CHAPTER XXI.

Complex Multiplication of Theta Functions. Correspondence of Points on a Riemann Surface.

376. In the present chapter some account is given of two theories; the former is a particular case of the theory of transformation of theta functions; the latter is intimately related with the theory of transformation of Riemann theta functions. Not much more of the results of these theories is given than is necessary to classify the references which are given to the literature.

377. In the transformation of the function $\Theta(u; \tau)$, to a function of the arguments w, with period τ' (§ 324, Chap. XVIII.), the following equations have arisen

$$u = Mw, \quad M = \alpha + \tau\alpha', \quad M\tau' = \beta + \tau\beta';$$

there* are cases, for special values of τ, in which τ' is equal to τ. We investigate necessary conditions for this to be so; and we prove, under a certain hypothesis, that they are sufficient. The results are stated in terms of the matrix of integers associated with the transformation; we do not enter into the investigation of the values of τ to which the results lead. We limit ourselves throughout to the function $\Theta(u; \tau)$; the change to the function $\vartheta(u; 2\omega, 2\omega', 2\eta, 2\eta')$ can easily be made.

Suppose that, corresponding to a matrix $\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$, of $2p$ rows and columns, for which

$$\alpha\bar{\beta} = \beta\bar{\alpha}, \quad \alpha'\bar{\beta}' = \beta'\bar{\alpha}', \quad \alpha\bar{\beta}' - \beta\bar{\alpha}' = r = \beta'\bar{\alpha} - \alpha'\bar{\beta},$$

where r is a positive integer, there exists a matrix τ satisfying the equation

$$(\alpha + \tau\alpha')\,\tau = \beta + \tau\beta',$$

which is such that, for real values of $n_1, \ldots, n_p$, the imaginary part of the quadratic form τn^2 is positive.

* References to the literature for the case $p=1$ are given below (§ 383). For higher values of p, see Kronecker, *Berlin. Monatsber.* 1866, p. 597, or *Werke*, Bd. I. (Leipzig, 1895), p. 146; Weber, *Ann. d. Mat.*, Ser. 2, t. IX. (1878—9), p. 140; Frobenius, *Crelle*, XCV. (1883), p. 281, where other references are given; Wiltheiss, *Bestimmung Abelscher Funktionen mit zwei Argumenten* u. s. w. Habilitationsschrift, Halle, 1881 (E. Karras), and *Math. Annal.* XXVI. (1886), p. 130.

In that case, as follows from Chap. XX., the function $\Theta\,[(\alpha+\tau\alpha')\,w\,;\,\tau]$, when multiplied by a certain exponential of the form $e^{\gamma w^2}$, is expressible as an integral polynomial of the r-th degree in 2^p functions $\Theta\,[w\,;\,\tau]$; on this account we say that there exists a *complex multiplication**, or a *special transformation*, belonging to the matrix Δ. The equation $(\alpha+\tau\alpha')\,\tau=\beta+\tau\beta'$ is equivalent to $(\bar{\beta}'-\tau\bar{\alpha}')\,\tau=-\bar{\beta}+\tau\bar{\alpha}$; this arises from the supplementary matrix

$$r\Delta^{-1}=\begin{pmatrix}\bar{\beta}' & -\bar{\beta}\\ -\bar{\alpha}' & \bar{\alpha}\end{pmatrix},$$

just as the former equation arises from Δ; we put $M=\alpha+\tau\alpha'$, $N=\bar{\beta}'-\tau\bar{\alpha}'$; we denote by $|\,\Delta-\lambda\,|$ the determinant of the matrix $\Delta-\lambda E$, where E is the matrix unity of $2p$ rows and columns, and λ is a single quantity; similarly we denote by $|\,M-\lambda\,|$ the determinant of the matrix $M-\lambda E'$, where E' is the matrix unity of p rows and columns.

Then we prove first, that *when there exists such a complex multiplication, to every root of the equation in λ of order p given by $|\,M-\lambda\,|=0$, there corresponds a conjugate complex root of the equation $|\,N-\lambda\,|=0$; that the $2p$ roots of the equation $|\,\Delta-\lambda\,|=0$ are constituted by the roots of the two equations $|\,M-\lambda\,|=0$, $|\,N-\lambda\,|=0$, or $|\,\Delta-\lambda\,|=|\,M-\lambda\,|\;|\,N-\lambda\,|$; and that all these roots are of modulus $\sqrt{r}$. Hence when $r=1$, they can be shewn to be all roots of unity.*

378. The equations of the general transformation, of order r, and its supplementary transformation, namely

$$M=a+\tau a',\quad M\tau'=\beta+\tau\beta',\quad N=\bar{\beta}'-\tau'\bar{a}',\quad N\tau=-\bar{\beta}+\tau'\bar{a},$$

give

$$(a+\tau a')\,\tau'=\beta+\tau\beta'\,;$$

hence, if $\tau=\tau_1+i\tau_2$, where τ_1 and τ_2 are matrices of real quantities, and similarly $\tau'=\tau_1'+i\tau_2'$, we have by equating imaginary parts

$$(a+\tau_1 a')\,\tau_2'=\tau_2\,(\beta'-a'\tau_1')\,;$$

therefore the two matrices

$$M\tau_2'=(a+\tau_1 a')\,\tau_2'+i\tau_2 a'\tau_2',\quad \tau_2\bar{N}=\tau_2\,(\beta'-a'\tau_1')-i\tau_2 a'\tau_2'$$

are conjugate imaginaries, $=f+ig$ and $f-ig$, say.

Now suppose $\tau'=\tau$; then from

$$M\tau_2=f+ig,\quad \tau_2\bar{N}=f-ig,$$

we have, if λ be any single quantity, and M_0 be the matrix whose elements are the conjugate complexes of the elements of M,

$$(M_0-\lambda)\,\tau_2=f-ig-\lambda\tau_2=\tau_2\,(\bar{N}-\lambda),$$

and hence, as $|\,\tau_2\,|$ is not zero,

$$|\,M_0-\lambda\,|=|\,\bar{N}-\lambda\,|,$$

* The name *principale Transformation* has been used (Frobenius, *Crelle*, xcv.).

which shews that to any root of the equation $|M-\lambda|=0$ there corresponds a conjugate complex root of the equation $|N-\lambda|=0$. Further we have, if $\tau_0=\tau_1-i\tau_2$,

$$\begin{pmatrix}1 & \tau\\ 1 & \tau_0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}=\begin{pmatrix}M & M\tau\\ M_0 & M_0\tau_0\end{pmatrix}=\begin{pmatrix}M & 0\\ 0 & M_0\end{pmatrix}\begin{pmatrix}1 & \tau\\ 1 & \tau_0\end{pmatrix},$$

and writing this equation in the form

$$t\Delta=\mu t,$$

where

$$t=\begin{pmatrix}1 & \tau\\ 1 & \tau_0\end{pmatrix},\quad \mu=\begin{pmatrix}M & 0\\ 0 & M_0\end{pmatrix},$$

it easily follows that the determinant of the matrix t is not zero, and that, if λ be any single quantity, we have

$$t(\Delta-\lambda)=(\mu-\lambda)t,$$

so that

$$|\Delta-\lambda|=|\mu-\lambda|=|M-\lambda|\,|M_0-\lambda|=|M-\lambda|\,|N-\lambda|.$$

Thus the roots of the equation $|\Delta-\lambda|=0$ are constituted by the roots of the equations

$$|M-\lambda|=0,\quad |N-\lambda|=0.$$

Further, from a result previously obtained (Chap. XVIII., § 325, Ex.), when, as here, $\tau'=\tau$ and $2\omega=1$, $2v=1$, we have

$$M_0\tau_2\overline{M}=r\tau_2 \text{ or } M\tau_2\overline{M}_0=r\tau_2;$$

also as, for real values of $n_1, \ldots, n_p$, the form $\tau_2 n^2$ is a positive form, it can be put into the shape $m_1^2+\ldots\ldots+m_p^2$, $=Em^2$, say, E being the matrix unity of p rows and columns, and m being a row of quantities given by $m=Sn$, where S is a matrix of real elements; the equation $\tau_2n^2=E.Sn.Sn$ gives $\tau_2=\overline{S}ES=\overline{S}S$; for distinctness we shall write

$$\tau_2=\overline{K}K_0,$$

$K=K_0=S$ being conjugate complex matrices. Take now a matrix $R=K\overline{M}K^{-1}$; then

$$\overline{R}R_0=\overline{K}^{-1}M\overline{K}K_0\overline{M}_0K_0^{-1}=\overline{K}^{-1}M\tau_2\overline{M}_0K_0^{-1}=r\overline{K}^{-1}\tau_2K_0^{-1}=r;$$

thus if λ be a root of $|M-\lambda|=0$, and therefore, as $R-\lambda=K(\overline{M}-\lambda)K^{-1}$, also a root of $|R-\lambda|=0$, and if z, $=x+iy$, be a row of p quantities such that $Rz=\lambda z=E\lambda z$, where E is the matrix unity of p rows and columns, we have

$$\overline{R}R_0z_0z=R_0z_0\,.\,Rz=E\lambda_0z_0\,.\,E\lambda z=\lambda\lambda_0\,.\,Ez_0z$$

or

$$(\lambda\lambda_0-r)\,Ez_0z=0.$$

Therefore as Ez_0z, which is equal to $\sum_{m=1}^{p}(x_m^2+y_m^2)$, is not zero, it follows that $\lambda\lambda_0=r$; in other words, all the roots of the equations $|M-\lambda|=0$, $|\Delta-\lambda|=0$, are of modulus $\sqrt{r}$.

Suppose now that $r=1$, so that the roots of the equation $|\Delta-\lambda|=0$ are all of modulus unity; then we prove for an equation

$$x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+N=0,$$

of any order, wherein the coefficients $A, B, \ldots, N$ are rational integers, and the coefficient of the highest power of x is unity, that if all the roots be of modulus unity, they are also roots of unity*; so that, for instance, there is no root of the form $e^{i\sqrt{2}}$.

* Kronecker, *Crelle*, LIII. (1857), p. 173; *Werke*, Bd. I. (1895), p. 103.

Let the roots be $e^{i\alpha}$, $e^{i\beta}$, ..., so that

$$A = -(\cos\alpha + \cos\beta + \ldots), \quad B = \cos(\alpha+\beta) + \cos(\alpha+\gamma) + \ldots, \ \ldots;$$

then A lies between $-n$ and n, and B lies between $\pm\frac{1}{2}n(n-1)$, etc.; hence there can only be a finite number, say k, of equations of the above form, whereof all the roots are roots of unity. Thus, if $x_1, \ldots, x_n$ be the roots of our equation, so that, for any positive integer μ, the roots of the equation

$$F_\mu(x) = (x - x_1^\mu)(x - x_2^\mu)\ldots(x - x_n^\mu) = 0,$$

are also roots of unity, it follows that, of the equations

$$F_1(x) = 0, \quad F_2(x) = 0, \ \ldots, \quad F_{k+1}(x) = 0,$$

there must be two at least which are identical. Hence, supposing $F_\mu(x) = 0$, $F_\nu(x) = 0$ to be identical, we have n equations of the form

$$x_1^\mu = x_{r_1}^\nu, \quad x_2^\mu = x_{r_2}^\nu, \ \ldots.$$

Choosing from these equations the cycle given by

$$x_1^\mu = x_{r_1}^\nu, \quad x_{r_1}^\mu = x_{s_1}^\nu, \ \ldots, \quad x_{m_1}^\mu = x_1^\nu,$$

consisting, suppose, of σ equations, we infer that

$$x_1^{\mu^\sigma} = x_1^{\nu^\sigma},$$

and, hence, that x_1 is a $(\mu^\sigma - \nu^\sigma)$-th root of unity.

Ex. Prove that, when $M = \alpha + \tau\alpha'$, $M\tau' = \beta + \tau\beta'$,

$$\begin{pmatrix} 1 & \tau \\ 1 & \tau_0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M_0 \end{pmatrix}\begin{pmatrix} 1 & \tau' \\ 1 & \tau_0' \end{pmatrix};$$

and deduce*, if $\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$ and

$$H = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ \tau_0 & \tau \end{pmatrix}\begin{pmatrix} \tau_2^{-1} & 0 \\ 0 & \tau_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & \tau \\ 1 & \tau_0 \end{pmatrix} = \begin{pmatrix} \tau_2^{-1} & \tau_2^{-1}\tau_1 \\ \tau_1\tau_2^{-1} & \tau_1\tau_2^{-1}\tau_1 + \tau_2 \end{pmatrix},$$

that

$$\bar\Delta H \Delta = rH'.$$

Hence, when $\tau' = \tau$, if z be a row of $2p$ elements, and $x = \Delta z$, we have

$$Hx^2 = rHz^2,$$

which expresses a self-transformation of the quadratic form Hz^2, which has real coefficients. Cf. Hermite, *Compt. Rendus*, XL. (1855), p. 785; Laguerre, *Journ. de l'éc. pol.*, t. XXV., cah. XLII. (1867), p. 215; Frobenius, *Crelle*, XCV. (1883), p. 285.

379. Conversely, let

$$\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$$

be a matrix of integers of $2p$ rows and columns, such that

$$\bar\alpha\alpha' = \bar\alpha'\alpha, \quad \bar\beta\beta' = \bar\beta'\beta, \quad \bar\alpha\beta' - \bar\alpha'\beta = r = \bar\beta'\alpha - \bar\beta\alpha',$$

* Cf. Chap. XVIII. § 325, Ex.

where r is a positive integer; and suppose that the roots of the equation $|\Delta-\lambda|=0$ are all complex and of modulus $\sqrt{r}$. *Under the special hypothesis* that the roots of* $|\Delta-\lambda|=0$ *are all different, we prove now that a matrix* τ *can be determined such that* (i) τ *is a symmetrical matrix,* (ii) *for real values of* $n_1, \ldots, n_p$ *the imaginary part of the quadratic form* τn^2 *is positive,* (iii) *the equation*

$$(\alpha+\tau\alpha')\,\tau=\beta+\tau\beta'$$

is satisfied. Thus every such matrix Δ gives rise to a complex multiplication.

380. We utilise the following lemma, of which we give the proof at once.—If h be a matrix of n rows and columns, such that the determinant $|h+\lambda|$, wherein λ is a single quantity, vanishes to the first order when λ vanishes, and if x, y be rows of n quantities other than zero, such that

$$hx=0, \quad \bar{h}y=0,$$

then the quantity $xy, =x_1y_1+\ldots\ldots+x_ny_n$, is not zero.

Denoting the row x by ξ_1, its elements being $\xi_{11}, \ldots, \xi_{1n}$, determine other $n(n-1)$ quantities $\xi_{i,j}$ $(i=2, \ldots, n;\ j=1, \ldots, n)$ such that the determinant $|\xi|$ does not vanish; similarly, denoting y by η_1, determine $n(n-1)$ further quantities $\eta_{i,j}$ such that the determinant $|\eta|$ does not vanish. Then consider the determinant of the matrix $\eta(h+\lambda)\bar{\xi}$; the (r, s)-th element of this matrix is

$$\sum_i \eta_{r,i} \sum_j h_{i,j}\xi_{s,j}+\lambda\sum_i \eta_{r,i}\xi_{s,i}=\sum_j \xi_{s,j}\sum_i h_{i,j}\eta_{r,i}+\lambda\sum_i \eta_{r,i}\xi_{r,i},$$

$(i=1, \ldots, n;\ j=1, \ldots, n)$, and when $r=1$ we have

$$\sum_i h_{i,j}\eta_{r,i}=h_{1,j}\eta_{1,1}+\ldots\ldots+h_{n,j}\eta_{1,n}=(\bar{h}y)_j=0,$$

while when $s=1$, we have

$$\sum_j h_{i,j}\xi_{s,j}=h_{i,1}\xi_{1,1}+\ldots\ldots+h_{i,n}\xi_{1,n}=(hx)_i=0;$$

thus the (1, 1)-th element of this matrix is λxy, and every other element in the first row and column has the factor λ; thus the determinant of the matrix is of the form $\lambda\,[Axy+\lambda B]$. But the determinant of the matrix is equal to $|h+\lambda|\,|\xi|\,|\eta|$, and therefore by hypothesis vanishes only to the first order when λ vanishes. Thus xy is not zero.

381. Suppose now that $\lambda, \lambda_0, \mu, \mu_0, \ldots$ are the roots of the equation $|\Delta-\lambda|=0$, where λ and λ_0, and μ and μ_0, etc. are conjugate complexes. It is possible to find two rows x, x', each of p quantities, to satisfy the equations

$$\alpha x+\beta x'=\lambda x, \quad \alpha' x+\beta' x'=\lambda x', \text{ or, say, } (\Delta-\lambda)(x, x')=0, \qquad \text{(i)},$$

and similarly two rows z, z', each of p quantities, to satisfy the equations

$$\alpha z+\beta z'=\mu z, \quad \alpha' z+\beta' z'=\mu z', \qquad \text{(ii)};$$

from equations (i), if x_0 be the conjugate imaginary to x, etc., it follows, since $\lambda\lambda_0=r$, that

$$\alpha x_0+\beta x_0'=\frac{r}{\lambda}x_0, \quad \alpha' x_0+\beta' x_0'=\frac{r}{\lambda}x_0',$$

and hence, in virtue of the relations satisfied by the matrices $\alpha, \beta, \alpha', \beta'$, we have

$$\bar{\beta}' x_0-\bar{\beta} x_0'=\lambda x_0, \quad -\bar{\alpha}' x_0+\bar{\alpha} x_0'=\lambda x_0',$$

* For the general case, see Frobenius, *Crelle*, xcv. (1883).

which belong to the supplementary matrix $r\Delta^{-1}$ just as the equations (i) belong to the matrix Δ; for our purpose however they are more conveniently stated by saying that $t=x_0'$, $t'=-x_0$, satisfy the equations

$$(\bar{\Delta}-\lambda)(t, t')=0;$$

hence as x, x' satisfy the equations

$$(\Delta-\lambda)(x, x')=0,$$

it follows from the lemma just proved, putting $n=2p$, that $tx+t'x'$ is not zero; in other words the quantity

$$xx_0'-x'x_0$$

is not zero. Further from the equations (i), (ii) we infer

$$\lambda\mu\,(xz'-x'z)=(ax+\beta x')(a'z+\beta'z')-(a'x+\beta'x')(az+\beta z');$$

and by the equations satisfied by the matrices a, β, a', β' this is easily found to be the same as

$$(\lambda\mu-r)(xz'-x'z)=0;$$

thus, as the equation $\lambda\mu=r$ would be the same as $\lambda=\lambda_0$, we have

$$xz'-x'z=0.$$

Also we have

$$az_0+\beta z_0'=\mu_0 z_0, \quad a'z_0+\beta'z_0'=\mu_0 z_0';$$

thus we deduce, as in the case just taken, that

$$(\lambda\mu_0-r)(xz_0'-x'z_0)=0;$$

and hence as $\lambda\mu_0-r$, $=r(\lambda/\mu-1)$, is not zero, we have

$$xz_0'-x'z_0=0.$$

If we put $x=x_1+ix_2$, $x_0=x_1-ix_2$, $x'=x_1'+ix_2'$, $x_0'=x_1'-ix_2'$, the quantity

$$xx_0'-x'x_0=-2i\,(x_1x_2'-x_1'x_2)$$

is seen to be a pure imaginary; if in equations (i) λ be replaced by λ_0, the sign of $xx_0'-x'x_0$ is changed, but the quantity is otherwise unaltered; since then the equations (i) determine only the ratios of the constituents of the rows x, x', *we may suppose the sign of the imaginary part of* λ *in equations* (i), *and the resulting values of the constituents of* x *and* x', *to be so taken that*

$$xx_0'-x'x_0=-2i;$$

this we shall suppose to be done; and we shall suppose that the conditions for the $(p-1)$ similar equations, such as

$$zz_0'-z'z_0=-2i,$$

are also satisfied. With this convention, let the constituents of x and x' be denoted by

$$\xi_{1,1}, \ldots, \xi_{1,p}, \xi'_{1,1}, \ldots, \xi'_{1,p};$$

similarly let the constituents of the rows z, z', which are taken corresponding to the root μ, be denoted by

$$\xi_{2,1}, \ldots, \xi_{2,p}, \xi'_{2,1}, \ldots, \xi'_{2,p},$$

and so on for all the p roots $\lambda, \mu, \ldots$. Then the equations $xx_0'-x'x_0=-2i$, $zz_0'-z'z_0=-2i$, etc., are all expressed by the statement that the diagonal elements of the matrix

$$\xi\bar{\xi}_0'-\xi'\bar{\xi}_0$$

are each equal to $-2i$. When r is not equal to s $(r, s<p+1)$, the (1, 2)-th element of this matrix is

$$xz_0'-x'z_0,$$

which we have shewn to be zero; similarly every element of the matrix, other than a diagonal element, is zero; we may therefore write

$$\xi\bar{\xi}_0' - \xi'\bar{\xi}_0 = -2i.$$

Take now a row of p quantities, t, and define the rows X, X' by the equations

$$X = \bar{\xi}t, \quad X' = \bar{\xi}'t,$$

so that

$$X_0 = \bar{\xi}_0 t_0, \quad X_0' = \bar{\xi}_0' t_0;$$

then

$$-2it_0 t = \xi\bar{\xi}_0' t_0 t - \xi'\bar{\xi}_0 t_0 t = XX_0' - X'X_0;$$

hence it follows that the determinant of the matrix ξ' is not zero, since otherwise it would be possible to determine t, with constituents other than zero, so that $X'=0$, and therefore also $X_0'=0$; as this would involve $-2it_0t=0$, it is impossible.

382. If now the matrix τ be determined from the equations

$$x + \tau x' = 0, \quad z + \tau z' = 0, \ldots,$$

where x, x' are determined, as explained, from a proper value of λ, etc., or, what is the same thing, if τ be defined by

$$\xi + \xi'\tau = 0,$$

then

$$\xi\bar{\xi}' - \xi'\bar{\xi} = \xi'\tau\bar{\xi}' - \xi'\bar{\tau}\bar{\xi}' = \xi'(\tau - \bar{\tau})\bar{\xi}';$$

but the equations of the form $xz' - x'z = 0$ are equivalent to

$$\xi\bar{\xi}' - \xi'\bar{\xi} = 0;$$

now, since the determinant $|\xi'|$ does not vanish, a row of quantities t can be determined so that $X' = \bar{\xi}'t$, for an arbitrary value of X'; thus for this arbitrary value we have

$$(\tau - \bar{\tau}) X'^2 = 0,$$

and therefore

$$\tau = \bar{\tau},$$

or the matrix τ is symmetrical.

Further, from the equation $\xi + \xi'\tau = 0$, we have

$$\xi\bar{\xi}_0' - \xi'\bar{\xi}_0 = \xi'\tau_0\bar{\xi}_0' - \xi'\tau\bar{\xi}_0' = \xi'(\tau_0 - \tau)\bar{\xi}_0',$$

and hence, if $\tau = \rho + i\sigma$, since $\xi\bar{\xi}_0' - \xi'\bar{\xi}_0 = -2i$, we have

$$1 = \xi'\sigma\bar{\xi}_0', \text{ or } t_0 t = \sigma X_0' X',$$

where t is a row of any p quantities and $X' = \bar{\xi}'t$; hence, since the determinant $|\xi'|$ does not vanish, it follows, if X' be any row of p quantities, that $\sigma X_0' X'$ is positive; in particular when $n_1, \ldots, n_p$ are real, the imaginary part of the quadratic form τn^2 is positive.

Finally from the equations

$$\alpha x + \beta x' = \lambda x, \quad \alpha' x + \beta' x' = \lambda x',$$

putting $x = -\tau x'$, we infer

$$(\beta - \alpha\tau)\, x' = -\lambda\tau x', \quad (\beta' - \alpha'\tau)\, x' = \lambda x',$$

and therefore

$$\tau\,(\beta' - \alpha'\tau)\, x' + (\beta - \alpha\tau)\, x' = 0,$$

or

$$[\beta + \tau\beta' - (\alpha + \tau\alpha')\,\tau]\, x' = 0,$$

and hence

$$[\beta + \tau\beta' - (\alpha + \tau\alpha')\,\tau]\, \bar{\xi}' = 0,$$

from which, as $|\,\xi'\,|$ is not zero, we obtain

$$\beta + \tau\beta' - (\alpha + \tau\alpha')\,\tau = 0.$$

We have therefore completely proved the theorem stated.

It may be noticed, as follows from the equation $\xi + \xi'\tau = 0$, that we may form a theta function with associated constants given by

$$2\omega = 2\xi', \quad 2\omega' = -2\xi\,;$$

these will then satisfy the equations

$$\omega'\bar{\omega} - \omega\bar{\omega}' = 0, \quad \omega\bar{\omega}_0{}' - \omega'\bar{\omega}_0 = -2i\,;$$

the former equation always holds; the matrix ω can be determined so that the latter holds, as is easy to see.

Ex. Prove that by cogredient linear substitutions of the form

$$u' = cu, \quad w' = cw,$$

we can reduce the equations $u = Mw$ to the form

$$u_1{}' = \mu_1 w_1{}', \;\ldots,\; u_p{}' = \mu_p w_p{}',$$

where $\mu_1, \ldots, \mu_p$ are the roots of $|\,M - \lambda\,| = 0$.

383. For an example we may take the case $p = 1$; suppose that a, β, a', β' are such integers that $a\beta' - a'\beta = r$, a positive integer, and that the roots of the equation

$$(a + \tau a')\,\tau = \beta + \tau\beta'$$

are imaginary; if $a' = 0$, the condition that τ should not be a rational fraction requires that

$$\begin{pmatrix} a & \beta \\ a' & \beta' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a^2 = r$, and then the equation for τ is satisfied by all values of τ; this case is that of a multiplication by the rational number a, and we may omit it here; when a' is not zero we have

$$2a'\tau = -(a - \beta') \pm \sqrt{(a + \beta')^2 - 4r},$$

and therefore $(a + \beta')^2 < 4r$; this of itself is sufficient to ensure that the roots of the equation

$$\begin{pmatrix} a - \lambda & \beta \\ a' & \beta' - \lambda \end{pmatrix} = \lambda^2 - \lambda\,(a + \beta') + r = 0$$

are unequal, conjugate imaginaries, of modulus $\sqrt{r}$.

If then r be any given positive integer and h be a positive or negative integer numerically less than $2\sqrt{r}$, and a, a' be any integers such that $(a^2-ha+r)/a'$ is integral, $=-\beta$, we obtain a special transformation corresponding to the matrix

$$\Delta=\begin{pmatrix} a & \beta \\ a' & h-a \end{pmatrix}$$

for a value of τ given by

$$\tau=\frac{h-2a}{2a'}+i\frac{\sqrt{4r-h^2}}{2\,|a'|},$$

where $|a'|$ is the absolute value of a', and the square root is to be taken positively; the corresponding value of M is $a+\tau a'$. Hence by the results of Chap. XX., the function

$$\Theta\left[\tfrac{1}{2}(h\pm i\sqrt{4r-h^2})\,w\,;\ \frac{h-2a\pm i\sqrt{4r-h^2}}{2a'}\right],$$

when multiplied by a certain exponential of the form $e^{\lambda w^2}$, is expressible as an integral polynomial of order r in two functions $\Theta\left[w\,;\ \dfrac{h-2a\pm i\sqrt{4r-h^2}}{2a'}\right]$ with different characteristics.

The expression for the elliptic functions is obtainable independently as in the general case of transformation. When

$$Mv=\omega a+\omega' a',\quad Mv'=\omega\beta+\omega'\beta',\quad a\beta'-a'\beta=r,\quad u=Mw,$$

if to any two integers m, m' we make correspond two integers n, n' and two integers k, k', each positive (or zero) and less than r, by means of the equations

$$rn+k=m\beta'-m'\beta,\quad rn'+k'=-ma'+m'a,$$

or the equivalent equations

$$m=na+n'\beta+\frac{1}{r}(ak+\beta k'),\quad m'=na'+n'\beta'+\frac{1}{r}(a'k+\beta' k'),$$

then we immediately infer from the equation

$$\wp(u)=u^{-2}+\sum_{m}\sum_{m'}{}'\,[(u+2m\omega+2m'\omega')^{-2}-(2m\omega+2m'\omega')^{-2}],$$

by using n, n', instead of m, m', as summation letters, that

$$M^2\wp(Mw\,|\,2\omega,\,2\omega')=\wp(w\,|\,2v,\,2v')+\sum_{k}\sum_{k'}{}'\left[\wp\left(w+\frac{2vk+2v'k'}{r}\,\Big|\,2v,\,2v'\right)-\wp\left(\frac{2vk+2v'k'}{r}\right)\right],$$

wherein the summation refers to the $r-1$ sets k, k' other than $k=k'=0$, for which (§ 357, p. 589) the congruences

$$ak+\beta k'\equiv 0,\quad a'k+\beta' k'\equiv 0 \pmod{r}$$

are satisfied*.

This formula is immediately applicable to the case when there is a complex multiplication; we may then put

$$2\omega=2v=1,\quad 2\omega'=2v'=\tau,\quad \beta'=h-a,\quad -\beta=(a^2-ha+r)/a',\quad \tau=(h-2a\pm i\sqrt{4r-h^2})/2a',$$

* When these congruences have a solution (k_0, k_0'), in which k_0, k_0' have no common factor, i.e. (Appendix II., § 418) when a, a', β, β' have no common factor, the remaining solutions are of the form $(\lambda k_0, \lambda k_0')$, where $\lambda<r$; in that case taking integers x, x' such that $k_0x'-k_0'x=1$, it is convenient to take $2vk_0+2v'k_0'$ and $2vx+2v'x'$ as the periods of the functions $\wp$ on the right side.

and $M=(h\pm i\sqrt{4r-h^2})/2$, as above, where $h^2<4r$. The application of the resulting equation is sufficiently exemplified by the case of $r=2$ given below (Exx. ii., iii.).

In the particular case where $r=1$, the condition $h^2<4r$ shews that h can have only the values 0 or $+1$ or -1; in this case the values n, n' given by

$$m=na-n'\frac{a^2-ha+r}{a'},\quad m'=na'+n'(h-a)$$

are integral when m and m' are integral; hence as $-\frac{a^2-ha+r}{a'}+(h-a)\tau=M\tau$, we immediately find

$$g_2=60\sum_m\sum_n{}'\frac{1}{(m+m'\tau)^4}=\left(\frac{2}{h+i\sqrt{4-h^2}}\right)^4 g_2;\ g_3=140\sum_m\sum_n{}'\frac{1}{(m+m'\tau)^6}=\left(\frac{2}{h+i\sqrt{4-h^2}}\right)^6 g_3.$$

Thus when $h=0$ we have $g_3=0$, and if a, a' be any integers such that $(a^2+1)/a'$ is integral, we have $\tau=(\pm i-a)/a'$, the upper or lower sign being taken according as a' is positive or negative. In this case the function $\wp(u)$ satisfies the equation

$$(\wp' u)^2=4(\wp u)^3-g_2\wp(u),$$

where

$$g_2=60\sum_m\sum_n{}'\frac{1}{[m+n(\pm i-a)/a']^4}.$$

When $h=1$ we have $g_2=0$, and if a, a' be any integers such that $(a^2-a+1)/a'$ is integral, we have $\tau=(1-2a\pm i\sqrt{3})/a'$; in this case

$$(\wp' u)^2=4(\wp u)^3-g_3.$$

When $h=-1$, we have $g_2=0$, and, if $(a^2+a+1)/a'$ be integral, then $\tau=(-1-2a\pm i\sqrt{3})/a$.

Ex. i. Denoting the general function $\wp u$ by $\wp(u;\, g_2, g_3)$, it is easy to prove that the arc of the lemniscate $r^2=a^2\cos 2\theta$ is given by $a^2/r^2=\wp(s/a;\, 4, 0)$; when n is any prime number of the form $4k+1$ the problem of dividing the perimeter of the curve into n equal parts is reducible to the solution of an equation of order k—when n is a prime number of the form $2^\lambda+1$, the problem can be solved by the ruler and compass only. (Fagnano, *Produzioni Matematiche*, (1716), Vol. II.; Abel, *Œuvres*, 1881, t. I., p. 362, etc.) It is also easy to prove that the arc of the curve $r^3=a^3\cos 3\theta$ is given by $a^2/r^2=\wp(s/a;\, 0, 4)$; when n is a prime number of the form $6k+1$, the problem of dividing the perimeter of this curve into n equal parts is reducible to the solution of an equation of order k (Kiepert, *Crelle*, LXXIV. (1872), etc.). These facts are consequences of the linear special transformations of the theta functions connected with the curves.

Ex. ii. In case $r=2$, taking $a=4$, $a'=9$, $h=0$, we have $\tau=(-4+i\sqrt{2})/9$, and

$$-2\wp(i\sqrt{2}\,.\,w)=\wp(w)+\wp\left(w+\frac{\tau}{2}\right)-\wp(\tau/2).$$

By expanding this equation in powers of w, and equating the coefficients of w^2, we find easily that, if $\wp(\tau/2)=e$, then $g_2=\frac{15}{2}e^2$, and $g_3=-\frac{7}{2}e^3$; hence we infer that by means of the transformation

$$-2\xi=x+\frac{9}{8(x-1)}$$

we obtain

$$\int_\xi^\infty\frac{d\xi}{\sqrt{8\xi^3-15\xi+7}}=i\sqrt{2}\int_x^\infty\frac{dx}{\sqrt{8x^3-15x+7}},$$

which can be directly verified. *It is manifest that when $r=2$, $h=0$, we are led to this equation for all values of a and a'.*

Ex. iii. Prove that if $m=\frac{1}{2}(h+i\sqrt{8-h^2})$, the substitution

$$m^2\xi=x+\frac{3m^4-3}{m^4+4}\frac{1}{x-1}$$

gives the equation

$$\int_\xi^\infty \frac{d\xi}{\sqrt{(m^4+4)\xi^3-15\xi-(m^4-11)}}=m\int_x^\infty \frac{dx}{\sqrt{(m^4+4)x^3-15x-(m^4-11)}}.$$

This includes all such equations obtainable when $r=2$. Complex multiplication arises for the five cases $h=0$, $h=\pm 1$, $h=\pm 2$.

Ex. iv. When $r=3$ and $p=1$, we see by considering the matrix

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}=\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

that the function $\Theta_{1,1}[(1+i\sqrt{2})w;\ i\sqrt{2}]$ is expressible as a cubic polynomial in the functions $\Theta_{0,1}(w;\ i\sqrt{2})$, $\Theta_{1,1}(w;\ i\sqrt{2})$. The actual form of this polynomial is calculable by the formulae of Chap. XXI. (§§ 366, 372), by applying in order the linear substitutions $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and then the cubic transformation $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$. Hence deduce that $k=\sqrt{2}-1$ and

$$\operatorname{sn}[(1+i\sqrt{2})\,W]=(1+i\sqrt{2})\operatorname{sn} W[1-\operatorname{sn}^2 W/\operatorname{sn}^2\gamma]/[1-k^2\operatorname{sn}^2 W.\operatorname{sn}^2\gamma],$$

where $\gamma=2(K-iK')/3$, K being (§ 365, Chap. XXI.) $=\pi\Theta_{00}^2$, and $iK'=\tau K$.

For the complex multiplication of elliptic functions the following may be consulted: Abel, *Œuvres*, t. I. (1881), p. 379; Jacobi, *Werke*, Bd. I., p. 491; Sohnke, *Crelle*, XVI. (1837), p. 97; Jordan, *Cours d'Analyse*, t. II. (1894), p. 531; Weber, *Elliptische Functionen* (1891), Dritter Theil; Smith, *Report on the Theory of Numbers*, British Assoc. Reports, 1865, Part VI.; Hermite, *Théorie des équations modulaires* (1859); Kronecker, *Berlin. Sitzungsber.* (1857, 1862, 1863, 1883, etc.), *Crelle*, LVII. (1860); Joubert, *Compt. Rendus*, t. L. (1860), p. 774; Pick, *Math. Annal.* XXV., XXVI.; Kiepert, *Math. Annal.* XXVI. (1886), XXXII. (1888), XXXVII., XXXIX.; Greenhill, *Proc. Camb. Phil. Soc.* IV., V. (1882—3), *Quart. Journal*, XXII. (1887), *Proc. Lond. Math. Soc.* XIX. (1888), XXI. (1890); Halphen, *Liouville*, (1888); Weber, *Acta Math.* XI. (1887), *Math. Annal.* XXIII., XXXIII. (1889), XLIII. (1893); Etc.

384. We come now to a different theory*, leading however in one phase of it, to the fundamental equations which arise for the transformation of theta functions, that namely of the correspondence of places on a Riemann surface. The theory has a geometrical origin; we shall therefore speak either of a Riemann surface, or of the plane curve which may be supposed to be represented by the equation associated with the Riemann surface, according to convenience. The nature of the points under consideration may be illustrated by a simple example. If at a point x of a curve the tangent be drawn, intersecting the curve again in $z_1, z_2, \ldots, z_{n-2}$, we may say that to the point x, regarded as a variable point, there correspond the $n-2$ points

* For references to the literature of the geometrical theory, see below, § 387, Ex. iv., p. 647. The theory is considered from the point of view of the theory of functions by Hurwitz, *Math. Annal.* XXVIII. (1887), p. 561; *Math. Annal.* XXXII. (1888), p. 290; *Math. Annal.* XLI. (1893), p. 403. See also, Klein-Fricke, *Modulfunctionen*, Bd. II. (Leipzig, 1892), p. 518, and Klein, *Ueber Riemann's Theorie* (Leipzig, 1882), p. 67. For (1, 1) correspondence in particular see the references given in § 393, p. 654.

$z_1, \ldots, z_{n-2}$. To any point z of the curve, regarded as arising as one of a set $z_1, \ldots, z_{n-2}$, there will reciprocally correspond all the points, $x_1, x_2, \ldots, x_{m-2}$, which are points of contact of tangents drawn to the curve from z. Such a correspondence is described as an $(n-2, m-2)$ correspondence. A point of the curve for which x coincides with one of the points $z_1, \ldots, z_{n-2}$ corresponding to it, is called a coincidence; such points are for instance the inflexions of the curve.

In general an (r, s) correspondence on a Riemann surface involves that any place x determines uniquely r places $z_1, \ldots, z_r$, while any place z, regarded as arising as one of a set $z_1, \ldots, z_r$, determines uniquely s places $x_1, \ldots, x_s$. The investigation of the possible methods of this determination is part of the problem.

385. Suppose such an (r, s) correspondence to exist; let the positions of z that correspond to any variable position of x be denoted by $z_1, \ldots, z_r$, and the positions of x that correspond to any variable position of z be denoted by $x_1, \ldots, x_s$; and denote by $c_1, \ldots, c_r$ the positions of $z_1, \ldots, z_r$ when x is at the particular place a, and by $a_1, \ldots, a_s$ the positions of $x_1, \ldots, x_s$ when z is at the particular place c; denoting linearly independent Riemann normal integrals of the first kind by $v_1, \ldots, v_p$, consider the sum

$$v_i^{z_1, c_1} + \ldots\ldots + v_i^{z_r, c_r}$$

as a function of x; since it is necessarily finite we clearly have equations of the form

$$M_{i,1} v_1^{x, a} + \ldots\ldots + M_{i,p} v_p^{x, a} \equiv v_i^{z_1, c_1} + \ldots\ldots + v_i^{z_r, c_r}, \qquad (i = 1, \ldots, p),$$

where $M_{i,1}, \ldots, M_{i,p}$ are constants. On the dissected surface the omitted aggregate of periods of the integral v_i indicated by the sign $\equiv$ is self-determinative; if the paths of integration be not restricted from crossing the period loops the sign $\equiv$ can be replaced by the sign of equality (cf. Chap. VIII. §§ 153, 158).

If now x describe the kth period loop of the second kind, from the right to the left side of the kth period loop of the first kind, the places $z_1, \ldots, z_r$ will describe corresponding curves and eventually resume, in some order, the places they originally occupied; since, on the dissected Riemann surface $v_i^{z_1, c_1} + v_i^{z_2, c_2} = v_i^{z_2, c_1} + v_i^{z_1, c_2}$, we may suppose each of them actually to resume its original position; hence we have an equation

$$M_{i,k} = \alpha_{i,k} + \tau_{i,1} \alpha'_{1,k} + \ldots\ldots + \tau_{i,p} \alpha'_{p,k},$$

wherein $\alpha_{i,k}, \alpha'_{i,k}, \ldots$ are integers; similarly by taking x round the kth period loop of the first kind we obtain

$$M_{i,1} \tau_{1,k} + \ldots\ldots + M_{i,p} \tau_{p,k} = \beta_{i,k} + \tau_{i,1} \beta'_{1,k} + \ldots\ldots + \tau_{i,p} \beta'_{p,k};$$

we have therefore $2p^2$ equations expressible in the form

$$M = \alpha + \tau\alpha', \qquad M\tau = \beta + \tau\beta',$$

wherein α, α', β, β' are matrices of integers, of p rows and columns.

Consider next, as a function of x, the integral

$$\int v_m^{x, a}\, d\,[\Pi_{z_1, c_1}^{z, c} + \ldots\ldots + \Pi_{z_r, c_r}^{z, c}],$$

wherein z, c are, primarily, arbitrary positions, independent of x, and $\Pi_{z_1, c_1}^{z, c}$ is the Riemann normal integral of the third kind. The subject of integration becomes infinite when any one of the places $z_1, \ldots, z_r$ coincides with z, or, in other words, when z is among the places corresponding to x, and this happens when x is at any one of the places $x_1, \ldots, x_s$, which correspond to z; the subject of integration similarly becomes infinite when x is at any one of the places $a_1, \ldots, a_s$, which correspond to the particular position of z denoted by c; it is assumed that c does not coincide with any one of the places $c_1, \ldots, c_r$. The sum of the values obtained when the integral is taken, in regard to x round the infinities $x_1, \ldots, x_s$, $a_1, \ldots, a_s$, is, save for an additive aggregate* of periods of the integral v_m, equal to

$$2\pi i\,(v_m^{x_1, a_1} + \ldots\ldots + v_m^{x_s, a_s}).$$

This quantity is then equal to the value obtained when x is taken round the period loops on the Riemann surface. Consider first, for the sake of clearness, the contribution arising as x describes the kth period loop of the second kind; if x described the left side of this period loop in the negative direction, from the right to the left side of the kth period loop of the first kind, the aggregates of the paths described by $z_1, \ldots, z_r$ would, in the notation just previously adopted, be equivalent to $\alpha_{\lambda, k}$ negative circuits of the λth period loop of the second kind, and $\alpha'_{\lambda, k}$ positive circuits of the λth period loop of the first kind ($\lambda = 1, \ldots, p$). In the actual contour integration under consideration the description by x of the left side of the kth period loop of the second kind is to be in the positive direction; hence the contribution arising for the integral as x describes both sides of the kth period loop of the second kind is

$$-\,2\pi i \tau_{m, k} \sum_{\lambda=1}^{p} \alpha'_{\lambda, k}\, v_\lambda^{z, c};$$

similarly the contribution as x describes the sides of the kth period loop of the first kind is

$$2\pi i E_{m, k} \sum_{\lambda=1}^{p} \beta'_{\lambda, k}\, v_\lambda^{z, c},$$

* Which vanishes when paths can be drawn on the dissected surface connecting $a_1, \ldots, a_s$ respectively to $x_1, \ldots, x_s$, so that simultaneous positions on these paths are simultaneous positions of $x_1, \ldots, x_s$. Cf. Chap. VIII. § 153; Chap. IX. § 165.

where $E_{m,k}=0$ unless $m=k$, and $E_{m,m}=1$. Taking therefore all the period loops into consideration, that is, $k=1,\ldots,p$, we obtain

$$v_m^{x_1,a_1}+\ldots\ldots+v_m^{x_s,a_s}\equiv\sum_{\lambda=1}^{p}\beta'_{\lambda,m}\,v_\lambda^{z,c}-\sum_{k=1}^{p}\sum_{\lambda=1}^{p}\tau_{m,k}\,\alpha'_{\lambda,k}\,v_\lambda^{z,c}\equiv\sum_{\lambda=1}^{p}N_{m,\lambda}\,v_\lambda^{z,c},$$

where
$$N_{m,\lambda}=\beta'_{\lambda,m}-\sum_{k=1}^{p}\tau_{m,k}\,\alpha'_{\lambda,k},$$

so that $N_{m,\lambda}$ is the (m,λ)th element of the matrix

$$N=\bar{\beta}'-\tau\bar{\alpha}';$$

since the equations $M=\alpha+\tau\alpha'$, $M\tau=\beta+\tau\beta'$ give

$$-\bar{\beta}+\tau\bar{\alpha}=(\bar{\beta}'-\tau\bar{\alpha}')\,\tau,$$

we have also

$$N\tau=-\bar{\beta}+\tau\bar{\alpha}.$$

These equations express the sum $v_m^{x_1,a_1}+\ldots+v_m^{x_s,a_s}$ in terms of integrals $v_\lambda^{z,c}$ in a manner analogous to the expression originally taken for $v_i^{z_1,c_1}+\ldots+v_i^{z_r,c_r}$ in terms of integrals $v_\lambda^{x,a}$, the difference being the substitution, for the matrix $\begin{pmatrix}\alpha & \beta\\ \alpha' & \beta'\end{pmatrix}$, of the matrix $\begin{pmatrix}\bar{\beta}' & -\bar{\beta}\\ -\bar{\alpha}' & \bar{\alpha}\end{pmatrix}$.

386. The theory of correspondence of points of a Riemann surface now divides into two parts according as the equation, which arises by elimination, either of the matrix M or the matrix N, namely,

$$\tau\alpha'\tau+\alpha\tau-\tau\beta'-\beta=0,$$

is true independently of the matrix τ, in virtue of special values for the matrices α, β, α', β', or, on the other hand, is true for more general values of these matrices, in virtue of a special value for the matrix τ.

We take the first possibility first; it is manifest that for any value of τ the equation is satisfied if

$$\alpha=-\gamma E,\ \beta=0,\ \alpha'=0,\ \beta'=-\gamma E,$$

where γ is any single integer, and E is the matrix unity of p rows and columns; conversely, if the equations are to hold independently of the value of τ, we must have the n^2 equations

$$\sum_{i,j}^{1\ldots p}\alpha'_{i,j}\tau_{m,i}\tau_{\lambda,j}=0,\quad \sum_{i=1}^{p}\alpha_{m,i}\tau_{i,\lambda}=\sum_{j=1}^{p}\tau_{m,j}\beta'_{j,\lambda},\quad \beta_{m,\lambda}=0,\quad (m,\lambda=1,\ldots,p),$$

and, for general values of τ, these give

$$\alpha'_{i,j}=0,\quad \alpha_{m,m}=\beta'_{\lambda,\lambda},\quad \alpha_{m,m'}=\beta'_{\lambda,\lambda'}=0,\quad \beta_{m,\lambda}=0,\qquad (m\neq m',\ \lambda\neq\lambda'),$$

which are equivalent to the results taken above.

With these values we have, as the particular forms of the general equations of § 385,

$$v_i^{z_1, c_1} + \ldots\ldots + v_i^{z_r, c_r} + \gamma v_i^{x, a} \equiv 0,$$

$$v_m^{x_1, a_1} + \ldots\ldots + v_m^{x_s, a_s} + \gamma v_m^{z, c} \equiv 0. \qquad (i, m = 1, \ldots, p).$$

Let the value on the dissected surface of the left side of the first of these equivalences be

$$g_i + g_1' \tau_{i,1} + \ldots\ldots + g'_p \tau_{i,p},$$

where $g_1, \ldots, g_p, g_1', \ldots, g'_p$ are integers. Consider now the function

$$\phi(x, z;\ a, c) = e^{\Pi_{z_1, c_1}^{z, c} + \ldots\ldots + \Pi_{z_r, c_r}^{z, c} + \gamma \Pi_{x, a}^{z, c} - 2\pi i (g_1' v_1^{z, c} + \ldots\ldots + g'_p v_p^{z, c})},$$

wherein $z_1, \ldots, z_r$ are the places corresponding to x, and $c_1, \ldots, c_r$ their positions when x is at a, and z, c are arbitrary places. In virtue of the equations just obtained it is a rational function of z, and rational in the place c (cf. Chap. VIII., § 158). Regarded as a function of x it is also rational; for the quotient of its values at the two sides of a period loop of the second kind, which, by what has just been shewn, must be rational in z, is, by the properties of the integral of the third kind, necessarily of the form

$$e^{2\pi i (K_1 v_1^{z, c} + \ldots\ldots + K_p v_p^{z, c})},$$

where $K_1, \ldots, K_p$ are integers; this quotient, as a function of z, has no infinities; being a rational function of z, it is therefore a constant, and therefore unity, since it reduces to unity when z is at c; hence $\phi(x, z;\ a, c)$, as a function of x, has no factors at the period loops; as it can have no infinities but poles it is therefore a rational function of x; it is similarly rational in a. As a function of x it vanishes when one of $z_1, \ldots, z_r$ coincides with z, that is, when x coincides with one of $x_1, \ldots, x_s$.

We have therefore the result. *Associated with any (r, s) correspondence which can exist upon a perfectly general Riemann surface, it is possible to construct a function $\phi(x, z;\ a, c)$, rational in the variable places x, z and the fixed places a, c, which, regarded as a function of x vanishes to the first order at the places $x_1, \ldots, x_s$, which correspond to z, and vanishes to order γ (if γ be positive), at the place z; which, as a function of x, is infinite to the first order when x coincides with any one of the places $a_1, \ldots, a_s$ which correspond to c, and is infinite to order γ (γ being positive) when x is at c; which, as a function of z, has similarly (for γ positive) the zeros $z_1, \ldots, z_r$, x^γ and the poles $c_1, \ldots, c_r$, a^γ.* An analogous statement can be made when γ is negative.

Ex. i. Some examples may be given to illustrate the form of this rational function. On a plane cubic curve we do in fact obtain a (1, 4) correspondence, for which $\gamma = 2$, by taking for the point z_1 which corresponds to x, the point in which the tangent at

x meets the curve again, and therefore, for the points x_1, x_2, x_3, x_4 which correspond to z, the points of contact of tangents to the curve drawn from z. The value $\gamma=2$ is obtained from Abel's theorem, which clearly gives the equation

$$v^{z_1, c_1}+2v^{x, a}\equiv 0$$

as representative of the fact that a straight line meets the curve twice at x and once at z_1. Denote the equation of the curve in the ordinary symbolical way by $A_x^3=0$; then the equation $A_x^2A_z=0$, for a fixed position of x, represents the tangent at x; and for a fixed position of z, represents the polar conic of the point z, which vanishes once in the points of contact, x_1, x_2, x_3, x_4, of tangents drawn from z and vanishes also twice at z, where it touches the curve; then consider the function

$$\frac{A_x^2A_z}{A_x^2A_c \,.\, A_a^2A_z};$$

when z, a, c are fixed, this function of x vanishes to the first order at x_1, x_2, x_3, x_4 and to the second order at z, and is infinite to the first order at the places a_1, a_2, a_3, a_4 which correspond to c, and infinite to the second order at c; when z, a, c are fixed, this function of z vanishes to the first order at z_1, and to the second order at x, and is infinite to the first order at the place c_1, which corresponds to a, and infinite to the second order at a.

Ex. ii. More generally for any plane curve of order n, and deficiency p, if to a point x we make correspond the $r=n-2$ points $z_1, \ldots, z_{n-2}$, in which the tangent at x meets the curve again, and to a point z the $s=2n+2p-4$ points of contact $x_1, \ldots, x_s$ of tangents drawn to the curve from z (so that, for instance, when the curve has κ cusps, κ of the points $x_1, \ldots, x_s$ will be the same for all positions of z), we shall have an (r, s) correspondence for which $\gamma=2$. If $A_x^n=0$ be the equation of the curve, the function

$$\frac{A_x^{n-1}A_z}{A_x^{n-1}A_c \,.\, A_a^{n-1}A_z},$$

regarded as a function of x, for fixed positions of z, a, c (of which a and c are not to be multiple points), has for zeros the places $x_1, \ldots, x_s$, z^2, for poles the places $a_1, \ldots, a_s$, c^2, and regarded as a function of z, has for zeros the places $z_1, \ldots, z_r$, x^2, and for poles the places $c_1, \ldots, c_r$, a^2.

Ex. iii. If from a point x a tangent be drawn to a plane curve, and the corresponding points be the points other than the point of contact, in which the tangent meets the curve again, we have

$$v^{z_1, c_1}+\ldots\ldots+v^{z_{n-3}, c_{n-3}}+2v^{z', c'}+v^{x, a}\equiv 0,$$

where z' is the point of contact of one of the tangents drawn from x, there being as many such equations as tangents to the curve from x; since the $2n+2p-4$ points z' lie on the first polar of x, it follows by Abel's theorem that

$$\Sigma v^{z', c'}+2v^{x, a}\equiv 0;$$

therefore

$$v^{z_1, c_1}+\ldots\ldots+v^{z_r, c_r}+(2n+2p-8)\, v^{x, a}\equiv 0,$$

so that $\gamma=2n+2p-8$. As a function of z the function $\phi(x, z;\, a, c)$ has therefore the $(n-3)(2n+2p-4)$ zeros $z_1, \ldots, z_r$, which correspond to x, as well as the zero x, of the $(2n+2p-8)$th order, and has as poles the places $c_1, \ldots, c_r$, which correspond to a, as well as the zero a, of the $(2n+2p-8)$th order.

For instance for a plane quartic, there are 10 places corresponding to x, one for each of the tangents that can be drawn from x to the curve; the function $\phi(x, z;\, a, c)$, as a function of z, vanishes to the first order at each of these ten places, and vanishes to the sixth order at x; its infinities are the places similarly derived from the fixed position, a, of x. We can build up this function in the manner suggested by the use already made of Abel's theorem in the determination of the value of γ; for a fixed position of x, let $T(z)=0$ be the equation, in the variable z, for the ten tangents to the quartic drawn from z; let $P(z)=0$ be the first polar of x; the quotient

$$T(z)/P^2(z)$$

vanishes when z is at the places $z_1, \ldots, z_{10}$, and vanishes when z is at x to order $10-2(2)=6$; let $T_a(z)$, $P_a(z)$ represent what $T(z)$, $P(z)$ become when x is at a; then the function of z

$$\frac{T(z)}{P^2(z)} \bigg/ \frac{T_a(z)}{P_a^2(z)}$$

has the same behaviour as has the function $\phi(x, z;\ a, c)$ as a function of z. From this function, by multiplication by a factor involving x but independent of z, we can form a symmetrical expression in x and z; this will be the function $\phi(x, z;\ a, c)$. In fact, denoting the equation of the quartic curve by $A_x^4=0$, and expressing the fact that the line joining the point x of the curve to the point ζ not on the curve should touch the curve, *viz.*, by equating to zero the discriminant in λ of $(A_x+\lambda A_\zeta)^4-A_x^4$, we obtain an equation of the form

$$A_\zeta^4[\zeta^6, x^6]=(A_xA_\zeta^3)^2[9(A_x^2A_\zeta^2)^2-16A_xA_\zeta^3\,.\,A_x^3A_\zeta],$$

which represents the tangents to the curve drawn from x. Replacing ζ by z, a point on the curve, so that $A_z^4=0$, we have, since $A_xA_\zeta^3=0$ is the first polar of x,

$$T(z)/P^2(z)=9(A_x^2A_z^2)^2-16A_xA_z^3\,.\,A_x^3A_z;$$

hence

$$\phi(x, z;\ a, c)=\frac{9(A_x^2A_z^2)^2-16A_xA_z^3\,.\,A_x^3A_z}{[9(A_a^2A_z^2)^2-16A_aA_z^3\,.\,A_a^3A_z][9(A_x^2A_c^2)^2-16A_xA_c^3\,.\,A_x^3A_c]}.$$

Ex. iv. If a (1, 1) correspondence exists, the rational function of x, denoted by $\phi(x, z;\ a, c)$, is of order $\gamma+1$.

387. A problem of great geometrical interest is to determine the number of positions of x, in which x coincides with one of the places $z_1, \ldots, z_r$, which correspond to it. This is called the number of *coincidences*.

A simple way to determine this number is to consider the rational function of x obtained as the limit when $z=x$, of the ratio $\phi(x, z;\ a, c)/(x-z)^2$; putting

$$\phi(x;\ a, c)=\lim_{z=x}[\phi(x, z;\ a, c)/(x-z)^2],$$

and bearing in mind that if t be the infinitesimal on the Riemann surface, dx/dt vanishes to the first order at every finite branch place, and is infinite to the second order at every infinite place of the surface, we immediately find from the properties of the function $\phi(x, z;\ a, c)$, on the hypothesis that none of the branch places of the surface are at infinity, the following result; the rational function of x denoted by $\phi(x;\ a, c)$ vanishes to the first order at every place x of the surface at which x coincides with one of the places

$z_1, \ldots, z_r$ which correspond to it, vanishes also to order 2γ at each of the n infinite places of the surface, and is infinite to order γ at each of the branch places of the surface and at each of the places a, c, while it is infinite to the first order at each of the places $c_1, \ldots, c_r$ which correspond to a, and at each of the places $a_1, \ldots, a_s$ which correspond to c; hence, denoting the number of coincidences by C we have

$$C + 2n\gamma = (2n + 2p - 2)\gamma + 2\gamma + r + s,$$

so that*

$$C = r + s + 2p\gamma.$$

The same result is obtained when there are branch places at infinity. The argument has assumed γ to be positive; a similar argument, when γ is negative, leads to the same result.

Ex. i. The number, i, of inflexions of a plane curve of order n and deficiency p is given (Ex. ii. § 386) by

$$i + h = n - 2 + 2n + 2p - 4 + 4p = 3(n + 2p - 2),$$

where h is the number of coincidences arising other than inflexions, as for instance at the multiple points of the curve. In determining h it must be remembered that we have not excluded the possibility of there being fixed positions of x which correspond to z for all positions of z; for instance in the case of a curve with cusps all these cusps have been reckoned among the places $x_1, \ldots, x_s$ which correspond to z. Therefore for a curve with κ cusps, h will contain a term 2κ; for a curve with only δ double points and κ cusps, the formula is the well-known one

$$i - \kappa = 3(m - n),$$

where m is the class of the curve, equal to $n(n-1) - 2\delta - 3\kappa$.

Ex. ii. Obtain the expression of the function $\phi(x; a, c)$ determined by the limit

$$\{A_x^{n-1} A_z/(x-z)^2 \,.\, A_x^{n-1} A_c \,.\, A_a^{n-1} A_z\}_{z=x},$$

where $A_x^n = 0 = A_z^n = A_a^n = A_c^n$. (Cf. Ex. ii. § 386.)

Ex. iii. The number of double tangents of a curve of order n and deficiency p may be obtained from Ex. iii. § 386, if we notice that a double tangent, touching at P and Q, will arise both when P is a coincidence, and when Q is a coincidence; hence if τ be the number of double tangents, and h the number of coincidences not giving rise to double tangents, we have

$$2\tau + h = 2(n-3)(2n + 2p - 4) + 2p(2n + 2p - 8) = 4\sigma(\sigma + 1) - 8p,$$

where $\sigma = n + p - 3$. For instance for a curve with no singular points other than δ double points and κ cusps, there will be a contribution to h equal to twice the number of those improper double tangents which are constituted by the tangents to the curve from the cusps and the lines joining the cusps in pairs. The number of tangents, t, from a cusp is given (cf. § 9, Chap. I., Ex.) by

$$t + \kappa - 1 = 2(n-2) + 2p - 2, \text{ or } t = 2n - 5 - \kappa + 2p = n^2 - n - 3 - 2\delta - 3\kappa.$$

There will not arise any such contribution corresponding to a double point, since the two

* This result was first given by Cayley; see, for references, Ex. iv. below.

points of the curve that there correspond are different places (cf. § 2, Chap. I.); hence we have

$$h = 2\kappa t + \kappa^2 - \kappa\,;$$

and therefore $$\tau = 2\sigma(\sigma+1) - 4p - \kappa t - \tfrac{1}{2}(\kappa^2 - \kappa)\,;$$

substituting the values for σ, p and t, we find the ordinary formula equivalent to

$$\tau = \delta + \tfrac{1}{2}(m-n)(m+n-9),$$

where m is the class of the curve.

Ex. iv. The points of contact of the double tangents of a quartic curve $A_x^4 = 0$ lie upon a curve whose equation is obtainable by determining the limit, when $z = x$, of the expression

$$[9(A_x^2 A_z^2)^2 - 16 A_x A_z^3 \,.\, A_x^3 A_z]/(x-z)^2.$$

For the result, cf. Dersch, *Math. Annal.* VII. (1874), p. 497.

For the general geometrical theory the reader will consult geometrical treatises; the following references may be given here; Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie* (Paris, 1879—1883), t. I. p. 261, t. II. p. 146, t. III. p. 76; Chasles, *Compt. Rendus*, t. LVIII. (1864); Chasles, *Compt. Rendus*, t. LXII. (1866), p. 584; Cayley, *Compt. Rendus*, t. LXII. (1866), p. 586, and *London Math. Soc. Proc.* t. I. (1865—6), and *Phil. Trans.* CLVIII. (1868) (or *Coll. Works*, v. 542; VI. 9; VI. 263); Brill, *Math. Annal.* t. VI. (1873), and t. VII. (1874). See also Lindemann, *Crelle*, LXXXIV. (1878); Bobek, *Sitzber. d. Wiener Akad.*, XCIII. (ii. Abth.), (1886), p. 899; Brill, *Math. Annal.* XXXI. (1887), XXXVI. (1890); Castelnuovo, *Rend. Acc. d. Lincei*, 1889; Zeuthen, *Math. Annal.* XL. (1892), and the references there given.

Ex. v. If we use the equation (Chap. X. § 187)

$$e^{\Pi_{x,a}^{z,c}} = \frac{\Theta(v^{x,z}+\frac{1}{2}\Omega)\,\Theta(v^{a,c}+\frac{1}{2}\Omega)}{\Theta(v^{x,c}+\frac{1}{2}\Omega)\,\Theta(v^{a,z}+\frac{1}{2}\Omega)},$$

where Ω is an odd half-period, equal to $\lambda + \tau\lambda'$ say, λ, λ' being each rows of p integers, and form the rational function of x and a,

$$R(x,a) = lim_{\substack{z=x\\c=a}} (-1)^\gamma \frac{\Theta^\gamma(v^{x,z}+\frac{1}{2}\Omega)\,\Theta^\gamma(v^{a,c}+\frac{1}{2}\Omega)}{\phi(x,z\,;\,a,c)}$$

$$= (-1)^\gamma \frac{[\sum\limits_m \Theta'_m(\frac{1}{2}\Omega)\,.\,Dv_m^{x,}]^\gamma\,[\sum\limits_m \Theta'_m(\frac{1}{2}\Omega)\,.\,Dv_m^{a,}]^\gamma}{[\phi(x,z\,;\,a,c)/(x-z)^\gamma(a-c)^\gamma]_{x=z,\,a=c}},$$

we have

$$\Theta(v^{x,a}+\tfrac{1}{2}\Omega)\,.\,e^{\pi i\lambda' v^{x,a}} = [R(x,a)]^{\frac{1}{2\gamma}}\, e^{\frac{1}{2\gamma}(\Pi_{z_1,c_1}^{x,a} + \dots\dots + \Pi_{z_r,c_r}^{x,a} - 2\pi i g' v^{x,a})},$$

which is a generalisation of the equation (i), p. 427.

The function $R(x,a)$ vanishes when x is at any one of the places $c_1, \dots, c_r$, which correspond to a, and when x is at any one of the places $a_1, \dots, a_s$ which correspond to the position a of the place c; it vanishes also 2γ times at each of the zeros of the function $\Theta(v^{x,a}+\frac{1}{2}\Omega)$. It is infinite C times, namely when x has any of the positions in which it coincides with one of the places $z_1, \dots, z_r$ which correspond to it. In the particular case of Ex. i. p. 427, the function $R(x,a)$ is $(x-a)^2 X(x)$, and the equation $C = r + s + 2p\gamma$ expresses that the number of branch places (where two places for which x is the same coincide) is $2(n-1)+2p$.

Ex. vi. Determine the periods of the function of x expressed by

$$\Pi_{z_1,c_1}^{x,a} + \dots\dots + \Pi_{z_r,c_r}^{x,a},$$

where $z_1, \ldots, z_r$ are the places corresponding to x, and $c_1, \ldots, c_r$ are the places corresponding to a.

Ex. vii. If there be upon the same Riemann surface two correspondences, an (r, s) correspondence and an (r', s') correspondence, then to any place z will correspond, in virtue of the first correspondence, the places $x_1, \ldots, x_s$, and to any one of these latter, say x_i, will correspond, in virtue of the second correspondence, say $z'_{i,1}, \ldots, z'_{i,r'}$; conversely to any place z' will correspond, in virtue of the second correspondence, the places $x_1, \ldots, x_{s'}$, and to any one of these latter, say x_i, will correspond, in virtue of the first correspondence, say $z_{i,1}, \ldots, z_{i,r}$; we have therefore an $(r's, rs')$ correspondence of the points (z, z'). In virtue of the equations

$$v^{x_1, a_1'} + \ldots\ldots + v^{x_{s'}, a'_{s'}} + \gamma' v^{z', c'} \equiv 0,$$

$$v^{z_{i,1}, c_{i,1}} + \ldots\ldots + v^{z_{i,r}, c_{i,r}} + \gamma v^{x_i, a'_i} \equiv 0, \qquad (i = 1, \ldots, s'),$$

we have

$$\sum_{i=1}^{s'} \sum_{j=1}^{r} v^{z_{i,j}, c_{i,j}} - \gamma\gamma' v^{z', c'} \equiv 0.$$

Hence* we can make the inference. *If upon the same Riemann surface there be two correspondences, an (r, s) correspondence of places x, z, and an (r', s') correspondence of places x', z', then the number of common corresponding pairs of these two correspondences, for which both x, x' coincide, and also z and z', is*

$$r's + rs' - 2\gamma\gamma' p.$$

388. We have so far considered only those correspondences† which can exist on any Riemann surface. We give now some results‡ relating to correspondences which can only exist on Riemann surfaces of special character, more particularly (1, 1) correspondences.

We prove first that any $(1, s)$ correspondence is associated with equations which are identical in form with those which have arisen in considering the special transformation of theta functions. For any such correspondence, in which to any place x corresponds the single place z, and to any position of z the places $x_1, \ldots, x_s$, we have shewn that we have the equations $(i = 1, \ldots, p)$

$$v_i^{z, c} \equiv M_{i,1} v_1^{x, a} + \ldots\ldots + M_{i,p} v_p^{x, a}, \quad M = \alpha + \tau\alpha', \quad M\tau = \beta + \tau\beta';$$

$$v_i^{x_1, a_1} + \ldots\ldots + v_i^{x_s, a_s} \equiv N_{i,1} v^{z, c} + \ldots\ldots + N_{i,p} v_p^{z, c}, \quad N = \bar{\beta}' - \tau\bar{\alpha}', \quad N\tau = -\bar{\beta} + \tau\bar{\alpha};$$

hence

$$s v_i^{z, c} \equiv M_{i,1} \sum_{m=1}^{s} v_1^{x_s, a_s} + \ldots\ldots + M_{i,p} \sum_{m=1}^{s} v_p^{x_s, a_s}$$

$$\equiv \sum_{k=1}^{p} M_{i,k} (N_{k,1} v_1^{z, c} + \ldots\ldots + N_{k,p} v_p^{z, c})$$

$$\equiv L_{i,1} v_1^{z, c} + \ldots\ldots + L_{i,p} v_p^{z, c},$$

* Provided the $(r's, rs')$ correspondence is not an identity.

† Called by Hurwitz, Werthigkeit-correspondenzen, γ being the Werthigkeit.

‡ For other results, see Klein-Fricke, *Modulfunctionen*, Bd. II. (Leipzig, 1892), pp. 540 ff.

where $L_{i,m}$ is the (i, m)th element of the matrix L, $= MN$. This matrix is therefore equal to s. Now

$$MN = M(\ \bar{\beta}' - \tau\bar{\alpha}') = \ (\alpha + \tau\alpha')\bar{\beta}' - (\beta + \tau\beta')\bar{\alpha}' = \ \alpha\bar{\beta}' - \beta\bar{\alpha}' + \tau(\alpha'\bar{\beta}' - \beta'\bar{\alpha}'),$$
$$MN\tau = M(-\bar{\beta} + \tau\bar{\alpha}) = -(\alpha + \tau\alpha')\bar{\beta} + (\beta + \tau\beta')\bar{\alpha} = -(\alpha\bar{\beta} - \beta\bar{\alpha}) + \tau(\beta'\bar{\alpha} - \alpha'\bar{\beta}),$$

which we may write in the form

$$MN = H + \tau B, \quad MN\tau = -A + \tau\bar{H};$$

if now $\tau = \tau_1 + i\tau_2$, where τ_1, τ_2 are matrices of real quantities, it follows by equating to zero the imaginary part in the equation

$$MN - s = H - s + \tau B = 0$$

that $\tau_2 B = 0$; since for real values of $n_1, \ldots, n_p$ the quadratic form $\tau_2 n^2$ is necessarily positive, the determinant of the matrix τ_2 is not zero; hence we must have $B = 0$; hence also $H = s$ and $A = 0$; or

$$\alpha\bar{\beta} = \beta\bar{\alpha}, \ \alpha'\bar{\beta}' = \beta'\bar{\alpha}', \ \alpha\bar{\beta}' - \beta\bar{\alpha}' = \beta'\bar{\alpha} - \alpha'\bar{\beta} = s;$$

and these equations, with the equation $(\alpha + \tau\alpha')\tau = \beta + \tau\beta'$, are identical in form with those already discussed in this chapter (§§ 377, ff.).

We are able then as in the former case to deduce certain conditions for the matrices $\alpha, \beta, \alpha', \beta'$, which in their general form necessarily involve special values for the matrix τ.

389. In particular, in order that a (1, 1) correspondence* may exist, the roots of the equation $|M - \lambda| = 0$ must be conjugate imaginaries of the roots of the equation $|N - \lambda| = 0$, must be all of modulus unity, and must be roots of the equation $|\Delta - \lambda| = 0$, where $\Delta = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}$. They must therefore be roots of unity. For the sake of definiteness we shall suppose $p > 1$ and that Δ and τ are such that the roots of $|M - \lambda| = 0$ are all different; this excludes the case already considered when $\Delta = \begin{pmatrix} -\gamma & 0 \\ 0 & -\gamma \end{pmatrix}$. Supposing a (1, 1) correspondence to exist, for which this condition is satisfied, if in the fundamental equations $(i = 1, \ldots, p)$

$$v_i^{z, c} \equiv M_{i, 1} v_1^{x, a} + \ldots\ldots + M_{i, p} v_p^{x, a},$$

we introduce other integrals of the first kind, say $V_1^{x, a}, \ldots, V_p^{x, a}$, where

$$V_i^{x, a} = c_{i, 1} v_1^{x, a} + \ldots\ldots + c_{i, p} v_p^{x, a},$$

and therefore also

$$V_i^{z, c} = c_{i, 1} v_1^{z, c} + \ldots\ldots + c_{i, p} v_p^{z, c},$$

* The (1, 1) correspondence for the case $p = 1$ is considered in an elementary way in § 394. The reader may prefer to consult that Article before reading the general investigation.

then we can put the fundamental equations into the form

$$V_i^{z,c} = \lambda_i V_i^{x,a};$$

for this it is necessary that λ_i should be a root of the equation $|M-\lambda| = 0$, and that the p quantities $c_{i,1}, \ldots, c_{i,p}$ should be determined from the equations

$$c_{i,1} M_{1,r} + \ldots\ldots + c_{i,p} M_{p,r} = \lambda_i c_{i,r}, \qquad (r = 1, \ldots, p);$$

under the prescribed conditions the determinant of the matrix c will be different from zero.

Hence as λ_i is a root of unity, it can be shewn, when $p > 1$, that *every such* (1, 1) *correspondence is periodic, with a finite period*; that is, if the place corresponding to x be z_1, the place corresponding to the position z_1, of x, be z_2, the place corresponding to the position z_2, of x, be z_3, and so on, then after a finite number of stages one of the places $z_1, z_2, z_3, \ldots$ coincides with x. In order to prove this, suppose that all the roots of the equation $|M-\lambda| = 0$ are k-th roots of unity; then denoting the place x by z_0 and the place a by c_0, the equations of the correspondence may be written

$$dV_i^{z_1, c_1} = \lambda_i dV_i^{z_0, c_0}, \; dV_i^{z_2, c_2} = \lambda_i dV_i^{z_1, c_1}, \ldots\ldots, dV_i^{z_k, c_k} = \lambda_i dV_i^{z_{k-1} c_{k-1}};$$

these give

$$dV_i^{z_k, c_k} = \lambda_i^k dV_i^{z_0, c_0} = dV_i^{z_0, c_0},$$

and therefore

$$c_{i,1}[dv_1^{z_k, c_k} - dv_1^{z_0, c_0}] + \ldots\ldots + c_{i,p}[dv_p^{z_k, c_k} - dv_p^{z_0, c_0}] = 0;$$

hence on the dissected Riemann surface we have equations of the form

$$v_r^{z_k, c_k} - v_r^{z_0, c_0} = \lambda_r + \lambda_1' \tau_{r,1} + \ldots\ldots + \lambda_p' \tau_{r,p}, \qquad (r = 1, \ldots, p),$$

where $\lambda_1, \ldots, \lambda_p'$ are integers. Thus either $z_k = z_0$ and $c_k = c_0$, which is the result we wish to obtain, or else there is a rational function expressed by

$$e^{\Pi_{z_k, c_k}^{x, a} - \Pi_{z_0, c_0}^{x, a} - 2\pi i (\lambda_1' v^{x, a} + \ldots\ldots + \lambda_p' v_p^{x, a})},$$

which is of the second order, having z_k, c_0 as zeros and z_0, c_k as poles; now a surface on which there is a rational function of the second order is necessarily hyperelliptic (Chap. V. § 55)—but, on a hyperelliptic surface, for which $p > 1$, of the two poles of such a function either determines the other, and of the two zeros either determines the other; it is not possible to construct such a function whereof, as here, one pole c_k is fixed, and the other arbitrary and variable (§ 52).

Hence we must have $z_k = z_0$, and $c_k = c_0$, which proves the result enunciated.

There is no need to introduce the integrals V in order to establish this result. It is known (Cayley, *Coll. Works*, Vol. II. p. 486) that if $\lambda_1, \lambda_2, \ldots$ be the roots of the equation $|M-\lambda| = 0$, the matrix M satisfies the equation $(M-\lambda_1)(M-\lambda_2)\ldots\ldots = 0$; when the roots

$\lambda_1, \lambda_2, \ldots$ are different k-th roots of unity it can thence be inferred that the matrix M satisfies the equation $M^k=1$; then from the successive equations $dv^{z_1, c_1}=Mdv^{z_0, c_0}$, $dv^{z_2, c_2}=Mdv^{z_1, c_1}$, etc., we can infer $dv^{z_k, c_k}=dv^{z_0, c_0}$, and hence as before that $z_k=z_0$, $c_k=c_0$.

A proof of the periodicity of the (1, 1) correspondence, following different lines, and not assuming that the roots of the equation $|M-\lambda|=0$ are different, is given by Hurwitz, *Math. Annal.* XXXII. (1888), p. 295, for the cases when $p>1$. It will be seen below that the cases $p=0$, $p=1$ possess characteristics not arising for higher values of p (§ 394).

390. Assuming the periodicity of the (1, 1) correspondence, we can shew that all Riemann surfaces upon which a (1, 1) correspondence exists, can be associated with an algebraic equation of particular form. As before let k be the index of the periodicity, and let $\omega = e^{2\pi i/k}$; let S, T be any two rational functions on the surface, and let the values of S at the successive places $x, z_1, z_2, \ldots, z_{k-1}, x$ which arise by the correspondence be denoted by $S, S_1, \ldots, S_{k-1}, S$, and similarly for T; then the values of the functions

$$s = S + \omega^{-1} S_1 + \ldots\ldots + \omega^{-(k-1)} S_{k-1}$$
$$t = T + \quad T_1 + \ldots\ldots + \quad T_{k-1}$$

at the place z_r are respectively

$$s_r = S_r + \omega^{-1} S_{r+1} + \ldots\ldots + \omega^{-(k-1)} S_{r+k-1} = \omega^r s, \text{ and } t;$$

hence it can be inferred (cf. Chap. I., § 4) that there exists a rational relation connecting s^k and t. Conversely S and T can be chosen of such generality that any given values of s and t arise only at one place of the original Riemann surface. Thus the surface can be associated with an equation of the form

$$(s^k, t) = 0,$$

wherein every power of s which enters is a multiple of k.

Such a surface is clearly capable of the periodic (1, 1) transformation expressed by the equations

$$s' = \omega s, \ t' = t.$$

The following further remarkable results may be mentioned*:

(α) The index of periodicity k cannot be greater than $10(p-1)$.

(β) When $k > 2p-2$ the Riemann surface can be associated with an equation of the form

$$s^k = t^{k_1}(t-1)^{k_2}(t-c)^{k_3}.$$

(γ) When $k > 4p-4$, the Riemann surface can be associated with an equation of the form

$$s^k = t^{k_1}(t-1)^{k_2}.$$

Herein k_1, k_2, k_3 are positive integers less than k.

* Hurwitz, *Math. Annal.* XXXII. (1888), p. 294.

391. We can deduce from § 389 that in the case of a (1, 1) correspondence the number of coincidences is not greater than $2p+2$. In the case of a hyperelliptic surface, when the correspondence is that in which conjugate places—of the canonical surface of two sheets—are the corresponding pairs, the coincidences are clearly the branch places, and their number is $2p+2$; for all other (1, 1) correspondences on a hyperelliptic surface, the number of coincidences cannot be greater than 4.

For, when the surface is not hyperelliptic, let g denote a rational function which is infinite only at one place z_0 of the surface, to an order $p+1$; and let g' be the value of the same function at the place z_1, which corresponds to z_0; then the function $g'-g$ is of order $2p+2$, being infinite to order $p+1$ at z_0 and to order $p+1$ at the place z_{-1} to which z_0 corresponds; now every coincidence of the correspondence is clearly a zero of $g'-g$; thus the number of coincidences is not greater than $2p+2$. In the case of a hyperelliptic surface

$$y^2=(x,1)_{2p+1},$$

we may similarly consider the function $x'-x$, of order 4;—unless the correspondence be that given by $y'=-y$, $x'=x$, for which $x'-x$ is identically zero. We thus obtain the result that the number of coincidences cannot be greater than 4, except for the (1, 1) correspondence $y'=-y$, $x'=x$.

It can be shewn for the most general possible (r, s) correspondence, associated with the equations

$$v_i^{z_1, c_1}+\dots\dots+v_i^{z_r, c_r}\equiv M_{i,1}v_1^{x, a}+\dots\dots+M_{i,p}v_p^{x, a},\quad M=a+\tau a',\quad M\tau=\beta+\tau\beta',$$

by equating the value obtained for the following integral, taken round the period loops,

$$\int d\left(\Pi_{z_1, c_1}^{x, c}+\dots\dots+\Pi_{z_r, c_r}^{x, c}\right),$$

to the value obtained for the integral taken round the infinities of the subject of integration, that the number of coincidences is

$$C=r+s-(a_{11}+\dots\dots+a_{pp}+\beta'_{11}+\dots\dots+\beta'_{pp}).$$

Since $a_{11}+\dots\dots+\beta'_{pp}$ is the sum of the roots of the equation $|\Delta-\lambda|=0$, it follows for a (1, 1) correspondence, in which all the $2p$ roots of $|\Delta-\lambda|=0$ are roots of unity, that $C \not> 2p+2$. For any (r, s) correspondence belonging to a matrix $\Delta=\begin{pmatrix}-\gamma & 0\\ 0 & -\gamma\end{pmatrix}$, the same formula gives $C=r+s+2p\gamma$, as already found.

We have remarked (§ 386, Ex. iv.) for the case of a (1, 1) correspondence associated with a matrix Δ of the form $\begin{pmatrix}-\gamma & 0\\ 0 & -\gamma\end{pmatrix}$, the existence of a rational function of order $1+\gamma$. For any such (1, 1) correspondence, if p be >1, γ must be equal to $+1$ in order that the number $1+1+2p\gamma$ of coincidences may be $\not> 2p+2$. Thus such a correspondence involves the existence of a rational function of order 2, and involves therefore that the surface be hyperelliptic. This is also obvious from the fact that such a correspondence is associated with equations of the form

$$v_i^{z, c}+\gamma v_i^{x, a}\equiv 0,\qquad (i=1, \dots, p);$$

conversely, for $\gamma=1$, equations of this form are known to hold for any hyperelliptic surface, associated with the correspondence of the conjugate places of the surface. From the considerations here given, it follows for $p>1$ that for a (1, 1) correspondence the number of coincidences can in no case be $>2p+2$.

392. In conclusion it is to be remarked that on any Riemann surface for which $p>1$, there cannot be an infinite number of (1, 1) correspondences. For consider the places of the Riemann surface that can be the poles of rational functions of order $<(p+1)$ which have no other poles (§§ 28, 31, 34—36, Chap. III.). Denote these places momentarily as g-places. As such a (1, 1) correspondence is associated with a linear transformation of integrals of the first kind, which does not affect the zeros of the determinant Δ, of § 31, it follows that the place corresponding to a g-place must also be a g-place. Now, when the surface is not hyperelliptic, every g-place cannot be a coincidence of the correspondence; for we have shewn (Chap. III., § 36) that then the number of distinct g-places is greater than $2p+2$; and we have shewn in this chapter (§ 391) that the number of coincidences in a (1, 1) correspondence, when $p>1$, can in no case be $>2p+2$. Therefore, when the surface is not hyperelliptic, a (1, 1) correspondence must give rise to a permutation among the g-places; since the number of such permutations is finite, the number of (1, 1) correspondences must equally be finite. But the result is equally true for a hyperelliptic surface; for we have shewn (§ 391) that for such a surface the number of coincidences of a (1, 1) correspondence cannot be greater than 4, except in the case of a particular one such correspondence; since the number of distinct g-places is $2p+2$, every (1, 1) correspondence other than this particular one must give rise to a permutation of these g-places. As the number of such permutations is finite, the number of (1, 1) correspondences must equally be finite.

It is proved by Hurwitz* that the number of (1, 1) correspondences, when $p>1$, cannot be greater than $84(p-1)$. In case $p=3$, a surface is known to exist having this number of (1, 1) correspondences†.

393. The preceding proof§ (§ 392) is retained on account of its ingenuity. It can however be replaced by a more elementary proof‡ by means of the remark that a (1, 1) correspondence upon a Riemann surface can be represented by a rational, reversible transformation of the equation of the surface into itself. Let the equation of the surface be $f(x, y)=0$; let (z, s) be the place corresponding to (x, y); then z, s are each rational functions of x and y such that $f(z, s)=0$; conversely x, y are each

* *Math. Annal.* XLI. (1893), p. 424.

† Klein, *Math. Annal.* XIV. (1879), p. 428; *Modulfunctionen*, t. I., 1890, p. 701.

§ Hurwitz, *Math. Annal.* XLI. (1893), p. 406.

‡ Weierstrass, *Math. Werke*, Bd. II. (Berlin, 1895), p. 241.

rational functions of z, s. To give a formal demonstration we may proceed as follows; supposing the number of sheets of the Riemann surface to be n, let $z_1, \ldots, z_n$ denote the places corresponding to the n places $x_1^{(0)}, \ldots, x_n^{(0)}$ for which $x=0$, and let $z_1', \ldots, z'_n$ denote the n places corresponding to the places $x_1^{(\infty)}, \ldots, x_n^{(\infty)}$ for which x is infinite; as x is a rational function on the surface we have, for suitable paths of integration (cf. Chap. VIII. § 154)

$$v_i^{x_1^{(0)}, x_1^{(\infty)}} + \ldots\ldots + v_i^{x_n^{(0)}, x_n^{(\infty)}} = 0, \qquad (i=1, \ldots, p);$$

hence from the equations

$$v_i^{z, c} \equiv M_{i,1} v_1^{x, a} + \ldots\ldots + M_{i,p} v_p^{x, a},$$

we have

$$v_i^{z_1', z_1} + \ldots\ldots + v_i^{z'_n, z_n} \equiv 0, \qquad (i=1, \ldots, p);$$

there exists therefore (Chap. VIII., § 158) a rational function having the places $z_1, \ldots, z_n$ as zeros, and the places $z_1', \ldots, z_n'$ as poles; regarding this as a function of z, s and denoting it by $\phi(z, s)$, it is clear therefore that $x/\phi(z, s)$ is a constant, which may be taken to be 1. Hence $x = \phi(z, s)$, etc.

For the theorem that for $p>1$ the number of (1, 1) correspondences is limited the reader may consult, Schwarz, *Crelle*, LXXXVII. (1879), p. 139, or *Gesamm. Math. Abhand.*, Bd. II. (Berlin, 1890), p. 285; Hettner, *Götting. Nachr.* (1880), p. 386; Noether, *Math. Annal.*, XX. (1882), p. 59; Poincaré, after Klein, *Acta Math.*, VII. (1885); Klein, *Ueber Riemann's Theorie u. s. w.* (Leipzig, 1882), p. 70 etc.; Noether, *Math. Annal.*, XXI. (1883), p. 138; Weierstrass, *Math. Werke*, Bd. II. (Berlin, 1895), p. 241; Hurwitz, *Math. Annal.*, XLI. (1893), p. 406.

394. In regard to the (1, 1) correspondence for the case $p=1$, some remarks may be made. The case $p=0$ needs no consideration here; any (1, 1) correspondence is expressible by an equation of the form

$$Att' + Bt + Ct' + D = 0;$$

thus there exists a triply infinite number of (1, 1) correspondences.

In case $p=1$, if there be a (1, 1) correspondence, whereby the variable place x corresponds to x', and a, a' be simultaneous positions of x and x', it is immediately shewn, if $v^{x, a}$ denote the normal integral of the first kind, that there exists an equation of the form

$$v^{x', a'} \equiv \mu v^{x, a},$$

wherein μ is a constant independent both of a and x. From this equation, by supposing x to describe the period loops, we deduce equations of the form

$$\mu = \alpha + \tau\alpha', \quad \mu\tau = \beta + \tau\beta', \qquad \text{(i)},$$

where α, α', β, β' are integers. By supposing x' to describe the period loops we deduce equations of the form

$$1 = \mu(\gamma + \tau\gamma'), \quad \tau = \mu(\delta + \tau\delta'), \qquad \text{(ii)},$$

where γ, γ', δ, δ' are integers. The expression of these integers in terms of α, α', β, β' is known from the general considerations of this chapter; it is however interesting to consider the equations independently. From the equations (ii) we deduce

$$\delta' - \tau\gamma' = \mu(\gamma\delta' - \gamma'\delta), \quad \delta - \tau\gamma = -\tau\mu(\gamma\delta' - \gamma'\delta);$$

if now $\gamma\delta' - \gamma'\delta = 0$, either γ' and γ are zero, which is inconsistent with $1 = \mu(\gamma + \tau\gamma')$, or else τ is a rational fraction; it is known that in that case the deficiency of the surface is not 1 but 0; we may therefore exclude that case; if $\gamma\delta' - \gamma'\delta$ be not zero, we have

$$\mu = \frac{\delta' - \tau\gamma'}{\gamma\delta' - \gamma'\delta} = a + \tau a', \quad \tau\mu = \frac{\tau\gamma - \delta}{\gamma\delta' - \gamma'\delta} = \beta + \tau\beta';$$

hence, unless τ be a rational fraction, we have

$$\frac{\delta'}{\gamma\delta' - \gamma'\delta} = a, \quad \frac{-\gamma'}{\gamma\delta' - \gamma'\delta} = a', \quad \frac{\gamma}{\gamma\delta' - \gamma'\delta} = \beta', \quad \frac{-\delta}{\gamma\delta' - \gamma'\delta} = \beta,$$

and therefore

$$1 = (a\beta' - a'\beta)(\gamma\delta' - \gamma'\delta);$$

thus $a\beta' - a'\beta = \gamma\delta' - \gamma'\delta = +1$ or -1; let ϵ denote their common value; then we deduce

$$\delta' = \epsilon a, \quad \gamma' = -a'\epsilon, \quad \gamma = \beta'\epsilon, \quad \delta = -\beta\epsilon;$$

by these the equations (ii) lead to

$$a + \tau a' = \mu, \quad \beta + \tau\beta' = \mu\tau,$$

that is, to the equations (i).

Further, from the equations (i) we deduce in turn

$$\tau^2 a' + \tau(a - \beta') - \beta = 0, \quad \mu^2 - \mu(a + \beta') + \epsilon = 0,$$

so that μ is a root of the equation

$$\begin{vmatrix} a - \mu & \beta \\ a' & \beta' - \mu \end{vmatrix} = 0;$$

now if a' be zero, the first of equations (i) gives $\mu = a$, and, therefore, as τ cannot be the rational fraction $\beta/(a - \beta')$, the second of equations (i) gives $a = \beta'$, $\beta = 0$; the equations

$$\mu = a = \beta', \quad a' = \beta = 0, \quad a\beta' - a'\beta = \epsilon$$

give $\mu^2 = \epsilon$, or, since μ, $= a$, is an integer, they require $\epsilon = +1$ and $\mu = +1$ or $\mu = -1$; the equations corresponding to $\mu = +1$ and $\mu = -1$ are

$$v^{x', a'} \equiv v^{x, a} \text{ and } v^{x', a'} + v^{x, a} \equiv 0;$$

these do belong to existing correspondences—of the kind considered in §§ 386, 387, *the coefficient* γ *being* ± 1*. *But they differ from the* (1, 1) *correspondences which are possible when* $p > 1$, *in each containing an arbitrary parameter;*

if next, a' be not zero, the equation for τ gives

$$2\tau a' = -(a - \beta') \pm \sqrt{(a + \beta')^2 - 4\epsilon},$$

so that, as τ cannot be real, we must have

$$(a + \beta')^2 - 4\epsilon < 0,$$

* For instance, on a plane cubic curve, the former equation is that in which to a point of argument u we make correspond the point of argument u + constant; the line joining these two points envelopes a curve of the sixth class, which in case the difference of arguments be a half-period becomes the Cayleyan, doubled; while the latter equation is that in which we make correspond the two variable intersections of a variable straight line passing through a fixed point of the cubic.

and this shews that, in this case also, $\epsilon=1$. Hence the equations are reduced to precisely the same form as those already considered for the special transformation of theta functions (§ 383); and the result is that the only special surfaces, having $p=1$, for which there exists a (1, 1) correspondence are those which may be associated with one of the two equations

$$y^2=4x^3-g_2x, \quad y^2=4x^3-g_3;$$

the former has the obvious (1, 1) correspondence given by $x'=-x$, $y'=iy$; the latter has the obvious correspondence given by $x'=e^{\frac{2i\pi}{3}}x$, $y'=y$; the index of periodicity is 2 in the former case and 3 in the latter case.

Ex. Consider the (1, 2) correspondence on a surface for which $p=1$ in a similar way.

For the equation

$$y^2=8x^3-15x+7$$

shew that a (1, 2) correspondence is given (cf. Ex. ii. § 383) by

$$-2\xi=x+\frac{9}{8(x-1)}, \quad \eta=y\frac{i\sqrt{2}}{4}\frac{x^2-2x-\frac{1}{8}}{(x-1)^2}.$$