems concerning celestial orbits. The theory of inversions was apparently not carried on in a systematic way until the 19th century, when the theory was developed purely geometrically from Euclid's Book III, but this (as far as we know) was not done in ancient times. We suggest that this was because Euclid's *Elements* and Apollonius' circles were parts of different historical strands. In Problem **16.4** we will explore a problem of Apollonius that uses inversions for its solutions.

PROBLEM 16.1 INVERSIONS IN CIRCLES

DEFINITIONS. An ***inversion with respect to a circle*** Γ is a transformation from the extended plane (the plane with ∞, the "point at infinity," added) to itself that takes *C*, the center of the circle, to ∞ and vice versa, and that takes a point at a distance *s* from the center to the point on the same ray (from the center) that is at a distance of r^2/s from the center, where *r* is the radius of the circle. See Figure 16.1. We call (*P*, *P'*) an ***inversive pair*** because (as the reader can check) *P* and *P'* are taken to each other by the inversion. The circle Γ is called the ***circle of inversion***.

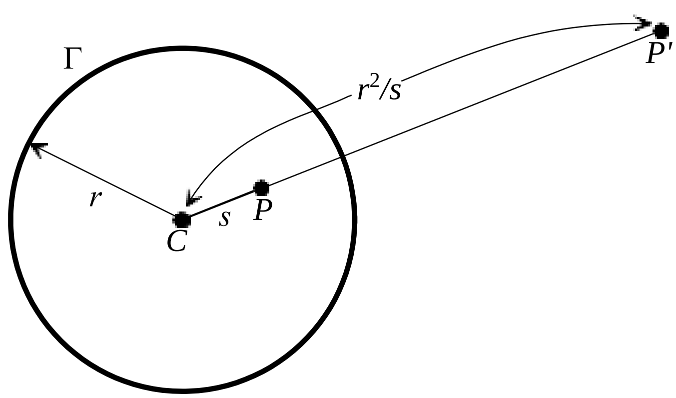

Figure 16.1 Inversion with respect to a circle

Note that an inversion takes the inside of the circle to the outside and vice versa and that the inversion takes any line through the center to itself. Because of this, inversion can be thought of as a reflection in the circle. See also part **c** for the close connection between inversions in the plane and reflections on a sphere.

We strongly suggest that the reader play with inversions by using some dynamic geometry software such as *GeoGebra*, for example. You may construct the image, P', of P under the inversion through the circle Γ as follows (see Figure 16.2):

> *If P is inside* Γ*, then draw through P the line perpendicular to the ray CP. Let S and R be the intersections of this line with* Γ*. Then P′ is the intersection of the lines tangent to* Γ *at S and R.* (To construct the tangents, note that lines tangent to a circle are perpendicular to the radius of the circle.)
>
> *If P is outside* Γ*, then draw the two tangent lines from P to* Γ*. Let S and R be the points of tangency on* Γ*. Then P′ is the inter- section of the line SR with CP.* (The points *S* and *R* are the intersections of Γ with the circle with diameter *CP*. *Why*?)

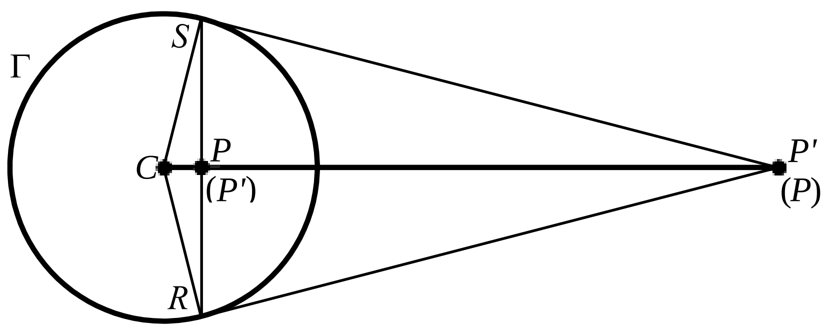

Figure 16.2 Constructing inversive images

a. *Prove that these constructions do construct inversive pairs.*

The purpose of this part is to explore and better understand inversion, but it will not be directly used in the other parts of this problem. When *P* is inside Γ, you need to prove that *C-P-P′* are collinear. When *P′* is outside Γ, you need to prove that *SR* is perpendicular to *CP*.

Next, we prove

b. Apollonius' Theorem. *Define the curve ck(A, B) to be the locus of points P such that PA=k×PB, where A and B are two points in the plane and k is a positive constant. A circle c (with center C and radius r) belongs to the family {ck(A, B)} if and only if A and B are an inversive pair with respect to c.*

The following result demonstrates the close connection between inversion through a circle in the plane and reflections through great circles on a sphere. If you have studied Problem **14.4** (or assume it), then you can use this part **c** in your analysis of inversions in Problem **16.2**.

c. *Let Σ be a sphere tangent at its south pole to the plane Π and let $f\colon \Sigma \to \Pi$ be a stereographic projection from the north pole. If Γ is the circle that is the image under f of the equator and if g is the intrinsic (or extrinsic) reflection of the sphere through its equator (or equatorial plane), then show that the transformation $f \circ g \circ f^{-1}$ is the inversion of the plane with respect to the circle Γ.* See Figure 16.3.

Imagine a sphere tangent to a plane at its south pole, *S*. Now use ***stereographic projection*** to project the sphere from the north pole, *N*, onto the plane; see Problem **14.4**. Stereographic projection was known already to Hipparchus (Greek, second century B.C.). Show that the triangle ΔSNP is similar to Δ*RNS*, which is congruent to Δ*QSN*, which is similar to Δ*SP'N*.

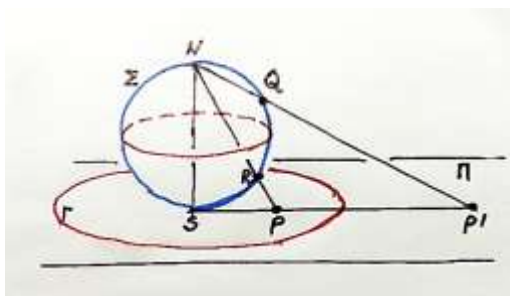

Figure 16.3 Stereographic projection and inversion

PROBLEM 16.2 INVERSIONS PRESERVE ANGLES AND PRESERVE CIRCLES (AND LINES)

Part **c** provides another route to prove that inversions are conformal (preserves angles, see Problem **14.4**). In Problem **14.4** you showed that *f*, stereographic projection, is conformal. In addition, *g* (being an isometry) is conformal. Thus, the inversion $f \circ g \circ f^{-1}$ is conformal.

a. *Show that an inversion takes each circle orthogonal to the circle of inversion to itself.* See Figure 16.4.

Two ***circles are orthogonal*** if, at each point of intersection, the angle between the tangent lines is 90°. (Note that, at these points, the radius of one circle is tangent to the other circle.)

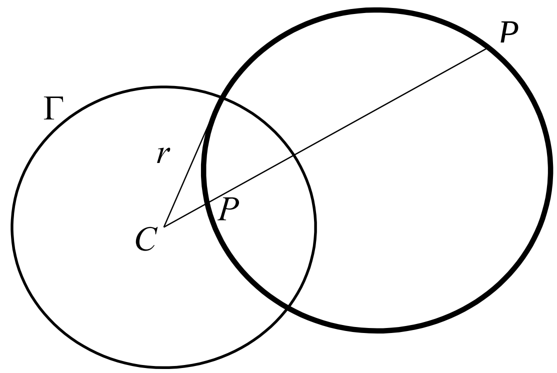

Figure 16.4 A circle orthogonal to Γ inverts to itself

b. *Show that an inversion takes a circle through the center of inversion to a line not through the center, and vice versa. What happens in the special cases when either the circle or the straight line intersects the circle of inversion*?

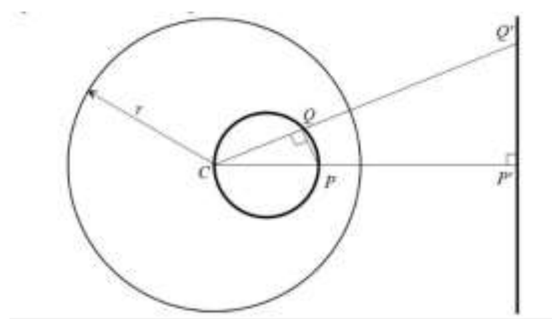

Figure 16.5 Circles through the center invert to lines

Look at Figure 16.5 (where *CP* is a diameter of the circle) and prove that Δ*CPQ* and Δ*CQ′P′* are similar triangles. Note that the line is parallel to the line tangent to the circle at *C*.

c. *An inversion takes circles not through the center of inversion to circles not through the center.* Note: The (circumference of a) circle inverts to another circle but the centers of these circles are on the same ray from *C* though **not** an inversive pair.

Look at Figure 16.6, where *PQ* is a diameter of the circle. If *P*, *Q*, *X* invert to *P′*, *Q′*, *X′*, then show that $\angle P' X' Q' = \angle P X Q$ = right angle by looking for similar triangles. Thus, argue that as *X* varies around the circle with diameter *PQ*, then *X′* varies around the circle with diameter *Q′P′*.

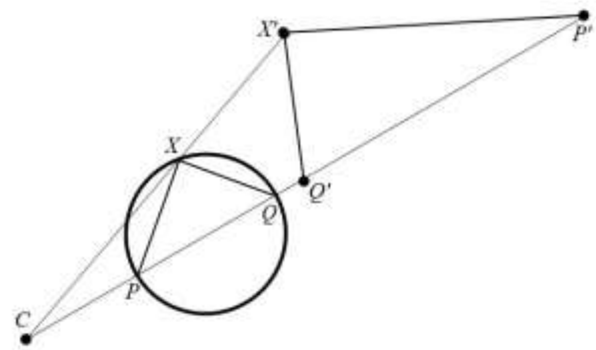

Figure 16.6 Circles invert to circles

d. *Inversions are conformal.*

Look at two lines that intersect and form an angle at P. Look at the images of these lines.

Inversions were used in the 19th century to solve a long-standing engineering problem (from the Motion/Machines Strand) that is the subject of Problem **16.3**. Other applications (that started with Apollonius) are discussed in Problem **16.4**. More history and further expansions of the notion of inversion are contained in the last section.

PROBLEM 16.3 USING INVERSIONS TO DRAW STRAIGHT LINES

At the beginning of Chapter 1, there is a brief history of attempts to find linkages that would draw a straight line. In this problem we explore the mathematics behind this linkage.

a. *Show that for the linkage in* Figure 16.7 *the points P and Q are the inversions of each other through the circle of inversion with center at C and radius* $r = \sqrt{s^2 - d^2}$.

Draw the circle with center R and radius d and note that C, P, Q are collinear.

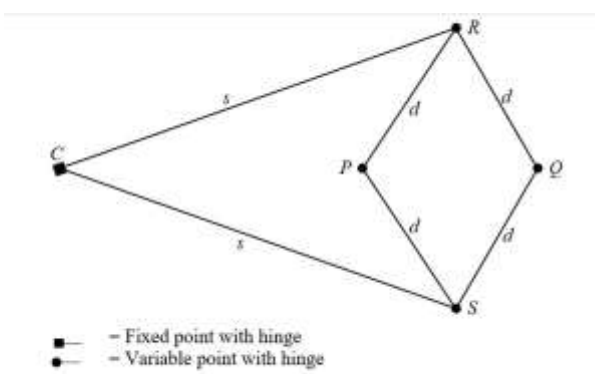

Figure 16.7 A linkage for constructing an inversion

b. *Show that the point Q in the linkage in Figure* 16.8 *always traces a straight line.*

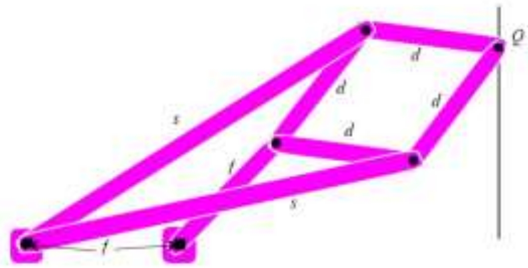

Figure 16.8 Linkage for drawing a straight line

If we modify the Peaucellier-Lipkin linkage (see Figure 1.5) by changing the distance between the anchor points, then

c. *The point Q in the linkage in Figure* 16.9 *always traces the arc of a circle. Why? Show that the radius of the circle is expressed by* $r^2 f / (g^2 - f^2)$, *where r is as in part* **a**.

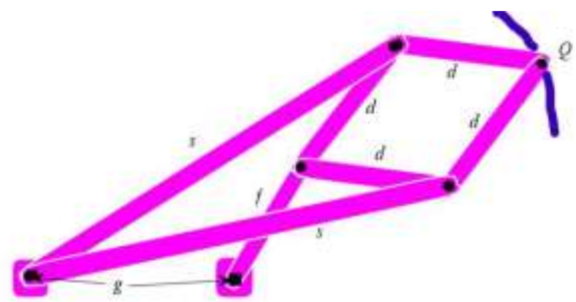

Figure 16.9 Peaucellier-Lipkin linkage modified to draw the arc of a circle

PROBLEM 16.4 APOLLONIUS' PROBLEM

In Book IV of the *Elements*, Euclid shows how to construct the circle that passes through three given (non-collinear) points and also how to construct a circle tangent to three given straight lines (not passing the same point). Apollonius of Perga (c. 250–175 B.C.) generalized this to

> **Apollonius' Problem:** *Given three objects, each of which may be a point, a line, or a circle, construct a circle that passes through each of the given points and is tangent to the given lines and circles.*

Solutions to this problem are discussed in Apollonius' *On Tangencies* (*De Tactionibus*). Unfortunately, Apollonius' work has not survived, but it has been "reconstructed" from both Arabic and Greek commentaries especially through the description of its contents from Pappus of Alexandria (ca. A.D. 300). In Book 7 of the *Mathematical Collection*, Pappus described the contents of various works by Apollonius. Pappus presents a list of problems in Apollonius' lost work, and on the basis of this information the work has been reconstructed at least four times.

Francois Viete (1540–1603) restored Apollonius *De tactionibus* and published it under the title *Apollonius Gallus* in 1600. Frans van Schooten, in a 1657 reconstruction, showed that Apollonius' problem can be solved by the algebraic methods of Descartes' *Geometrie* (1637). Joachim Jungius and Woldeck Weland (1622 –1641), in a reconstruction titled *Apollonius Saxonicus*, used a purely geometrical method that they called "metagoge," that is, the reduction of the general case of a problem to a special case of the same problem or a simpler problem (such as in Problem **16.4c** below). Another reconstruction was completed by Robert Simson (1749). The authors of these reconstructions were doing mathematics, not the history of mathematics, as can be inferred from the fact that the "reconstructions" differ from each other and sometimes deal with generalizations of the problems that had actually been treated by Apollonius.

In addition to the attempted reconstructions, there were many and varied solutions of the Apollonius problem produced by later mathematicians, including Isaac Newton (1643–1727), A. van Roomen (1561–1615), J. Casey (1820–1881), R. Descartes, P. Fermat, Princess Elizabeth (1596–1662), L. Euler (1707–1783), N. Fuss (1755–1826), L. N. M. Carnot (1753–1823), J. D. Gergonne (1771–1859), C. F. Gauss (1777–1855), J. V. Poncelet (1788–1867), A. L. Cauchy (1789–1857), and Eduard Study (1862–1930). You should recognize some of these names. Poncelet and Cauchy solved Apollonius' problem while first year students at the École Polytechnique.

In 1679 P. Fermat formulated and solved an extension of Apollonius problem to 3-space: Construct a sphere tangent to four given spheres. This is called *Fermat's problem* by some later authors. A large number of mathematicians were discussing this problem in 19th century. Some further generalizations of Apollonius' problem are discussed in H. S. M. Coxeter, "The Problem of Apollonius," *American Mathematical Monthly* 75 (1968), pp. 5–15.

In 2003, R. H. Lewis and S. Bridgett, in a paper entitled "Conic tangency equations and Apollonius problems in biochemistry and pharmacology" (*Mathematics and Computers in Simulation*, vol. 61; Jan. 2003, pp. 101–114), discuss current applications of Apollonius' problems. The applications involve bonding interactions in human bodies between protein molecules and hormone, drug, and other molecules.

Apollonius' problem can be discussed in 10 possible cases (letting P= point, L = line, C = circle):

PPP, PPL, PPC, PLL, PLC, PCC, LLL, LLC, LCC, CCC.

a. *Solve the case* **PPP**. *Show that the solution implies that the perpendicular bisectors of the sides of a triangle all pass through the same point*. We call this point the ***circumcenter*** of the triangle. See Figure 16.10.

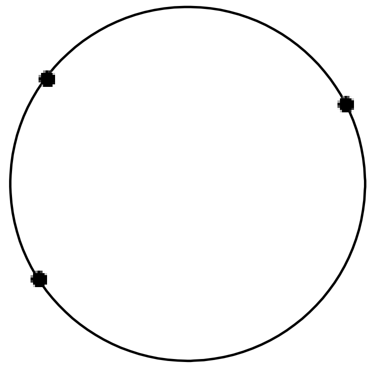

Figure 16.10 **PPP**

Hint: If the three points are on a line, then that line (as a circle with infinite radius) is the solution. Otherwise the three points determine a triangle.

b. *Solve the case* **LLL**. *Show that the solution implies that the angle bisectors of a triangle pass all through the same point.* This point is called the ***incenter*** of the triangle.

Hint: If the three lines intersect in the same point, then there is no solution. If the three lines form a triangle, then there are four solutions. See Figure 16.11. What happens in other cases?

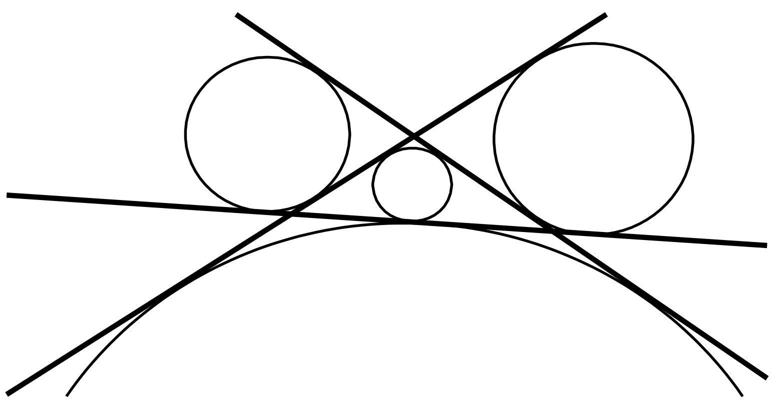

Figure 16.11 **LLL**

c. *Solve the cases* **PPL** *and* **PPC**.

Outline of solution: If both points are on the given line or circle, the only solution is the given line or circle. If point A is not on the given line (or circle), then we can take A as the center of an inversion. Denote the other point B and the given line or circle Γ. Under the inversion, Γ goes to another circle Γ' (never a line, *Why?*) and B goes to another point B'. Let l be a line through B' that is tangent to Γ'. Inverting this tangent l back to the original picture, we will have a solution. (*Why?*) If B is on Γ, then there exists one solution; if neither A or B are on Γ, then there are two solutions.

d. *Solve the cases* **PLL**, **PLC**, *and* **PCC**.

Hint: Choose the given point as the center of an inversion. After the inversion the solution will be a line (in order to contain the inversive image of the point). See Figure 16.12. Depending on the original location of the given point, we get the following subcases:

1. The point is on neither of the lines (line and circle, circles). Then those circles (line and circle, lines) in inversion will go to two circles and the problem reduces to constructing a tangent to two given circles. In this subcase there are either 0, 1, 2, 3, or 4 solutions possible. (Why?)
2. The point is the point of tangency of circles (or circle and line).
3. The point is the point of intersection of circles (or circle and line).
4. The point is on one circle (or line) but not on the other.

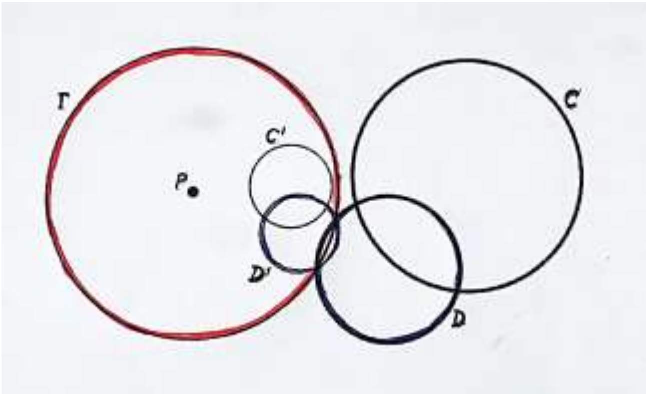

Figure 16.12 Case **PCC**: invert through Γ and then find tangent to *C′* and *D′*

e. **Application of PCC.** *There is a story that in World War I troops used* **PCC** *to pinpoint the location of large enemy guns. The three separate observation points synchronize their clocks to the second and then note the exact second that they hear the gun's sound. How could these timings be used to pinpoint the location of the gun?*

Hint: The speed of sound is approximately 340 meters/second (it varies ±5% in somewhat predictable ways with temperature and atmospheric conditions). Draw a picture of the instant in time when the sound reaches the first observation point.

f. *Solve cases* **LLC**, **LCC**, *and* **CCC**.

Outline of solution: Each of these three cases can be reduced to either **CCC** or **PCC** by using an appropriate inversion. (*Do you see how?*) There are many subcases depending on how the circles and lines relate to each other: inside, outside, intersecting. However, the overall strategy is to reduce these cases to **PLC**, **PLL**, or **PCC** in part **d**. We illustrate this with the subcase of **LCC**, where all circles and lines are disjoint, and the two circles are on the same side of the line and exterior to each other. See Figure 16.13.

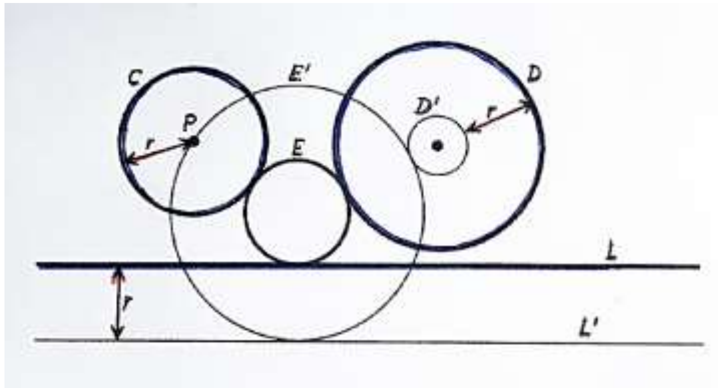

Figure 16.13 Reducing **LCC** to **PLC**

Let r be the radius of the smaller circle, C. Shrink the other circle, D, to the circle, D', with the same center but with radius reduced by r. Construct another line parallel to L, on the side opposite to C and D, and at a distance of r. Now apply **PLC** to the point P, the circle D', and the line L', to get a circle E' that is tangent to P, D', and L'. The required solution to the original **LCC** is the circle E with the same center as E' but with radius decreased by r. See Figure 16.13.

EXPANSIONS OF THE NOTION OF INVERSIONS

There was a rebirth of interest in inversions in the 19th century. Jakob Steiner (1796–1863) was among the first to start extensively using the technique of inversions in circles to solve geometric problems. Steiner had no early schooling and did not learn to read or write until he was age 14. Against the wishes of his parents, at age 18 he went to the Pestalozzi School at Yverdon, Switzerland, where his extraordinary geometric intuition was discovered. By age 28 he was making many geometric discoveries using inversions. At age 38 he occupied the chair of geometry established for him at the University of Berlin, a post he held until his death.

Steiner defined an ***inversive transformation*** to be any transformation that is the composition of inversions and initiated ***inversive geometry***, which is the study of properties of the extended plane that are preserved by inversive transformations. It follows from Problem **16.2** that inversive transformations preserve angles and take circles and lines to circles and lines.

Jean Victor Poncelet (1788–1867) showed that an inversion is a ***birational transformation***, that is, a one-to-one transformation of the extended plane such that both the transformation and its inverse are of the form

$$x' = f(x, y),\ y' = g(x, y), \text{ where } f \text{ and } g \text{ are rational functions.}$$

There were many other European mathematicians in the 19th century who studied inversions and inversive geometry. Applications of inversions in physics were used by

Lord Kelvin (Sir William Thomson) (1824–1907) in 1845, and also by Joseph Liouville (1809–1882) in 1847, who called inversions *the transformations by reciprocal radii*.

In 1854 Luigi Cremona (1830–1903) made a systematic study of birational transformations that carry the entire extended plane onto itself. These transformations are now often called ***Cremona transformations***. Since inversions take the entire extended plane to itself, they are Cremona transformations. These were subsequently studied by Max Noether (1844–1921), who proved that a plane Cremona transformation (and thus inversions) could be constructed by a sequence of quadratic and linear transformations.

In 1855, August Ferdinand Möbius (1790–1868) undertook a systematic study of circular transformations (conformal transformations that map points on a circle to points on a circle) by purely geometrical means. He defined what are now called ***Möbius transformations***, which are often studied today in courses on complex analysis:

> *A* ***Möbius transformation*** *is any transformation of the extended complex plane onto itself of the form*
>
> $M(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers and $ad - bc \neq 0$.

The following properties of Möbius transformations are proved in Sections 5.3 and 5.4 of [**TX**: Brannan]:

- *Every Möbius transformation is an inversive transformation* (but no inversion can be a Möbius transformation because Möbius transformations preserve orientation and inversions do not).
- Möbius *transformations form a subgroup of the group of inversive transformation.*
- *Every inversion F can be written in the form* $F(z) = \overline{M(z)}$, *where M is a Möbius transformation*.
- *Given any two sets of three points,* z_1, z_2, z_3, *and* w_1, w_2, w_3, *there is a unique Möbius transformation that maps* z_1 *to* w_1, z_2 *to* w_2, *and* z_3 *to* w_3.

Möbius geometry is also connected to Laguerre geometry initiated by Edmond Laguerre (1834–1868) and Minkowskian geometry initiated by Hermann Minkowski (1864–1909), which is the geometry that Einstein used in the Theory of Relativity and Space/Time. In 1900, Edward Kasner (1878–1955) was apparently the first to study inversive geometry in accordance with Klein's Erlanger Program (see Chapter 11, page 153). Research on transformations that preserve circles continued into at least the middle of the 20th century. Continuing to today, inver- sions are used in the two Poincaré models of hyperbolic geometry (see Problems **17.2** through **17.5**).

For more discussion of inversive and related geometries, see [**TX**: Brannan], Chapter 5, and [**HI**: Kline], Section 39.3. Kline also makes connections to algebraic geometry. See also [**HM**: Marchisotto] for related history in the 20th century.