CHAPTER XXII.

DEGENERATE ABELIAN INTEGRALS.

395. THE present chapter contains references to parts of the existing literature dealing with an interesting application of the theory of transformation of theta functions.

It was remarked by Jacobi* for the case $p=2$, that if the fundamental algebraic equation be of the form

$$y^2 = x(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda),$$

an hyperelliptic integral of the first kind is reducible to elliptic integrals; in fact, putting $\xi = x + \kappa\lambda/x$, we immediately verify that

$$\frac{(x \pm \sqrt{\kappa\lambda})\,dx}{\sqrt{x(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda)}} = \frac{d\xi}{\sqrt{(\xi \mp 2\sqrt{\kappa\lambda})(\xi - 1 - \kappa\lambda)(\xi - \kappa - \lambda)}}.$$

396. Suppose more generally that for any value of p there exists an integral of the first kind

$$U = \lambda_1 u_1 + \ldots\ldots + \lambda_p u_p,$$

wherein $u_1, \ldots, u_p$ denote the normal integrals of the first kind, which is reducible to the form

$$\int \frac{d\xi}{\sqrt{R(\xi)}},$$

$R(\xi)$ being a cubic polynomial in ξ, such that ξ and $\sqrt{R(\xi)}$ are rational functions on the original Riemann surface; then there exist p pairs of equations of the form

$$\lambda_i = b_i'\Omega - a_i'\Omega', \quad \lambda_1\tau_{i,1} + \ldots\ldots + \lambda_p\tau_{i,p} = -b_i\Omega + a_i\Omega', \qquad (i = 1, \ldots, p),$$

wherein a_i, b_i, a_i', b_i' are integers; we may suppose Ω' to be chosen so that the $2p$ integers

$$a_1, \ldots, a_p, a_1', \ldots, a_p'$$

have no common factor and so that

$$a_1b_1' + a_2b_2' + \ldots\ldots + a_pb_p' - a_1'b_1 - a_2'b_2 - \ldots\ldots - a_p'b_p = r,$$

* *Crelle*, VIII. (1832), p. 416.

where r is a *positive* integer; we assume that r is not zero. Eliminating the quantities $\lambda_1, \ldots, \lambda_p$, and putting $\omega = \Omega'/\Omega$, we have the p equations

$$b_i + b_1' \tau_{i,1} + \ldots + b_p' \tau_{i,p} = \omega (a_i + a_1' \tau_{i,1} + \ldots + a_p' \tau_{i,p}), \qquad (i = 1, \ldots, p);$$

if therefore the matrix of integers, $\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$, of $2p$ rows and columns, wherein the first column consists of the integers $a_1, \ldots, a_p'$ in order, and the $(p+1)$th column consists of the integers $b_1, \ldots, b_p'$ in order, be determined to satisfy the conditions for a transformation of order r,

$$\bar{\alpha}\alpha' = \bar{\alpha}'\alpha, \quad \bar{\beta}\beta' = \bar{\beta}'\beta, \quad \bar{\alpha}\beta' - \bar{\alpha}'\beta = r,$$

(§ 420, Appendix II.), then it immediately follows from the equation for the transformed period matrix τ', namely

$$(\alpha + \tau\alpha')\tau' = \beta + \tau\beta',$$

that $\tau'_{11} = \omega$, $\tau'_{12} = 0, \ldots, \tau'_{1p} = 0$; to see this it is sufficient to compare the elements of the first columns of the two matrices $\beta + \tau\beta'$, $(\alpha + \tau\alpha')\tau'$. In other words, when there exists such a degenerate integral of the first kind as here supposed, it is possible*, by a transformation of order r, to arrive at periods τ' for which the theta function $\vartheta(w, \tau' \mid q)$ is a product of an elliptic theta function, in the variable w_1, and a theta function of $(p-1)$ variables, $w_2, \ldots, w_p$.

397. It can however be shewn that in the same case it is possible by a *linear* transformation to arrive at a period matrix τ'' for which

$$\tau''_{13} = 0, \; \tau''_{14} = 0, \; \ldots, \; \tau''_{1p} = 0,$$

while τ''_{12}, $= 1/r$, is a rational number. We shall suppose† two rows x, x', each of p integers, to be determined satisfying the equations

$$a x' - a' x = 1, \quad b x' - b' x = 0,$$

such that the $2p$ elements of $rx - b$, $rx' - b'$ have unity as their greatest common factor, a denoting the row $a_1, \ldots, a_p$, etc., and suppose (§ 420) a matrix of integers, of $2p$ rows and columns,

$$\Delta = \begin{pmatrix} \gamma & \delta \\ \gamma' & \delta' \end{pmatrix} = \left(\begin{array}{l|l} a, \; rx - b, \; \ldots & x, \; \ldots \\ a', \; rx' - b', \; \ldots & x', \; \ldots \end{array}\right)$$

to be determined, satisfying the conditions for a linear transformation,

$$\bar{\gamma}\gamma' = \bar{\gamma}'\gamma, \quad \bar{\delta}\delta' = \bar{\delta}'\delta, \quad \bar{\gamma}\delta' - \bar{\gamma}'\delta = 1,$$

wherein the first column consists of the elements of a and a', the second column consists of the elements of $rx - b$ and $rx' - b'$, and the $(p+1)$th

* This theorem is due to Weierstrass, see Königsberger, *Crelle*, LXVII. (1867), p. 73; Kowalevski, *Acta Math.* IV. (1884), p. 395. See also Abel, *Œuvres*, t. I. (1881), p. 519.

† The proof that this is possible is given in Appendix II., § 419. It may be necessary, beforehand, to make a linear transformation of the periods Ω, Ω'.

column consists of the elements of x and x'; the conditions for a linear transformation, so far as they affect these three columns only, are

$$a(rx'-b')-a'(rx-b)=0,\ ax'-a'x=1,\ (rx-b)x'-(rx'-b')x=0,$$

and these are satisfied in virtue of the equation $ab'-a'b=r$. Then the equation for the transformed period matrix τ'', namely

$$(\gamma+\tau\gamma')\tau''=\delta+\tau\delta',$$

leads to $\tau''_{3,1}=0, \dots, \tau''_{p,1}=0$ if only the p equations

$$[\gamma_{i,1}+(\tau\gamma')_{i,1}]\tau''_{1,1}+[\gamma_{i,2}+(\tau\gamma')_{i,2}]\tau''_{2,1}=\delta_{i,1}+(\tau\delta')_{i,1}, \qquad (i=1, \dots, p),$$

which are obtained by equating corresponding elements of the first columns of the matrices $\delta+\tau\delta'$, $(\gamma+\tau\gamma')\tau''$, are satisfied; these p equations are included in the single equation

$$\tau''_{1,1}[a+\tau a']+\tau''_{2,1}[rx-b+\tau(rx'-b')]=x+\tau x',$$

and are satisfied* by $\tau''_{1,1}=\omega/r$, $\tau''_{2,1}=1/r$; for we have, as the fundamental condition, the equation

$$\omega(a+\tau a')=b+\tau b'.$$

398. It follows therefore in case $p=2$ that the matrix τ'' has the form

$$\begin{pmatrix}\tau''_{11}, & 1/r\\ 1/r, & \tau''_{22}\end{pmatrix};$$

hence it immediately follows that beside the integral of the first kind already considered, which is expressible as an elliptic integral, there is another having the same property. In virtue of the equations here obtained the first integral having this property can be represented, after division by Ω, in the form

$$U=(b'-r\tau''_{1,1}a')u,$$

where u denotes the row of 2 integrals u_1, u_2; consider now the integral

$$V=[rt'-a'-r\tau''_{2,2}(rx'-b')]u,$$

where t' is a row of two elements, these being the constituents of the first column of the matrix δ'; the periods of V at the first set of period loops are given by the row of quantities

$$rt'-a'-r\tau''_{2,2}(rx'-b'),$$

* See Kowalevski, *Acta Math.* IV. (1884), p. 400; Picard, *Bulletin de la Soc. Math. de France*, t. XI. (1882—3), p. 25, and *Compt. Rendus*, XCII. XCIII. (1881); Poincaré, *Bulletin de la Soc. Math. de France*, t. XII. (1883—4), p. 124; Poincaré, *American Journal*, vol. VIII. (1886), p. 289.

and are linear functions of the two quantities 1, $r\tau''_{2,2}$; the periods of V at the second set of period loops are given by

$$[\tau(rt'-a')]_i - r\tau''_{2,2}[\tau(rx'-b')]_i, \qquad (i=1, 2);$$

now the equation $(\gamma+\tau\gamma')\tau''=\delta+\tau\delta'$ gives

$$(\gamma+\tau\gamma')_{i,1}\tau''_{1,2}+(\gamma+\tau\gamma')_{i,2}\tau''_{2,2}=(\delta+\tau\delta')_{i,2}, \qquad (i=1, 2),$$

and hence we have

$$\tau''_{1,2}[a+\tau a']+\tau''_{2,2}[rx-b+\tau(rx'-b')]=t+\tau t',$$

where t is the row formed by the constituents of the first column of the matrix δ; therefore, as $\tau''_{1,2}=1/r$, the periods of V at the second set of period loops are expressible in the form

$$-(rt-a)_i+r\tau''_{2,2}(rx-b)_i, \qquad (i=1, 2),$$

and these are also linear functions of the two quantities 1, $r\tau''_{2,2}$. Hence it may be inferred that the integral V is reducible to an elliptic integral.

399. It has been shewn in the last chapter that for special values of the periods τ there exist transformations of the theta functions into theta functions for which the transformed periods are equal to the original periods. It can be shewn* that for the special case now under consideration such a transformation holds. Suppose that a theta function ϑ, with period τ, is transformed, as described above, into a theta function ϕ, with period τ', for which $\tau'_{1,2}=0=\ldots=\tau'_{1,p}$, by a transformation associated with the matrix $\Delta=\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}$; suppose further that there exists, associated with a matrix $H=\begin{pmatrix}\lambda & \mu\\ \lambda' & \mu'\end{pmatrix}$, a transformation whereby the theta function ϕ is transformed into another theta function with the same period τ'; then it is easy to prove that there exists a corresponding transformation of the theta function ϑ whereby it becomes changed into a theta function with the same period τ, namely the transformation is that associated with the matrix

$$\begin{pmatrix}f & g\\ f' & g'\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}\begin{pmatrix}\lambda & \mu\\ \lambda' & \mu'\end{pmatrix}\begin{pmatrix}\bar\beta' & -\bar\beta\\ -\bar\alpha' & \bar\alpha\end{pmatrix};$$

to prove this it is only necessary to shew that the equations

$$(\lambda+\tau'\lambda')\tau'=\mu+\tau'\mu', \quad (\alpha+\tau\alpha')\tau'=\beta+\tau\beta'$$

give the equation

$$(f+\tau f')\tau=g+\tau g'.$$

* Wiltheiss, *Math. Annal.* XXVI. (1886), p. 127.

Hence it follows that in order to determine a transformation of the function ϑ which leaves the period τ unaltered, it is sufficient to determine a transformation of the function ϕ which leaves the period τ' unaltered; this determination is facilitated by the special values of $\tau'_{1,2}, \ldots, \tau'_{1,p}$; and in fact we immediately verify that the equation $(\lambda + \tau'\lambda')\tau' = \mu + \tau'\mu'$ is satisfied by taking $\lambda' = \mu = 0$ and by taking each of λ and μ' to be the matrix in which every element is zero except the elements in the diagonal, each of these elements being 1 except the first, which is -1.

400. Thus for the case $p = 2$, supposing $r = 2$, the original function ϑ is transformed into a theta function with unaltered period τ, by means of the transformation of order 4 associated with the matrix,

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \bar{\beta}', & -\bar{\beta} \\ -\bar{\alpha}', & \bar{\alpha} \end{pmatrix}, \quad = \Delta M \nabla \text{ say,}$$

where m denotes the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; the matrix ∇ is equal to $2\Delta^{-1}$, and it is easy to see that this transformation of order 4 is equivalent to a multiplication, with multiplier 2, together with a *linear* transformation associated with the matrix

$$\Delta M (\tfrac{1}{2}\nabla).$$

We have therefore the result; when, in case $p = 2$, there exists a transformation of the second order whereby the periods τ are changed into periods τ' for which $\tau'_{1,2} = 0$, then there exists a linear transformation whereby the periods τ are changed into the same periods τ, or what we have called in the last chapter a complex multiplication.

401. The transcendental results thus obtained enable us to specify the algebraic conditions for the existence of an integral of the first kind which is reducible to an elliptic integral.

Thus for instance when $p = 2$, to determine all the cases in which an integral of the first kind can be reduced to an elliptic integral by means of a transformation of the second order, $\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$, it is sufficient to consider the conditions that the transformed even theta function $\vartheta\left(w;\ \tau' \,\middle|\, \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$ may vanish for zero values of w; for when $\tau'_{1,2} = 0$ this function breaks up into the product of two odd elliptic theta functions. By means of the formulae* for transformation of the second order, it can be shewn† that this condition leads to the equation

$$-\vartheta_2^2\vartheta_{01}^2\vartheta_{34}^2 + \vartheta_{14}^2\vartheta_0^2\vartheta_{23}^2 = 0,$$

* Chap. XX. § 364.

† Königsberger, *Crelle*, LXVII. (1867), p. 77.

and by means of the relations expressing the constants of the fundamental algebraic equation in terms of the zero values of the even theta functions* it can be shewn that this is equivalent to the condition that the fundamental algebraic equation may be taken to be of the form

$$y^2 = x(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda),$$

so that the case obtained by Jacobi is the only one possible for transformations of the second order.

In the same case of $p=2$, $r=2$, the same result follows more easily from the existence, deduced above, of a complex multiplication belonging to a transformation of the first order. For it follows from this fact that the algebraic equation can be taken in a form in which it can be transformed into itself by a transformation in which the independent variable is transformed by an equation of the form

$$x = \frac{A\xi + B}{C\xi - A},$$

and this leads† to the form, for the fundamental algebraical equation,

$$s^2 = (z^2 - a^2)(z^2 - b^2)(z^2 - c^2),$$

which is immediately identified with the form above by putting

$$x = \sqrt{\kappa\lambda}\,(z+1)/(z-1),$$

the quantities a, b, c being respectively

$$1, \quad (\sqrt{\kappa\lambda}+1)^2/(\sqrt{\kappa\lambda}-1)^2, \quad (\sqrt{\kappa}+\sqrt{\lambda})^2/(\sqrt{\kappa}-\sqrt{\lambda})^2.$$

Similarly for $p=3$, when the surface is not hyperelliptic, it can be shewn‡ from the relations connecting the theta functions when a theta function is the product of an elliptic theta function and a theta function of two variables, that the only cases in which an integral of the first kind can be reduced to an elliptic integral are those in which the fundamental algebraic equation can be taken to be of the form

$$\sqrt{x(Ax+By)} + \sqrt{y(Cx+Dy)} + \sqrt{1+Fx+Gy} = 0.$$

The Riemann surface associated with this equation possesses a (1, 1) correspondence given by the equations

$$\xi = -x/(1+Fx+Gy), \quad \eta = -y/(1+Fx+Gy).$$

* Cf. Ex. v. p. 341. By means of the substitution $x = c_1 + (a_1 - c_1)\xi$, the branch places can be taken at $\xi = 0, 1, \kappa, \lambda, \mu$, wherein, if c_1, a_1, c_2, a_2, c be real and in ascending order, $0, 1, \kappa, \lambda, \mu$ are in ascending order of magnitude. For complete formulae, when the theta functions are regarded as primary, and the algebraic equation as derived, see Rosenhain, *Mém. p. divers Savants*, XI. (1851), p. 416 ff.

† Wiltheiss, *Math. Annal.* XXVI. (1886), p. 134.

‡ Kowalevski, *Acta Math.* IV. (1884), p. 403.

402. But the problem of determining the algebraic equations for which an associated integral of the first kind reduces to an elliptic integral may be considered algebraically, by beginning with an elliptic integral and transforming it into an Abelian integral. The reader may consult Richelot, *Crelle*, XVI. (1837); Malet, *Crelle*, LXXVI. (1873), p. 97; Brioschi, *Compt. Rendus*, LXXXV. (1877), p. 708; Goursat, *Bulletin de la Soc. Math. de France*, t. XIII. (1885), p. 143, and *Compt. Rendus*, C. (1885), p. 622; Burnside, *Proc. Lond. Math. Soc.* vol. XXIII. (1892), p. 173.

403. The paper of Königsberger already referred to (*Crelle*, LXVII.) deals with the case of a transformation of the second order, for $p=2$. For the case of a transformation of the third order, when $p=2$, consult, beside the papers of Goursat (*loc. cit.* § 402), also Hermite, *Ann. de la Soc. Scient. de Bruxelles*, 1876, and Burkhardt, *Math. Annal.* XXXVI. (1890), p. 410. For the case $p=2$, and a transformation of the fourth order, see Bolza, *Ueber die Reduction hyperelliptischer Integrale u. s. w.*, Götting. Dissertation (Berlin, Schade, 1885), or *Sitzungsber. der Naturforsch. Ges. zu Freiburg* (1885). The paper of Kowalevski (*Acta Math.* IV.) deals with the case of a transformation of the second order for $p=3$. See further the references given in this chapter, and Poincaré, *Compt. Rendus*, t. XCIX. (1884), p. 853; Biermann, *Sitzungsber. der Wiener Akad.* Bd. LXXXVII. (ii. Abth.) (1883), p. 983.