APPENDIX I.

ON ALGEBRAIC CURVES IN SPACE.

404. GIVEN an algebraic curve (C) in space, let a point O be found, not on the curve, such that the number of chords of the curve that pass through O is finite; let the curve be projected from O on to any arbitrary plane, into the plane curve (f), and referred to homogeneous coordinates ξ, η, τ in that plane, whose triangle of reference has such a position that the curve does not pass through the angular point η, and has no multiple points on the line $\tau=0$; let the curve (C) be referred to homogeneous coordinates ξ, η, ζ, τ of which the vertex ζ of the tetrahedron of reference is at O. Putting $x=\xi/\tau$, $y=\eta/\tau$, $z=\zeta/\tau$, it is sufficient to think of x, y, z as Cartesian coordinates, the point O being at infinity. Thus the plane curve (f) is such that y is not infinite for any finite value of x, and its equation is of the form $f(y, x)=y^m+A_1y^{m-1}+\ldots\ldots+A_m=0$, where $A_1, \ldots, A_m$ are integral polynomials in x; the curve (C) is then of order m; we define its deficiency to be the deficiency of (f); to any point (x, y) of (f) corresponds in general only one point (x, y, z) of (C), and, on the curve (C), z is not infinite for any finite values of x, y.

Now let $f'(y)=\partial f(y, x)/\partial y$; let ϕ be an integral polynomial in x and y, so chosen that at every finite point of (f) at which $f'(y)=0$, say at $x=a$, $y=b$, the ratio $(x-a)\phi/f'(y)$ vanishes to the first order at least; let $\alpha=\Pi(x-a)$ contain a simple factor corresponding to every finite value of x for which $f'(y)=0$; let $y_1, \ldots, y_m$ be the values of y which, on the curve (f), belong to a general value of x, so that to each pair (x, y_i) there belongs, on the curve (C), only one value of z; considering the summation

$$\sum_{i=1}^{m}\frac{(c-y_1)\ldots\ldots(c-y_m)}{c-y_i}\,\alpha\left[\frac{z\phi}{f'(y)}\right]_{y=y_i},$$

where c is an arbitrary quantity, we immediately prove, as in § 89, Chap. VI., that it has a value of the form

$$\alpha\,(c^{m-1}u_1+c^{m-2}u_2+\ldots\ldots+u_m),$$

where $u_1, \ldots, u_m$ are integral polynomials in x; putting y_i for c, after division by α, we therefore infer that z can be represented in the form

$$z=\psi/\phi,$$

where ϕ, ψ are integral polynomials in x and y, whereof ϕ is arbitrary, save for the conditions for the fractions $(x-a)\phi/f'(y)$. This is Cayley's monoidal expression of a curve in space with the adjunction of the theorem, described by Cayley as the capital theorem of Halphen, relating to the arbitrariness of ϕ (Cayley, *Collect. Works*, Vol. v. 1892, p. 614).

It appears therefore that a curve in space may be regarded as arising as an interpretation of the relations connecting three rational functions on a Riemann surface; and, within a finite neighbourhood of any point of the curve in space, the coordinates of the points of the curve may be given by series of integral powers of a single quantity t, this being the quantity we have called the infinitesimal for a Riemann surface; to represent the whole curve only a finite number of different infinitesimals is necessary. More generally the representation by means of automorphic functions holds equally well for curves in space. And the theory of Abelian integrals can be developed for a curve in space precisely as for a plane curve, or can be deduced from the latter case; the identity of the deficiency for the curve in space and the plane curve may be regarded as a corollary. Also we can deduce the theorem that, of the intersections with a curve in space of a variable surface, not all can be arbitrarily assigned, the number of those whose positions are determined by the others being, for a surface of sufficiently high order, equal to the deficiency of the curve.

Ex. If through $p-1$ of the generators of a quadric surface, of the same system, a surface of order $p+1$ be drawn, the remaining curve of intersection is representable by two equations of the form

$$y^2=(x,\ 1)_{2p+2},\quad zu_1=u_2,$$

where $(x,\ 1)_{2p+2}$ is an integral polynomial in x of order $2p+2$, and u_1, u_2 are respectively linear and quadric polynomials in x and y.

For the development of the theory consult, especially, Noether, *Abh. der Akad. zu Berlin vom Jahre* 1882, pp. 1 to 120; Halphen, *Journ. École Polyt.*, Cah. LII. (1882), pp. 1—200; Valentiner, *Acta Math.*, t. II. (1883), pp. 136—230. See also, Schubert, *Math. Annal.* XXVI. (1885); Castelnuovo, *Rendiconti della R. Accad. dei Lincei*, 1889; Hilbert, *Math. Annal.*, XXXVI. (1890).