Chapter 3

Extrinsic Descriptions of Intrinsic Curvature

PROBLEM 3.1. Smooth Surfaces and Tangent Planes

***a.**

If the surface is infinitesimally planar at p and the tangent planes vary continuously, then, for every tolerance $\tau/4$ there is a field of view with center p and radius ρ, such that, if p and x are both on the surface in the field of view, then each point on the surface is within $\tau\rho/4$ of the tangent plane T_p and each point on T_p is within $\tau\rho/4$ of the tangent plane T_x. Then, for every q on the surface such that $|p - q| < \rho/2$, every point on the surface within $\rho/2$ of q is within $\tau\rho/4$ of T_p which in turn is within $\tau\rho/4$ of T_q. Thus every point on the surface within $\rho/2$ of q is within $\tau\rho/2$ of the tangent plane T_q. Thus continuously infinitesimally planar implies smooth (uniformly infinitesimally planar).

If the surface is smooth then for every tolerance $\tau/4$ there is a radius ρ such that if p and x are on the surface and $|p-x| < \rho$ then we have that x is indistinguishable from a point on the tangent plane T_p and the same ρ works for each point q in the field of view. Thus if q is on the surface within $\rho/2$ of p then in a f.o.v. of radius $\rho/2$ centered at q every point on the surface is within $\tau\rho/4$ of both the tangent plane T_p and the tangent plane T_q. Thus, in the field of view, the tangent planes T_p and T_q are within $2(\tau\rho/4) = \tau\rho/2$ of each other. Thus the tangent planes are varying continuously.

b.

If the partial derivatives are linearly independent then they span a plane which is the tangent plane. Since the partial derivatives vary continuously the tangent planes must also vary continuously and thus by Part **a** the surface is smooth.

The function $\mathbf{x}(x,y) = (x^3,y)$ is a coordinate patch for the plane but the partial derivatives with respect to x are zero when $x = 0$ and thus, the partial derivatives are not linearly independent.

c.

One may use the coordinate patches for these surfaces from Chapter 1 and check that the partial derivatives are linearly independent, using Part **b**. Or, one may argue geometrically that each of these surfaces has tangent planes and for every tolerance τ the same ρ will work for all points except on the cone. On the cone there are obvious tangent planes which are tangent to the cone along a generator and for every tolerance τ the amount of zooming necessary is dependent (in a linear fashion) on how far the point is from the cone point, and thus, the amount of zooming is uniform over neighborhoods whose closures miss the cone point.

d.

If the function f is smooth then the partial derivatives are $\mathbf{x}_1(\theta,x) = (1, f'(x) \cos \theta, f'(x) \sin \theta)$ and $\mathbf{x}_2(\theta,x) = (0, -f(x) \sin \theta, f(x) \cos \theta)$, which vary continuously and are nonzero and perpendicular (and thus linearly independent) as long as $f(x)$ is not zero. Thus, by Part **b** the surface is smooth.

If the surface is smooth, then the tangent planes along the curve $\theta = 0$ are perpendicular to the plane $\theta = 0$ and thus intersect that in lines which are tangent to the graph of f. Thus, by **2.2**, f is continuously differentiable.

***e.**

In this case the coordinate patch is $\mathbf{x}(x,y) = (x,y,g(x,y))$. If g has continuous partial derivatives then the partial derivatives of $\mathbf{x}$ are $\mathbf{x}_1(x,y) = (1,0,g_1(x,y))$ and $\mathbf{x}_2(x,y) = (0,1,g_2(x,y))$, which are always linearly independent because neither can be a linear multiple of the other. Thus, by Part **b** the surface is smooth. The tangent planes must project one-to-one onto the (x,y)-plane because the partial derivatives which span the tangent planes project to the standard basis $\{(0,1),(1,0)\}$.

If the surface is smooth with every tangent plane projecting one-to-one onto the (x,y)-plane, then the tangent planes intersected with the planes $x = a$ or $y = b$ are the tangent lines to the graphs of the functions $g(a,y)$ and $g(x,b)$. By Part **a** these tangent planes and tangent lines vary continuously and thus, (using **2.2**) the partial derivatives of g exist and are continuous.

f.

Let the vertical axis be the z-axis and then at each z we have the following picture:

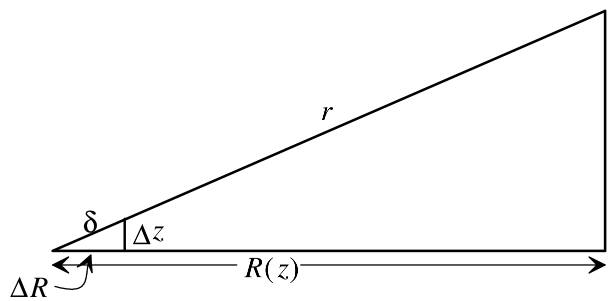

Figure 3.A. Relating $R(z)$, r, Δz, and ΔR.

Thus $\frac{\Delta R}{\Delta z} = \frac{-R(z)}{\sqrt{(r+\delta)^2 - R(z)^2}}$. In the limit as δ (and ΔR and Δz) go to zero we get $\frac{dR}{dz} = \frac{-R(z)}{\sqrt{r^2 - R(z)^2}}$.

We can get the same differential equation by using Problem **1.8.c** which implies that the circle at height z has circumference $2\pi r e^{-s/r}$, where s is the arclength along the surface from $(0,r)$ to $(z,R(z))$. Then, solving for s, we get $s = -r\ln\left(\frac{R(z)}{r}\right) = \int_0^z \sqrt{1 + \left(\frac{dR}{dz}\right)^2}\, dz$. Then differentiating both sides we get $\sqrt{1 + \left(\frac{dR}{dz}\right)^2} = -r\left(\frac{1}{R(z)/r}\right)\left(\frac{1}{r}\right)\frac{dR}{dz}$. Squaring both sides and collecting terms we get $\frac{dR}{dz} = \frac{-R(z)}{\sqrt{r^2 - R(z)^2}}$.

If we separate variables we get $\frac{\sqrt{r^2 - R(z)^2}}{R(z)} dR = dz$. A Table of Integrals like those in most calculus books yields $z = \sqrt{r^2 - R^2} - r\ln\left|\frac{r + \sqrt{r^2 - R^2}}{R}\right| + C$, where r is the constant radius of the annulus and R is a variable. When $z = 0$ we have $R = r$ and thus $C = 0$. So $z = \sqrt{r^2 - R^2} - r\ln\left|\frac{r + \sqrt{r^2 - R^2}}{R}\right|$. Here z is a continuously differentiable function of R and the derivative (for $z \neq 0$) is never zero, hence R is also a continuously differentiable function of z. Since R is never zero, Part **d** applies and, thus, we can conclude that this hyperbolic surface of revolution is a smooth surface.

Problem 3.2. Extrinsic Curvature - Geodesic on Sphere

a.

One can check this by using the explicit extrinsic parametrization of the geodesics which we developed in Chapter 1. On the cylinder the geodesics are the helixes and generators and in each case (except for the vertical generators which have no extrinsic curvature) we have already calculated (in the solution to Problem **2.5**) that the curvature vector point in a direction which is perpendicular to the surface. On the cone one can also use the extrinsic parametrization of geodesics on the cone given in the solution to Problem **1.4.e**, but this is an algebraically messy computation. Alternatively, we can argue geometrically that for any geodesic on the cone or cylinder if there were a component of the extrinsic curvature in the plane tangent to the surface then there would be an intrinsic experience of curving.

b.

If the extrinsic curvature did not point towards the center of the sphere then the osculating circle would not lie in a plane through the center of the sphere and thus the osculating plane will intersect the sphere in a non-great circle and we would intrinsically experience it as curving.

c.

Now, for a curve on a sphere which is not planar, since the osculating plane pivots around the tangent line, it is not possible (by Problem **2.6a**) for two nearby osculating planes to both contain the center of the sphere and, thus, both points cannot have extrinsic curvature vectors that point towards the center of the sphere. It follows from Problem **3.2.b** that any geodesic on the sphere must lie in a plane and again **3.2.b** implies that the plane must contain the center of the sphere. Since the intersection of such a plane with the sphere is a great circle it follows that the only geodesics on the sphere are arcs of great circles.

Problem 3.3. Intrinsic Curvature - Curves on Sphere

a.

Locally and intrinsically, we can do on the cone and cylinder exactly what we did on the plane. In particular, when we subtract $\mathbf{T}(p + h) - \mathbf{T}(p - h)$ we must use the fact that the cone and the cylinder are locally isometric to the plane to move the two tangent vectors to parallel copies that can be subtracted. One way to do this is to do all constructions and parallel transporting in the covering space. Again, we must interpret the normal vector and the angle θ intrinsically or in the covering space. We can figure out what to do here intrinsically only because of the planar covering space. For a general surface we wish to do the same things but it will take us until Chapter 8 to accomplish.

b.

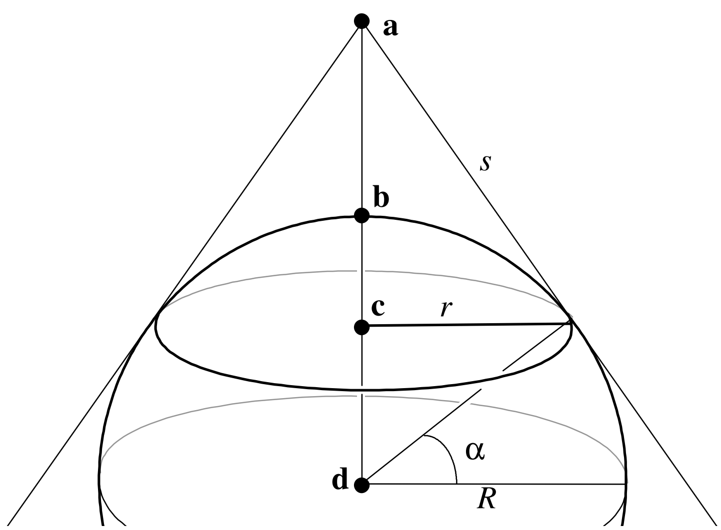

Figure 3.B. Circle on sphere with four centers.

Consider a cone that is tangent to the sphere with radius R along the latitude circle. We see from the similar triangles in Figure 3.B that $R\cos a = r =$ extrinsic radius of the latitude circle, and then $\sin a = \frac{r}{s} = \frac{R\cos a}{s}$ or $s = R\cot a$. Only the great circles have no intrinsic curvature.

Note that the latitude circle in Figure 3.B has four different centers: The ***extrinsic center*** (or ***center of extrinsic curvature***) is the point **c** in the plane of the latitude at the center of the circle. The ***intrinsic center*** is the point **b** which is the center of the circle with respect to the surface of the sphere — it is the center of the circle from the point of view of a 2-dimensional bug on the surface. Then there is the point **a** which is the intersection of all the planes tangent to the sphere along the latitude circle. The point **a** can be called the ***center of intrinsic curvature***. Then the point **d** is the center of the sphere and is the ***center of normal curvature*** for the latitude circle.

c.

For cylinders the extrinsic curvature is perpendicular to the surface and thus projects to zero. This is appropriate because the extrinsic circles on the cylinder are geodesics on the cylinder.

For cones and spheres we have the situation of Part **b** and Figure 3.B. We now redraw the relevant parts (Figure 3.3 of the text). We see that the projection is in the direction towards the intrinsic center of the circle on the cone, and we see that the projection has length $|\boldsymbol{\kappa}|\sin a = \frac{\sin a}{r} = \frac{\sin a}{R\cos a} = \frac{1}{R\cot a} = \frac{1}{s}$. Since s is the intrinsic radius of the circle on the cone, the projection must be the intrinsic curvature vector.

d.

The length of the projection onto the normal is $|\boldsymbol{\kappa}|\cos a = \frac{\cos a}{r} = \frac{\cos a}{R\cos a} = \frac{1}{R}$. The fact that we get $1/R$ for every latitude circle makes sense because this is the component of the curvature of the curve that is due to the surface and the sphere curves the same in all directions at every point.

PROBLEM 3.4. Geodesics on Surfaces — the Ribbon Test

a.

If the curve is extrinsically straight then its extrinsic curvature is zero and thus the projection onto the tangent plane is zero and there is no intrinsic curvature. Examples include generators of cones and cylinders, horizontal segments on the strake, the center line of the helicoid.

b.

i. When the ribbon is laid tangent on a plane the center line is (intrinsically and extrinsically) straight and, thus, the intrinsic curvature is zero.

ii. When the ribbon is isometrically embedded into another space then intrinsically there is no change to the ribbon and, thus, the intrinsic curvature $\boldsymbol{\kappa}_g$ of the centerline is still zero. Therefore, since $\boldsymbol{\kappa} = \boldsymbol{\kappa}_n + \boldsymbol{\kappa}_g$, we conclude that $\boldsymbol{\kappa} = \boldsymbol{\kappa}_n$ and, thus, that $\boldsymbol{\kappa}$ is perpendicular to the tangent space T_pR of the ribbon at each point p.

iii. When the (isometrically embedded) ribbon is tangent to a surface M along the centerline then, at every point p along the centerline, T_pR coincides with the tangent space of the surface T_pM. Thus, by **ii,** the curvature $\boldsymbol{\kappa}$ is perpendicular to T_pM and thus $\boldsymbol{\kappa}_g = 0$ on the surface and the centerline is a geodesic on the surface.

c.

Clearly, each curve with constant θ is the center line of ribbon tangent to the surface of revolution. The same is true for each generating circle, z = constant, at which $r'(z) = 0$, but note that in the case when this value of z is an inflection point (thus, not a local extrema), then the ribbon (though tangent) will lie partly inside and partly outside the surface.

d.

Clearly, each curve which runs radially along the annuli is the center line of a ribbon laid tangent to the surface in the extrinsic embedding produced in Problem **3.1.f** is a geodesic. In addition, since any extrinsically straight curve is a geodesic, we can find the geodesic joining any two points by holding the annular hyperbolic plane by these points and pulling until there is an extrinsically straight line between them.