SOLUTIONS

Chapter 7

Applications of Gaussian Curvature

Problem 7.1. Gaussian Curvature in Local Coordinates

***a.**

We calculate: $\mathbf{x}_1(x,y) = (1,0,f_x(x,y))$, $\mathbf{x}_1(0,0) = (1,0,0)$ and $\mathbf{x}_2(x,y) = (0,1,f_y(x,y))$, $\mathbf{x}_2(0,0) = (0,1,0)$. Thus,

$$\mathbf{x}_{11}(x,y) = (0,0,f_{xx}(x,y)),\ \mathbf{x}_{12}(x,y) = (0,0,f_{xy}(x,y)) = (0,0,f_{yx}(x,y)) = \mathbf{x}_{21}(x,y), \text{ and } \mathbf{x}_{22}(x,y) = (0,0,f_{xx}(x,y)).$$

But $\mathbf{n}(0,0) = (0,0,1)$ and so

$$K(0,0) = \kappa_1(0,0)\kappa_2(0,0) = (\det(g_{ij}(0,0)))^{-1} \det\begin{pmatrix} \langle \mathbf{x}_{11}(0,0), \mathbf{n}(0,0)\rangle & \langle \mathbf{x}_{12}(0,0), \mathbf{n}(0,0)\rangle \\ \langle \mathbf{x}_{21}(0,0), \mathbf{n}(0,0)\rangle & \langle \mathbf{x}_{22}(0,0), \mathbf{n}(0,0)\rangle \end{pmatrix} =$$

$$(1)\det\begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix} = f_{xx}(0,0)\,f_{yy}(0,0) - (f_{xy}(0,0))^2.$$

b.

Outline of a proof of Problem 7.1.b:

1. Since $h = |\mathbf{x}_1| = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle}$ we can calculate that

$$h_2 = \tfrac{1}{2}\frac{\mathbf{x}_2\langle \mathbf{x}_1, \mathbf{x}_1\rangle}{\sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle}} = \tfrac{1}{2}\frac{\langle \mathbf{x}_{21}, \mathbf{x}_1\rangle + \langle \mathbf{x}_1, \mathbf{x}_{21}\rangle}{\sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle}} = \frac{\langle \mathbf{x}_{21}, \mathbf{x}_1\rangle}{\sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle}}$$

and that (using the quotient rule)

$$h_{22} = \frac{(\langle \mathbf{x}_{221}, \mathbf{x}_1\rangle + \langle \mathbf{x}_{21}, \mathbf{x}_{21}\rangle)\sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} - \langle \mathbf{x}_{21}, \mathbf{x}_1\rangle \tfrac{1}{2}(\langle \mathbf{x}_1, \mathbf{x}_1\rangle)^{-1/2}(2\langle \mathbf{x}_{21}, \mathbf{x}_1\rangle)}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle}$$

$$-\frac{h_{22}}{h} = -\frac{(\langle \mathbf{x}_{221}, \mathbf{x}_1\rangle + \langle \mathbf{x}_{21}, \mathbf{x}_{21}\rangle)\langle \mathbf{x}_1, \mathbf{x}_1\rangle - \langle \mathbf{x}_{21}, \mathbf{x}_1\rangle^2}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle^2}.$$

2. Next we show that

$$\langle \mathbf{x}_{221}, \mathbf{x}_1\rangle = \langle \mathbf{x}_2\mathbf{x}_{21}, \mathbf{x}_1\rangle = \langle \mathbf{x}_2\mathbf{x}_{12}, \mathbf{x}_1\rangle = \langle (\mathbf{x}_2\mathbf{x}_1)\mathbf{x}_2, \mathbf{x}_1\rangle = \langle (\mathbf{x}_1\mathbf{x}_2)\mathbf{x}_2, \mathbf{x}_1\rangle = \langle \mathbf{x}_{122}, \mathbf{x}_1\rangle =$$
$$= \mathbf{x}_1\langle \mathbf{x}_{22}, \mathbf{x}_1\rangle - \langle \mathbf{x}_{22}, \mathbf{x}_{11}\rangle = 0 - \langle \mathbf{x}_{22}, \mathbf{x}_{11}\rangle,$$

where the "0" is because $\mathbf{x}_{22}(a,b)$ is the curvature of the geodesic curve $\gamma(s) = \mathbf{x}(a,s)$ at $s=b$ (which is assume to be parametrized by arclength), and thus is perpendicular to $\mathbf{x}_1$.

3. We can then calculate that: $-\dfrac{h_{22}}{h} = \dfrac{\langle \mathbf{x}_{22}, \mathbf{x}_{11}\rangle}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} - \dfrac{\langle \mathbf{x}_{21}, \mathbf{x}_{12}\rangle\langle \mathbf{x}_1, \mathbf{x}_1\rangle - \langle \mathbf{x}_{21}, \mathbf{x}_1\rangle^2}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle^2} =$

$$= \frac{\langle \mathbf{x}_{22}, \mathbf{x}_{11}\rangle}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} - \frac{\langle \mathbf{x}_{21}, \mathbf{x}_{12}\rangle\langle \mathbf{x}_1, \mathbf{x}_1\rangle - |\mathbf{x}_{21}|^2|\mathbf{x}_1|^2\cos^2\theta}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle^2} = \frac{\langle \mathbf{x}_{22}, \mathbf{x}_{11}\rangle}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} - \frac{|\mathbf{x}_{21}|^2(1-\cos^2\theta)}{\langle \mathbf{x}_1, \mathbf{x}_1\rangle},$$

where θ is the angle from $\mathbf{x}_1$ to $\mathbf{x}_{21}$.

4. At the same time $K = (\det(g_{ij}))^{-1} \times \det\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix} = \dfrac{\langle \mathbf{x}_{11},\mathbf{n}\rangle\langle \mathbf{x}_{22},\mathbf{n}\rangle - \langle \mathbf{x}_{12},\mathbf{n}\rangle^2}{\langle \mathbf{x}_1,\mathbf{x}_1\rangle} =$
$= \dfrac{\langle \mathbf{x}_{22},\mathbf{x}_{11}\rangle}{\langle \mathbf{x}_1,\mathbf{x}_1\rangle} - \dfrac{|\mathbf{x}_{21}|^2\cos^2\phi}{\langle \mathbf{x}_1,\mathbf{x}_1\rangle}$, where, since $\mathbf{x}_{22}$ is in the same direction as the normal $\mathbf{n}$, $\langle \mathbf{x}_{11},\mathbf{n}\rangle\langle \mathbf{x}_{22},\mathbf{n}\rangle = \langle \mathbf{x}_{11},\mathbf{x}_{22}\rangle$, and where ϕ is the angle from $\mathbf{x}_{21}$ to $\mathbf{n}$.
5. Differentiating $\langle \mathbf{x}_1,\mathbf{x}_2\rangle = 0$ we get get $\langle \mathbf{x}_{21},\mathbf{x}_2\rangle = -\langle \mathbf{x}_1,\mathbf{x}_{22}\rangle = 0$ because $\mathbf{x}_{22}$ is parallel to $\mathbf{n}$. Thus, $\mathbf{x}_{21}$ lies in the plane of $\mathbf{x}_1$ and $\mathbf{n}$. Therefore, $\theta + \phi =$ (the angle from $\mathbf{x}_1$ to $\mathbf{n}$) $= \pi/2$ and $\cos\phi = \sin\theta$, and the above expressions imply that the Gaussian curvature is given by $K = -\dfrac{h_{22}}{h}$.

c.

The base curve (being parametrized by arclength) has velocity vector $\mathbf{x}_1(u^1,0)$ whose length is 1, thus, $f(0) = 1$. Then we can calculate

$$f'(0) = \mathbf{x}_2\sqrt{\langle \mathbf{x}_1,\mathbf{x}_1\rangle}\Big|_{t=0} = \frac{1}{2}\frac{2\langle \mathbf{x}_{21}(u^1,0),\mathbf{x}_1(u^1,0)\rangle}{\sqrt{\langle \mathbf{x}_1(u^1,0),\mathbf{x}_1(u^1,0)\rangle}} = \langle \mathbf{x}_{21}(u^1,0),\mathbf{x}_1(u^1,0)\rangle =$$
$$\langle \mathbf{x}_{12}(u^1,0),\mathbf{x}_1(u^1,0)\rangle = -\langle \mathbf{x}_2(u^1,0),\mathbf{x}_{11}(u^1,0)\rangle = 0$$

because the base curve is a geodesic and thus, $\mathbf{x}_{11}(u^1,0)$ is in the direction of the normal. Using the second derivative test for local extrema $f''(0) = h_{22}(u^1,0) = -K$. Thus, $f(t)$ has a local maximum at $t=0$ when $K > 0$ and a local minimum at $t = 0$ when $K < 0$.

***PROBLEM 7.2. Curvature on Sphere, Strake & Catenoid**

a.

We must use geodesic rectangular coordinates on the sphere, that is the equator and the longitude must be parametrized by arclength, $u^1 = R\theta$ and $u^2 = R\phi$. These coordinates are

$$\boldsymbol{x}(u^1,u^2) = (R\cos\tfrac{u^1}{R}\cos\tfrac{u^2}{R}, R\sin\tfrac{u^1}{R}\cos\tfrac{u^2}{R}, R\sin\tfrac{u^2}{R}),$$

thus, $$h(u^1,u^2) = |\boldsymbol{x}_1(u^1,u^2)| = \left|(-\sin\tfrac{u^1}{R}\cos\tfrac{u^2}{R}, \cos\tfrac{u^1}{R}\cos\tfrac{u^2}{R}, 0)\right| = \cos\tfrac{u^2}{R}.$$

Then $$K = -\frac{h_{22}}{h} = -\frac{(1/R^2)(-\sin(u^2/R))}{\sin(u^2/R)} = \frac{1}{R^2}.$$

b.

The local coordinates for the strake are $\mathbf{x}(\theta,r) = (r\cos\theta, r\sin\theta, k\theta)$, which satisfy the hypotheses for Problem **7.2.b**. Thus, $\mathbf{x}_1(\theta,r) = (-r\sin\theta, r\cos\theta, k)$ and $h(\theta,r) = \sqrt{r^2+k^2}$, and

$$K(\theta,r) = -\frac{h_{22}}{h} = -\frac{1}{h}\frac{\partial}{\partial r}\frac{r}{\sqrt{r^2+k^2}} = -\frac{1}{h}\left[\frac{\sqrt{r^2+k^2} - r(r^2+k^2)^{-1/2}(r)}{r^2+k^2}\right] = -\frac{k^2}{(r^2+k^2)^2} < 0.$$

c.

The Gaussian curvature of the helicoid is the same as the strake since they have the same local coordinates (just different domain for r). But, if we use geodesic rectangular coordinates as in the solution to Problem **6.6.e**, then we have for the helicoid (denoting $g(s,r) = |\mathbf{y}_1(s,r)|$) $\mathbf{y}(s,r) = (r\cos\frac{s}{h}, r\sin\frac{s}{h}, s)$ and $g(s,r) = \sqrt{1+(\frac{r}{h})^2}$. From the solution to Problem **6.6.e** we have that for the catenoid

$$\mathbf{x}(w,s) = \left(\tfrac{1}{a}\sqrt{1+(as)^2}\cos aw, \tfrac{1}{a}\sqrt{1+(as)^2}\sin aw, \tfrac{1}{a}\sinh^{-1}as\right) \text{ and } .$$

Thus, for the catenoid,

$$K = -\frac{h_{22}}{h} = -\frac{1}{h}\frac{\partial}{\partial s}\frac{a^2 s}{\sqrt{1+(as)^2}} = -\tfrac{1}{h}\left[\frac{a^2\sqrt{1+(as)^2} - a^2 s(1+(as)^2)^{-1/2}(a^2 s)}{1+(as)^2}\right] = -\frac{a^2}{(1+(as)^2)^2},$$

with the same result for the helicoid if we replace a by $1/h$ and replace s by r.

d.

Note that r/R is the angle subtended at the center of the sphere by the circle's intrinsic radius. Thus, it is easy to see the first formula for the circumference. The second formula then follows by using the Taylor series for $\sin(r/R)$.

e.

The first formula was derived in the solution to Problem **4.5.b**. The power series comes from expanding $\cos(r/R)$ using the Taylor series.

Problem 7.3. Circles, Polar Coordinates, and Curvature

a.

Let $f(r) \equiv h(\theta,r) \equiv |\mathbf{y}_1(\theta,r)|$, then in order to determine the third Taylor approximation we need to find $f(0), f'(0), f''(0)$, and $f'''(0)$. Since $\mathbf{y}(\theta,0) = p$ is constant, it follows that $f(0) = 0$.

To determine $f'(0)$ we use the fact that M is a smooth surface and we can zoom in on p enough that the surface in indistinguishable from the tangent plane. Expanding the derivatives we have

$$f'(0) = \lim_{r\to 0^+} \tfrac{|\mathbf{y}_1(\theta,r)|}{r} = \left|\lim_{r\to 0^+} \tfrac{1}{r}\left(\lim_{h\to 0} \tfrac{\mathbf{y}(\theta+h,r)-\mathbf{y}(\theta,r)}{h}\right)\right| = \left|\lim_{r\to 0^+} \lim_{h\to 0} \tfrac{\mathbf{y}(\theta+h,r)-\mathbf{y}(\theta,r)}{rh}\right|.$$

Now fix h and pick a tolerance $0 < \varepsilon < h^2 < 1$ and zoom in further, if necessary, so that the intrinsic circles in M are indistinguishable from circles in the tangent plane, and that intrinsic radii of the circles in M are indistinguishable from straight. Then, for $r = \delta$ (the radius of the field of view),

$$(r-\delta\varepsilon)2\sin\tfrac{h}{2} - 2\delta\varepsilon \le \mathbf{y}(\theta+h,r) - \mathbf{y}(\theta,r) \le (r+\delta\varepsilon)2\sin\tfrac{h}{2} + 2\delta\varepsilon.$$

Then $$\left|\lim_{r\to 0^+} \tfrac{(r-\delta\varepsilon)}{r} \lim_{h\to 0} \tfrac{2\sin\frac{h}{2}-2\delta\varepsilon}{h}\right| \le f'(0) \le \left|\lim_{r\to 0^+} \tfrac{(r+\delta\varepsilon)}{r} \lim_{h\to 0} \tfrac{2\sin\frac{h}{2}+2\delta\varepsilon}{h}\right|,$$

and thus, independent of r, $\quad (1-\varepsilon)\lim_{h\to 0} \tfrac{2\sin\frac{h}{2}-2\delta\varepsilon}{h} \le f'(0) \le (1+\varepsilon)\lim_{h\to 0} \tfrac{2\sin\frac{h}{2}+2\delta\varepsilon}{h}.$

We conclude that $f'(0) = 1$.

Problem **7.1.b** gives us information about $f''(0)$, $\; f''(0) = \lim_{r\to 0} h_{22}(\theta,r) = -\lim_{r\to 0} K(\theta,r)h(\theta,r) = 0$.
Find $f'''(0)$ by differentiating the result from Problem **7.1.b** and then (carefully) taking the limit:

$$f'''(0) = \lim_{r\to 0} h_{222}(\theta,r) = -\lim_{r\to 0}[K(\theta,r)h_2(\theta,r) + K_2(\theta,r)h(\theta,r)] = -K(\theta,0) = -K(p)$$

Now according to the theory of Taylor polynomials $h(\theta,r) = f(r) = r - \tfrac{K(p)r^3}{6} + R(\theta,r)$, where $\lim\limits_{r\to 0} \tfrac{R(\theta,r)}{r^3} = 0$.
The limit is uniform in θ because $R(\theta,r)$ is continuous in θ over the compact interval $[0,2\pi]$.

b.

We integrate around the intrinsic circle

$$C(r) = \int_0^{2\pi} |\mathbf{y}_1(\theta,r)| d\theta = \int_0^{2\pi} h(\theta,r)\, d\theta = \int_0^{2\pi}\left[r - \tfrac{K(p)r^3}{6} + R(\theta,r)\right]d\theta =$$
$$\int_0^{2\pi}\left(r - \tfrac{K(p)r^3}{6}\right)d\theta + \int_0^{2\pi} R(\theta,r)d\theta = 2\pi\left(r - \tfrac{K(p)r^3}{6}\right) + R_C(r),$$

where $$\lim_{r\to 0} \tfrac{R_C(r)}{r^3} = \lim_{r\to 0} \int_0^{2\pi} \tfrac{R(\theta,r)}{r^3} d\theta = \int_0^{2\pi} \lim_{r\to 0} \tfrac{R(\theta,r)}{r^3} d\theta = 0.$$

You can interchange the limit and the integration because the limit is uniform in θ.

c.

We integrate using Problem **4.5.a**:

$$\int_0^r \int_0^{2\pi} \sqrt{\det g(\theta,r)}\, d\theta dr = \int_0^r \int_0^{2\pi} h\, d\theta dr = \int_0^r\left[\int_0^{2\pi}\left(r - \tfrac{K(p)r^3}{6} + R(\theta,r)\right)d\theta\right]dr =$$

$$\int_0^r\left[2\pi\left(r - \tfrac{K(p)r^3}{6}\right) + R_C\right]dr = \int_0^r 2\pi\left(r - \tfrac{K(p)r^3}{6}\right)dr + \int_0^r R_C\, dr = \pi r^2 - \pi\tfrac{K(p)r^4}{24} + R_A$$

where, using L'Hôpital's Rule, $\lim_{r\to 0} \tfrac{R_A}{r^4} = \lim_{r\to 0} \tfrac{1}{r^4}\int_0^r R_C\, dr = \lim_{r\to 0} \tfrac{R_C(r)}{4r^3} = 0$

Problem 7.4. Exponential Map and Shortest Is Straight

a.

Let $\alpha(s)$ be the geodesic (parametrized by arclength) in the direction of $\mathbf{U}(\theta)$ starting at $\alpha(0) = p$. Then $\alpha'(rs) = r\mathbf{U}(s)$ and $\mathbf{y}(\theta,r) = \exp(r\mathbf{U}(\theta)) = \alpha(r1) = \alpha(r)$, which is the definition of geodesic polar coordinates.

It follows directly from Problem **4.9** that all the geodesics in U_p that pass through p are perpendicular to the level curves $\{ \exp_p(\mathbf{V}) \mid |\mathbf{V}| = \text{constant} \}$.

b.

Proof of Problem 7.4.b:

1. Assume that there is a piecewise smooth path $\boldsymbol{\alpha}: [0,b] \to U_p$ from p to p^* which is shorter than γ. Then using geodesic polar coordinates $\mathbf{y}(\theta,r)$ we can write $\boldsymbol{\alpha}(t) = \mathbf{y}(\theta(t),r(t))$. Differentiate

$$\boldsymbol{\alpha}'(t) = \theta'(t)\mathbf{y}_1(\theta(t),r(t)) + r'(t)\mathbf{y}_2(\theta(t),r(t)).$$

Then, for $0<a\le t\le b$, $|\alpha'(t)| = \sqrt{(\theta'(t))^2h^2 + (r'(t))^2} \ge |(r'(t))|$, with equality if and only if $\theta'(t) = 0$.

2. Then integrate: $\int_a^b |\boldsymbol{\alpha}'(t)|dt \ge \int_a^b |r'(t)|dt \ge |r(a) - r(b)|$, with equality if and only if $r(t)$ is monotone and $\theta(t)$ is constant.
3. Then the length of α from 0 to b is $\lim_{a\to 0} \int_a^b |\boldsymbol{\alpha}'(t)|dt \ge \lim_{a\to 0} |r(a) - r(b)| = |r(0) - r(b)| = |0 - r(b)|$, which is the length of the geodesic from $p = \mathbf{y}(\theta,0)$ to $p^* = \mathbf{y}(\theta(b),r(b))$. This is the desired result.

c.

***Proof of Problem 7.4.c*:**

1. Look at the path α marked in Figure 7.2 (in the text) with parametrization $\boldsymbol{\alpha}(\theta) = \mathbf{y}(\theta,r(\theta))$, $-\phi/2 \le \theta \le \phi/2$ where $r(\theta) = a\frac{\cos\phi/2}{\cos\theta}$ and $r'(\theta) = a\cos\phi/2\left(\frac{-\sin\theta}{\cos^2\theta}\right) = -r(\theta)\tan\theta$. Thus

$$|\boldsymbol{\alpha}'(\theta)| = \left|\mathbf{y}_1(\theta,r(\theta)) + r'(\theta)\mathbf{y}_2(\theta,r(\theta))\right| = \sqrt{h^2 + (r'(\theta))^2} = \sqrt{h^2 + r^2(\theta)\tan^2\theta}\,.$$

2. Look at the integral that expresses the length of α:

$$\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)|d\theta = \int_{-\phi/2}^{\phi/2} \sqrt{h^2 + r^2(\theta)\tan^2\theta}\,d\theta = \int_{-\phi/2}^{\phi/2} \sqrt{\left(r(\theta) - \frac{K(p)r^3(\theta)}{6} + R(\theta,r(\theta))\right)^2 + r^2(\theta)\tan^2\theta}\,d\theta =$$
$$= \int_{-\phi/2}^{\phi/2} \sqrt{\left(r - \tfrac{Kr^3}{6} + R\right)^2 + r^2\tan^2\theta}\,d\theta = \int_{-\phi/2}^{\phi/2} \frac{r}{\cos\theta}\sqrt{\left(1 - \tfrac{Kr^2}{6} + \tfrac{R}{r}\right)^2\cos^2\theta + \sin^2\theta}\,d\theta =$$
$$= \int_{-\phi/2}^{\phi/2} \frac{r}{\cos\theta}\sqrt{1 + \left[-\tfrac{Kr^2}{3} + \tfrac{K^2r^4}{36} + \tfrac{R}{r}\left(2 - \tfrac{Kr^2}{3} + \tfrac{R}{r}\right)\right]\cos^2\theta}\,d\theta =$$
$$= \int_{-\phi/2}^{\phi/2} \frac{r}{\cos\theta}\sqrt{1 + a^2\left[-\frac{K\cos^2\phi/2}{3} + \frac{K^2a^2\cos^4\phi/2}{36\cos^2\theta}\right] + \frac{R\cos^2\theta}{a^2}\left(\frac{2a\cos\theta}{\cos\phi/2} - \frac{Ka^3\cos\phi/2}{3\cos\theta} + \frac{R\cos^2\theta}{\cos^2\phi/2}\right)}\,d\theta$$

3. Thus, we have shown that $\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)| = \int_{-\phi/2}^{\phi/2} \frac{a\cos\phi/2}{\cos^2\theta}\sqrt{1 + a(A(a,\theta)) + \frac{R(\theta,r(\theta))}{a^2}(B(a,\theta))}\,d\theta$, where $A(a,\theta)$ and $B(a,\theta)$ are bounded for $-\pi/2 < -\phi/2 \le \theta \le \phi/2 < \pi/2$ and $0 < a \le 1$. Then, we have

$$0 = \lim_{r\to 0}\frac{R(\theta,r(\theta))}{r^3(\theta)} = \lim_{a\to 0}\frac{R(\theta,r(\theta))}{a^2}\frac{\cos^3\theta}{a\cos^3\phi/2} \Rightarrow \lim_{a\to 0}\frac{R(\theta,r(\theta))}{a^2} = 0.$$

Thus, for sufficiently small a, $\sqrt{1 + a(A(a,\theta)) + \frac{R(\theta,r(\theta))}{a^2}(B(a,\theta))} \le C < \frac{1}{\sin\phi/2}$, and

$$\int_{-\phi/2}^{\phi/2} |\boldsymbol{\alpha}'(\theta)| \le \int_{-\phi/2}^{\phi/2} \frac{a\cos\phi/2}{\cos^2\theta}C\,d\theta = 2aC\sin\phi/2 < 2a.$$

d.

Proof of Problem 7.4.d:

1. Let p, q be any two points in M with their distance $d(p,q) = b$. Let C be a circle of radius δ and center p so that $C \subset U_p$. There is a point p^* on C such that $d(p^*,q) \le d(x,q)$, for all $x \in C$. Now $p^* = \exp_p(\delta \mathbf{V})$, for some unit tangent vector $\mathbf{V} \in T_{p\,M}$. CLAIM: $\exp_p(b\mathbf{V}) = q$; this will show that the geodesic $\boldsymbol{\gamma}(t) = \exp_p(t\mathbf{V})$ is a geodesic of length b joining p to q.
2. The claim will be true if $b \in A \equiv \{\, t \mid d(\boldsymbol{\gamma}(t),q) = b - t \,\}$, because then $d(\boldsymbol{\gamma}(b),q) = b - b = 0$ and thus, $\boldsymbol{\gamma}(b) = q$.
3. Since every curve from p to q must cross C, we have $d(p,q) = \min_{x\in C}[d(p,x) + d(x,q)] = \delta + d(p^*,q)$. So $d(p^*,q) = b-\delta$ and $\delta \in A$.
4. Let t^* be the least upper bound of all t in A. Then $t^* \in A$. Suppose that $t^* < b$. Let C^* be the circle of radius δ^* around $\gamma(t^*)$. Let q^* be the point on C^* which is closest to q and let q^{**} be the point on C^* which is closest to p^*. See Figure 7.3 in the text.
5. But then Problem **7.4.c** tells us that we can obtain a shorter path by going directly from q^{**} to q^* unless the angle ϕ in Figure 7.3 is equal to π. But if $\phi = \pi$ then q^* is in A and therefore t^* is not the least upper bound of elements in A.

Thus, it must be true that b is in A and the result is established.

***e.**

If M is Cauchy complete and $\gamma(s)$ is a geodesic in M which is defined for all $0 \le s < b$, then choose t_n converging to b. Clearly, $\gamma(t_n)$ is a Cauchy sequence in M and so $\gamma(t_n)$ converges to some point p in M and we can define $\gamma(b) = p$. Then, by Problem **7.4.a**, we can continue this geodesic past b for at least a little bit. This continues indefinitely.

If M is geodesically complete and $\{x_i\}$ is a Cauchy sequence, then pick x_i so that for $j > i$ we have $|\, x_i - x_j \,| < 1/m$. Then for each $j > i$, use **7.4.d** to find a geodesic γ_j which joins x_i to x_j. Assume each of the geodesics is parametrized by arclength with $\gamma(0) = x_i$ and $\gamma_j'(0) = \mathbf{T}_j$, for each $j > i$. Since the $\mathbf{T}_j$ are unit tangent vectors at x_i and since the unit circle is compact, there is a subsequence $\{\mathbf{T}_{j_i}\}$ which converges to the unit vector $\mathbf{T}$. Let $\gamma(s)$ be the geodesic starting at x_i with velocity vector $\mathbf{T}$. For each $j > i$, let $x_j = \gamma(s_j)$. The sequence $\{s_j\}$ is a Cauchy sequence on the real line since $|s_j - s_k| \le d(x_j,x_k)$. Let s_o be the limit of $\{s_{j_i}\}$, where we take a subsequence again if necessary. Since M is geodesically complete the exponential map is continuously defined on the *whole* tangent space at x_i, $\mathrm{T}_{x_i}\mathrm{M}$. It is easy to see that $\{s_{j_i}\mathbf{T}_{j_i}\}$ is a Cauchy sequence in $\mathrm{T}_{x_i}\mathrm{M}$ that must converge to $s_o\mathbf{T}$. But then $\exp_{x_i}(s_o\mathbf{T}) = \gamma(s_o) = p$, some point in M which is the limit of $\{x_{j_i}\}$. It is then easy to check that p is also the limit of the whole sequence $\{x_j\}$.

Problem 7.5. Surfaces with Constant Curvature

a.

According to Problem **4.9** the Riemannian metric is as stated in **7.5a**. Problem **7.1.b** asserts that $K = -\dfrac{h_{22}(a,b)}{h(a,b)}$. Problem **7.1.c** tells us that for the function $f(t) = h(u^1,t) = |\mathbf{x}_1(u^1,t)|$ satisfies, for each u^1: $f(0) = 1$ *and* $f'(0) = 0$. Thus, for surfaces with constant Gaussian curvature, the function f satisfies the differential equation $f''(t) = -Kf(t)$ with initial conditions $f(0) = 1, f'(0) = 0$. By ordinary calculus the general solutions of this equation are $f(t) = a\sin\sqrt{K}\,t + b\cos\sqrt{K}\,t$, and applying the initial conditions we get $f(t) = \cos\sqrt{K}\,t$. When the Gaussian curvature K is negative then $f(t) = \cos\left(i\sqrt{|K|}\,t\right) = \cosh\sqrt{|K|}\,t$.

b.

Denote the surfaces by M and N and let $p \in M$ and $q \in N$ be any two points on the surfaces. Construct geodesic rectangular coordinates (with the base curve a geodesic) on both surfaces ($\mathbf{x}$ on M and $\mathbf{y}$ on N) with $p = \mathbf{x}(0,0)$ and $q = \mathbf{y}(0,0)$. Define a map that takes the point $\mathbf{x}(a,b)$ on M to the point $\mathbf{y}(a,b)$ on N. Then, by Problem **7.5.a**, the Riemannian metrics of the two surfaces are equal at the corresponding points. The arclength of any curve γ on a surface is given by the integral $\int |\gamma'(t)|\, dt = \int \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}\, dt$, and angles between tangent vectors are also determined by the Riemannian metric. Thus, since the Riemannian metrics are the same at corresponding points the above map must preserve all lengths and angles and, thus, be an isometry wherever the geodesic rectangular coordinates are defined.

c.

In Problem **7.5.b** we could have $M = N$ and choose p and q to be any two points and the base curves be any geodesics emanating from p or q. If γ is any geodesic in M and p and q are any two points on γ, then we can apply the construction in Problem **7.5.b** with γ being the base curve at both p and q and the second coordinate curves being positive on the same sides of γ – the result will be a locally defined translation along γ that takes p to q. In order to construct a local rotation through angle ϕ about the point p, pick $p = q$ in Problem **7.5.b** and pick as base geodesics two geodesics emanating from p in directions that are ϕ apart and pick the positive direction for the second coordinate consistently. A local reflection can be constructed by using in Problem **7.5.b** $p = q$ and the same base curve but with the positive direction of the second coordinate reversed.

PROBLEM 7.6. Ruled Surfaces and Ribbons

a.

In Problem **7.1**, we showed that $K = \kappa_1 \kappa_2 = (\det(g_{ij}))^{-1} \det\begin{pmatrix} \langle \mathbf{x}_{11}, \mathbf{n} \rangle & \langle \mathbf{x}_{12}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{n} \rangle & \langle \mathbf{x}_{22}, \mathbf{n} \rangle \end{pmatrix}$.

We now calculate $\quad \mathbf{x}_{11}(t,s) = \boldsymbol{\alpha}''(t) + s\mathbf{r}''(t);\ \ \mathbf{x}_{12}(t,s) = \mathbf{r}'(t);\ \ \mathbf{x}_{22}(t,s) = \mathbf{0}$,

$\mathbf{n} = \mathbf{x}_1 \times \mathbf{x}_2 = (\boldsymbol{\alpha}' + s\mathbf{r}') \times \mathbf{r} = \boldsymbol{\alpha}' \times \mathbf{r}(t) + s(\mathbf{r}' \times \mathbf{r})$.

Since for C^1 local coordinates $\det(g_{ij})$ is never zero, we conclude

$$K = 0 \Leftrightarrow \det\begin{pmatrix} \langle \mathbf{x}_{11}, \mathbf{n} \rangle & \langle \mathbf{x}_{12}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{n} \rangle & \langle \mathbf{x}_{22}, \mathbf{n} \rangle \end{pmatrix} = 0 \Leftrightarrow 0 = \langle \mathbf{x}_{12}, \mathbf{n} \rangle$$

$$\langle \mathbf{x}_{12}, \mathbf{n} \rangle = \langle \mathbf{r}', \boldsymbol{\alpha}' \times \mathbf{r} + s(\mathbf{r}' \times \mathbf{r}) \rangle = \langle \mathbf{r}', \boldsymbol{\alpha}' \times \mathbf{r} \rangle + \langle \mathbf{r}', s(\mathbf{r}' \times \mathbf{r}) \rangle = \langle \mathbf{r}', \boldsymbol{\alpha}' \times \mathbf{r} \rangle,$$

since $\langle \mathbf{V}, \mathbf{V} \times \mathbf{r} \rangle = 0$. Thus, M is developable if and only if (see Problem **7.5.b**)

$$\mathbf{0} = \langle \mathbf{r}'(t), \boldsymbol{\alpha}'(t) \times \mathbf{r}(t) \rangle = [\mathbf{r}'(t), \boldsymbol{\alpha}'(t), \mathbf{r}(t)] = -[\mathbf{r}(t), \mathbf{r}'(t), \boldsymbol{\alpha}'(t)],$$

see *Appendix* **A.5** for result on triple products.

b.

First we check that this ruled surface is regular along $\boldsymbol{\alpha}$, that is when $s = 0$:

$$\mathbf{x}_1(t,s) = \boldsymbol{\alpha}' + s\left(\tfrac{-\langle \mathbf{n}'', \mathbf{n}' \rangle}{\langle \mathbf{n}', \mathbf{n}' \rangle^{3/2}} (\mathbf{n} \times \mathbf{n}') + \tfrac{(\mathbf{n}' \times \mathbf{n}') + (\mathbf{n} \times \mathbf{n}'')}{|\mathbf{n}'|} \right) = \boldsymbol{\alpha}' + s\left(\tfrac{-\langle \mathbf{n}'', \mathbf{n}' \rangle}{\langle \mathbf{n}', \mathbf{n}' \rangle^{3/2}} (\mathbf{n} \times \mathbf{n}') + \tfrac{(\mathbf{n} \times \mathbf{n}'')}{|\mathbf{n}'|} \right) \quad \text{and} \quad \mathbf{x}_2(t,s) = \tfrac{\mathbf{n} \times \mathbf{n}'}{|\mathbf{n}'|}.$$

If $\mathbf{x}_1(t,0) = \boldsymbol{\alpha}'(t)$ were parallel to $\mathbf{x}_2(t,0)$ then $\boldsymbol{\alpha}'(t)$ would be perpendicular to $\mathbf{n}'(t)$; but notice that $\mathbf{n}'(t)$ is exactly the directional derivative $\boldsymbol{\alpha}'\mathbf{n}$. Thus, $\langle \boldsymbol{\alpha}'(t), \mathbf{n}'(\boldsymbol{\alpha}(t)) \rangle = \langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'\mathbf{n}' \rangle = -\kappa_{\mathbf{n}} \neq 0$. Thus, for $|s|$ near zero, it must be that $\mathbf{x}(t,s)$ define a regular surface.

This ruled surface is developable if and only if $0 = [\boldsymbol{r}(t), \boldsymbol{r}'(t), \boldsymbol{\alpha}'(t)]$. So we calculate, using Theorems **A.5.2** and **A.5.3** and the fact that $\boldsymbol{\alpha}'$ is perpendicular to $\mathbf{n}$:

$$[\boldsymbol{r}(t), \boldsymbol{r}'(t), \boldsymbol{a}'(t)] = \left\langle \tfrac{(\mathbf{n}\times\mathbf{n}'')}{|\mathbf{n}'|} - \tfrac{\langle \mathbf{n}'',\mathbf{n}'\rangle}{\langle \mathbf{n}',\mathbf{n}'\rangle^{3/2}} (\mathbf{n}\times\mathbf{n}'), \boldsymbol{a}' \times \tfrac{\mathbf{n}\times\mathbf{n}'}{|\mathbf{n}'|} \right\rangle =$$
$$= c\langle \mathbf{n}\times\mathbf{B}, \boldsymbol{a}'\times(\mathbf{n}\times\mathbf{n}')\rangle = c\langle \mathbf{n}\times\mathbf{B}, \langle \mathbf{n}', \boldsymbol{a}'\rangle\mathbf{n} - \langle \mathbf{n}, \boldsymbol{a}'\rangle\mathbf{n}'\rangle = c\langle \mathbf{n}\times\mathbf{B}, \langle \mathbf{n}', \boldsymbol{a}'\rangle\mathbf{n}\rangle = 0.$$

Thus, the ruled surface is developable.

Along α (that is, $s = 0$), the normal to the ruled surface is parallel to

$$\mathbf{x}_1 \times \mathbf{x}_2 = \left(\boldsymbol{a}' + s\tfrac{-\langle \mathbf{n}'',\mathbf{n}'\rangle}{\langle \mathbf{n}',\mathbf{n}'\rangle^{3/2}} (\mathbf{n}\times\mathbf{n}') + s\tfrac{(\mathbf{n}\times\mathbf{n}'')}{|\mathbf{n}'|}\right) \times \tfrac{\mathbf{n}\times\mathbf{n}'}{|\mathbf{n}'|} =$$
$$= a(\boldsymbol{a}'\times(\mathbf{n}\times\mathbf{n}')) + b((\mathbf{n}\times\mathbf{n}')\times(\mathbf{n}\times\mathbf{n}')) + c((\mathbf{n}\times\mathbf{n}'')\times(\mathbf{n}\times\mathbf{n}')) = a\langle \mathbf{n}', \boldsymbol{a}'\rangle\mathbf{n} + \mathbf{0} + c\langle \mathbf{n}\times\mathbf{n}'', \mathbf{n}'\rangle\mathbf{n} ..$$

Thus the ruled surface is tangent to the curve along α.

c.

If α is a geodesic on M, then it is also a geodesic on the ruled surface because the two surface have parallel normals which are thus both parallel the (extrinsic) curvature vector of α. Since α is geodesic on the ruled surface and since the ruled surface is developable near α, then the ruled surface is locally isomorphic to a neighborhood of a straight line in the plane – this neighborhood contains a ribbon with center line on α.

Problem 7.7. Curvature of the Hyperbolic Plane

a.

In Problem **3.1.f** we showed that the hyperbolic plane was locally isometric to a certain surface of revolution ($R(z)\cos\theta$, $R(z)\sin\theta$, z). From Problem **6.2.f** we have that $K = \kappa_1\kappa_2 = \frac{-R''(z)}{R(z)[1+(R'(z))^2]^2}$. Unfortunately we do not have an explicit expression for $R(z)$, but from Problem **3.1.f** we do have $R'(z) = \frac{R(z)}{\sqrt{r^2-R(z)^2}}$, and thus, $R''(z) = \frac{r^2R'(z)}{(r^2-R(z)^2)^{3/2}} = \frac{r^2R(z)}{(r^2-R(z)^2)^2}$. Putting these into the expression for K we get

$$K = \frac{-r^2R(z)}{(r^2-R(z)^2)^2}\left(R(z)\left[1+\left(\frac{R(z)}{\sqrt{r^2-R(z)^2}}\right)^2\right]^2\right)^{-1} = \frac{-r^2R(z)\,(r^2-R(z)^2)^2}{(r^2-R(z)^2)^2\,R(z)r^4} = \frac{-1}{r^2}.$$

b.

Pick geodesic rectangular coordinates so that the coordinate curves $\mathbf{x}(x,b)$ follow the annular strips and the coordinate curves $\mathbf{x}(a,y)$ are perpendicular to the annular strips. From Problem **1.8.c** it easily follows that, for the annular hyperbolic plane, we have $h(x,y) = |\mathbf{x}_1(x,y)| = \exp\frac{-y}{r}$. Then we can calculate

$$K = \frac{-h_{22}(x,y)}{h(x,y)} = \frac{-(\frac{-1}{r})^2\exp\frac{-y}{r}}{\exp\frac{-y}{r}} = \frac{-1}{r^2}.$$

c.

In Problem **5.7.d** we used holonomy and area to calculate that the intrinsic curvature is $-1/r^2$.

d.

The extrinsic computation in part **a** is the most algebraic and analytically involved. The intrinsic computation in part **b** is straight forward but involves using the formula from Problem **7.1** that was difficult to derive. The intrinsic and geometric computation in Problem **5.6.d** is the most elementary and the most directly related to the construction of the annular hyperbolic plane.