

Chapter 4

Tangent Space, Metric, and Directional Derivative

In this chapter we will begin to set up the formal machinery that will allow us to talk about curvature on a surface. We will use this terminology and formalism to obtain an equation for the normal curvature of any curve on the surface. (See Problem 4.7.) In Chapter 5 we will use this formalism as a part of our intrinsic description of intrinsic curvature. The expression in Problem 4.7 for the normal curvature at a point on a curve depends only on the direction of the curve at that point. Thus it will be the starting point for our investigation of the curvature of the surface in Chapter 6. Note from Chapter 3 we know that the normal curvature is due to the curving of the surface and not due to any intrinsic curving of curves in the surface.

PROBLEM 4.1. The Tangent Space

Go back to Problem 3.1 for the discussion of smooth surfaces and their tangent spaces and normal spaces.

If a curve C intersects a plane Π at a point p , we say that C is *tangent* to the plane at p , if when we zoom in on p sufficiently [that is, given any tolerance τ there is radius ρ such that in any f.o.v. with radius $< \rho$] the portion of the curve in the f.o.v. is indistinguishable from a subset of the plane [and the projection of the curve to the plane is 1-1]. But clearly this does not mean that C lies in the plane. Thus, in general, for a curve that is tangent to the plane at p , as we zoom in, the portion of the curve in the f.o.v. becomes closer and closer to the plane until it becomes indistinguishable from it. However, when the curve is straight then, as we zoom in, we see the same picture at all magnifications. (See Figure 4.1.) Which angles we can distinguish depend on the tolerance. With decreasing tolerances we will be able to distinguish smaller and smaller angles. Put this discussion together to show that:

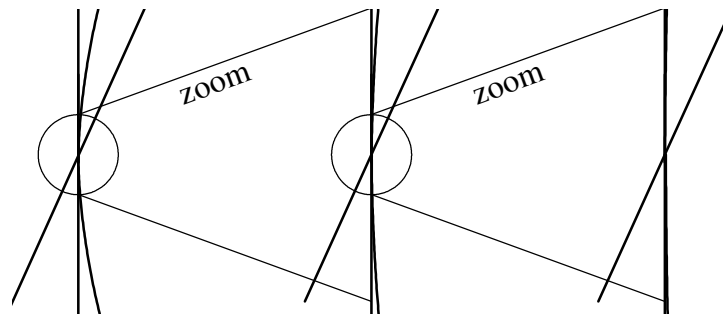


Figure 4.1. Zooming In on Two Straight Lines and a Tangent Curve

- a. A straight line that is tangent to a plane is contained in the plane (not merely indistinguishable from it).

Thus, note that tolerances must increase as we zoom in. This is not the same as “zooming in” in computer graphics because the normal computer screen has fixed pixels and thus fixed tolerance. To model on a

computer the type of zooming in we are considering here you would need a computer screen capable of decreasing tolerances (or of decreasing the size of the pixels and increasing their number). In addition, you would need to use “vector graphics” (common in drawing programs), not “pixel graphics” (common in paint programs).

Note that a tangent line [or plane] approximates a curve [or surface] in a very different way than $22/7$ approximates π . As we zoom in on $22/7$ and π , they will become further and further apart. On the contrary, as we zoom in on a point of tangency, the two objects concerned become closer and closer in the f.o.v. and eventually become indistinguishable. It would be useful at this point for the reader to experiment with a favorite function graphing program that is capable of zooming.

If p is a point on a smooth surface M in \mathbf{R}^n and p is taken to be the origin of \mathbf{R}^n , then:

- b.** *Show that, for every parametrized curve $\mathbf{p}(t)$ that lies in M with $\mathbf{p}(0) = p$, the velocity vector $\mathbf{p}'(0)$ is contained in $T_p M$.*

[Hint: It is not true in a single f.o.v. that, if A is indistinguishable from B and B is indistinguishable from C, then A is indistinguishable from C. However, you can argue that “being tangent” is a transitive relation, at least in this case.]

- c.** *Show that every vector lying in $T_p M$ is the velocity vector of some parametrized curve lying in M .*

For Part **c** consider the intersection of M with $(n-1)$ -dimensional subspaces determined by a tangent vector in $T_p M$ and the whole normal space $N_p M$. Start by looking only at surfaces in \mathbf{R}^3 and note that in \mathbf{R}^3 the normal space is a normal line. (See Figure 4.2 for a picture of this situation in \mathbf{R}^3 .)

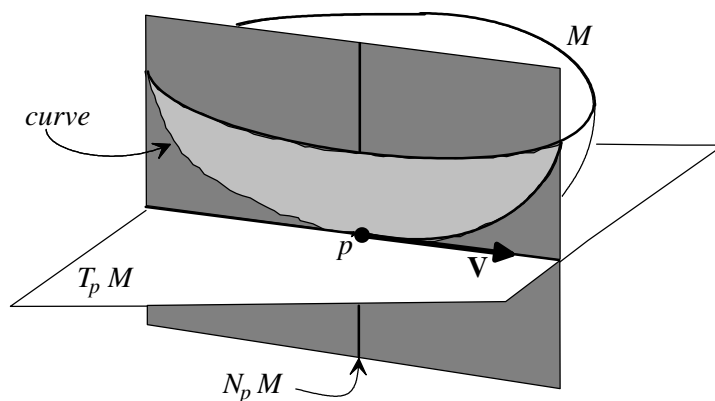


Figure 4.2. Finding a curve, on the surface, with velocity vector V .

We are talking about *parametrized* curves because the velocity of a curve only makes sense if the curve is parametrized. Different parametrizations will, in general, have different speeds but their directions will be the same (or opposite).

The tangent plane is extrinsic since, in general, the tangent vectors do not stay in the surface. But Problem 4.1 allows us to describe a tangent plane intrinsically by saying that the (*intrinsic*) **tangent plane** $T_p M$ is the collection of velocity vectors of parametrized curves in M , that pass through p . It is still easier for us to think of these velocity vectors extrinsically, but they also would have intrinsic meaning to a 2-dimensional bug in the surface. In a small neighborhood of p the bug would experience the surface as a flat plane and would have no problem visualizing velocity of curves in that plane. This is the same as

our experience in the physical 3-dimensional universe. We experience the portion of the universe around us as being a Euclidean space in which we can imagine velocity and other vectors. It does not matter to our imagination what the global geometry of the universe is because for vectors that lie in our immediate space, we can imagine addition and scalar multiplication according to the usual rules.

See Problem 8.2 for further intrinsic descriptions of tangent planes (and tangent spaces).

PROBLEM 4.2. Mean Value Theorem - Curves - Surfaces

In Problem 2.6.b we pointed out that there was a:

- a. Mean Value Theorem for Planar Curves.** Given two (not equal) points \mathbf{p} and \mathbf{q} on a differentiable curve in the plane, there is some point \mathbf{r} on the curve between \mathbf{p} and \mathbf{q} such that the tangent vector \mathbf{T}_r at \mathbf{r} is parallel to $\mathbf{p} - \mathbf{q}$. Prove this theorem.

[Hint: Look at the line determined by \mathbf{p} and \mathbf{q} and then move this line parallel to itself until it last touches the curve.]

As we pointed out in 2.6.b this mean value theorem is not true for curves in 3-space. However, we can prove:

- b. Mean Value Theorem for Space Curves.** Let λ be a smooth curve in \mathbf{R}^n with two distinct points a and b and let L be any $((n-1)$ -dimensional) hyperplane in \mathbf{R}^n which contains a and b . Then there is at least one point c on λ between a and b such that the line tangent to λ at c is parallel to some line in L .

[Hint: Use the same idea as in Part a.]

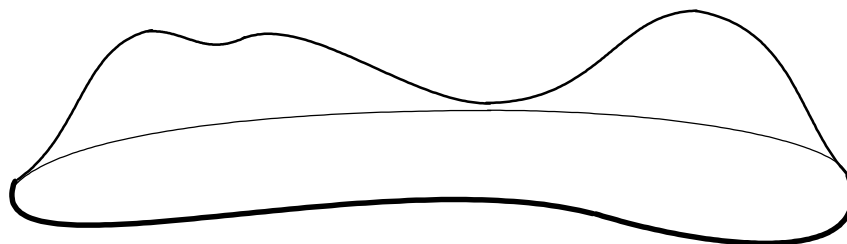


Figure 4.3. Smooth surface with boundary.

There is also an extension of this for surfaces with boundary in 3-space. A **smooth surface with boundary** is intuitively a “smooth surface with an edge” and can be defined by specifying that away from the boundary, the surface is smooth as defined above and that the boundary is a smooth curve, and at each point along the boundary the surface has a tangent half-plane (that is, as you zoom in on the point, the surface becomes indistinguishable from a half-plane whose bounding line is the tangent line to the bounding curve). (See Figure 4.3.)

- c. Mean Value Theorem for Surfaces with Planar Boundary.** If M is a differentiable surface in 3-space whose boundary is a planar curve, then some point on the surface has a tangent plane which is parallel to the plane containing the bounding curve.

[Hint: Use an argument similar to the one used in part a..]

Some more terminology: A surface is said to be **bounded** if it is contained in the interior of a finite sphere. A surface is called **bounding** if it is the boundary of a volume in 3-space. If a bounded surface

has no boundary, then it is called a *closed surface*. You can extend the above Mean Value Theorems to closed surfaces and prove:

- d. If M is a closed differentiable surface and P is any plane in 3-space, then there are at least two points on the surface whose tangent plane is parallel to P .

Natural Parametrizations of Curves

If you wish to find extrinsically a curve on a smooth surface in \mathbf{R}^3 that has a given tangent vector, \mathbf{T}_p , then you may proceed as in Problem 4.1.b and intersect the surface with the plane determined by \mathbf{T}_p and \mathbf{n} , the normal to the surface. If you wish to find intrinsically a curve on a smooth surface that has a given tangent vector, \mathbf{T}_p , then imagine starting at p and proceeding straight in the direction of the tangent vector along a geodesic. The geodesic may be determined by using the ribbon test or the local intrinsic notions of symmetry.

The most natural intrinsic parametrization of a curve is by arc length. Start at some point on the curve and choose a positive direction along the curve. The parameter of a point on the curve is the distance (measured along the curve) from the starting point to that point. Or, if you want a parametrization with constant (nonzero) speed then let the parameter of a point be the time it takes to go at the constant speed from the starting point to that point; that is, $\gamma(t)$ is the point on the curve that is a distance vt from the starting point $\gamma(0)$, where v is the constant speed.

Alternatively, if we have a smooth curve C in \mathbf{R}^n , then we can use any vector \mathbf{X} that is tangent to the curve at p to determine a natural extrinsic parametrization of the curve in a neighborhood of p . Pick rectangular coordinates for \mathbf{R}^n so that

$$p = (0,0,0,\dots,0)$$

and

$$\mathbf{X} = (v,0,0,\dots,0).$$

Then let

$$g(a,b,c,\dots,z) = (a,0,0,\dots,0)$$

be the projection onto the tangent line at p . (See Figure 4.4.) Suppose $g|C$ (the projection restricted to C) is not one-to-one in a neighborhood of p . Then there is a sequence of point pairs $\{a_n, b_n\}$ on C such that $g(a_n) = g(b_n)$, for all n . Let l_n be the line segment joining a_n to b_n . Applying Problem 4.2.b, there is a point c_n on l_n between a_n and b_n such that a vector tangent to C projects to a point on the tangent line. But then the tangent lines to C cannot be varying continuously. Thus, $g|C$ will be one-to-one in some neighborhood of p . In that neighborhood there is a function

$$\mathbf{x}: \mathbf{R} \rightarrow C \text{ such that } g(\mathbf{x}(t)) = (vt,0,0,\dots,0).$$

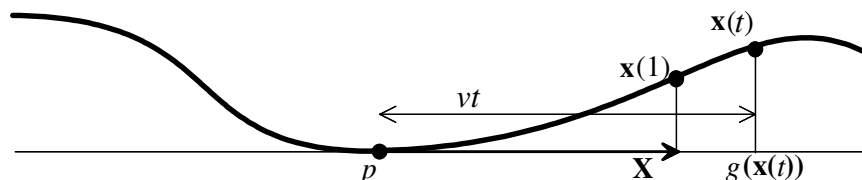


Figure 4.4. Natural extrinsic parametrization for curves.

Note that $\mathbf{x}(t)$ is the intersection of the curve with the $(n-1)$ -dimensional subspace that is perpendicular to the tangent line at the point $(vt,0,0,\dots,0)$. In practice it is usually very difficult, if not impossible, to find

an analytic expression for this function. This \mathbf{x} gives a parametrization for a neighborhood of p in C and $\mathbf{x}'(0) = \mathbf{X}$.

Note that every smooth surface has an extrinsic Monge patch \mathbf{x} (Problem 1.9) that is C^1 (Problem 3.1.e). If (our convention is to use superscripts on coefficients and subscripts on basis vectors)

$$\mathbf{X}_p = X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2,$$

then

$$\gamma(t) = \mathbf{x}(X^1 t, X^2 t)$$

is a curve on M with velocity vector \mathbf{X}_p . In \mathbf{R}^3 , this curve is a parametrized version of the intersection of the surface with the plane spanned by \mathbf{X}_p and \mathbf{n} .

PROBLEM 4.3. Riemannian Metric

If M is a smooth surface in \mathbf{R}^n , then the (induced) *Riemannian metric* (or *first fundamental form*) at $p \in M$ is defined by

$$\langle \mathbf{X}, \mathbf{Y} \rangle = |\mathbf{X}| |\mathbf{Y}| \cos \theta_{\mathbf{X}\mathbf{Y}},$$

for \mathbf{X} and \mathbf{Y} vectors in $T_p M$, where $\theta_{\mathbf{X}\mathbf{Y}}$ is the angle from \mathbf{X} to \mathbf{Y} . This is an intrinsic definition, because vectors in $T_p M$ can be intrinsically described as velocity vectors, and thus, their lengths are speeds and the angle θ between the directions of two vectors is a quantity that can be intrinsically measured. The reason for the term ‘*metric*’ is that it will allow us (see Problem 4.5) to express lengths of curves and areas of regions. The fact that the Riemannian metric is bilinear (4.3.a) will allow us later to use local coordinates in powerful ways.

It is precisely 4.3.a that allows us to express the Riemannian metric in local coordinates. Thus it would lead to circular reasoning if we used local coordinates to prove 4.3.a. We need to know that the important properties in 4.3.a are geometric properties that follow directly from the geometric definition and hold regardless of the coordinate system.

a. Show that the Riemannian metric is:

symmetric, that is, $\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{Y}, \mathbf{X} \rangle$;

bilinear, that is, $a\langle \mathbf{X}, \mathbf{Y} \rangle = \langle a\mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{X}, a\mathbf{Y} \rangle$, for $a \in \mathbf{R}$,
 $\langle \mathbf{X}, \mathbf{Y} + \mathbf{Z} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle + \langle \mathbf{X}, \mathbf{Z} \rangle$

positive definite, that is $\langle \mathbf{X}, \mathbf{X} \rangle$ is positive, if $\mathbf{X} \neq \mathbf{0}$.

[Hint: If $a < 0$ then

$$\theta_{a\mathbf{X}\mathbf{Y}} = \theta_{\mathbf{X}\mathbf{Y}} + \pi.$$

Be sure you see why and take this into account when showing the bilinearity. To show that

$$\langle \mathbf{X}, \mathbf{Y} + \mathbf{Z} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle + \langle \mathbf{X}, \mathbf{Z} \rangle$$

draw a picture and look geometrically at projections.]

b. If $\mathbf{X}_1, \mathbf{X}_2$ is an orthonormal basis for $T_p M$,

(that is, $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle = 0$ and $\langle \mathbf{X}_1, \mathbf{X}_1 \rangle = 1 = \langle \mathbf{X}_2, \mathbf{X}_2 \rangle$)

and

$$\mathbf{A} = a_1 \mathbf{X}_1 + a_2 \mathbf{X}_2 \text{ and } \mathbf{B} = b_1 \mathbf{X}_1 + b_2 \mathbf{X}_2,$$

then show that

$$\langle \mathbf{A}, \mathbf{B} \rangle = a_1 b_1 + a_2 b_2.$$

c. If $\mathbf{X}_1, \mathbf{X}_2$ is an arbitrary basis for $T_p M$, and

$$\mathbf{A} = a_1 \mathbf{X}_1 + a_2 \mathbf{X}_2 \text{ and } \mathbf{B} = b_1 \mathbf{X}_1 + b_2 \mathbf{X}_2,$$

then show that

$$\langle \mathbf{A}, \mathbf{B} \rangle = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

Note that $\langle \mathbf{A}, \mathbf{B} \rangle$ is equal to the inner product $a_1 b_1 + a_2 b_2$ only when the basis $\mathbf{X}_1, \mathbf{X}_2$ is orthonormal and thus the matrix (g_{ij}) is the identity matrix. It is always possible to pick a coordinate system that is orthonormal at a particular point, but it is *not* possible to find a coordinate system that is orthonormal at all points unless the surface is developable from the plane (locally isometric to the plane).

In most texts the matrix $g = (g_{ij})$ is called the *first fundamental form* or *metric* (with respect to the basis $\mathbf{X}_1, \mathbf{X}_2$). Clearly, with different bases the matrices g will, in general, be different.

Note that the above definition of $\langle \mathbf{X}, \mathbf{Y} \rangle$ makes sense in any dimension. In particular, $\langle \mathbf{X}, \mathbf{Y} \rangle$ depends only on a plane (2-dimensional subspace) that contains \mathbf{X} and \mathbf{Y} . If \mathbf{X} and \mathbf{Y} are linearly independent (do not lie in the same line), then there is a unique plane, $\text{sp}(\mathbf{X}, \mathbf{Y})$, which is determined (spanned) by \mathbf{X} and \mathbf{Y} . If we look at that plane, we see that

$$\langle \mathbf{X}, \mathbf{Y} \rangle = |\mathbf{X}| (|\mathbf{Y}| \cos \theta),$$

where $|\mathbf{Y}| \cos \theta$ is the length of the projection of \mathbf{Y} onto \mathbf{X} with a negative sign if \mathbf{X} and the projection of \mathbf{Y} point in opposite directions. (See Figure 4.5.)

Note that the picture in Figure 4.5 holds on the tangent plane (which has Euclidean geometry) and, in general, does not hold on a surface that does not have Euclidean geometry.

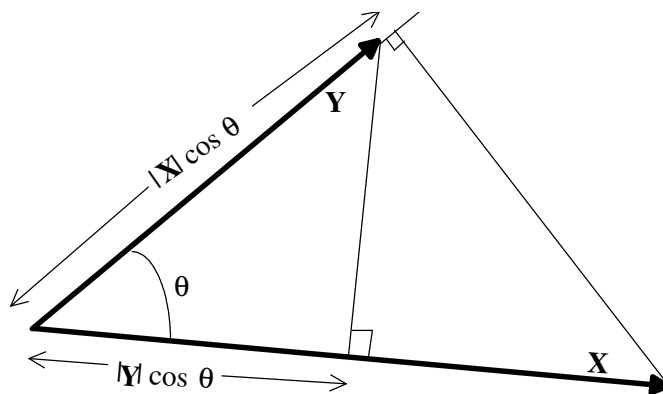


Figure 4.5. Riemannian metric in terms of projections.

It is usually best when working with the Riemannian metric to avoid manipulating the $\cos \theta$ because one is likely to get involved in the unnecessary complication of trigonometric identities. Usually, it is enough to look at the Riemannian metric geometrically in terms of projections as in Figure 4.5 or to use (after you have proved it!) the bilinearity of the metric.

It is particularly important to start with a coordinate free description of the Riemannian metric because, even if the surface is sitting in \mathbf{R}^n with a coordinate system, there will not, in general, be a natural coordinate system on a tangent plane, $T_p M$, to the surface. In different settings, we may choose a coordinate system on the tangent plane, but different settings will naturally lead to different coordinate systems. For example, if we want to focus on a particular curve through p , then we may want the basis vectors for $T_p M$ to be the unit tangent vector and the unit intrinsic normal to the curve at p . If we wish to focus on the curvature of the surface then it is often convenient to choose the principal directions as the coordinate directions (see discussion of curvature and principal directions in Chapter 6).

Note that $\langle \mathbf{X}, \mathbf{Y} \rangle$ can be interpreted as an inner product in any space in which it sits with respect to an orthonormal coordinate system. For example, if \mathbf{X} and \mathbf{Y} are tangent vectors at p on a surface M , which is in \mathbf{R}^n , then (as in Problem 4.3) $\langle \mathbf{X}, \mathbf{Y} \rangle$ is the inner product with respect to any two-dimensional orthonormal coordinate system on $T_p M$, but it is also the inner product with respect to any orthonormal coordinate system in \mathbf{R}^n .

Riemannian Metric in Local Coordinates on a Sphere

We use the sphere to develop an example of expressing the Riemannian metric in terms of (extrinsically defined) local coordinates.

A point on the sphere of radius r can be expressed in terms of two coordinates, θ, ϕ , by this formula (see Figure 4.6):

$$\mathbf{x}(\theta, \phi) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi).$$

Note that \mathbf{x} is a map from \mathbf{R}^2 into \mathbf{R}^3 . At the point $p = \mathbf{x}(\theta, \phi)$ the coordinate curves are

$$\boldsymbol{\lambda}(t) = \mathbf{x}(t, \phi) \text{ and } \boldsymbol{\gamma}(t) = \mathbf{x}(\theta, t).$$

We can obtain a basis for the tangent space at p by using the velocity vectors of these curves:

$$\frac{d}{dt} \boldsymbol{\lambda}(t)_{t=\theta} = \frac{\partial}{\partial t} \mathbf{x}(t, \phi)_{t=\theta} \equiv \mathbf{x}_1(\theta, \phi) = (-r \sin \theta \sin \phi, r \cos \theta \sin \phi, 0),$$

$$\frac{d}{dt} \boldsymbol{\gamma}(t)_{t=\phi} = \frac{\partial}{\partial t} \mathbf{x}(\theta, t)_{t=\phi} \equiv \mathbf{x}_2(\theta, \phi) = (r \cos \theta \cos \phi, r \sin \theta \cos \phi, -r \sin \phi).$$

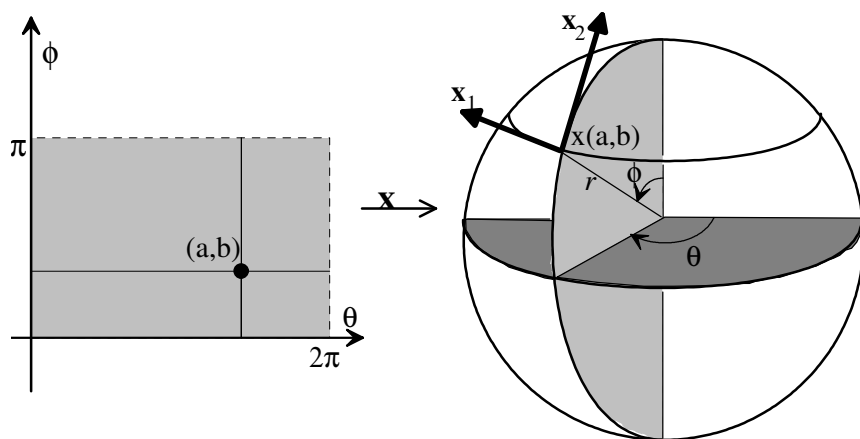


Figure 4.6. Local coordinates on the sphere.

We can now express the Riemannian metric in terms of these local coordinates using Problem 4.3.c:

$$\begin{aligned} g_{11}(\theta, \phi) &= \langle \mathbf{x}_1, \mathbf{x}_1 \rangle = |\mathbf{x}_1|^2 = r^2 \sin^2 \phi, \\ g_{22}(\theta, \phi) &= \langle \mathbf{x}_2, \mathbf{x}_2 \rangle = |\mathbf{x}_2|^2 = r^2, \\ g_{12}(\theta, \phi) &= \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0 = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = g_{21}(\theta, \phi), \end{aligned}$$

and thus the matrix of the Riemannian metric is:

$$g_{ij} = \begin{pmatrix} r^2 \sin^2 \phi & 0 \\ 0 & r^2 \end{pmatrix}.$$

Note that this shows that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis only on the equator of a unit sphere.

Riemannian Metric in Local Coordinates on a Strake

We use the strake to develop another example of expressing the Riemannian metric in terms of (extrinsically defined) local coordinates. (See Figure 4.7.)

A point on the strake can be expressed in terms of two coordinates, r, θ , by this formula:

$$\mathbf{x}(\theta, r) = (r \cos \theta, r \sin \theta, k\theta).$$

Note that \mathbf{x} is a map from \mathbf{R}^2 into \mathbf{R}^3 .

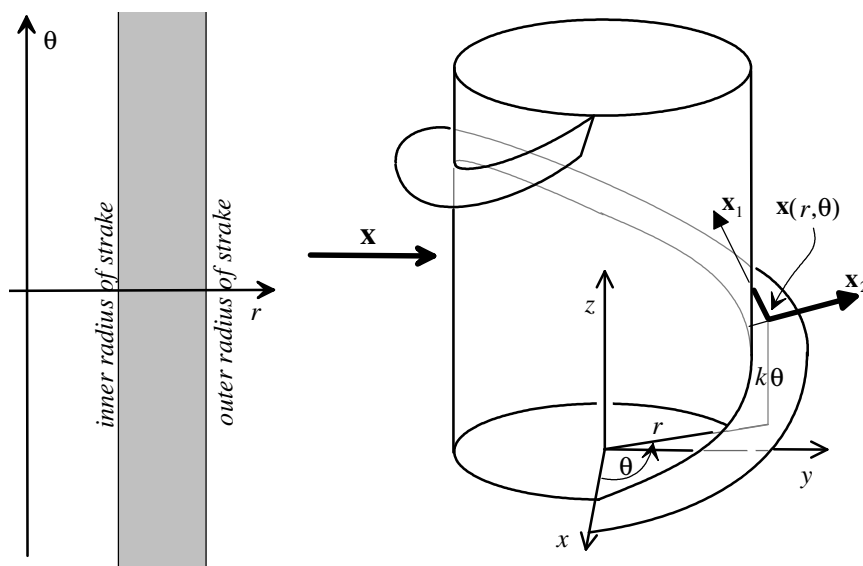


Figure 4.7. Local coordinates on a strake.

At the point $p = \mathbf{x}(\theta, r)$ the coordinate curves are $\lambda(t) = \mathbf{x}(t, r)$ and $\gamma(t) = \mathbf{x}(\theta, t)$. We can obtain a basis for the tangent space at p by using the velocity vectors of these curves:

$$\frac{d}{dt} \lambda(t)_{t=\theta} = \frac{\partial}{\partial t} \mathbf{x}(t, r)_{t=\theta} \equiv \mathbf{x}_1(\theta, r) = (-r \sin \theta, r \cos \theta, k)$$

$$\frac{d}{dt} \gamma(t)_{t=r} = \frac{\partial}{\partial t} \mathbf{x}(\theta, t)_{t=r} \equiv \mathbf{x}_2(\theta, r) = (\cos \theta, \sin \theta, 0).$$

Now we can express the Riemannian metric in terms of these coordinates using Problem 4.3.c:

$$g_{11}(\theta, r) = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle = |\mathbf{x}_1|^2 = r^2 + k^2,$$

$$g_{22}(\theta, r) = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle = |\mathbf{x}_2|^2 = 1,$$

$$g_{12}(\theta, r) = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0 = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = g_{21}(\theta, r).$$

Thus the matrix of the Riemannian metric in these coordinates is:

$$[g_{ij}] = \begin{pmatrix} r^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that this is an orthogonal coordinate system and, although it is impossible to make it orthonormal at every point, it is possible that at some point it will be orthonormal (that is, when $r^2 + k^2 = 1$).

Thus if

$$\mathbf{X}_p = a\mathbf{x}_1 + b\mathbf{x}_2 \text{ and } \mathbf{Y}_p = c\mathbf{x}_1 + d\mathbf{x}_2$$

are two tangent vectors at p , then we can write

$$\langle \mathbf{X}_p, \mathbf{Y}_p \rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} r^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ac(r^2 + k^2) + bd.$$

Intrinsic Riemannian Metric in Geodesic Rectangular Coordinates on an Annular Hyperbolic Plane

In Problem 1.8.b we described geodesic rectangular coordinates on the annular hyperbolic plane H . This coordinate system is defined by picking a point O and a direction along the annular base curve (horocycle) through O and setting $\mathbf{x}(w,s)$ to be the point gotten by traveling a distance of w along the base curve and then traveling a distance s along the radial geodesic at that point (with the positive direction being in the direction toward the center of the annular strip). See Figure 4.7a.

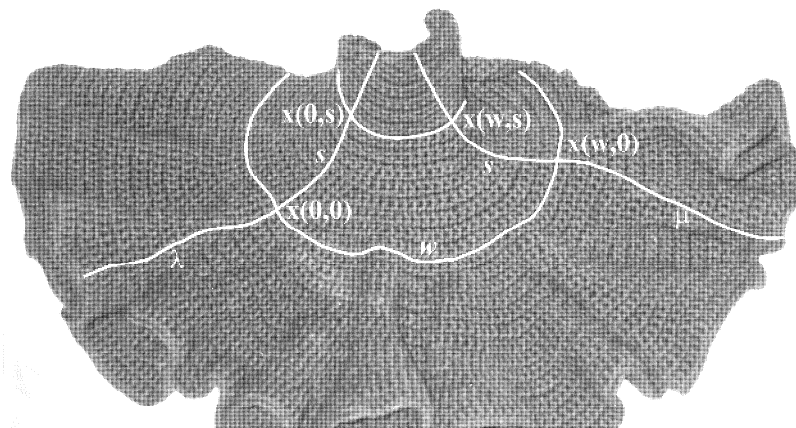


Figure 4.7a. Intrinsic coordinates on the annular hyperbolic plane.

There is no extrinsic description of these coordinates, but we can interpret the tangent plane $T_p H$ intrinsically (see the discussion at end of Problem 4.1) as the collection of velocity vectors of parametrized curves in M , that pass through p . Then the coordinate curves through $p = \mathbf{x}(w,s)$ are: $\boldsymbol{\gamma}(t) = \mathbf{x}(t,s)$ and $\boldsymbol{\mu}(t) = \mathbf{x}(w,t)$ and the basis vectors for $T_p H$ are:

$$\frac{d}{dt}\boldsymbol{\gamma}(t)_{t=w} = \frac{\partial}{\partial t}\mathbf{x}(t,s)_{t=w} \equiv \mathbf{x}_1(w,s) \text{ and } \frac{d}{dt}\boldsymbol{\mu}(t)_{t=s} = \frac{\partial}{\partial t}\mathbf{x}(w,t)_{t=s} \equiv \mathbf{x}_2(w,s),$$

where these are velocities of the coordinate curves. And on the annular hyperbolic plane we have a notion of length and angle. Thus we can define the Riemannian metric on H . Since the second coordinate curves are parametrized by arclength, $|\mathbf{x}_2(w,s)|=1$; and it follows directly from Problem 1.8.c that

$|\mathbf{x}_1(w,s)| = \exp(-s/r)$, where r is the radius of the hyperbolic plane (the extrinsic radius of the annular strips). Note also that the basis vectors are orthogonal. Thus the matrix of the (intrinsic) Riemannian metric in these coordinates is:

$$[g_{ij}] = \begin{pmatrix} [\exp(-s/r)]^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will treat the intrinsic description of Riemannian metrics in more detail in Chapter 8.

PROBLEM 4.4. Vectors in Extrinsic Local Coordinates

If M is a smooth surface in \mathbf{R}^n and $\mathbf{x}: U \rightarrow M$ is a one-to-one function defined on an open region U in the (u^1, u^2) -plane with values in M , then we call \mathbf{x} a \mathbf{C}^1 *coordinate patch* (or \mathbf{C}^1 *local coordinates*) for M if, as a function from \mathbf{R}^2 to \mathbf{R}^n , \mathbf{x} is \mathbf{C}^1 (that is, \mathbf{x} is differentiable and the partial derivatives are continuous), and the vectors $\mathbf{x}_1(a,b)$, $\mathbf{x}_2(a,b)$ are linearly independent for each (a,b) in U , where, if $p = \mathbf{x}(a,b)$, then $\mathbf{x}(a, u^2)$ and $\mathbf{x}(u^1, b)$ are curves on M , and their velocity vectors at p are

$$\mathbf{x}_1(a,b) = \frac{\partial}{\partial u^1} \mathbf{x}(u^1, b)_{u^1=a}; \quad \mathbf{x}_2(a,b) = \frac{\partial}{\partial u^2} \mathbf{x}(a, u^2)_{u^2=b}.$$

In \mathbf{R}^3 , the normal to the surface at $p = \mathbf{x}(a,b)$ is perpendicular to both of the tangent vectors $\mathbf{x}_1(a,b)$ and $\mathbf{x}_2(a,b)$ and thus the normal can be expressed as the unit vector:

$$\mathbf{n}(a,b) = \frac{\mathbf{x}_1(a,b) \times \mathbf{x}_2(a,b)}{|\mathbf{x}_1(a,b) \times \mathbf{x}_2(a,b)|}.$$

Note that at every point on the surface there are two possible normals (in opposite directions to each other), and the above expression picks one of these continuously over the coordinate patch. However, most surfaces cannot be covered by a single coordinate patch. Spheres and cylinders need at least two coordinate patches, yet it is possible to make a continuous selection of normal over the whole surface—such surfaces are said to be *orientable*. But on some surfaces (for example, a Moebius band), it is not possible to continuously pick a normal at every point—such a surface is said to be a *non-orientable*. For further discussion of orientable and non-orientable surfaces see Jeff Weeks' delightful book, *The Shape of Space*, [DG: Weeks].

- a.** For each of the surfaces [cylinder, cone, sphere, strake, surfaces of revolution (defined by a smooth positive-valued function), and the graph of a smooth function $z = f(x,y)$] use the (extrinsically defined) local coordinates from Chapter 1 and find in each case an expression for \mathbf{x}_1 and \mathbf{x}_2 . Check that the local coordinates are a \mathbf{C}^1 coordinate patch.

Since $\mathbf{x}_1(a,b)$, $\mathbf{x}_2(a,b)$ are linearly independent, they form a basis for the tangent (vector) space $T_p M$. Therefore a vector \mathbf{X}_p in $T_p M$ can be expressed as

$$\mathbf{X}_p = X^1 \mathbf{x}_1(a,b) + X^2 \mathbf{x}_2(a,b) = \sum X^i \mathbf{x}_i(a,b),$$

where X^1, X^2 are real numbers. Our convention is to use superscripts on coefficients and subscripts on basis vectors.

If we have a vector field defined on M [that is, a vector-valued function

$$\mathbf{X}_p = \mathbf{X}(p) = \mathbf{X}(\mathbf{x}(u^1, u^2)),$$

where $\mathbf{X}(\mathbf{x}(u^1, u^2))$ is in the tangent space at $p = \mathbf{x}(u^1, u^2)$], then we can write

$$\begin{aligned} \mathbf{X}(p) &= \mathbf{X}(\mathbf{x}(u^1, u^2)) = \\ &= X^1(u^1, u^2) \mathbf{x}_1(u^1, u^2) + X^2(u^1, u^2) \mathbf{x}_2(u^1, u^2) \\ &= \sum X^i(u^1, u^2) \mathbf{x}_i(u^1, u^2). \end{aligned}$$

But usually we implicitly assume the coordinate variables and simply write

$$\mathbf{X}_p = X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2 = \sum X^i \mathbf{x}_i.$$

In many texts this expression is simplified even further by using the so-called “summation convention.” In the summation convention there is an implied summation over all repeated indices with one as a subscript and one as a superscript: $\mathbf{X}_p = X^i \mathbf{x}_i$.

Now we can use Problem 4.3.c to express the Riemannian metric applied to the vector fields

$$\mathbf{X}_p = \sum X^i \mathbf{x}_i \text{ and } \mathbf{Y}_p = \sum Y^i \mathbf{x}_i$$

in local coordinates as

$$\langle \mathbf{X}_p, \mathbf{Y}_p \rangle = \begin{pmatrix} X^1 & X^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \sum X^i g_{ij} Y^j,$$

where $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$.

b. For each of the coordinate patches in Part a, determine the matrix of the Riemannian metric.

Warning: Some texts call (g_{ij}) simply the Riemannian metric, but this only makes sense if a local coordinate system is being assumed (either implicitly or explicitly). When you change the local coordinates, then the matrix (g_{ij}) usually changes.

PROBLEM 4.5. Measuring Using the Riemannian Metric

The goal of this problem is to connect with previous knowledge and notation.

To find the distance (arclength) along any path γ we may integrate the speed $|\gamma'|$ along the path. Thus, the arclength between $\gamma(a)$ and $\gamma(b)$ is equal to

$$\int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt.$$

Note that this is an ordinary integral studied in first-year calculus. So, in principle (or, theoretically), once the Riemannian metric is known we can determine the arclength of any path and thus (using geodesics segments) the distance along the surface between points. I say “in principle” because, in general, the presence of the radical makes solving the integral very difficult. There is only a small special class of curves for which it is possible to evaluate the integral exactly. You can find examples of these curves in the exercises at the end of the arclength section of standard calculus texts. In a recent article [Z: Pottmann, p.183], there is a description of all planar curves with parametrizations, which are rational functions and for which the arclength integral is solvable as a rational function. For most curves the arclength can be calculated only approximately.

If $\mathbf{x}(u^1, u^2)$ gives local coordinates for a region $V = \mathbf{x}(U)$ on the surface, then, in a f.o.v. in which the surface is indistinguishable from the tangent plane, a change of coordinates from

$$u^1 \text{ to } u^1 + \Delta u^1 \text{ [or, from } u^2 \text{ to } u^2 + \Delta u^2]$$

will produce a segment on the surface of length

$$\int_{u^1}^{u^1 + \Delta u^1} |\mathbf{x}_1(t, u^2)| dt \quad \text{[or, } \int_{u^2}^{u^2 + \Delta u^2} |\mathbf{x}_2(u^1, t)| dt].$$

If we let Δu^1 and Δu^2 be small enough, then these segments on the surface will become indistinguishable from straight line segments and we get a small parallelogram on the surface. (See Figure 4.8.)

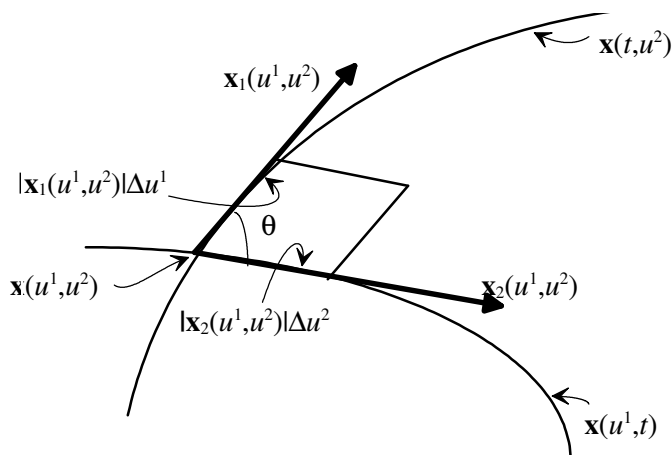


Figure 4.8. Calculating Surface Area

- a. Show that this parallelogram has area:

$$(|\mathbf{x}_1(u^1, u^2)|\Delta u^1)(|\mathbf{x}_2(u^1, u^2)|\Delta u^2)(\sin \theta) = \sqrt{\det g(u^1, u^2)} \Delta u^1 \Delta u^2,$$

where $g(u^1, u^2) = [g_{ij}(u^1, u^2)]$ is the matrix of the Riemannian metric with respect to the coordinates $\mathbf{x}(u^1, u^2)$.

[Hint: Use the definition of the Riemannian metric to find an expression for $\sin \theta$.]

We can now let Δu^1 and Δu^2 go to zero and integrate over the region $V = \mathbf{x}(U)$ to get the following expression for the area of V :

$$\iint_U \sqrt{\det g(u^1, u^2)} du^1 du^2.$$

- b. Use the above expression to find the surface area of an intrinsic circular disk of (intrinsic) radius r on a sphere of radius R . Compare this to the area of a circular disk of the same radius on a plane in the case of $r = 1$ km and $R = 6360$ km (the approximate radius of the earth).

[Hint: Use spherical coordinates from Problem 4.4, and choose the North Pole as the center of the circle. You will need to evaluate the integral as an improper integral (Why?). Relate what you find to the formula πr^2 .]

- c. On a cone with angle α , use the above expression to find the surface area of an intrinsic circular disk of (intrinsic) radius r with the (intrinsic) center of the circle at the cone point.

[Hint: Similar to Part b.]

- d. Find the area of one turn of the striae, and compare this area to the area of the annular strip that approximates it. Use the dimensions given in Problem 2.5.

[Hint: This is a somewhat tedious computation, but it can be done, especially if you use an integral table or computer algebra system to evaluate the integrals that appear.]

- e. Find the area of a region, V , on the annular hyperbolic plane bounded by the base curve (annular curve) and the two radial geodesics that are a distance c apart along the base curve. (See Figure 4.8b.)

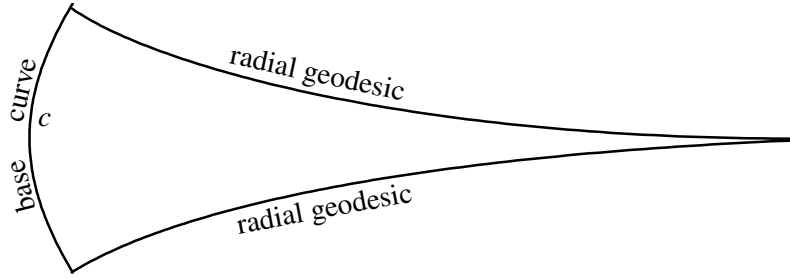


Figure 4.8a. Area of a coordinate sector in the annular hyperbolic plane

Directional Derivatives

If f is a function (real-valued or vector-valued) defined on the smooth surface M , and \mathbf{X}_p is a tangent vector in the tangent plane $T_p M$, then we can define the (extrinsic) **directional derivative of f with respect to \mathbf{X}_p** by

$$\mathbf{X}_p f = \frac{d}{dt} f(\gamma(t))_{t=0} = \lim_{h \rightarrow 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h},$$

where $\gamma(t)$ is any parametrized curve with $\gamma(0) = p$ and $\gamma'(0) = \mathbf{X}_p$. (Such a γ exists by Problem 4.1.) Thus $\mathbf{X}_p f$ is the rate of change of f as one travels along the curve γ at the point p . The function f is said to be **differentiable at p** if $\mathbf{X}_p f$ exists independent of the choice of the curve γ (such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{X}_p$) for each tangent vector \mathbf{X}_p in $T_p M$. Note that in order for f to be differentiable, it must be a function defined on some neighborhood of p , not just on γ or just at p . Below we will show that $\mathbf{X}_p f$ is independent of the choice of curve whenever there is a local coordinate patch \mathbf{x} such that $f \circ \mathbf{x}$ is differentiable.

Many texts use the terminology *directional derivative in the direction of \mathbf{X}* . That terminology originated because at first people only considered directional derivatives with respect to unit vectors and thus did not consider the lengths of the vectors. We do not follow that terminology because, in a literal sense, we are not differentiating *in the direction of a vector* because the directional derivative depends on both the direction and the length of the vector.

If our surface M is the plane, then the rate of change of f along a curve γ is the derivative of $f(\gamma(t))$ with respect to t . But by the chain rule, at $t = 0$, this is

$$f'(\gamma(0)) \gamma'(0) = f'(\gamma(0)) \mathbf{X}_p.$$

This result is one of the motivations for the notation. In addition, we will show in Problem 4.8 that $\mathbf{X}_p f$ depends linearly on \mathbf{X}_p , and thus the notation makes it convenient to express this linearity as

$$(\mathbf{X}_p + (a\mathbf{Y}_p)) f = \mathbf{X}_p f + a(\mathbf{Y}_p f).$$

Examples using the (global) coordinates in \mathbf{R}^n :

1. If M is the sphere

$$x^2 + y^2 + z^2 = 4$$

in 3-space and

$$f(x, y, z) = 5x^2 + y + z$$

and \mathbf{X}_p is the tangent vector at $p = (2, 0, 0)$, which is parallel to $(0, 1, 0)$, then pick

$$\gamma(t) = (2 \cos t/2, 2 \sin t/2, 0).$$

Note that $\gamma'(0) = \mathbf{X}_p$, then

$$f(\gamma(t)) = (5)(4 \cos^2 t/2) + (2 \sin t/2) + (0)$$

and

$$\mathbf{X}_p f = (5)(4)(2)(\cos 0)(-\sin 0)(1/2) + (2)(\cos 0)(1/2) + (0) = 1.$$

2. For the same M , \mathbf{X}_p , p , we can also differentiate the vector-valued function

$$\mathbf{n}(x, y, z) = 1/2(-x, -y, -z),$$

which gives the unit (inward) normal to M and thus

$$\mathbf{n}(\gamma(t)) = 1/2(-2 \cos t/2, -2 \sin t/2, 0)$$

and

$$\mathbf{X}_p \mathbf{n} = (1/2 \sin 0, -1/2 \cos 0, 0) = (0, -1/2, 0) = -1/2 \mathbf{X}_p.$$

Note that in $\mathbf{X}_p \mathbf{n}$, the \mathbf{X}_p is a tangent vector at p but \mathbf{n} is a vector-valued function. (If it were not a function, you would not be able to differentiate it!)

Example in Local Coordinates:

Let us redo Example 2, above, in local coordinates:

2. If M is the sphere

$$x^2 + y^2 + z^2 = 4$$

in 3-space and \mathbf{X}_p is the tangent vector at $p = (2, 0, 0)$, which is parallel to $(0, 1, 0)$, then pick

$$\gamma(t) = (2 \cos t/2, 2 \sin t/2, 0).$$

Then we can differentiate the vector-valued function

$$\mathbf{n}(x, y, z) = 1/2(-x, -y, -z),$$

which gives the unit (inward) normal to M and thus

$$\mathbf{n}(\gamma(t)) = 1/2(-2 \cos t/2, -2 \sin t/2, 0)$$

and

$$\mathbf{X}_p \mathbf{n} = (1/2 \sin 0, -1/2 \cos 0, 0) = (0, -1/2, 0) = -1/2 \mathbf{X}_p.$$

Let

$$\mathbf{x}(\theta, \phi) = (2 \cos \theta \cos \phi, 2 \sin \theta \cos \phi, 2 \sin \phi)$$

be the local coordinate system on the sphere M . Then the point

$$p = (2, 0, 0) = \mathbf{x}(0, 0)$$

and at this point:

$$\begin{aligned} \mathbf{x}_1 = \mathbf{x}_1(0, 0) &= \frac{\partial}{\partial \theta} \mathbf{x}(\theta, 0)_{\theta=0} = \\ &= (-2 \sin \theta \cos 0, 2 \cos \theta \cos 0, 2 \sin 0)_{\theta=0} = (0, 2, 0), \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_2 = \mathbf{x}_2(0, 0) &= \frac{\partial}{\partial \phi} \mathbf{x}(0, \phi)_{\phi=0} = \\ &= (-2 \cos 0 \sin \phi, -2 \sin 0 \sin \phi, 2 \cos \phi)_{\phi=0} = (0, 0, 2) \end{aligned}$$

Now

$$\mathbf{X}_p = \frac{1}{2} \mathbf{x}_1 + 0 \mathbf{x}_2, \text{ so } X^1 = \frac{1}{2} \text{ and } X^2 = 0.$$

Then

$$(\mathbf{n} \circ \mathbf{x})(\theta, \phi) = (-\cos \theta \cos \phi, -\sin \theta \cos \phi, -\sin \phi)$$

and we can write $\gamma(t) = \mathbf{x}(t/2, 0)$ and at $p = (0, 0)$:

$$\begin{aligned} \mathbf{X}_p \mathbf{n} &= \frac{d}{dt} \mathbf{n}(\gamma(t))_{t=0} = \frac{d}{dt} (\mathbf{n} \circ \mathbf{x})(t/2, 0)_{t=0} = \\ &= \frac{\partial(\mathbf{n} \circ \mathbf{x})}{\partial \theta} \left(\frac{1}{2} \right) + \frac{\partial(\mathbf{n} \circ \mathbf{x})}{\partial \phi} (0) = \\ &= -X^1 \mathbf{x}_1 \mathbf{n} + X^2 \mathbf{x}_2 \mathbf{n} = -(1/2)(0, 1, 0) + 0 = -\frac{1}{2} \mathbf{X}_p. \end{aligned}$$

Directional Derivative in Local Coordinates

More generally, if f is a function (vector-valued or real-valued) defined on M such that $f \circ \mathbf{x}$ is C^1 (continuously differentiable), then the directional derivatives in the directions of \mathbf{x}_1 and \mathbf{x}_2 are:

$$\mathbf{x}_1 f = \frac{d}{du^1} f(\mathbf{x}(u^1, b))_{u^1=a} = \frac{\partial(f \circ \mathbf{x})}{\partial u^1}(a, b)$$

and

$$\mathbf{x}_2 f = \frac{d}{du^2} f(\mathbf{x}(a, u^2))_{u^2=b} = \frac{\partial(f \circ \mathbf{x})}{\partial u^2}(a, b).$$

Now we can express $\mathbf{X}_p f$ in terms of these coordinates. Let $\boldsymbol{\alpha}(t)$ be a curve on M with

$$\boldsymbol{\alpha}(0) = p \quad \text{and} \quad \boldsymbol{\alpha}'(0) = \mathbf{X}_p.$$

Then, near p (or when t is near 0) we may write

$$\boldsymbol{\alpha}(t) = \mathbf{x}(\alpha^1(t), \alpha^2(t)).$$

Here $\alpha^1(t), \alpha^2(t)$ are the coordinates of $\boldsymbol{\alpha}(t)$. In this context $\mathbf{x}(u^1, u^2)$ gives the location of the point with coordinates u^1, u^2 . We can then write the velocity of $\boldsymbol{\alpha}$ as:

$$\begin{aligned} \boldsymbol{\alpha}'(0) &= \frac{d}{dt} \mathbf{x}(\alpha^1(t), \alpha^2(t))_{t=0} = \\ &= (\mathbf{x}_1) \left(\frac{d\alpha^1}{dt} \right)_{t=0} + (\mathbf{x}_2) \left(\frac{d\alpha^2}{dt} \right)_{t=0} = \\ &= X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2. \end{aligned}$$

Thus

$$\left(\frac{d\alpha^1}{dt} \right)_{t=0} = X^1 \quad \text{and} \quad \left(\frac{d\alpha^2}{dt} \right)_{t=0} = X^2.$$

Therefore,

$$\begin{aligned} \mathbf{X}_p f &= \frac{d}{dt} f(\boldsymbol{\alpha}(t))_{t=0} = \frac{d}{dt} f(\mathbf{x}(\alpha^1(t), \alpha^2(t)))_{t=0} = \\ &= \frac{d}{dt} (f \circ \mathbf{x})(\alpha^1(t), \alpha^2(t))_{t=0} = \\ &= \frac{\partial(f \circ \mathbf{x})}{\partial u^1} \left(\frac{d\alpha^1}{dt} \right)_{t=0} + \frac{\partial(f \circ \mathbf{x})}{\partial u^2} \left(\frac{d\alpha^2}{dt} \right)_{t=0} = \\ &= X^1 \mathbf{x}_1 f + X^2 \mathbf{x}_2 f. \end{aligned}$$

Thus we have shown:

THEOREM 4.5. *If $f \circ \mathbf{x}$ is C^1 , then the directional derivatives of f do not depend on the choice of curve, and $[X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2]f = X^1 \mathbf{x}_1 f + X^2 \mathbf{x}_2 f$.*

PROBLEM 4.6. Differentiating a Metric

Local coordinates are not necessary in any of these parts and the proofs are easier if you do not use local coordinates; but you do use the properties of the Riemannian metric from Problem 4.3 and note that for a surface M in \mathbf{R}^3 , $\langle \cdot, \cdot \rangle_M = \langle \cdot, \cdot \rangle_{\mathbf{R}^3}|_M$.

- a. Show that if \mathbf{X} and \mathbf{Y} are differentiable vector-valued functions defined on a curve C with parametrization $\gamma(t)$, then

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{X}(\gamma(t)), \mathbf{Y}(\gamma(t)) \rangle &= \\ &= \left\langle \frac{d}{dt} \mathbf{X}(\gamma(t)), \mathbf{Y}(\gamma(t)) \right\rangle + \left\langle \mathbf{X}(\gamma(t)), \frac{d}{dt} \mathbf{Y}(\gamma(t)) \right\rangle. \end{aligned}$$

[Hint: $\langle \mathbf{X}(\gamma(t)), \mathbf{Y}(\gamma(t)) \rangle$ is a continuous (Why?) real-valued function of a real variable, so you can apply the limit definition of derivative from first-semester calculus and the ideas of the usual calculus proof of the Product Rule. Avoid expressions where “ $\cos \theta$ ” shows but do use the properties of the metric from 4.3.]

- b. Show that if \mathbf{X} and \mathbf{Y} are defined on a neighborhood of p in M , and \mathbf{Z}_p is a vector in $T_p M$, then

$$\mathbf{Z}_p \langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{Z}_p \mathbf{X}, \mathbf{Y}(p) \rangle + \langle \mathbf{X}(p), \mathbf{Z}_p \mathbf{Y} \rangle.$$

[Hint: Use Part a.]

- c. If \mathbf{X} and \mathbf{Y} are defined (and differentiable) on a neighborhood of p in M and are everywhere perpendicular, then show that:

$$\langle \mathbf{Z}_p \mathbf{X}, \mathbf{Y} \rangle = - \langle \mathbf{X}, \mathbf{Z}_p \mathbf{Y} \rangle.$$

[Hint: Use Part b.]

PROBLEM 4.7. Expressing Normal Curvature

In Chapter 3 we defined the normal curvature of a curve C on a surface M in \mathbf{R}^3 to be the projection of the (extrinsic) curvature vector onto the normal line. Now, we express the normal curvature in local coordinates and prepare for the second fundamental form, which will be an important tool in studying the curvature of the surface.

- a. Let C be a smooth curve on a smooth surface M in \mathbf{R}^3 , let \mathbf{T}_p be a unit tangent vector at p on C , and let \mathbf{n} be a vector-valued function defined in a neighborhood of p that gives a continuous choice of unit normal vector. If \mathbf{n} is differentiable along C , then the directional derivative $\mathbf{T}_p \mathbf{n}$ is in $T_p M$ and the normal curvature κ_n of C at the point p is given by

$$\kappa_n = \langle \mathbf{T}_p, -\mathbf{T}_p \mathbf{n} \rangle \mathbf{n}.$$

[Hint: Use parametrization by arclength and note that $\kappa_n = \langle \kappa, \mathbf{n} \rangle \mathbf{n}$. Let $\gamma(s)$ be a parametrization of the curve C by arclength such that $\gamma(0) = p$. Then the curvature of the curve at p is, by definition,

$$\kappa = \frac{d}{ds} \mathbf{T}(\gamma(s))_{s=0}.$$

And

$$\mathbf{T}_p \mathbf{n} = \frac{d}{ds} \mathbf{n}(\gamma(s))_{s=0}.$$

Also, note that along the curve:

$$\langle \mathbf{T}(\gamma(s)), \mathbf{n}(\gamma(s)) \rangle = 0.$$

Note that this Riemannian metric must be the metric on \mathbf{R}^3 , not $T_p M$, since \mathbf{n} is not in the tangent plane.] *Be sure you see why these statements are true* and then differentiate the last expression with respect to s .]

- b.** Find geometric meaning in the expression $\langle \mathbf{T}_p, -\mathbf{T}_p \mathbf{n} \rangle$ by relating it to Problems 2.4.b and 4.1.c.

Here we see a first hint of why it may be possible for the normal curvature (the curvature due to the curving of the surface) to produce an intrinsic quantity because, even though \mathbf{n} is an extrinsic quantity, its derivative $\mathbf{T}_p \mathbf{n}$ (being the derivative of a unit vector) is a tangent vector at p (Why?) and thus is intrinsic.

- *c. Find a simple smooth surface on which, at some point p , the normal \mathbf{n} is not differentiable in some directions.
- d. Use Part a to show that on a sphere of radius R , the (scalar) normal curvature of every curve at every point is $1/R$.

Since κ_n depends only on the unit tangent vector \mathbf{T}_p we see that the normal curvature is the same for all curves through p that have the same unit tangent vector (that is, that go in the same direction). Thus we can speak of the normal curvature in the direction \mathbf{T} and write

$$|\kappa_n(\mathbf{T})| = \kappa_n(\mathbf{T}) = \langle \mathbf{T}, -\mathbf{Tn} \rangle.$$

Let us illustrate by applying this to the strake using the coordinates from pages 56-57. We can calculate the normal to strake at the point p as

$$\mathbf{n}(\theta, r) = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{(-k \sin \theta, k \cos \theta, -r)}{\sqrt{k^2 + r^2}}.$$

There is, of course, another unit normal in the opposite direction, but it is conventional to use the above as the choice of normal when there is a local coordinate system.

Now we can differentiate the normal in the coordinate directions:

$$\begin{aligned} \mathbf{x}_1 \mathbf{n} &= \frac{d}{dt} \mathbf{n}(\gamma(t))_{t=\theta} = \frac{\partial}{\partial t} \mathbf{n}(t, r)_{t=\theta} = \\ &= \frac{(-k \cos \theta, -k \sin \theta, 0)}{\sqrt{r^2 + k^2}} = \frac{-k}{\sqrt{r^2 + k^2}} \mathbf{x}_2, \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_2 \mathbf{n} &= \frac{d}{dt} \mathbf{n}(\lambda(t))_{t=r} = \frac{\partial}{\partial t} \mathbf{n}(\theta, t)_{t=r} = \\ &= \frac{((-r)(-k \sin \theta), (-r)(k \cos \theta), -k^2)}{(r^2 + k^2)^{\frac{3}{2}}} = \frac{-k}{(r^2 + k^2)^{\frac{3}{2}}} \mathbf{x}_1. \end{aligned}$$

Now from Problem 4.7.a we can calculate the normal curvatures of the strake in the directions

$$\mathbf{T}_1 = \mathbf{x}_1 / |\mathbf{x}_1| = (r^2 + k^2)^{-\frac{1}{2}} \mathbf{x}_1 \text{ and } \mathbf{T}_2 = \mathbf{x}_2 / |\mathbf{x}_2| = \mathbf{x}_2 :$$

$$\kappa_n(\mathbf{T}_1) = (r^2 + k^2)^{-1} \langle \mathbf{x}_1, -\mathbf{x}_1 \mathbf{n} \rangle = (r^2 + k^2)^{-1} \left\langle \mathbf{x}_1, \frac{k}{\sqrt{r^2+k^2}} \mathbf{x}_2 \right\rangle = 0$$

$$\kappa_n(\mathbf{T}_2) = \langle \mathbf{x}_2, -\mathbf{x}_2 \mathbf{n} \rangle = \left\langle \mathbf{x}_2, \frac{k}{(r^2+k^2)^{3/2}} \mathbf{x}_1 \right\rangle = 0.$$

The reader can also find a geometric argument for why these normal curvatures are zero. (See Problem 6.4.a.) As we shall see in Chapter 6, the fact that both of these normal curvatures are zero does not imply that all normal curvatures are zero. In particular, $\kappa_n(\mathbf{T})$ is not a linear function; in fact, the reader can check (using the bilinearity of the Riemannian metric) that:

$$\kappa_n(a\mathbf{T}) = \langle a\mathbf{T}, -(a\mathbf{T})\mathbf{n} \rangle = a^2 \kappa_n(\mathbf{T})$$

and

$$\begin{aligned} \kappa_n(\mathbf{T}+\mathbf{V}) &= \langle \mathbf{T}+\mathbf{V}, -(\mathbf{T}+\mathbf{V})\mathbf{n} \rangle = \\ &= \kappa_n(\mathbf{T}) + \kappa_n(\mathbf{V}) + \langle \mathbf{T}, -\mathbf{V}\mathbf{n} \rangle + \langle \mathbf{V}, -\mathbf{T}\mathbf{n} \rangle. \end{aligned}$$

This will motivate later the definition of the bilinear *second fundamental form* as:

$$\mathbb{I}(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, -\mathbf{Y}\mathbf{n} \rangle.$$

Geodesic Local Coordinates

If you wish to find intrinsically a local coordinate chart at the point p on a smooth surface M in \mathbf{R}^n , then you may construct *geodesic polar coordinates*, $\mathbf{p}(\theta, r)$, or *geodesic (rectangular) coordinates*, $\mathbf{c}(x, y)$, as follows. (Refer to Figure 4.9.) Choose a base geodesic, γ , with tangent vector \mathbf{T} at p . Then define $\mathbf{p}(\theta, r)$ to be the point that is a distance r from p along the geodesic that is in the direction making an angle θ (measured counterclockwise from the point of view of the normal to the surface) with \mathbf{T} . Assign $\mathbf{c}(x, y)$ to be the point that is obtained by going a distance x along γ in the direction of \mathbf{T} and then going a distance y along the geodesic that is perpendicular to γ at $\mathbf{c}(x, 0)$, turning left if y is positive and right if y is negative.

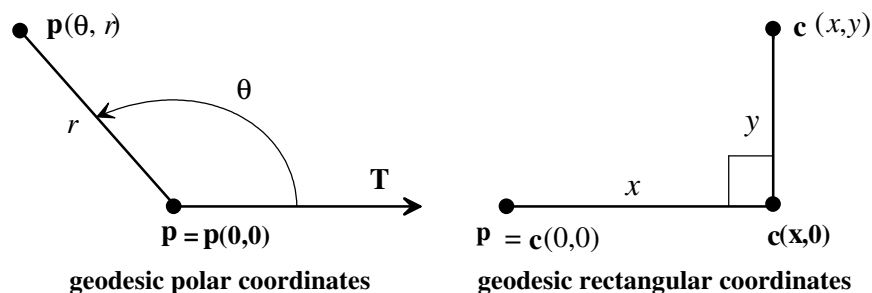


Figure 4.9.

In geodesic rectangular coordinates, the base curve and the geodesics perpendicular to it are parametrized by arclength: Thus, for every a, b , basis vectors $\mathbf{c}_2(a, b)$ and $\mathbf{c}_1(a, 0)$ are unit vectors. However, the coordinate curves $\mathbf{c}(t, b)$, for fixed $b \neq 0$, are in general not parametrized by arclength and $\mathbf{c}_1(a, b) \neq 0$. On the earth the standard north-south-east-west coordinates have the equator (the only latitude circle which is a geodesic) as the base curve. This coordinate system can be changed to be geodesic rectangular coordinates by replacing the angle coordinates with arclength coordinates. The resulting geodesic rectangular coordinate patch will cover all the earth except for the longitude 180°W ($= 180^\circ\text{E}$). A geodesic polar coordinate patch will cover the whole sphere except for one point (the antipodal point to the origin).

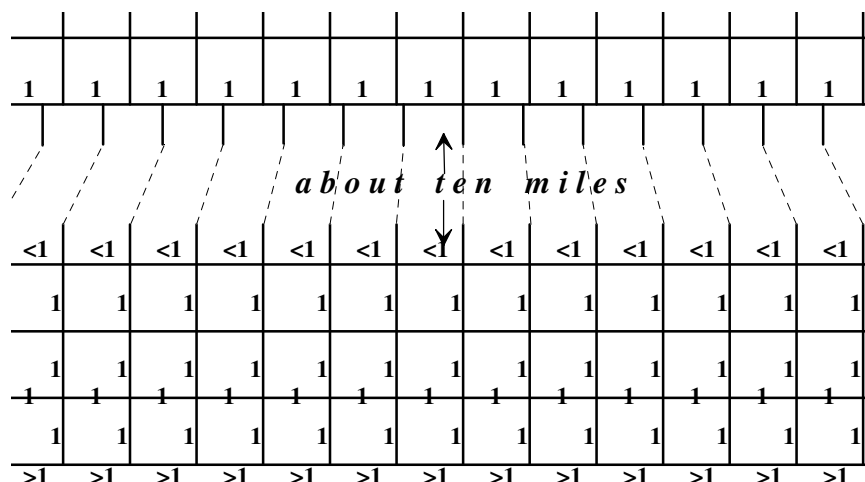


Figure 4.10. Grid of roads in the Midwest (1 mile = 1.61 km).

You can see concrete examples of this phenomenon in many places in the Midwest region of the USA. In the Midwest, typically one road is laid out east-west (as a base curve) and then from this road, at one mile (= 1.61 km) intervals, north-south roads are constructed, and then additional east-west roads at one-mile intervals along these north-south roads. (See Figure 4.10 or maps showing the county roads in some area of the Midwest.) However, as one travels north from the base east-west road, the distance between successive north-south roads becomes, more and more, less than one mile. (If you travel south the opposite happens.) Thus, every ten miles or so, one must make a correction as indicated in Figure 4.10.

PROBLEM 4.8. Differential Operator

We call \mathbf{x} a C^2 *coordinate patch* (or C^2 *local coordinates*) for M if \mathbf{x} is a C^1 coordinate patch and, as a function from \mathbf{R}^2 to \mathbf{R}^n , \mathbf{x} is C^2 (that is, \mathbf{x} is twice differentiable, and the second partial derivatives are continuous).

You can now prove:

Let $\mathbf{x}(u^1, u^2)$ be a C^2 coordinate patch for the smooth surface M . Let \mathbf{F} be a real-valued or vector-valued function defined on a neighborhood of $p = \mathbf{x}(a, b)$ in M such that $\mathbf{F} \circ \mathbf{x}$ is C^1 . Given tangent vectors \mathbf{X}_p and \mathbf{Y}_p in $T_p M$, express these tangent vectors in terms of the coordinates and show that:

- a. $\mathbf{X}_p \mathbf{F}$ does not depend on the choice of curve $\gamma(t)$ such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{X}_p$.

[Use Theorem 4.5.]

- b. For any scalar a ,

$$(\mathbf{X}_p + \mathbf{Y}_p)\mathbf{F} = \mathbf{X}_p\mathbf{F} + \mathbf{Y}_p\mathbf{F} \text{ and } (a\mathbf{X}_p)\mathbf{F} = a(\mathbf{X}_p\mathbf{F}).$$

[Use local coordinates. Note: If α, β, γ are curves on the surface such that

$$\alpha'(0) = \mathbf{X}_p, \beta'(0) = \mathbf{Y}_p, \gamma'(0) = \mathbf{X}_p + \mathbf{Y}_p,$$

and

$$\alpha(0) = \beta(0) = \gamma(0) = p,$$

then in \mathbf{R}^n it is possible to specify that $\gamma(s) = \alpha(s) + \beta(s)$, but this is NOT possible in general on a surface. *Be sure you see why.*]

c. If f is a C^1 real-valued function, then

$$\mathbf{X}_p(f\mathbf{F}) = (\mathbf{X}_p f)\mathbf{F} + f(\mathbf{X}_p \mathbf{F}).$$

[Hint: this is the product rule.]

d. $\mathbf{x}_1(a,b)\mathbf{x}_2 = \mathbf{x}_2(a,b)\mathbf{x}_1$.

Each part of this problem is almost just a matter of notation. If you rewrite the equations in terms of the local coordinates, then you will see that they hold. This is more a notational problem and not a technical problem. Note that \mathbf{x}_j is a (tangent) vector-valued function of the coordinates (u^1, u^2) or, equivalently, is a function of the points $q = \mathbf{x}(u^1, u^2)$ in M . Thus $\mathbf{x}_1(a, b)$ can also be written $\mathbf{x}_1(\mathbf{x}(a, b))$ as a tangent vector at the point $p = \mathbf{x}(a, b)$ and

$$\begin{aligned} \mathbf{x}_1(a, b)\mathbf{x}_2 &= \lim_{h \rightarrow 0} \frac{\mathbf{x}_2(\mathbf{x}(a+h, b)) - \mathbf{x}_2(\mathbf{x}(a, b))}{h} = \\ &= \lim_{h \rightarrow 0} \frac{\mathbf{x}_2(a+h, b) - \mathbf{x}_2(a, b)}{h} = \frac{\partial}{\partial u^1} \mathbf{x}_2(u^1, b)_{u^1=a} \end{aligned}$$

is the directional derivative of the function \mathbf{x}_2 with respect to $\mathbf{x}_1(a, b)$. Thus,

$$\mathbf{x}_1(a, b)\mathbf{x}_2 = \frac{\partial}{\partial u^1} \mathbf{x}_2(u^1, b)_{u^1=a} = \frac{\partial}{\partial u^1} \left(\frac{\partial}{\partial u^2} \mathbf{x}(u^1, u^2) \right)_{(u^1, u^2)=(a, b)}.$$

It is usual to define:

$$\mathbf{x}_{ij} = \mathbf{x}_i \mathbf{x}_j = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} \mathbf{x}.$$

e. Calculate \mathbf{x}_{12} and \mathbf{x}_{21} for the standard coordinate system on a sphere and on a strake.

On the sphere of radius r we calculate that

$$\mathbf{x}_{12} = \mathbf{x}_{21} = (-r \sin\theta \cos\phi, r \cos\theta \cos\phi, 0) = (\cot\phi)\mathbf{x}_1.$$

The length of the tangent vector in the latitudinal (east-west) direction of \mathbf{x}_1 starts off at r^2 on the equator but decreases as you move toward either pole; and \mathbf{x}_{21} is the rate of change of \mathbf{x}_1 as you move southward along a longitude. Also, note that \mathbf{x}_{12} is the rate of change of the tangent vector in the longitudinal direction of \mathbf{x}_2 as you move westward along a latitude circle and, even though the length of \mathbf{x}_2 is constantly r^2 , its direction is changing. I urge the reader to investigate this phenomenon on the sphere until it becomes as natural and comfortable as possible.

PROBLEM 4.9. Metric in Geodesic Coordinates

Explain each step in the following argument.

Let $\mathbf{x}(u^1, u^2)$ be geodesic rectangular coordinates, $\mathbf{c}(x, y)$, or geodesic polar coordinates, $\mathbf{p}(\theta, r)$, as in Figure 4.9 above. According to Problem 4.3 the Riemannian metric can be expressed in local coordinates as the matrix:

$$g = (g_{ij}) = \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle & \langle \mathbf{x}_2, \mathbf{x}_2 \rangle \end{pmatrix}.$$

a. From the definition of geodesic coordinates, for constant a , the geodesic curves $\mathbf{x}(a, u^2)$ are parametrized by arc length and thus

$$g_{22}(u^1, u^2) = 1. \text{ (Why?)}$$

b. We now need to find $g_{12}(u^1, u^2)$. To do this we first differentiate:

$$\begin{aligned} \frac{\partial}{\partial u^2} g_{12}(u^1, u^2) &= \mathbf{x}_2 \langle \mathbf{x}_1(u^1, u^2), \mathbf{x}_2(u^1, u^2) \rangle = \\ &= \langle \mathbf{x}_{21}, \mathbf{x}_2 \rangle + \langle \mathbf{x}_1, \mathbf{x}_{22} \rangle. \text{ (Why?)} \end{aligned}$$

Now, since \mathbf{x}_2 is a unit vector

$$\langle \mathbf{x}_{21}, \mathbf{x}_2 \rangle = \langle \mathbf{x}_{12}, \mathbf{x}_2 \rangle = \left\langle \frac{\partial}{\partial u^1} \mathbf{x}_2, \mathbf{x}_2 \right\rangle = 0. \text{ (Why?)}$$

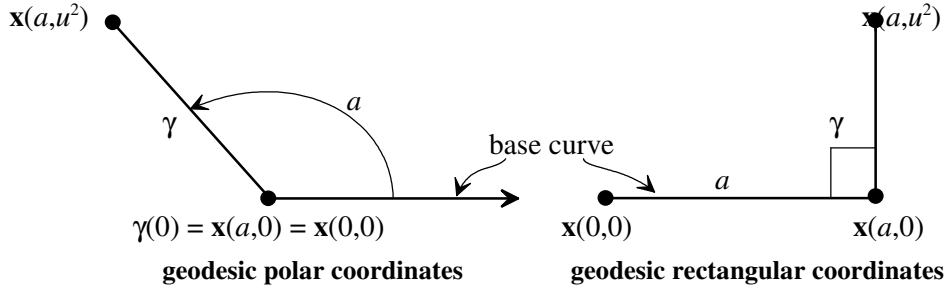


Figure 4.11. Second coordinate curves.

c. Now we focus on the second coordinate curve

$$\gamma(u^2) = \mathbf{x}(a, u^2),$$

(see Figure 4.11). Since $\gamma(u^2) = \mathbf{x}(a, u^2)$ is parametrized by arc length, its unit tangent vector is $\mathbf{x}_2(a, u^2)$ and thus

$$\mathbf{x}_{22} = \frac{\partial^2}{(\partial u^2)^2} \mathbf{x}(a, u^2) = \frac{\partial}{\partial u^2} \mathbf{x}_2(a, u^2) = \boldsymbol{\kappa}$$

is its (extrinsic) curvature vector. (Why?) Since the curve is a geodesic, its curvature vector must be parallel to the normal to the surface. Thus

$$\langle \mathbf{x}_1, \mathbf{x}_{22} \rangle = 0 \text{ and therefore } \frac{\partial}{\partial u^2} g_{12}(a, u^2) = 0.$$

We can then conclude that $g_{12}(a, u^2)$ is a constant independent of u^2 . (Why?)

d. By definition of geodesic rectangular coordinates,

$$g_{12}(u^1, 0) = \langle \mathbf{x}_1(u^1, 0), \mathbf{x}_2(u^1, 0) \rangle = 0. \text{ (Why?)}$$

(Remember that $\mathbf{x}_2(u^1, 0) = \frac{\partial}{\partial u^2} \mathbf{x}(u^1, u^2)|_{u^2=0}$.) For geodesic polar coordinates, $\mathbf{x}(u^1, 0) = \mathbf{p}(\theta, 0) = \mathbf{p}(0, 0)$, a constant. Thus, again,

$$g_{12}(u^1, 0) = \langle \mathbf{x}_1(u^1, 0), \mathbf{x}_2(u^1, 0) \rangle = 0. \text{ (Why?)}$$

e. We can now conclude that

$$g_{12}(u^1, u^2) = 0, \text{ for all } u^1 \text{ and } u^2. \text{ (Why?)}$$

Thus, for geodesic rectangular or polar coordinates:

$$g(u^1, u^2) = \begin{pmatrix} (h(u^1, u^2))^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $h(u^1, u^2) = |\mathbf{x}_1(u^1, u^2)| > 0$.

We will use this representation of the Riemannian metric to find explicit intrinsic calculations of the Gaussian curvature in local coordinates in Chapter 7.