<center>Chapter 9</center>

MORE TRIANGLE CONGRUENCIES

Things which coincide with one another are equal to one another.

— Euclid, Elements, Common Notion 4

This chapter is a continuation of the triangle congruence properties studied in Chapter 6.

PROBLEM 9.1 SIDE-SIDE-SIDE (SSS)

a. *Are two triangles congruent if the two triangles have congruent corresponding sides? Look at plane, spheres, and hyperbolic planes.* See Figure 9.1.
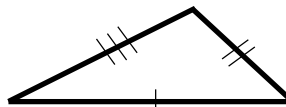

Figure 9.1 SSS

SUGGESTIONS

Start investigating SSS by making two triangles coincide as much as possible and see what happens. For example, in Figure 9.2, if we line up one pair of corresponding sides of the triangles, we have two different orientations for the other pairs of sides as depicted in Figure 9.2. Of course, it is up to you to determine if each of these orientations is actually possible, and to prove or disprove SSS. Again, symmetry can be very useful here.

Figure 9.2 Are these congruencies possible?

On a sphere, SSS doesn't work for all triangles. The counterexample in Figure 9.3 shows that no matter how small the sides of the triangle are, SSS does not hold because the three sides always determine two different triangles on a sphere. Thus, it is necessary to restrict the size of more than just the sides for SSS to hold on a sphere. Whatever argument you used for the plane should work for *suitably defined* small triangles on the sphere and all triangles on a hyperbolic plane. Make sure you see what it is in your argument that doesn't work for large triangles on a sphere.
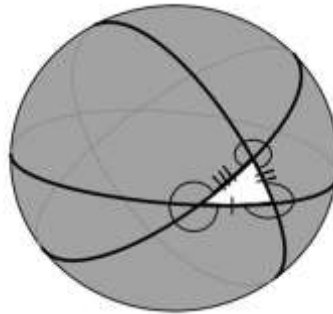

Figure 9.3 A large triangle with small sides

There are also other types of counterexamples to SSS on a sphere. Can you find them?

PROBLEM 9.2 ANGLE-SIDE-SIDE (ASS)

a. *Are two triangles congruent if an angle, an adjacent side, and the opposite side of one triangle are congruent to an angle, an adjacent side, and the opposite side of the other? Look at plane, spheres, and hyperbolic planes.* See Figure 9.4.
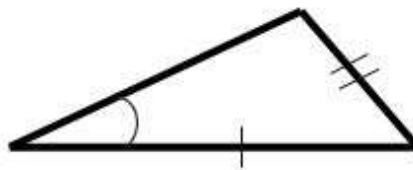

Figure 9.4 ASS

SUGGESTIONS

Suppose you have two triangles with the above congruencies. We will call them ASS triangles. We would like to see if, in fact, the triangles are congruent. We can line up the angle and the first side, and we know the length of the second side (*BC* or *B'C'*), but we don't know where the second and third sides will meet. See Figure 9.5.
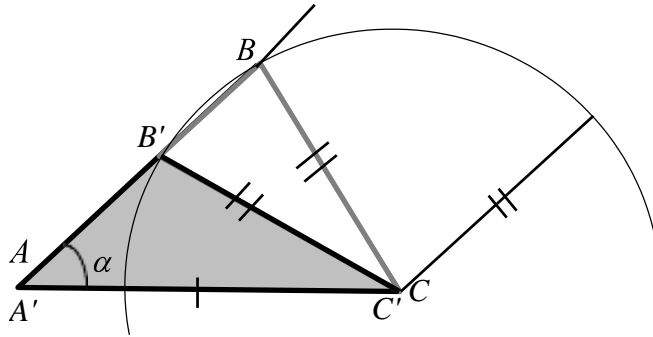

Figure 9.5 ASS is not true, in general

Here, the circle that has as its radius the second side of the triangle intersects the ray that goes from *A* along the angle α to *B* twice. You see that ASS doesn't work for all triangles on the plane or spheres or hyperbolic planes. Try this for yourself on these surfaces to see what happens. Can you make ASS work for an appropriately restricted class of triangles? On a sphere, also look at triangles with multiple right angles, and, again, define "small triangles" as necessary. Your definition of "small triangle" here may be very different from your definitions in Problems **6.4** and **6.5**. There are numerous collections of triangles for which ASS is true.

Explore. See what you find on all three surfaces.

> *b.* *Show that ASS holds for right triangles on the plane (where the Angle in Angle-Side-Side is right).*

This result is often called the *Right-Leg-Hypotenuse Theorem* (RLH), which can be expressed in the following way:

> **RLH:** *On the plane, if the leg and hypotenuse of one right triangle are congruent to the leg and hypotenuse of another right triangle, then the triangles are congruent. What happens on a sphere and a hyperbolic plane?*

At this point, you might conclude that RLH is true for small triangles on a sphere. But there *are* small triangle counterexamples to RLH on spheres! The counterexample in Figure 9.6 will help you to see some ways in which spheres are intrinsically very different from the plane. We can see that the second leg of the triangle intersects the geodesic that contains the third side an infinite number of times. So, on a sphere there are small triangles that satisfy the conditions of RLH although they are not congruent. What about RLH on a hyperbolic plane?
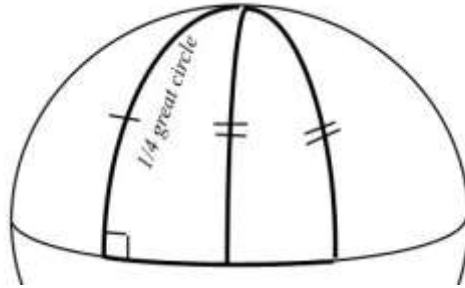
Figure 9.6 Counterexample to RLH on a sphere

However, if you look at your argument for RLH on the plane, you should be able to show the following:

On a sphere, RLH is valid for a triangle with all sides less than 1/4 of a great circle.

RLH is also true for a much larger collection of triangles on a sphere. Can you find such a collection? What about on a hyperbolic plane?

PROBLEM 9.3 SIDE-ANGLE-ANGLE (SAA)

Are two triangles congruent if one side, an adjacent angle, and the opposite angle of one triangle are congruent, respectively, to one side, an adjacent angle, and the opposite angle of the other triangle? Look at plane, spheres, and hyperbolic planes.

SUGGESTIONS

As a general strategy when investigating these problems, start by making the two triangles coincide as much as possible. You did this when investigating SSS and ASS. Let us try it as an initial step in our proof of SAA. Line up the first sides and the first angles. Because we don't know the length of the second side, we might end up with a picture similar to Figure 9.7.
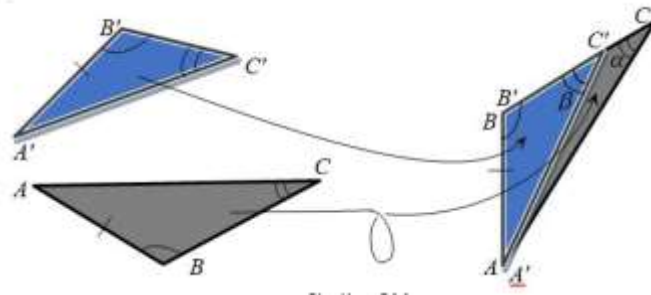

Figure 9.7 Starting SAA

The situation shown in Figure 9.7 may seem to you to be impossible. You may be asking yourself, "Can this happen?" If your temptation is to argue that α and β cannot be congruent angles and that it is not possible to construct such a figure, behold Figure 9.8.
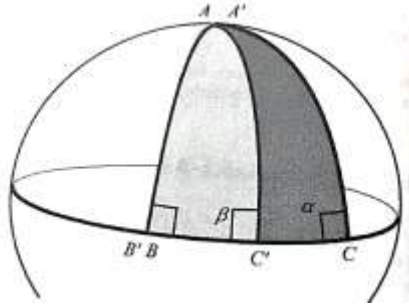
Figure 9.8 A counterexample to SAA

You may be suspicious of this example because it is not a counterexample on the plane. You may feel certain that it is the only counterexample to SAA on a sphere. In fact, we can find other counterexamples for SAA on a sphere.

With the help of parallel transport, you can construct many counter- examples for SAA on a sphere. If you look back to the first counterexample given for SAA, you can see how this problem involves parallel transport, or similarly how it involves Euclid's Exterior Angle Theorem, which we looked at in Problem **8.1**.

Can we make restrictions such that SAA *is* true on a sphere? You should be able to answer this question by using the fuller understanding of parallel transport you gained in Problems **8.1** and **8.2**. You may be tempted to use the result, *the sum of the interior angles of a triangle is 180°*, in order to prove SAA on the plane. This result will be proven later (Problem **10.2**) for the plane, but we saw in Problem **7.3** that it does not hold on spheres and hyperbolic planes. Thus, we encourage you to avoid using it and to use the concept of parallel transport instead. This suggestion stems from our desire to see what is common between the plane and the other two surfaces, as much as possible. In addition, before we can prove that the sum of the angles of a triangle is 180°, we will have to make some additional assumptions on the plane that are not needed for SAA.

PROBLEM 9.4   ANGLE-ANGLE-ANGLE (AAA)

Are two triangles congruent if their corresponding angles are congruent? Look at plane, spheres, and hyperbolic planes.
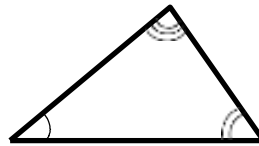

Figure 9.9 AAA

Two triangles that have corresponding angles congruent are called *similar triangles*. We will discuss similar triangles in Problem **13.4**.

As with the three previous problems, make the two AAA triangles coincide as much as possible. We know that we can line up one of the angles, but we don't know the lengths of either of the sides coming from this angle. There are two possibilities (1) Both sides of one are longer than both sides of the other, as the example in Figure 9.10 shows on the plane, or (2) one side of the first triangle is longer than the corresponding side of the second

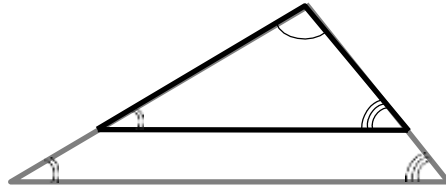triangle and vice versa, as the example in Figure 9.11 shows on a sphere.

Figure 9.10 Is this possible?

As with Problem 9.3, you may think that the example in Figure 9.11 cannot happen on a plane, a sphere, or a hyperbolic plane. The possible existence of a counterexample relies heavily on parallel transport — you can identify the parallel transports in each of the examples given. Try each counterexample on the plane, on a sphere, and on a hyperbolic plane and see what happens. If these examples are not possible, explain why, and if they are possible, see if you can restrict the triangles sufficiently so that AAA does hold.
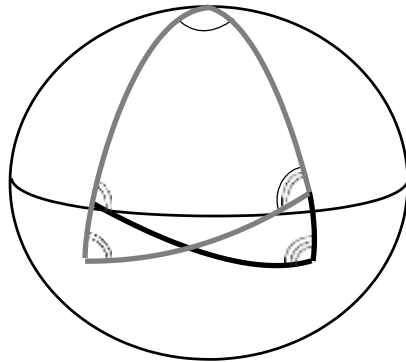

Figure 9.11 Is this possible?

Parallel transport shows up in AAA, similar to how it did in SAA, but here it happens simultaneously in two places. In this case, you will recognize that parallel transport produces similar triangles that are not necessarily congruent. Are these constructions (in Figures 9.10 and 9.11) possible? How? Are the triangles not congruent? Why? These constructions and non-congruencies may seem intuitively possible to you, but you should justify why in each case. Again, you may need properties of angle sums from Problem **7.3** and properties of parallel transport from Problems **8.1** through **8.4**. You may also use the property of parallel transport on the plane stated in Problem **10.1** — you can assume this property now as long as you are sure not to use AAA when proving it later.

On a sphere or on a hyperbolic plane, is it possible to make the two parallel transport constructions shown in Figure 9.11 and thus get two non-congruent triangles? Try it and see. It is important that you make such constructions and that you study them on a model of a sphere and on a model of a hyperbolic plane.