SOLUTIONS

Chapter 4

Tangent Space, Metric, and Directional Derivative

PROBLEM 4.1. The Tangent Space

a.

If we have a curve C which intersects a plane Π at a point p, then, if we zoom in on p and find that the portion of the curve in the f.o.v. becomes indistinguishable from a subset of the plane, then we would say that C is *tangent* to the plane at p. But clearly this does not mean that C lies in the plane. Thus, in general, for a curve that is tangent to the plane at p, as we zoom in, the portion of the curve in the f.o.v. becomes closer and closer to the plane until it becomes indistinguishable from it. However, when the curve is straight (such as a vector) then as we zoom in, we see the same picture at all magnifications. See Figure 4.1 of the text. Which angles we can distinguish depend on the tolerance. With decreasing tolerances we will be able to distinguish smaller and smaller angles. In fact the tolerance is essentially a measure of the smallest angle (subtended at the center) that is not indistinguishable from a line segment. Thus, straight line is tangent to the plane only if it forms a zero angle with the plane and, thus, is in the plane.

b.

The velocity vector will be tangent to the surface and, thus, to the tangent plane. By Part **a** this vector then must lie in the plane.

c.

Consider the intersection of M with $(n-1)$-dimensional subspaces determined by a tangent vector in $T_{\mathbf{p}}M$ and the whole normal space $N_{\mathbf{p}}M$. See Figure 4.2 (in the text) for a picture of this situation in $\mathbf{R}^3$. By the definition of smooth surface, near p the surface M projects one-to-one onto the tangent plane; thus, the intersection of the $(n-1)$-dimensional subspace with M is a curve (near p), which we call C.

Pick rectangular coordinates for $\mathbf{R}^n$ so that $p=(0,0,0,...,0)$ and $\mathbf{V}=(|\mathbf{V}|,0,0,...,0)$. Then let $g(a,b,c,...,z) = (a,0,0,...,0)$ be the projection onto the tangent line at p. Then $g|C$ (the projection restricted to C) is one-to-one and there is a function $\gamma\colon \mathbf{R} \to C$ such that $g(\gamma(t)) = (|\mathbf{V}|t,0,0,...,0)$. This γ gives a parametrization for a neighborhood of p in C and $\gamma'(0) = \mathbf{V}$.

PROBLEM 4.2. Mean Value Theorem - Curves - Surfaces

a.

Look at the line determined by **p** and **q** and then move this line parallel to itself (in one or the other direction) until it last touches the curve. Call this parallel transported line of last contact l. The point **r** of last contact has a tangent line t. If t is the same as l then we are done. If t is different from l, then pick a tolerance τ so that in every f.o.v. the angle between t and l can be seen. Then with this tolerance zoom in until the curve and t are indistinguishable. In this f.o.v. **r** is not the last point of contact with the parallel transported line.

b.

Move L parallel to itself (in one direction or the other) to a parallel hyperplane L' that last touches the curve. Call this last point of contact c. Let t be the tangent line to λ at c. By an argument the same as in Part **a**, this tangent line must lie in the parallel transported L (and then we are done) or there are points on λ near c on both sides of L' and, thus, it was not the last point of contact with λ.

c.

Move the plane containing the bounding curve until it last contacts the surface. This parallel transported plane at the last point of last contact is the tangent plane by an entirely similar argument.

d.

Move P parallel to itself until it totally misses M (this is possible because M is closed and, thus, bounded. Now move P parallel to itself until it first and last contacts the M. These are two points whose tangent planes are parallel to P.

Problem 4.3. Riemannian Metric

a.

symmetric: The definition is entirely symmetric.

bilinear:

If $a > 0$, then $\theta_{a\mathbf{XY}} = \theta_{\mathbf{XY}} = \theta_{\mathbf{X}a\mathbf{Y}}$ and we get

$$a\langle \mathbf{X},\mathbf{Y}\rangle = a|\mathbf{X}|\,|\mathbf{Y}|\cos\theta_{\mathbf{XY}} = |a\mathbf{X}|\,|\mathbf{Y}|\cos\theta_{a\mathbf{XY}} = \langle a\mathbf{X},\mathbf{Y}\rangle = |\mathbf{X}|\,|a\mathbf{Y}|\cos\theta_{\mathbf{X}a\mathbf{Y}} = \langle \mathbf{X},a\mathbf{Y}\rangle\ .$$

If $a < 0$, then $\theta_{a\mathbf{XY}} = \theta_{\mathbf{XY}} + \pi = \theta_{\mathbf{X}a\mathbf{Y}}$ and we get

$$a\langle \mathbf{X},\mathbf{Y}\rangle = a|\mathbf{X}|\,|\mathbf{Y}|\cos\theta_{\mathbf{XY}} = -|a\mathbf{X}|\,|\mathbf{Y}|\,(-\cos\theta_{a\mathbf{XY}}) = \langle a\mathbf{X},\mathbf{Y}\rangle = |\mathbf{X}|\,(-|a\mathbf{Y}|)\,(-\cos\theta_{\mathbf{X}a\mathbf{Y}}) = \langle \mathbf{X},a\mathbf{Y}\rangle\ .$$

If $a = 0$, then it is easy to check that all three terms are zero.

Now look at the following Figure 4.A.

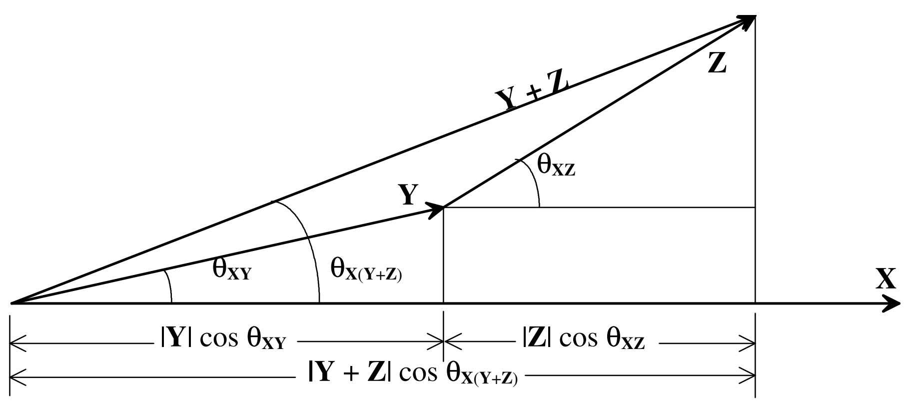


Figure 4.A. Rieman metric is linear.

Thus, $\langle \mathbf{X},\mathbf{Y+Z}\rangle = |\mathbf{X}|\,|\mathbf{Y+Z}|\cos\theta_{\mathbf{X(Y+Z)}} = |\mathbf{X}|\,\{|\mathbf{Y}|\cos\theta_{\mathbf{XY}} + |\mathbf{Z}|\cos\theta_{\mathbf{XZ}}\} = \langle \mathbf{X},\mathbf{Y}\rangle + \langle \mathbf{X},\mathbf{Z}\rangle$

***positive definite*:**

If $\mathbf{X} \neq \mathbf{0}$, then $|\mathbf{X}| \neq 0$ and $\langle \mathbf{X},\mathbf{X}\rangle = |\mathbf{X}|\,|\mathbf{X}|\cos 0 = |\mathbf{X}|^2 > 0$.

b. We use Part **a** to calculate: $\langle \mathbf{A},\mathbf{B}\rangle = \langle\, a_1\mathbf{X}_1 + a_2\mathbf{X}_2\,,\, b_1\mathbf{X}_1 + b_2\mathbf{X}_2 \,\rangle =$

$$= \langle a_1\mathbf{X}_1\,, b_1\mathbf{X}_1\rangle + \langle a_2\mathbf{X}_2\,, b_1\mathbf{X}_1\rangle + \langle a_1\mathbf{X}_1\,, b_2\mathbf{X}_2\rangle + \langle a_2\mathbf{X}_2\,, b_2\mathbf{X}_2\rangle$$

=

$$= a_1b_1\langle\mathbf{X}_1\,, \mathbf{X}_1\rangle + a_2b_1\langle\mathbf{X}_2\,, \mathbf{X}_1\rangle + a_1b_2\langle\mathbf{X}_1\,, \mathbf{X}_2\rangle + a_2b_2\langle\mathbf{X}_2\,, \mathbf{X}_2\rangle =$$

$$= a_1b_1(1) + a_2b_1(0) + a_1b_2(0) + a_2b_2(1) = a_1b_1 + a_2b_2$$

c. This follows immediately from the definition of matrix multiplication and the answer to Part **b**.

Problem 4.4. Vectors in Extrinsic Local Coordinates

a. b.

Cylinder: $$\boldsymbol{x}(\theta,z) = (R\cos\theta, R\sin\theta, z)$$

$$\boldsymbol{x}_1(\theta,z) = (-R\sin\theta, R\cos\theta, 0) \text{ and } \boldsymbol{x}_2(\theta,z) = (0,0,1)$$

These are clearly C^1 and linearly independent.

$$[g_{ij}] = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Cone: $$\mathbf{x}(\theta,r) = (r\sin\phi\cos\tfrac{2\pi\theta}{\alpha}, r\sin\phi\sin\tfrac{2\pi\theta}{\alpha}, r\cos\phi),$$

where α is the (intrinsic) cone angle and ϕ is the angle between the axis of the cone and a generator of the cone with $\alpha = 2\pi\sin\phi$.

$$\mathbf{x}_1(\theta,r) = (r(\sin\phi)\tfrac{2\pi}{\alpha}(-\sin\tfrac{2\pi\theta}{\alpha}), r(\sin\phi)\tfrac{2\pi}{\alpha}\cos\tfrac{2\pi\theta}{\alpha}, 0) = (r(-\sin\tfrac{2\pi\theta}{\alpha}), r\cos\tfrac{2\pi\theta}{\alpha}, 0) \quad ,$$

$$\mathbf{x}_2(\theta,r) = (\sin\phi\cos\tfrac{2\pi\theta}{\alpha}, \sin\phi\sin\tfrac{2\pi\theta}{\alpha}, \cos\phi) = \left(\tfrac{\alpha}{2\pi}\cos\tfrac{2\pi\theta}{\alpha}, \tfrac{\alpha}{2\pi}\sin\tfrac{2\pi\theta}{\alpha}, \sqrt{1-(\tfrac{\alpha}{2\pi})^2}\right)$$

For $r \neq 0$ and $\phi \neq 0$, these are linearly independent (since they are orthogonal and nonzero) and C^1.

$$[g_{ij}] = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, these are local coordinates for the cone except at the cone point.

Sphere: $$\mathbf{x}(\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi),$$

$$\mathbf{x}_1(\theta,\phi) = (-r\sin\theta\sin\phi, r\cos\theta\sin\phi, 0) \text{ and } \mathbf{x}_2(\theta,\phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, -r\sin\phi).$$

For $\phi \neq 0,\pi$, these are linearly independent (since they are orthogonal and nonzero) and C^1.

$$g_{ij} = \begin{pmatrix} r^2\sin^2\phi & 0 \\ 0 & r^2 \end{pmatrix}.$$

Thus, these are local coordinates except at the North and South Poles.

Strake: $$\mathbf{x}(\theta,r) = (r\cos\theta, r\sin\theta, k\theta),$$

$$\mathbf{x}_1(\theta,r) = (-r\sin\theta, r\cos\theta, k) \text{ and } \mathbf{x}_2(\theta,r) = (\cos\theta, \sin\theta, 0).$$

These are linearly independent (since they are orthogonal and nonzero) and C^1.

$$[g_{ij}] = \begin{pmatrix} r^2+k^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $2\pi k = h$, the height of one revolution of the strake.

Surface of Revolution: $$\mathbf{x}(\theta,x) = (x, f(x)\cos\theta, f(x)\sin\theta),$$

$$\mathbf{x}_1(\theta,x) = (0, -f(x)\sin\theta, f(x)\cos\theta) \text{ and } \mathbf{x}_2(\theta,x) = (1, f'(x)\cos\theta, f'(x)\sin\theta).$$

For $f(x) \neq 0$, these are linearly independent (since they are orthogonal and nonzero) and C^1.

$$[g_{ij}] = \begin{pmatrix} (f(x))^2 & 0 \\ 0 & 1+(f'(x))^2 \end{pmatrix}.$$

Graph of a Smooth Function: $\mathbf{x}(x,y) = (x,y,g(x,y))$,

$\mathbf{x}_1(x,y) = (1,0,g_1(x,y))$ and $\mathbf{x}_2(x,y) = (0,1,g_2(x,y))$.

These are linearly independent (since projections onto the (x,y)-plane are linearly independent) and C^1.

$$[g_{ij}] = \begin{pmatrix} 1+(g_1(x,y))^2 & g_1(x,y)g_2(x,y) \\ g_2(x,y)g_1(x,y) & 1+(g_2(x,y))^2 \end{pmatrix}.$$

Problem 4.5. Measuring Using the Riemannian Metric

a.

Using the definition of the Riemannian metric we can express

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\tfrac{\langle \mathbf{x}_1,\mathbf{x}_2\rangle}{|\mathbf{x}_1||\mathbf{x}_2|}\right)^2} = \sqrt{\tfrac{\langle \mathbf{x}_1,\mathbf{x}_1\rangle\langle \mathbf{x}_2,\mathbf{x}_2\rangle-\langle \mathbf{x}_1,\mathbf{x}_2\rangle^2}{|\mathbf{x}_1|^2|\mathbf{x}_2|^2}} = \sqrt{\tfrac{\det g}{|\mathbf{x}_1|^2|\mathbf{x}_2|^2}},$$

thus, $\sqrt{\det g(u^1,u^2)} = |\mathbf{x}_1(u^1,u^2)||\mathbf{x}_2(u^1,u^2)|\sin\theta$.

We can now let Δu^1 and Δu^2 go to zero and integrate over the region $V = \mathbf{x}(U)$ to get the following expression for the area of V: $\iint_U \sqrt{\det g(u^1,u^2)}\, du^1 du^2$.

b.

The surface area of this intrinsic disk on the sphere is (the limit is technically necessary because the coordinate patch is singular at the North Pole, $\phi = 0$)

$$\lim_{\delta\to 0} \int_\delta^{r/R}\left(\int_0^{2\pi} \sqrt{R^2R^2\sin^2\phi}\, d\theta\right)d\phi = R^2\left(\lim_{\delta\to 0}\int_\delta^{r/R}\sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right) = 2\pi R^2(1-\cos\tfrac{r}{R})$$

For r = 1 km and R = 6360 km, we calculate the surface area on the sphere as 3.14159223 (km)^2 as opposed to πr^2 = 3.14159265 (km)^2 for the disk of the same radius on the plane.

c.

The desired surface area integral is $\lim_{\delta\to 0}\int_\delta^r\left(\int_0^\alpha \sqrt{r^2}\, d\theta\right)dr = \lim_{\delta\to 0}\int_\delta^r r\left(\int_0^\alpha d\theta\right)dr = \int_0^r \alpha r dr = \frac{\alpha}{2}r^2$, which agrees with the calculation of the area of a sector of the circle in the covering space: $(\alpha/2\pi)\pi r^2$.

d.

From Problem **2.5**, the inner radius is 1m, the outer radius 1.2m, and the height of one turn is h = 10m (thus, $k = h/(2\pi)$ = 1.592m). The desired area is (see solution to Problem **4.4**.)

$$\begin{aligned} \int_0^{2\pi}\int_1^{1.2}\sqrt{r^2+k^2}\; dr\, d\theta &= \left(\int_0^{2\pi} d\theta\right)\left(\int_1^{1.2}\sqrt{r^2+k^2}\; dr\right) = \\ &= 2\pi\left(\tfrac{r}{2}\sqrt{r^2+k^2} + \tfrac{k^2}{2}\ln\left(r+\sqrt{r^2+k^2}\right)\right)\Big|_{r=1}^{r=1.2} = \\ = \tfrac{2\pi}{2}\Big[\Big(1.2\sqrt{(1.2)^2+k^2} &+ k^2\ln\Big(1.2+\sqrt{(1.2)^2+k^2}\Big)\Big) - \Big(1\sqrt{1^2+k^2} + k^2\ln\Big(1+\sqrt{1^2+k^2}\Big)\Big)\Big] = \\ = \pi\left[1.2\sqrt{(1.2)^2+(1.592)^2} \right.&\left. - \sqrt{1+(1.592)^2} + (1.592)^2\ln\left(\tfrac{1.2+\sqrt{(1.2)^2+(1.592)^2}}{1+\sqrt{1+(1.592)^2}}\right)\right] = 2.432\,\text{m}^2 \end{aligned}$$

The annular strip that approximates it has (from the solution to Problem **2.5.b**) inner radius equal the radius of curvature r_i = 3.533m and inner and outer arclengths

$$l_i = \sqrt{h^2+(2\pi r)^2} = \sqrt{100+4\pi^2} = 11.810\text{ m},$$
$$l_o = l_i\tfrac{3.533+0.2}{3.533} = 12.479\text{ m}.$$

Then, using the fact that the area of a circular sector is one-half the arclength times the radius, the area of the annulus is

$$\tfrac{1}{2}l_o(r_i+0.2)-\tfrac{1}{2}l_ir_i=\tfrac{1}{2}(12.479)(3.533+0.2)-\tfrac{1}{2}(11.810)(3.533)=2.430\,\mathrm{m}^2,$$

which is extremely close to the area of the strake.

e.

The region $V = \mathbf{x}(U)$, where $U=\{(w,s)\mid 0\le w\le c; 0\le s<\infty\}$

$$Area\,(V)=\iint_U \sqrt{\det g_{ij}(w,s)}\,dwds=\lim_{b\to\infty}\int_0^c\Big(\int_0^b \exp(-s/r)ds\Big)dw$$

$$=\lim_{b\to\infty}c(-r\exp(-b/r)+r\exp(-0/r))=\lim_{b\to\infty}c(r-r\exp(-b/r))=cr$$

PROBLEM 4.6. Differentiating a Metric

a.

Using the definition of derivative

$$\begin{aligned}\tfrac{d}{dt}\langle \boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t))\rangle &= \lim_{h\to 0}\tfrac{1}{h}[\langle \boldsymbol{X}(\gamma(t+h)),\boldsymbol{Y}(\gamma(t+h))\rangle-\langle \boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t))\rangle\,]=\\ &=\lim_{h\to 0}\tfrac{1}{h}[\langle \boldsymbol{X}(\gamma(t+h))\,-\boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t+h))\rangle+\langle \boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t+h))\,-\boldsymbol{Y}(\gamma(t))\rangle\,]=\\ &=\lim_{h\to 0}\Big[\Big\langle \tfrac{\boldsymbol{X}(\gamma(t+h))-\boldsymbol{X}(\gamma(t))}{h},\boldsymbol{Y}(\gamma(t+h))\Big\rangle+\Big\langle \boldsymbol{X}(\gamma(t)),\tfrac{\boldsymbol{Y}(\gamma(t+h))-\boldsymbol{Y}(\gamma(t))}{h}\Big\rangle\Big]=\\ &=\Big\langle \tfrac{d}{dt}\boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t))\Big\rangle+\Big\langle \boldsymbol{X}(\gamma(t)),\tfrac{d}{dt}\boldsymbol{Y}(\gamma(t))\Big\rangle\,.\end{aligned}$$

Since it is differentiable, it is continuous.

b.

Using the definition of directional derivative and Part **a** we have, for any curve $\gamma(t)$ such that $\gamma(0)=p$ and $\gamma'(0)=\mathbf{Z}_p$,

$$\boldsymbol{Z}_p\langle \boldsymbol{X},\boldsymbol{Y}\rangle=\tfrac{d}{dt}\langle \boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t))\rangle_{t=0}=\Big\langle \tfrac{d}{dt}\boldsymbol{X}(\gamma(t)),\boldsymbol{Y}(\gamma(t))\Big\rangle_{t=0}+\Big\langle \boldsymbol{X}(\gamma(t)),\tfrac{d}{dt}\boldsymbol{Y}(\gamma(t))\Big\rangle_{t=0}=$$
$$\Big\langle \boldsymbol{Z}_p\boldsymbol{X},\boldsymbol{Y}(\gamma(0))\Big\rangle+\Big\langle \boldsymbol{X}(\gamma(0)),\boldsymbol{Z}_p\boldsymbol{Y}\Big\rangle=\Big\langle \boldsymbol{Z}_p\boldsymbol{X},\boldsymbol{Y}(p)\Big\rangle+\Big\langle \boldsymbol{X}(p),\boldsymbol{Z}_p\boldsymbol{Y}\Big\rangle\,.$$

c.

Since **X** and **Y** are perpendicular everywhere, $0=\boldsymbol{Z}_p\langle \boldsymbol{X},\boldsymbol{Y}\rangle=\Big\langle \boldsymbol{Z}_p\boldsymbol{X},\boldsymbol{Y}(p)\Big\rangle+\Big\langle \boldsymbol{X}(p),\boldsymbol{Z}_p\boldsymbol{Y}\Big\rangle$. The desired result follows.

PROBLEM 4.7. Expressing Normal Curvature

a.

If **n** is differentiable along C, then $\mathbf{T}_p\,\mathbf{n}$ (being the derivative of a unit vector) is perpendicular to **n** and, thus, in T_pM. Let $\gamma(s)$ be a parametrization of C by arclength such that $\gamma(0)=p$. Then the curvature of C at p is, by definition, $\boldsymbol{\kappa}(p)=\tfrac{d}{ds}\boldsymbol{T}(\gamma(s))_{s=0}=\boldsymbol{T}_p\boldsymbol{T}$. And, by definition of directional derivative, $\boldsymbol{T}_p\boldsymbol{n}=\tfrac{d}{ds}\boldsymbol{n}(\gamma(s))_{s=0}$.
Also, note that **T** and **n** are perpendicular everywhere and, thus, $\boldsymbol{\kappa}_{\boldsymbol{n}}(p)=\langle\boldsymbol{\kappa}(p),\boldsymbol{n}(p)\rangle\boldsymbol{n}(p)$ using the Riemannian metric which is the usual inner product in $\mathbf{R}^3$. Then, by **4.6.c**,

$$\boldsymbol{\kappa}_{\boldsymbol{n}}(p)=\langle\boldsymbol{\kappa}(p),\boldsymbol{n}(p)\rangle\boldsymbol{n}(p)=\Big\langle \tfrac{d}{ds}\mathbf{T}(\gamma(s))_{s=0},\boldsymbol{n}(p)\Big\rangle\boldsymbol{n}(p)=\Big\langle \boldsymbol{T}_p\boldsymbol{T},\boldsymbol{n}(p)\Big\rangle\boldsymbol{n}(p)=$$
$$-\Big\langle \boldsymbol{T}(p),\boldsymbol{T}_p\boldsymbol{n}\Big\rangle\boldsymbol{n}(p)=\Big\langle \boldsymbol{T}(p),-\boldsymbol{T}_p\boldsymbol{n}\Big\rangle\boldsymbol{n}(p)\quad .$$

b.

Note that $\langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n}\rangle$ depends on $\mathbf{T}_p$ but not on γ. Thus, the normal curvature does not depend on the curve but only on the unit tangent vector. By **4.1.c** this normal curvature is the extrinsic curvature of the curve defined by intersecting the surface with the plane determined by **n** and $\mathbf{T}_p$. Here we see a first hint of why it may be possible for the normal curvature (the curvature due to the curving of the surface) to produce an intrinsic quantity because, even though **n** is an extrinsic quantity, its derivative $\mathbf{T}_p\,\mathbf{n}$ (being the derivative of a unit vector) is a tangent vector at p (*Why?*) and, thus, is intrinsic and depends only on $\mathbf{T}_p$.

***c.**

For example, take the curves in Figure **2.6** and cross them with the real line. Note that on a circle of radius R, the derivative of $\mathbf{n}$ (with respect to arclength) has magnitude equal to $1/R$.

d.

Since the normal (by Part **a**) does not depend on the curve but only on the tangent vector, it is enough to check this for great circles. Great circles have their normal curvature equal to their extrinsic curvature which is $1/R$.

PROBLEM 4.8. Differential Operator

a.

Just before Problem **4.6** in Theorem 4.5 we showed that for a real-valued function f if $f\circ\boldsymbol{x}$ is C^1 then $\mathbf{X}_p f$ does not depend on the curve chosen and if $\mathbf{X}_p = X^1\mathbf{x}_1 + X^2\mathbf{x}_2$, then $\mathbf{X}_p f = X^1\mathbf{x}_1 f + X^2\mathbf{x}_2 f$. Now apply this to each component of $\mathbf{F}(\mathrm{p}) = (f_1, f_2, \ldots)$.

b.

If $\mathbf{X}_p = X^1\mathbf{x}_1 + X^2\mathbf{x}_2$ and $\mathbf{Y}_p = Y^1\mathbf{x}_1 + Y^2\mathbf{x}_2$, then

$$(\mathbf{X}_p + \mathbf{Y}_p)\mathbf{F} = (X^1+Y^1)\mathbf{x}_1\mathbf{F} + (X^2+Y^2)\mathbf{x}_2\mathbf{F} = X^1\mathbf{x}_1\mathbf{F} + Y^1\mathbf{x}_1\mathbf{F} + X^2\mathbf{x}_2\mathbf{F} + Y^2\mathbf{x}_2\mathbf{F} = \mathbf{X}_p\mathbf{F} + \mathbf{Y}_p\mathbf{F},$$

and $(a\mathbf{X}_p)\mathbf{F} = aX^1\mathbf{x}_1\mathbf{F} + aX^2\mathbf{x}_2\mathbf{F} = a(X^1\mathbf{x}_1\mathbf{F} + X^2\mathbf{x}_2\mathbf{F}) = a(\mathbf{X}_p\mathbf{F})$. Note: If α, β, γ are curves on the surface such that $\alpha'(0) = \mathbf{X}_p$, $\beta'(0) = \mathbf{Y}_p$, $\gamma'(0) = \mathbf{X}_p + \mathbf{Y}_p$, and $\alpha(0) = \beta(0) = \gamma(0) = p$, then in $\mathbf{R}^n$ it is possible to specify that $\gamma(s) = \alpha(s) + \beta(s)$ but this is NOT possible in general on a surface because there is no global notion of addition.

c.

Let $\gamma(t)$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{X}_p$. Then

$$\mathbf{X}_p(f\mathbf{F}) = \tfrac{d}{dt}[f(\gamma(t))\boldsymbol{F}(\gamma(t))]_{t=0} = \left[\tfrac{d}{dt}f(\gamma(t))\right]\boldsymbol{F}(\gamma(0)) + f(\gamma(0))\tfrac{d}{dt}[\boldsymbol{F}(\gamma(t))] = (\mathbf{X}_p f)\mathbf{F} + f(\mathbf{X}_p\mathbf{F})$$

d.

Note that $\mathbf{x}_j$ is a (tangent) vector-valued function of the coordinates (u^1, u^2) or, equivalently, is a function of the points $q = \mathbf{x}(u^1, u^2)$ in M. Thus $\mathbf{x}_1(a,b)$ can also be written $\mathbf{x}_1(\mathbf{x}(a,b))$ as a tangent vector at the point $p = \mathbf{x}(a,b)$ and

$$\begin{aligned}\mathbf{x}_1(a,b)\mathbf{x}_2 &= \lim_{h\to 0}\frac{\mathbf{x}_2(\mathbf{x}(a+h,b)) - \mathbf{x}_2(\mathbf{x}(a,b))}{h} = \lim_{h\to 0}\frac{\mathbf{x}_2(a+h,b) - \mathbf{x}_2(a,b)}{h} = \\ &= \lim_{h\to 0}\tfrac{1}{h}\left[\lim_{g\to 0}\frac{\mathbf{x}(a+h,b+g) - \mathbf{x}(a+h,b)}{g} - \lim_{g\to 0}\frac{\mathbf{x}(a,b+g) - \mathbf{x}(a,b)}{g}\right] = \\ &= \lim_{h\to 0}\lim_{g\to 0}\tfrac{1}{h}\tfrac{1}{g}[\mathbf{x}(a+h,b+g) - \mathbf{x}(a+h,b) - \mathbf{x}(a,b+g) + \mathbf{x}(a,b)].\end{aligned}$$

Since the limits exist and are continuous we take the limits in either order. If we reverse the order of the limits then the last expression becomes

$$\begin{aligned}&= \lim_{g\to 0}\lim_{h\to 0}\tfrac{1}{h}\tfrac{1}{g}[\mathbf{x}(a+h,b+g) - \mathbf{x}(a+h,b) - \mathbf{x}(a,b+g) + \mathbf{x}(a,b)] = \\ &= \lim_{g\to 0}\frac{1}{g}\left[\lim_{h\to 0}\frac{\mathbf{x}(a+h,b+g) - \mathbf{x}(a,b+g)}{h} - \lim_{h\to 0}\frac{\mathbf{x}(a+h,b) - \mathbf{x}(a,b)}{h}\right] = \\ &= \lim_{g\to 0}\frac{\mathbf{x}_1(a,b+g) - \mathbf{x}_1(a,b)}{g} = \lim_{g\to 0}\frac{\mathbf{x}_1(\mathbf{x}(a,b+g)) - \mathbf{x}_1(\mathbf{x}(a,b))}{g} = \mathbf{x}_2(a,b)\mathbf{x}_1.\end{aligned}$$

e.

On the sphere $\mathbf{x}_1(\theta,\phi) = (-r\sin\theta\sin\phi, r\cos\theta\sin\phi, 0)$ and $\boldsymbol{x}_2(\theta,\phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, -r\sin\phi)$. Thus, $\mathbf{x}_{12} = \mathbf{x}_{21} = (-r \sin\theta \cos\phi, r \cos\theta \cos\phi, 0) = (\cot \phi)\mathbf{x}_1$. The length of the tangent vector in the latitudinal (east-west) direction of $\mathbf{x}_1$ starts off at r^2 on the equator but decreases as you move toward either pole; and $\mathbf{x}_{21}$ is the rate of change of $\mathbf{x}_1$ as you move southward along a longitude. Also, note that $\mathbf{x}_{12}$ is the rate of change of the tangent vector in the longitudinal direction of $\mathbf{x}_2$, as you move westward along a latitude circle and, even though the length of $\mathbf{x}_2$ is constantly r^2, its direction is changing. I urge the reader to investigate this on the sphere until it becomes as natural and comfortable as possible.

On the strake, $\boldsymbol{x}_1(\theta,r) = (-r\sin\theta, r\cos\theta, k)$ and $\boldsymbol{x}_2(\theta,r) = (\cos\theta, \sin\theta, 0)$. Thus,

$$\boldsymbol{x}_{12}(\theta,r) = (-\sin\theta, \cos\theta, 0) = \boldsymbol{x}_{21}(\theta,r).$$

Problem 4.9. Metric in Geodesic Coordinates

Explain each step in the following argument.

Let $\mathbf{x}(u^1,u^2)$ be geodesic rectangular coordinates, $\mathbf{c}(x,y)$, or geodesic polar coordinates, $\mathbf{p}(\theta,r)$, as in Figure 4.9 above. According to Problem **4.3** the Riemannian metric can be expressed in local coordinates as the matrix:

$$g = (g_{ij}) = \begin{pmatrix} \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle & \langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle \\ \langle \boldsymbol{x}_2, \boldsymbol{x}_1 \rangle & \langle \boldsymbol{x}_2, \boldsymbol{x}_2 \rangle \end{pmatrix}.$$

a. From the definition of geodesic coordinates, for constant a, the geodesic curves $\mathbf{x}(a,u^2)$ are parametrized by arclength and, thus,

$$g_{22}(u^1,u^2) = 1,$$

because

$$g_{22}(u^1,u^2) = \langle \boldsymbol{x}_2, \boldsymbol{x}_2 \rangle = |\boldsymbol{x}_2|^2,$$

the square of the length of the velocity vector of the curve $\mathbf{x}(a,u^2)$.

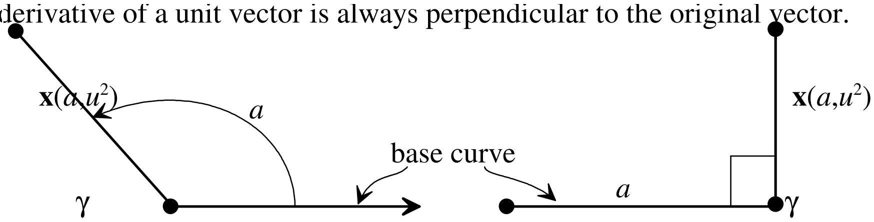
b. We now need to find $g_{12}(u^1,u^2)$. To do this we first differentiate:

$$\frac{\partial}{\partial u^2} g_{12}(u^1,u^2) = \boldsymbol{x}_2 \langle \boldsymbol{x}_1(u^1,u^2), \boldsymbol{x}_2(u^1,u^2) \rangle = \\ = \langle \boldsymbol{x}_{21}, \boldsymbol{x}_2 \rangle + \langle \boldsymbol{x}_1, \boldsymbol{x}_{22} \rangle. \text{ (By \textbf{4.6.b}.)}$$

Now, since $\mathbf{x}_2$ is a unit vector

$$\langle \boldsymbol{x}_{21}, \boldsymbol{x}_2 \rangle = \langle \boldsymbol{x}_{12}, \boldsymbol{x}_2 \rangle = \left\langle \frac{\partial}{\partial u^1} \boldsymbol{x}_2, \boldsymbol{x}_2 \right\rangle = 0$$

because the derivative of a unit vector is always perpendicular to the original vector.



geodesic polar coordinates **geodesic rectangular coordinates**

$\gamma(0) = \mathbf{x}(a,0) = \mathbf{x}(0,0)$ $\mathbf{x}(0,0)$ $\mathbf{x}(a,0)$

Figure 4.5. Second coordinate curves.

c. Now we focus on the second coordinate curve

$$\gamma(u^2) = \mathbf{x}(a,u^2).$$

(See Figure 4.5.) Since $\gamma(u^2) = \mathbf{x}(a,u^2)$ is parametrized by arclength, its unit tangent vector is $\mathbf{x}_2(a,u^2)$ and thus

$$\mathbf{x}_{22} = \tfrac{\partial^2}{(\partial u^2)^2}\mathbf{x}(a,u^2) = \tfrac{\partial}{\partial u^2}\mathbf{x}_2(a,u^2) = \boldsymbol{\kappa}$$

is its (extrinsic) curvature vector, since this is just the derivative (with respect to arclength) of the tangent vector. Since the curve is a geodesic, its curvature vector must be parallel to the normal to the surface. Thus,

$$\langle \mathbf{x}_1, \mathbf{x}_{22} \rangle = 0 \text{ and therefore } \tfrac{\partial}{\partial u^2} g_{12}(a,u^2) = 0.$$

We can then conclude that $g_{12}(a,u^2)$ is a constant independent of u^2, because from first semester calculus any real-valued function of a real variable is a constant if its derivative is equal to 0.

d. By definition of geodesic rectangular coordinates,

$$g_{12}(u^1,0) = \langle \mathbf{x}_1(u^1,0), \mathbf{x}_2(u^1,0) \rangle = 0,$$

because the second coordinate curves are perpendicular to the base curve ($u^2 = 0$).

For geodesic polar coordinates, $\mathbf{x}(u^1,0) = \mathbf{p}(\theta,0) = \mathbf{p}(0,0)$, a constant. Thus, again,

$$g_{12}(u^1,0) = \langle \mathbf{x}_1(u^1,0), \mathbf{x}_2(u^1,0) \rangle = 0,$$

since $\mathbf{x}_2(u^1,0) = 0$ being the derivative of a constant.

e. We can now conclude that

$$g_{12}(u^1,u^2) = 0, \text{ for all } u^1 \text{ and } u^2,$$

since we showed that $g_{12}(u^1,u^2)$ was a constant and is equal to 0 at $u^2 = 0$.

Thus, for geodesic rectangular or polar coordinates:

$$g(u^1,u^2) = \begin{pmatrix} (h(u^1,u^2))^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $h(u^1,u^2) = |\mathbf{x}_1(u^1,u^2)| > 0$.