CHAPTER XX.

TRANSFORMATION OF THETA FUNCTIONS.

363. IT has been shewn in Chapter XVIII. that a theta function of the first order, in the arguments u, with characteristic (Q, Q'), say $\vartheta(u, Q)$, may be regarded as a theta function of the r-th order in the arguments w, with characteristic (K, K'), provided certain relations, (I), (II), of § 322, p. 532, are satisfied. Let this theta function in w be denoted by $\Pi(w, K)$. We confine ourselves in this chapter, unless the contrary be stated, to the case when (Q, Q') is a half-integer characteristic. Then the function $\vartheta(u, Q)$ is odd or even; therefore, since $u = Mw$, the function $\Pi(w, K)$ is an odd or even function of the arguments w. Now we have shewn, in Chap. XV. (§ 287), that every such odd, or even, theta function of order r, is expressible as a linear function of functions of the form

$$\psi_r(w;\ K,\ K'+\mu) = \vartheta\left[rw;\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right]$$
$$+\epsilon\vartheta\left[-rw;\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right],$$

where ϵ is ± 1, according as the function is even or odd. The most important result of the present chapter is that the functions $\psi_r(w;\ K,\ K'+\mu)$ which occur can be expressed as integral polynomials of the r-th degree in 2^p theta functions $\vartheta\left(w;\ 2v,\ 2v',\ 2\zeta,\ 2\zeta' \,\middle|\, \begin{matrix}R'\\ R\end{matrix}\right)$, whose characteristics are those of a Göpel system of half-integer characteristics (Chap. XVII., § 297); the earlier part (§§ 364—370) of the chapter is devoted to proving this theorem.

The theory is different according as r is odd or even. When r is odd, ϵ is $e^{\pi i |Q|}$, and we have shewn (§ 327 Chap. XVIII.) that, for odd values of r, $|Q| \equiv |K|$, (mod. 2); the theory deals then only with functions

$$\psi_r(w;\ K,\ K'+\mu)$$

in which $\epsilon = e^{\pi i |K|}$. When r is even, ϵ, though still equal to $e^{\pi i |Q|}$, may or may not be equal to $e^{\pi i |K|}$, according to the integer matrix which determines the transformation; but in this case, also, the value of ϵ in the functions $\psi_r(w;\, K,\, K'+\mu)$ which occur is determinate.

The proof of the theorem is furnished by obtaining actual expressions for the functions $\psi_r(w;\, K,\, K'+\mu)$ as integral polynomials of the r-th degree in the 2^p functions $\vartheta\left(w;\, 2v,\, 2v',\, 2\zeta,\, 2\zeta' \,\middle|\, \begin{matrix} R' \\ R \end{matrix}\right)$; the coefficients arising in these polynomials are theta functions whose arguments are r-th parts of periods, of the form $(2vm + 2v'm')/r$. The completion of the theory of the transformation requires that these coefficients should be expressed in terms of constants depending on theta functions with half integer characteristics (§ 373).

Further the theory requires that the coefficients in the expression of the function $\Pi(w;\, K)$ by the functions $\psi_r(w;\, K,\, K'+\mu)$ should be assigned in general. In simple cases this is often an easy matter. The general case is reduced to simpler cases by regarding the general transformation of the r-th order as arising from certain standard transformations for which there is no difficulty as to the coefficients, by the juxtaposition of linear transformations (§§ 371—2)*.

364. It follows from § 332, Chap. XVIII. that any transformation may be obtained by composition of transformations for which the order r is a prime number. It is therefore sufficient theoretically to consider the two cases when $r = 2$, and when r is an odd prime number. We begin with the former case, and shew that the transformed theta function can be expressed as a quadric polynomial in 2^p theta functions belonging to a special Göpel system. A more general expression is given later (§ 370).

* For the transformation of theta functions, and of Abelian functions, the following may be consulted. Jacobi, *Crelle*, VIII. (1832), p. 416; Richelot, *Crelle*, XII. (1834), p. 181, and *Crelle*, XVI. (1837), p. 221; Rosenhain, *Crelle*, XL. (1850), p. 338, and *Mém. par divers Savants*, t. XI. (1851), pp. 396, 402; Hermite, *Liouville*, Ser. 2, t. III. (1858), p. 26, and *Comptes Rendus*, t. XL. (1855); Königsberger, *Crelle*, LXIV. (1865), p. 17, *Crelle*, LXV. (1866), p. 335, *Crelle*, LXVII. (1867), p. 58; Weber, *Crelle*, LXXIV. (1872), p. 69, and *Annali di Mat.* Ser. 2, t. IX. (1878); Thomae, *Ztschr. f. Math. u. Phys.*, t. XII. (1867), and *Crelle*, LXXV. (1872), p. 224; Kronecker, *Berlin. Monatsber.*, 1880, pp. 686, 854; H. J. S. Smith, *Report on the Theory of Numbers, British Association Reports*, 1865, Part VI., § 125 (cf. Weber, *Acta Math.*, VI. (1885), p. 342; Weber, *Elliptische Functionen* (1891), p. 103; Dirichlet, in *Riemann's Werke* (1876), p. 438; Cauchy, *Liouville*, V. (1841), and *Exer. de Math.*, II., p. 118; Gauss, *Werke* (1863), t. II., p. 11 (1808), etc.; Kronecker, *Berlin. Sitzungsber.* 1883; Frobenius, *Crelle*, LXXXIX. (1880), p. 40, *Crelle*, XCVII. (1884), pp. 16, 188, *Crelle*, CV. (1889), p. 35; Wiltheiss, *Crelle*, XCVI. (1884), p. 21; the books of Krause, *Die Transformation der Hyperelliptischen Functionen* (1886), (and the bibliography there given), *Theorie der Doppeltperiodischen Functionen* (1895); Prym u. Krazer, *Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen* (1892), Zweiter Teil. See also references given in Chap. XXI., of the present volume, and in Appendix II.

By means of the equations $u=Mw$, a function $\vartheta\left(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta' \,\middle|\, \begin{matrix} Q' \\ Q \end{matrix}\right)$, with half-integer characteristic $\begin{pmatrix} Q' \\ Q \end{pmatrix}$, becomes a theta function in w, $\Pi(w;\ K,\ K')$, of order 2, with the associated constants 2υ, $2\upsilon'$, 2ζ, $2\zeta'$ and the characteristic $(K,\ K')$, where (§ 324, Chap. XVIII.)

$$2M\upsilon = 2\omega\alpha + 2\omega'\alpha',\quad 2M\upsilon' = 2\omega\beta + 2\omega'\beta',\quad 2\bar{M}(\eta\alpha+\eta'\alpha') = 4\zeta,$$

$$2\bar{M}(\eta\beta+\eta'\beta') = 4\zeta',\quad K' = \bar{\alpha}Q' - \bar{\alpha}'Q - \tfrac{1}{2}d(\bar{\alpha}\alpha'),\quad -K = \bar{\beta}Q' - \bar{\beta}'Q - \tfrac{1}{2}d(\bar{\beta}\beta'),$$

and

$$\bar{\alpha}\alpha' = \bar{\alpha}'\alpha,\quad \bar{\beta}\beta' = \bar{\beta}'\beta,\quad \bar{\alpha}\beta' - \bar{\alpha}'\beta = \bar{\beta}'\alpha - \bar{\beta}\alpha' = 2\,;$$

this theta function in w, $\Pi(w;\ K,\ K')$, can by § 287, p. 463, be expressed as a linear aggregate of terms of the form

$$\psi_r(w;\ K,\ K'+\mu) = \vartheta\left[rw;\ 2\upsilon,\ 2r\upsilon',\ 2\zeta/r,\ 2\zeta' \,\middle|\, \begin{matrix} (K'+\mu)/r \\ K \end{matrix}\right]$$
$$+\ \epsilon\vartheta\left[-rw;\ 2\upsilon,\ 2r\upsilon',\ 2\zeta/r,\ 2\zeta' \,\middle|\, \begin{matrix} (K'+\mu)/r \\ K \end{matrix}\right],$$

r being equal to 2; here ϵ, $=e^{4\pi i QQ'}$, is ± 1, according as the original function, that is, according as the function $\Pi(w;\ K,\ K')$, is even or odd. For brevity we put $w = 2\upsilon W$, $\upsilon\tau' = \upsilon'$, and denoting by $\Theta(W,\ \tau')$ the series $\Sigma e^{2\pi i Wn + i\pi\tau' n^2}$, we consider the function

$$\Psi_r(W;\ K,\ K'+\mu) = \Theta\left[rW;\ r\tau' \,\middle|\, \begin{matrix} (K'+\mu)/r \\ K \end{matrix}\right] + \epsilon\Theta\left[-rW;\ r\tau' \,\middle|\, \begin{matrix} (K'+\mu)/r \\ K \end{matrix}\right],$$

which is equal to $e^{-\frac{1}{2}r\zeta\upsilon^{-1}w^2}\psi_r(w;\ K,\ K'+\mu)$. Throughout the chapter the symbols $\vartheta\left(w \,\middle|\, \begin{matrix} K' \\ K \end{matrix}\right)$, $\Theta\left(W \,\middle|\, \begin{matrix} K' \\ K \end{matrix}\right)$ denote respectively

$$\vartheta\left[w;\ 2\upsilon,\ 2\upsilon',\ 2\zeta,\ 2\zeta' \,\middle|\, \begin{matrix} K' \\ K \end{matrix}\right],\quad \Theta\left(W;\ \tau' \,\middle|\, \begin{matrix} K' \\ K \end{matrix}\right).$$

Taking the final formula of § 291, p. 472, replacing ω, ω', η, η', $\begin{pmatrix} q' \\ q \end{pmatrix}$, $\begin{pmatrix} r' \\ r \end{pmatrix}$ respectively by υ, υ', ζ, ζ', $\tfrac{1}{2}\begin{pmatrix} \alpha' \\ \alpha \end{pmatrix}$, $\tfrac{1}{2}\begin{pmatrix} \alpha' \\ \alpha \end{pmatrix} + \begin{pmatrix} K' \\ K \end{pmatrix}$, multiplying both sides of the equation by $e^{\pi i \alpha(\mu - K' - \alpha')}$, where μ is a row of integers each either 0 or 1, and adding the 2^p equations obtainable by giving α all values in which each of its elements is 0 or 1, we obtain

$$\sum_{\alpha} e^{\pi i\alpha(\mu-K')}\,\Theta\left[V-U;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}\right]\Theta\left[V+U;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}+\binom{K'}{K}\right]$$
$$=\sum_{\alpha} e^{\pi i\alpha(\epsilon'+\mu)}\sum_{\epsilon'}\Theta\left[2V;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}(\epsilon'+K')\\ K\end{matrix}\right]\Theta\left[2U;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}(\epsilon'+K'+\alpha')\\ K\end{matrix}\right],$$

and hence, replacing V, U respectively by W, 0,

$$2^p\Theta\left[2W;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}(\mu+K')\\ K\end{matrix}\right]\Theta\left[0;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}(\mu+K'+\alpha')\\ K\end{matrix}\right]$$
$$=\sum_{\alpha} e^{\pi i\alpha(\mu-K')}\,\Theta\left[W;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}\right]\Theta\left[W;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}+\binom{K'}{K}\right].$$

This may be regarded as the fundamental equation for quadric transformation; we consider various cases of it.

(i) When (K, K') is the zero characteristic we obtain

$$\Theta\left[2W;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}\mu\\ 0\end{matrix}\right]=2^{-p}\sum_{\alpha} e^{\pi i\alpha\mu}\Theta^2\left[W;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}\right]\Big/\Theta\left[0;\ 2\tau' \,\middle|\, \begin{matrix}\tfrac{1}{2}(\mu+\alpha')\\ 0\end{matrix}\right],$$

the right-hand side being independent of α', which for simplicity may be put $=0$.

We can infer that in any quadric transformation, *when the transformed function has zero characteristic, it can be expressed as a linear aggregate of the* 2^p *squares* $\vartheta^2\left(w \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}\right)$, *in which* α' *is an arbitrary row of integers (each* 0 *or* 1) *and* α *has all possible values in which its elements are either* 0 *or* 1.

(ii) When $K'=0$, $K=\tfrac{1}{2}n$ is not zero, we obtain

$$\Theta\left[2W;\ 2\tau' \,\middle|\, \tfrac{1}{2}\binom{\mu}{n}\right]\Theta\left[0;\ 2\tau' \,\middle|\, \tfrac{1}{2}\binom{\mu+\alpha'}{n}\right]$$
$$=2^{-p}\sum_{\alpha} e^{\pi i\alpha\mu}(1+e^{\pi i n(\mu+\alpha')})\,\Theta\left[W;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha}\right]\Theta\left[W;\ \tau' \,\middle|\, \tfrac{1}{2}\binom{\alpha'}{\alpha+n}\right],$$

where on the right side only 2^{p-1} terms are to be taken in the summation in regard to α, two values of α whose difference is a row of elements congruent (mod. 2) to the elements of n not being both admitted. When $\tfrac{1}{2}\binom{\mu}{n}$ is an even characteristic we may put $\alpha'=0$; when $\tfrac{1}{2}\binom{\mu}{n}$ is an odd characteristic we may put $\alpha'=\mu$.

In this case, as before, only 2^p theta functions enter on the right hand, and their characteristics form a special Göpel system.

The cases (i) and (ii) give the transformation of any theta function when the matrix, of 2^p rows and columns, associated with the transformation* is

* For the notation, cf. Chap. XVIII., §§ 322, 324.

$\binom{20}{01}$. It can be shewn that by adjunction of linear transformations every quadric transformation is reducible to this case (cf. § 415 below); so that theoretically no further formulae are required. As it may often be a matter of difficulty to obtain the linear transformations necessary to reduce any given quadric transformation to this one, it is proper to give the formulae for the functions

$$\Psi_2(W;\ K, K'+\mu) = \Theta\left[2W;\ 2\tau' \left| \begin{matrix} \tfrac{1}{2}(\mu+K') \\ K \end{matrix}\right.\right] + \epsilon\Theta\left[-2W;\ 2\tau' \left| \begin{matrix} \tfrac{1}{2}(\mu+K') \\ K \end{matrix}\right.\right];$$

by this means the problem is reduced to finding the coefficients in the expression of any theta function in w, of the second order, in terms of functions $\Psi_2(W;\ K, K'+\mu)$ (see § 372 below). Hence we add the following case.

(iii) When K' is not zero, we deduce, by changing the sign of W in the fundamental formula, the equation

$$2^p\Theta\left[0;\ 2\tau' \left| \begin{matrix} \tfrac{1}{2}(\mu+K'+\alpha') \\ K \end{matrix}\right.\right]\Psi_2(W;\ K, K'+\mu)$$
$$= \sum_{\alpha} e^{\pi i\alpha(\mu-K')}\, C_\alpha \Theta\left[W;\ \tau' \left| \tfrac{1}{2}\binom{\alpha'}{\alpha}\right.\right]\Theta\left[W;\ \tau' \left| \tfrac{1}{2}\binom{\alpha'}{\alpha} + \binom{K'}{K}\right.\right],$$

where, putting $K=\tfrac{1}{2}k$, $K'=\tfrac{1}{2}k'$, we have $C_\alpha = 1 + \epsilon e^{\pi ik(k'+\alpha')+\pi i\alpha k'}$. When ϵ is $+1$, there are 2^{p-1} values of α for which $\alpha k' \equiv k(k'+\alpha')+1$ (§ 295, Chap. XVII.); for these values $C_\alpha = 0$; when $\epsilon = -1$, there are 2^{p-1} values of α for which $\alpha k' \equiv k(k'+\alpha')$; for these values $C_\alpha = 0$. In either case it follows that the right side of the equation contains only 2^{p-1} terms, and contains only 2^p theta functions whose characteristics are a special Göpel system.

It is easy to see that the results of cases (ii) and (iii) can be summarised as follows: *when the characteristic* (K, K') *is not zero the transformed function is a linear aggregate of* 2^{p-1} *products of the form* $\vartheta[w;\ A, P_i]\,\vartheta[w;\ A, K, P_i]$ *wherein the* 2^{p-1} *characteristics* P_i *are of the form* $\tfrac{1}{2}\binom{0}{\alpha}$, $K = \binom{K'}{-K}$, *and* A, K *are such that** $e^{\pi i|K| + \pi i|A, K|} = \epsilon$.

These results are in accordance with § 288, Chap. XV.; there being $2^{p-1}(1+\epsilon)$ linearly independent theta functions of the second order with zero characteristic and of character ϵ, namely 2^p such *even* functions and no odd functions, and there being 2^{p-1} linearly independent theta functions of the second order with characteristic other than zero.

365. *Ex.* i. When $p=1$, the results of case (i), if we put $\Theta_{gh}(W;\ \tau')$ for $\Theta\left[W;\ \tau' \left| \tfrac{1}{2}\binom{g}{-h}\right.\right]$, as is usual, are

$$\Theta_{00}(2W;\ 2\tau') = \frac{\Theta_{00}^2(W;\ \tau') + \Theta_{01}^2(W;\ \tau')}{2\Theta_{00}(2\tau')} = \frac{\Theta_{10}^2(W;\ \tau') + \Theta_{11}^2(W;\ \tau')}{2\Theta_{10}(2\tau')},$$

* For the notation, see Chap. XVII., § 294.

and

$$\Theta_{10}(2W;\ 2\tau')=\frac{\Theta_{00}^2(W;\ \tau')-\Theta_{01}^2(W;\ \tau')}{2\Theta_{10}(2\tau')}=\frac{\Theta_{10}^2(W;\ \tau')-\Theta_{11}^2(W;\ \tau')}{2\Theta_{00}(2\tau')},$$

where $\Theta(2\tau')$ denotes $\Theta(0;\ 2\tau')$. If then we introduce the notations

$$\sqrt{k}=\frac{\Theta_{10}(2\tau')}{\Theta_{00}(2\tau')},\quad \sqrt{k'}=\frac{\Theta_{01}(2\tau')}{\Theta_{00}(2\tau')},\quad \sqrt{\lambda}=\frac{\Theta_{10}(\tau')}{\Theta_{00}(\tau')},\quad \sqrt{\lambda'}=\frac{\Theta_{01}(\tau')}{\Theta_{00}(\tau')},$$

$$\sqrt{x}=\frac{1}{\sqrt{k}}\frac{\Theta_{11}(2W;\ 2\tau')}{\Theta_{01}(2W;\ 2\tau')},\quad \sqrt{y}=\sqrt{\frac{k'}{k}}\frac{\Theta_{10}(2W;\ 2\tau')}{\Theta_{01}(2W;\ 2\tau')},\quad \sqrt{z}=\sqrt{k'}\frac{\Theta_{00}(2W;\ 2\tau')}{\Theta_{01}(2W;\ 2\tau')},$$

$$\sqrt{\xi}=\frac{1}{\sqrt{\lambda}}\frac{\Theta_{11}(W;\ \tau')}{\Theta_{01}(W;\ \tau')},\quad \sqrt{\eta}=\sqrt{\frac{\lambda'}{\lambda}}\frac{\Theta_{10}(W;\ \tau')}{\Theta_{01}(W;\ \tau')},\quad \sqrt{\zeta}=\sqrt{\lambda'}\frac{\Theta_{00}(W;\ \tau')}{\Theta_{01}(W;\ \tau')},$$

we find by multiplying the equations above that

$$\Theta_{00}^4(W;\ \tau')-\Theta_{01}^4(W;\ \tau')=\Theta_{10}^4(W;\ \tau')-\Theta_{11}^4(W;\ \tau'),$$

and therefore that

$$\lambda^2+\lambda'^2=1,$$

so that also

$$k^2+k'^2=1;$$

while, comparing the two forms for $\Theta_{00}(2W;\ 2\tau')$, putting $W=0$, we obtain

$$\sqrt{k}=\frac{\lambda}{1+\lambda'},\ \text{ or }\ k=\frac{1-\lambda'}{1+\lambda'},\ \text{ giving }\ \lambda=\frac{2\sqrt{k}}{1+k};$$

further the equations for $\Theta_{00}(2W;\ 2\tau')$ and $\Theta_{10}(2W;\ 2\tau')$ give the results

$$\frac{\zeta+\lambda'}{\eta+\lambda'\xi}=1+\lambda',\quad \frac{\zeta-\lambda'}{\eta-\lambda'\xi}=1-\lambda',$$

from which we find

$$\eta=1-\xi,\ \zeta=1-\lambda^2\xi;\ \text{thus also}\ y=1-x,\ z=1-k^2x.$$

Ex. ii. The equations of case (ii), also for $p=1$, give

$$\Theta_{01}(2W;\ 2\tau')=\frac{\Theta_{00}(W;\ \tau')\,\Theta_{01}(W;\ \tau')}{\Theta_{01}(2\tau')},\quad \Theta_{11}(2W;\ 2\tau')=\frac{\Theta_{10}(W;\ \tau')\,\Theta_{11}(W;\ \tau')}{\Theta_{01}(2\tau')}.$$

From these we have by division

$$\sqrt{x}=(1+\lambda')\frac{\sqrt{\xi(1-\xi)}}{\sqrt{1-\lambda^2\xi}},$$

while from these and the results of Ex. 1, we find

$$\sqrt{y}=[1-(1+\lambda')\,\xi]/\sqrt{1-\lambda^2\xi},\quad \sqrt{z}=[1-(1-\lambda')\,\xi]/\sqrt{1-\lambda^2\xi}.$$

Ex. iii. When $p=1$, by considering the change in the value of the function

$$\vartheta_{01}^2(w)\frac{d}{dw}\left[\frac{\vartheta_{11}(w)}{\vartheta_{01}(w)}\right]$$

when w is increased by a period, we immediately find that it is a theta function in w of the second order with characteristic $\frac{1}{2}\binom{1}{0}$; hence by the result of case (iii) above, the function is a constant multiple of $\vartheta_{10}(w)\,\vartheta_{00}(w)$; determining the constant by putting $w=0$, we obtain the equation

$$\begin{aligned}\Theta_{00}(\tau')\,\Theta_{10}(\tau')\,[\Theta'_{11}(W;\ \tau')\,\Theta_{01}(W;\ \tau')-\Theta'_{01}(W;\ \tau')\,\Theta_{11}(W;\ \tau')]\\ =\Theta'_{11}(\tau')\,\Theta_{01}(\tau')\,\Theta_{10}(W;\ \tau')\,\Theta_{00}(W;\ \tau'),\end{aligned}$$

which is immediately seen to be equivalent to

$$\frac{\Theta'_{11}(\tau')\,\Theta_{00}(\tau')}{\Theta_{01}(\tau')\,\Theta_{10}(\tau')}\,W=\int_0^\xi \frac{d\xi}{\sqrt{4\xi(1-\xi)(1-\lambda^2\xi)}}.$$

[We may obtain the theta relation, here deduced, from the addition formula of Ex. i., § 286, p. 457; taking therein $m=\tfrac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix}$, $a_1=\tfrac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$, $a_2=\tfrac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix}$, $w=0$, $q=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$, $r=p=\tfrac{1}{2}\begin{pmatrix}0\\0\end{pmatrix}$, we immediately derive

$$\vartheta_{10}(u)\,\vartheta_{00}(u)\,\vartheta_{11}(2v)\,\vartheta_{01}(0)=\vartheta_{00}(v)\,\vartheta_{10}(v)\,[\vartheta_{01}(u-v)\,\vartheta_{11}(u+v)-\vartheta_{01}(u+v)\,\vartheta_{11}(u-v)];$$

if, for small values of v, this equation be expanded in powers of v, and the coefficients of v on the two sides be put equal, there results the equation in question.]

Ex. iv. By differentiating the second result of Ex. ii., putting $W=0$, and putting $W=0$ in the first result of the same example and in the second value for $\Theta_{00}(2W;\ 2\tau')$ in Ex. i., we obtain

$$\frac{\Theta'_{11}(2\tau')}{\Theta_{00}(2\tau')\,\Theta_{01}(2\tau')\,\Theta_{10}(2\tau')}=\frac{\Theta'_{11}(\tau')}{\Theta_{00}(\tau')\,\Theta_{01}(\tau')\,\Theta_{10}(\tau')},$$

so that the second of these functions is unaltered by replacing τ' by $2^n\tau'$, n being as large as we please. Hence we immediately find from the series for the functions, by putting $\tau'=\infty$, that each of these fractions is equal to π. Hence if the integral occurring in the last example be denoted by J we have $J=\pi\Theta_{00}^2(\tau')\,W$. In precisely the same way we find $I=2\pi\Theta_{00}^2(2\tau')\,W$, where I is an integral differing only from J by the substitution of x for ξ and k for λ. Hence

$$I/J=2\Theta_{00}^2(2\tau')/\Theta_{00}^2(\tau'),\ =1+\lambda',$$

as follows from the first result of Ex. 1.

From these results we are justified in writing the formula of Ex. ii. in the form

$$\operatorname{sn}\left[(1+\lambda')\,J;\ \frac{1-\lambda'}{1+\lambda'}\right]=\frac{(1+\lambda')\operatorname{sn}(J,\lambda)\operatorname{cn}(J,\lambda)}{\operatorname{dn}(J,\lambda)};$$

and this is Landen's first transformation for Elliptic functions.

Ex. v. The preceding examples deal, in the case $p=1$, with the quadric transformation associated with the matrix $\begin{pmatrix}2&0\\0&1\end{pmatrix}$. Prove when $p=1$ that for any matrix of quadric transformation the transformed theta function is expressible linearly in terms of one or more of the eight functions

$$\Theta=\Theta_{00}(2W;\ 2\tau'),\quad \Theta_2=\Theta_{10}(2W;\ 2\tau'),\quad \Theta_0=\Theta_{01}(2W;\ 2\tau'),\quad \Theta_1=\Theta_{11}(2W;\ 2\tau'),$$

$$\Theta_4=\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}1/4\\0\end{matrix}\right)+\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}-1/4\\0\end{matrix}\right),\quad \Theta_5=\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}1/4\\0\end{matrix}\right)-\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}-1/4\\0\end{matrix}\right),$$

$$\Theta_6=\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}1/4\\1/2\end{matrix}\right)+i\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}-1/4\\1/2\end{matrix}\right),\quad \Theta_7=\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}1/4\\1/2\end{matrix}\right)-i\Theta\left(2W;\ 2\tau'\,\middle|\,\begin{matrix}-1/4\\1/2\end{matrix}\right).$$

Prove in particular that the functions arising for the transformation associated with the matrix $\begin{pmatrix}1&0\\0&2\end{pmatrix}$ are expressed as follows:

$$\Theta_{00}(W;\ \tfrac{1}{2}\tau')=\Theta+\Theta_2,\quad \Theta_{01}(W;\ \tfrac{1}{2}\tau')=\Theta-\Theta_2,\quad \Theta_{10}(W;\ \tfrac{1}{2}\tau')=\Theta_4,\quad \Theta_{11}(W;\ \tfrac{1}{2}\tau')=-i\Theta_5;$$

and that the functions arising for the transformation associated with the matrix $\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ are expressed as follows :

$$\Theta_{00}(W;\ \tfrac{1}{2}\tau'-\tfrac{1}{2})=\Theta-i\Theta_2,\quad \Theta_{01}(W;\ \tfrac{1}{2}\tau'-\tfrac{1}{2})=\Theta+i\Theta_2,$$

$$\Theta_{10}(W;\ \tfrac{1}{2}\tau'-\tfrac{1}{2})=e^{-\frac{3\pi i}{8}}\Theta_6,\quad \Theta_{11}(W;\ \tfrac{1}{2}\tau'-\tfrac{1}{2})=e^{\frac{\pi i}{8}}\Theta_7.$$

Obtain from the formulae of the text the expressions of the functions Θ_4, Θ_5, Θ_6, Θ_7 of the form

$$\Theta_4=C_4\Theta_{00}(W)\Theta_{10}(W),\ \Theta_5=C_5\Theta_{01}(W)\Theta_{11}(W),\ \Theta_6=C_6\Theta_{01}(W)\Theta_{10}(W),\ \Theta_7=C_7\Theta_{00}(W)\Theta_{11}(W),$$

where C_4, C_5, C_6, C_7 are constants.

Ex. vi. The reason why the matrices $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ are selected in Ex. v. will appear subsequently (§ 415) ; the matrix $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ gives the transformation which is *supplementary* to that given by $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$; it gives results leading to the equation

$$\operatorname{sn}[(1+k)u,\ 2\sqrt{k}/(1+k)]=(1+k)\operatorname{sn}(u, k)/[1+k\operatorname{sn}^2(u, k)]\ ;$$

by combination of these results with those for the matrix $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ we obtain the multiplication formula

$$\Theta_{11}(2W;\ \tau')=A\Theta_{11}(W;\ \tau')\Theta_{01}(W;\ \tau')\Theta_{10}(W;\ \tau')\Theta_{00}(W;\ \tau'),$$

where A is a constant (cf. Ex. vii., § 317, Chap. XVII. and § 332, Chap. XVIII.).

The matrix associated with any quadric transformation can be put into the form

$$\Omega\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\Omega',$$

where Ω, Ω' are matrices of linear transformations ; for instance we have

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

with the corresponding equations

$$U=\tau W_1,\quad W_1=2W_2,\quad W_2=-\tau_2 W_3;\qquad \tau_1=-1/\tau,\quad \tau_2=\tau_1/2,\quad \tau_3=-1/\tau_2,$$

from which we have, for instance,

$$\Theta_{10}(W_3;\ \tfrac{1}{2}\tau_3)=\Theta_{10}(U;\ \tau)=e^{\frac{-\pi i U^2}{\tau}}\sqrt{\frac{i}{\tau}}\,\Theta_{01}(W_1;\ \tau_1)=e^{\frac{-\pi i U^2}{\tau}}\sqrt{\frac{i}{\tau}}\,\Theta_{01}(2W_2;\ 2\tau_2)$$

$$=e^{\frac{-\pi i U^2}{\tau}}E\Theta_{00}(W_2;\ \tau_2)\Theta_{01}(W_2;\ \tau_2)=F\Theta_{00}(W_3;\ \tau_3)\Theta_{10}(W_3;\ \tau_3),$$

(E, F being constants) whereby the transformation formula for $\Theta_{10}(W_3;\ \tfrac{1}{2}\tau_3)$ is obtained from those for $\Theta_{10}(2W;\ 2\tau')$, with the help of those arising for linear transformation.

366. We pass now to the case when the order of transformation is any odd number, dealing with the matter in a general way. Simplifications that can theoretically be always introduced by means of linear transformations are considered later (§ 372).

We first investigate a general formula* whereby the function

$$\vartheta\left[rw\,;\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta'\,\middle|\,\begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right]$$

can be expressed in terms of products of functions with associated constants $2v$, $2v'$, 2ζ, $2\zeta'$. We shall then afterwards employ the formulae developed in Chap. XVII., to express these products in the required form.

Let σ, σ' be two matrices each of p rows and m columns, whose constituents are any constants; let the j-th columns of these be denoted respectively by $\sigma^{(j)}$ and $\sigma'^{(j)}$, so that the values of j are $1, 2, \ldots, m$; let Υ_σ denote the matrix $2v\sigma + 2v'\sigma'$, which has p rows and m columns, and let the j-th column of this matrix, which is given by $2v\sigma^{(j)} + 2v'\sigma'^{(j)}$, be denoted by $\Upsilon_\sigma^{(j)}$; also, K, K' being rows of any p real rational elements, let Υ_K, Z_K denote the rows $2vK + 2v'K'$, $2\zeta K + 2\zeta' K'$; and use the abbreviation

$$\varpi(w\,;\ K,\ K') = Z_K(w + \tfrac{1}{2}\Upsilon_K) - \pi i KK'\,;$$

finally, let $s = (s^{(1)}, \ldots, s^{(m)})$ be a column of m integers whose squares have the sum r, so that

$$\sum_j [s^{(j)}]^2 = r\,;$$

then, using always $\vartheta(w)$ for $\vartheta(w\,;\ 2v,\ 2v',\ 2\zeta,\ 2\zeta')$, *the function*

$$\Pi(w) = e^{-r\varpi[w;\ K/r,\ K'/r]} \prod_{j=1}^{m} \vartheta\left[s^{(j)}\left(w + \frac{\Upsilon_K - \Upsilon_\sigma s}{r}\right) + \Upsilon_\sigma^{(j)}\right]$$

is, in w, *a theta function of order* r *with associated constants* $2v$, $2v'$, 2ζ, $2\zeta'$ *and characteristic* (K, K').

For when the arguments w are increased by the elements of the row Υ_N, where N, N' are rows of p integers, the function

$$\vartheta\left[s^{(j)}\left(w + \frac{\Upsilon_K - \Upsilon_\sigma s}{r}\right) + \Upsilon_\sigma^{(j)}\right]$$

is multiplied by a factor e^{ψ_j}, where ψ_j is equal to

$$[2\zeta N s^{(j)} + 2\zeta' N' s^{(j)}]\left[s^{(j)}\left(w + \frac{\Upsilon_K - \Upsilon_\sigma s}{r}\right) + \Upsilon_\sigma^{(j)} + vNs^{(j)} + v'N's^{(j)}\right] - \pi i\,[Ns^{(j)}]\,[N's^{(j)}],$$

that is

$$[s^{(j)}]^2\left\{Z_N\left(w + \frac{\Upsilon_K - \Upsilon_\sigma s}{r} + \tfrac{1}{2}\Upsilon_N\right) - \pi i NN'\right\} + Z_N \Upsilon_\sigma^{(j)} s^{(j)}\,;$$

the sum of the m values of ψ_j is given by

$$\begin{aligned}\sum_{j=1}^{m} \psi_j &= r\,\{Z_N(w + \tfrac{1}{2}\Upsilon_N) - \pi i NN'\} + Z_N\Upsilon_K - Z_N\Upsilon_\sigma s + Z_N\Upsilon_\sigma s\\ &= r\varpi(w\,;\ N,\ N') + Z_N\Upsilon_K\,;\end{aligned}$$

* Königsberger, *Crelle*, LXIV. (1865), p. 28. See Rosenhain, *Crelle*, XL. (1850), p. 338, and *Mém. par divers Savants*, t. XI. (1851), p. 402.

also, when w is increased by Υ_N, the function $-r\varpi\,[w\,;\ K/r,\ K'/r)$ is increased by $-Z_K\Upsilon_N$; thus the complete resulting factor of $\Pi\,(w)$ is

$$e^{r\varpi\,(w\,;\ N,\ N')+Z_N\Upsilon_K-Z_K\Upsilon_N},$$

of which (§ 190, p. 285) the exponent is equal to

$$r\varpi\,(w\,;\ N,\, N')+2\pi i\,(NK'-N'K)\,;$$

thus (§ 284, p. 448) $\Pi\,(w)$ is a theta function in w, of the r-th order with $(K,\, K')$ as characteristic.

Therefore (§ 284, p. 452) we have an equation

$$\Pi\,(w)=\sum_{\mu} A_\mu \vartheta\left[rw\,;\ 2v,\, 2rv',\, 2\zeta/r,\, 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right],$$

where μ is a row of p integers each positive (including zero) and less than r, and the coefficients A_μ are independent of w. *The coefficients A_μ are independent of K, K'*, as we see immediately by first proving the equation which arises from this equation by putting K and K' zero, and then, in that equation, replacing w by $w+2vK/r+2v'K'/r$.

In this equation, replace K by $K+h$, where h is a row of p integers, each positive (including zero) and less than r; then, using the equation previously written (§ 190, p. 286), for integral M, in the form

$$\vartheta\,(u\,;\ q+M)=e^{2\pi i M q'}\,\vartheta\,(u\,;\ q),$$

we find

$$e^{-r\varpi\,[w\,;\ (K+h)/r,\ K'/r]-2\pi i\,(K'+\epsilon)\,h/r}\prod_{j=1}^{m}\vartheta\left[s^{(j)}\left(w+\frac{2vh+\Upsilon_K-\Upsilon_\sigma s}{r}\right)+\Upsilon_\sigma^{(j)}\right]$$
$$=\sum_{\mu} A_\mu e^{2\pi i(\mu-\epsilon)\,h/r}\,\vartheta\left[rw\,;\ 2v,\, 2rv',\, 2\zeta/r,\, 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right],$$

where ϵ is taken to be any row of p integers each positive (or zero) and less than r; ascribing now to h all the possible r^p values, and using the fact that

$$r^{-p}\sum_{h} e^{2\pi i(\mu-\epsilon)\,h/r}=1,\ \text{ or }\ 0,$$

according as $\mu-\epsilon\equiv 0$ or $\not\equiv 0$, (mod. r), we infer, by addition, the equation

$$C_\mu\vartheta\left[rw\,;\ 2v,\, 2rv',\, 2\zeta/r,\, 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r)\\ K\end{matrix}\right]$$
$$=\sum_{h} e^{\psi}\prod_{j=1}^{m}\vartheta\left[s^{(j)}\left(w+\frac{2vh+\Upsilon_K-\Upsilon_\sigma s}{r}\right)+\Upsilon_\sigma^{(j)}\right],$$

where

$$\psi=-r\varpi\,[w\,;\ (K+h)/r,\ K'/r]-2\pi i\,(K'+\mu)\,h/r,$$

and C_μ, $=r^p A_\mu$, is independent of w and of the characteristic $(K,\, K')$.

367. We put down now two cases of this very general formula:—

(α) if each of the matrices σ, σ' consist of zeros, and each of the m integers $s^{(1)}, \ldots, s^{(m)}$ be unity, so that $m = r$, we obtain

$$C_\mu \vartheta\left[rw;\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right]$$
$$= \sum_h e^{-r\varpi[w;\ (K+h)/r,\ K'/r] - 2\pi i (K'+\mu) h/r}\, \vartheta^r\left[w + \frac{2vh + \Upsilon_K}{r}\right].$$

In using this equation we shall make the simplification which arises by putting $w = 2vW$, $v^{-1}v' = \tau'$, and

$$\Theta(W, \tau') = e^{-\frac{1}{2}\zeta v^{-1} w^2}\, \vartheta(w) = \sum_n e^{2\pi i W n + i\pi\tau' n^2};$$

then the equation can be transformed without loss of generality, by means of the relations connecting the matrices v, v', ζ, ζ' (cf. § 284, p. 447), to the form

$$C_\mu e^{-2\pi i K'[W + \frac{1}{2}\tau' K'/r] - 2\pi i K K'/r}\, \Theta\left(rW;\ r\tau' \,\middle|\, \begin{matrix}(K'+\mu)/r\\ K\end{matrix}\right)$$
$$= \sum_h e^{-2\pi i \mu h/r}\, \Theta^r\left[W + \frac{h + K + \tau' K'}{r};\ \tau'\right], \qquad \text{(I)}$$

where C_μ is independent of W and of K and K'.

This equation is of frequent application in this chapter; it is of a different character from the multiplication formula given Chap. XVII., § 317, Ex. vii., whereby the function $\Theta(rW, \tau')$ was expressed by functions $\Theta(W, \tau')$ with different characteristics but the same *period*, τ'.

Ex. i. When $r=2$, $p=2$, we have

$$C_0\Theta(2W, 2\tau') = \Theta^2(W_1, W_2;\ \tau') + \Theta^2(W_1+\tfrac{1}{2}, W_2;\ \tau') + \Theta^2(W_1, W_2+\tfrac{1}{2};\ \tau')$$
$$+ \Theta^2(W_1+\tfrac{1}{2}, W_2+\tfrac{1}{2};\ \tau').$$

Ex. ii. If λ, μ, h be rows of p integers each less than r, prove that the ratio

$$\sum_h e^{-2\pi i \mu h/r}\, \Theta^r\left[W + \frac{h}{r} \,\middle|\, \begin{matrix}\lambda/r\\ 0\end{matrix}\right] \div \sum_\mu e^{-2\pi i \mu h/r}\, \Theta^r\left[W + \frac{h}{r}\right]$$

is independent of W.

(β) if the matrix σ' consist of zeros, and if each of the m integers $s^{(1)}, \ldots, s^{(m)}$ be unity, so that $m = r$, and if the matrix σ, of p rows and r columns, have, for the constituents of every one of its rows, the elements

$$0,\ \frac{1}{r},\ \frac{2}{r},\ \ldots\ldots,\ \frac{r-1}{r},$$

then the matrix Υ_σ will have, for the constituents of its i-th row, the elements

$$0,\ \frac{\Omega_i}{r},\ \frac{2\Omega_i}{r},\ \ldots\ldots,\ \frac{(r-1)\,\Omega_i}{r},$$

where Ω_i is the sum of the elements of the i-th row of the matrix $2v$, so that

$$\Omega_i = 2 \sum_{h=1}^{p} v_{i,h};$$

also the i-th of the p elements denoted by $\frac{1}{r}\Upsilon_\sigma s$ will be

$$\frac{1}{r}\left[\frac{\Omega_i}{r} + \dots\dots + \frac{(r-1)\Omega_i}{r}\right] = \frac{r-1}{2r}\Omega_i,$$

and therefore the i-th of the elements of $\Upsilon_\sigma^{(j)} - \frac{1}{r}\Upsilon_\sigma s$ will be

$$\frac{j-1}{r}\Omega_i - \frac{r-1}{2r}\Omega_i.$$

Thus, denoting the row $(\Omega_1, \dots, \Omega_p)$ by Ω, the theorem is

$$C_\mu \vartheta\left[rw,\, 2v,\, 2rv',\, 2\zeta/r,\, 2\zeta' \,\middle|\, \begin{matrix}(K'+\mu)/r \\ K\end{matrix}\right]$$
$$= \sum_h e^{\psi} \prod_{j=1}^{r} \vartheta\left[w + \frac{2vh + \Upsilon_K}{r} + \left(\frac{j-1}{r} - \frac{r-1}{2r}\right)\Omega\right],$$

where ψ has the same value as in § 366. And as before this result can be written without loss of generality in the form

$$C_\mu e^{-2\pi i K'[W + \frac{1}{2}\tau' K'/r] - 2\pi i K K'/r}\, \Theta\left[rW,\, r\tau' \,\middle|\, \begin{matrix}(K'+\mu)/r \\ K\end{matrix}\right]$$
$$= \sum_h e^{-2\pi i \mu h/r} \phi\left(U + \frac{h + K + \tau' K'}{r}\right), \qquad \text{(II)}$$

where $U = W - (r-1)/2r$ and, for any value of u,

$$\phi(u) = \Theta(u;\, \tau')\, \Theta\left(u + \frac{1}{r};\, \tau'\right) \dots\dots \Theta\left(u + \frac{r-1}{r};\, \tau'\right);$$

the number of different terms on the right side of this equation is r^{p-1}; for if m be a positive integer less than r, the two values of h expressed by $h = (h_1, \dots, h_p)$ and $h = (h_1', \dots, h_p')$, in which $h_1' \equiv h_1 + m, \dots, h_p' \equiv h_p + m$, (mod. r), give the same value for $\phi\left(U + \frac{h + K + \tau' K'}{r}\right)$.

Ex. i. For $p=2$, $r=2$, we obtain

$$\tfrac{1}{2}C_0\Theta(2W, 2\tau') = \Theta(W_1 - \tfrac{1}{4}, W_2 - \tfrac{1}{4};\, \tau')\,\Theta(W_1 + \tfrac{1}{4}, W_2 + \tfrac{1}{4};\, \tau')$$
$$+ \Theta(W_1 + \tfrac{1}{4}, W_2 - \tfrac{1}{4};\, \tau')\,\Theta(W_1 - \tfrac{1}{4}, W_2 + \tfrac{1}{4};\, \tau').$$

Ex. ii. For $p=2$, $r=3$, we obtain, omitting the period τ' on the right side,

$$\tfrac{1}{3}C_0\Theta(3W;\, 3\tau') = \Theta(W_1, W_2)\,\Theta(W_1 - \tfrac{1}{3}, W_2 - \tfrac{1}{3})\,\Theta(W_1 + \tfrac{1}{3}, W_2 + \tfrac{1}{3})$$
$$+ \Theta(W_1, W_2 - \tfrac{1}{3})\,\Theta(W_1 + \tfrac{1}{3}, W_2)\,\Theta(W_1 - \tfrac{1}{3}, W_2 + \tfrac{1}{3})$$
$$+ \Theta(W_1 + \tfrac{1}{3}, W_2 - \tfrac{1}{3})\,\Theta(W_1 - \tfrac{1}{3}, W_2)\,\Theta(W_1, W_2 + \tfrac{1}{3}).$$

368. We consider now the expression of the function

$$\Psi_r(W;\ K, K'+\mu) = \Theta\left[rW;\ r\tau' \,\middle|\, \begin{matrix}(K'+\mu)/r \\ K\end{matrix}\right] + \epsilon\Theta\left[-rW;\ r\tau' \,\middle|\, \begin{matrix}(K'+\mu)/r \\ K\end{matrix}\right],$$

in terms of functions $\Theta\left[W;\ \tau' \,\middle|\, \begin{matrix}R' \\ R\end{matrix}\right]$, in the case when r is odd. We suppose as before (K, K') to be a half-integer characteristic, and we suppose $\epsilon = e^{\pi i |K|}$, so that ϵ is ± 1 according as the characteristic (K, K') is even or odd*. It follows from § 327, Chap. XVIII., if (K, K') has arisen by transformation of order r from a characteristic (Q, Q'), that ϵ is also equal to $e^{\pi i |Q|}$ and is ± 1 according as the function is even or odd.

It is immediately seen that equation (I) (§ 367) can be put into the form

$$C_\mu e^{2\pi i (r-1) K' \left[W + \frac{K}{r} - \frac{r-1}{2r}\tau' K'\right]} \Theta\left[rW;\ r\tau' \,\middle|\, \begin{matrix}(K'+\mu)/r \\ K\end{matrix}\right]$$

$$= \sum_h e^{-2\pi i \left(K' + \frac{\mu}{r}\right) h}\, \Theta^r \left[W + \frac{h-(r-1)(K+\tau' K')}{r} \,\middle|\, \begin{matrix}K' \\ K\end{matrix}\right];$$

from this equation by changing the sign of W, we deduce the result

$$C_\mu e^{2\pi i \frac{r-1}{r} K' [K - (r-1)\tau' K'/2]}\, \Psi_r[W;\ K, K'+\mu]$$

$$= \sum_h e^{-2\pi i \left(K'+\frac{\mu}{r}\right) h} \left\{ e^{-2\pi i (r-1) K' W}\, \Theta^r\left[W + a \,\middle|\, \begin{matrix}K' \\ K\end{matrix}\right] + e^{2\pi i (r-1) K' W}\, \Theta^r\left[W - a \,\middle|\, \begin{matrix}K' \\ K\end{matrix}\right] \right\},$$

where we have replaced $\epsilon e^{-4\pi i r K K'}, = \epsilon e^{-\pi i r |K|}$, by unity, and a denotes the expression $[h-(r-1)(K+\tau' K')]/r$, which is an r-th part of a period. We proceed to shew that *the function*

$$e^{-2\pi i (r-1) K' W}\, \Theta^r\left[W + a \,\middle|\, \begin{matrix}K' \\ K\end{matrix}\right] + e^{2\pi i (r-1) K' W}\, \Theta^r\left[W - a \,\middle|\, \begin{matrix}K' \\ K\end{matrix}\right]$$

can be expressed as an integral polynomial of the r-th degree in 2^p *functions* $\Theta^r[W;\ \tau' \,|\, AP_i]$, where AP_i *are the characteristics of any Göpel system of half-integer characteristics whereof* (K, K') *is one characteristic.*

From the formula of § 311, p. 513, putting $C=0$, $A'=A$, $B=P=\frac{1}{2}\binom{q'}{q}$, and replacing U, V, W, $\binom{P_i}{A}\epsilon_i$, $\binom{P_j}{A}\epsilon_j$ respectively by W, a, b, ϵ_i, ϵ_j we obtain, if $P_a = \frac{1}{2}\binom{q_a'}{q_a}$,

* Thus, when $2(K'+\mu) = rm$, m being integral,

$$\epsilon = e^{2\pi i K(rm-2\mu)} = e^{2\pi i K m} = e^{4\pi i K \frac{K'+\mu}{r}},$$

as in § 287, Chap. XV., and

$$\Psi_r(W;\ K, K'+\mu) \text{ reduces to } 2\Theta\left[rW, r\tau' \,\middle|\, \begin{matrix}m/2 \\ K\end{matrix}\right].$$

$$2^p\Theta(W+a;\ A+P)\Theta(W+b;\ A)$$

$$=\sum_\epsilon \frac{\chi(a,b;\ P,\epsilon)}{\chi(a+b,0;\ P,\epsilon)}\sum_a \epsilon_a e^{-\frac{1}{2}\pi i q' q_a}\Theta(W+a+b;\ A+P+P_a)\Theta(W;\ A+P_a),$$

where

$$\chi(u,v;\ P,\epsilon)=\sum_a \epsilon_a e^{-\frac{1}{2}\pi i q' q_a}\Theta(u;\ A+P+P_a)\Theta[v;\ A+P_a];$$

the function $\chi(u,v;\ A,P,\epsilon)$ may be immediately shewn to be unaltered by the addition of an integral characteristic to the characteristic P_a of one of its terms; we may therefore suppose all these characteristics to be reduced characteristics, each element being 0 or $\frac{1}{2}$.

Hence we get

$$2^p\Theta^2(W+a;\ A)=\sum_\epsilon \frac{\chi(a,a;0,\epsilon)}{\chi(2a,0;0,\epsilon)}\sum_a \epsilon_a\Theta(W+2a;\ A+P_a)\Theta(W;\ A+P_a),$$

and hence $2^{2p}\Theta^3(W+a;\ A)$ is equal to

$$\sum_\epsilon H_1\sum_a \epsilon_a\Theta(W;\ A+P_a)\sum_{\epsilon'} H_2\sum_\beta \epsilon_\beta' e^{-\frac{1}{2}\pi i q_a' q_\beta}\Theta(W+3a;\ A+P_a+P_\beta)\Theta(W;A+P_\beta),$$

where

$$H_1=\frac{\chi(a,a;\ 0,\epsilon)}{\chi(2a,0;\ 0,\epsilon)},\quad H_2=\frac{\chi(2a,a;\ P_a,\epsilon')}{\chi(3a,0;\ P_a,\epsilon')};$$

proceeding in this way we obtain $2^{(r-1)p}\Theta^r(W+a;\ A)$

$$=\sum_{\epsilon_1}H_1\sum_{a_1}\Theta_1\sum_{\epsilon_2}H_2\sum_{a_2}\Theta_2\dots\sum_{\epsilon_{r-1}}H_{r-1}\chi(W+ra,\ W;\ P_{a_1}+\dots+P_{a_{r-2}};\ \epsilon_{r-1}),\qquad\text{(III)}$$

where each of P_{a_1}, P_{a_2}, ... becomes in turn all the characteristics of the group (P), and ϵ_1, ϵ_2, ... relate respectively to the groups described by P_{a_1}, P_{a_2}, ..., and further

$$H_m=\chi[ma,a;\ P_{a_1}+\dots+P_{a_{m-1}},\epsilon_m]\div\chi[(m+1)a,0;\ P_{a_1}+\dots+P_{a_{m-1}},\epsilon_m],\qquad(m=1,\dots,r-1),$$

$$\Theta_m=\epsilon_{a_m}e^{\lambda_m}\Theta(W;\ A+P_{a_m}),\quad \lambda_m=-\tfrac{1}{2}\pi i(q'_{a_1}+\dots+q'_{a_{m-1}})q_{a_m},\qquad(m=1,\dots,r-2).$$

The equation (III) expresses $\Theta^r(W+a;\ A)$ as an integral polynomial which is of the $(r-1)$th degree in functions $\Theta(W;\ A+P_a)$, whose characteristics belong to the Göpel system (AP), and is of the first degree in functions $\Theta[W+ra;\ A+P_a]$. But it does not thence follow when a is an r-th part of a period, that $\Theta^r(W+a;\ A)$ can be expressed as an integral polynomial of the r-th degree in functions $\Theta[W;\ A+P_a]$; for instance if the Göpel system be taken to be one of which all the characteristics are even (§ 299, Chap. XVII.), it is not the case that the function $\Theta^3(W+\frac{1}{3})$, which is neither odd nor even, or the function $\Theta^3(W+\frac{1}{3})-\Theta^3(W-\frac{1}{3})$, which is odd, can be expressed as an integral polynomial of the third degree in the functions of this Göpel system; differential coefficients of these functions will enter into the expression. The reason is found in the fact noticed in § 308, p. 510; *the denominator of H_{r-1} may vanish.*

Noticing however, when P is any characteristic of the Göpel group (P), that $\chi(-u, -v; P, \epsilon)=e^{\pi i|P|+\pi i|A, P|}\chi(u, v; P, \epsilon)$, so that the coefficients H_m are unaltered by change of the sign of a, and putting the characteristic $A=\begin{pmatrix}K'\\K\end{pmatrix}$, we infer, from the equation (III), that

$$2^{(r-1)p}\left[e^{-2\pi i(r-1)K'W}\Theta^r(W+a;\,A)+e^{2\pi i(r-1)K'W}\Theta^r(W-a;\,A)\right]$$

is equal to

$$\sum_{\epsilon_1} H_1 \sum_{a_1} \dots\dots \sum_{\epsilon_{r-1}} H_{r-1}\left[e^{-2\pi i(r-1)K'W}\chi(W+ra,\,W;\,P,\,\epsilon_{r-1})\right.$$
$$\left.+e^{2\pi i(r-1)K'W}\chi(W-ra,\,W;\,P,\,\epsilon_{r-1})\right],$$

where P denotes $P_{a_1}+\dots+P_{a_{r-2}}$; and it can be shewn that when a becomes equal to $[h-(r-1)(K+\tau'K')]/r$, the limit of the expression

$$U=H_{r-1}\left[e^{-2\pi i(r-1)K'W}\chi(W+ra,\,W;\,P,\epsilon_{r-1})+e^{2\pi i(r-1)K'W}\chi(W-ra, a;\,P,\epsilon_{r-1})\right],$$

if it is not a quadratic polynomial in functions $\Theta(W;\,AP_a)$, *is zero.* The consequence of this will be that $\Psi_r[W;\,K,K'+\mu]$ is expressible as a polynomial involving only the functions $\Theta(W;\,AP_a)$.

For the fundamental formula of § 309, p. 510, immediately gives*, for any values of a, b,

$$\chi(W+a,\,W+b;\,P,\,\epsilon)\chi(a+b,\,0;\,P,\,\epsilon)=\chi(a,\,b;\,P,\,\epsilon)\chi(W+a+b,\,W;\,P,\,\epsilon),$$

and hence, replacing ϵ_{r-1} simply by ϵ, the expression U is equal to

$$\sum_a \epsilon_a e^{-\frac{1}{2}\pi i q' q_a}\left\{e^{-2\pi i(r-1)K'W}\Theta(W+a;\,A+P_a)\Theta[W+(r-1)a;\,A+P+P_a]\right.$$
$$\left.+e^{2\pi i(r-1)K'W}\Theta(W-a;\,A+P_a)\Theta[W-(r-1)a;\,A+P+P_a]\right\},$$

where $P, =\frac{1}{2}\begin{pmatrix}q'\\q\end{pmatrix}$, is used for $P_{a_1}+\dots\dots+P_{a_{r-2}}$ and $\epsilon_1, \epsilon_2, \dots$ for $(\epsilon_{r-1})_1$, $(\epsilon_{r-1})_2, \dots$. Replacing ra in this expression by the period $h-(r-1)(K+\tau'K')$, and omitting an exponential factor depending only on r, h, K, K' and P, it becomes

$$\sum_a \zeta_a e^{-\frac{1}{2}\pi i q' q_a}\left\{\Theta[W+a;\,A+P_a]\Theta[W-a;\,A+P+P_a]\right.$$
$$\left.+\Theta[W-a;\,A+P_a]\Theta[W+a;\,A+P+P_a]\right\},$$

* We take the case when the characteristics B, A of § 309 are equal. It is immediately obvious from the equation here given that in the expressions here denoted by H_m the value of the half-integer characteristic A is immaterial.

A being as before taken $=\begin{pmatrix}K'\\K\end{pmatrix}$ and $\zeta_a = \epsilon_a e^{\pi i[h-(r-1)K]q'_a+\pi i(r-1)K'q_a}$; and this is immediately shewn to be the same as

$$\left(1+\zeta_P\begin{pmatrix}A\\P\end{pmatrix}e^{-\frac{1}{2}\pi i|P|}\right)\sum_a \zeta_a e^{-\frac{1}{2}\pi i q' q_a}\Theta(W+a;\ A+P_a)\,\Theta(W-a;\ A+P+P_a),$$

where ϵ_P is the fourth root of unity associated with the characteristic P of the Göpel group (P), which is to be taken equal to 1 in case $P=0$. Thus the expression vanishes when $\zeta_P = -e^{\frac{1}{2}\pi i|P|}\begin{pmatrix}A\\P\end{pmatrix}$. Hence, in order to prove that when the expression U is not a quadratic polynomial in functions $\Theta(W;\ AP_a)$, it is zero, it is sufficient to prove that the only case in which U is not such a quadratic polynomial is when $\zeta_P = -e^{\frac{1}{2}\pi i|P|}\begin{pmatrix}A\\P\end{pmatrix}$.

Now the denominator of H_{r-1} is

$$\sum_a \epsilon_a e^{-\frac{1}{2}\pi i q' q_a}\Theta[ra;\ A+P+P_a]\,\Theta[0;\ A+P_a],$$

where P still denotes $P_{a_1}+\ldots+P_{a_{r-2}}$ and ϵ_a has the set of values of ϵ_{r-2}; save for a non-vanishing exponential factor this is equal to

$$\sum_a \zeta_a \Theta^2(0;\ AP_a),$$

or $$\left(1+\zeta_P\begin{pmatrix}A\\P\end{pmatrix}e^{-\frac{1}{2}\pi i|P|}\right)\sum_\beta \zeta_\beta e^{-\frac{1}{2}\pi i q' q_\beta}\Theta[0;\ A+P+P_\beta]\,\Theta[0;\ A+P_\beta],$$

according as $P=0$ or not, where, in the second form, P_β is to describe a group of 2^{p-1} characteristics such that the combination of this group with the group $(0,P)$ gives the Göpel group (P). We shall assume that, when ζ_P is not equal to $-e^{\frac{1}{2}\pi i|P|}\begin{pmatrix}A\\P\end{pmatrix}$, neither of these expressions vanishes for general values of the periods τ'.

Since the function $\Psi_r(W;\ K, K'+\mu)$ is certainly finite, we do not examine the finiteness of the coefficients H_m when m is less than $r-1$, these coefficients being independent of W; further, in a Göpel system (AP), any one of the characteristics AP_a may be taken as the characteristic A; the change being only equivalent to adding the characteristic P_a to each characteristic of the group (P); hence (§ 327, Chap. XVIII.), our investigation gives the following result:—*Let any 2^p functions $\vartheta\left(u;\ 2\omega, 2\omega', 2\eta, 2\eta' \,\middle|\, \begin{matrix}Q'\\Q\end{matrix}\right)$, whose (half-integer) characteristics form a Göpel system, syzygetic in threes, be transformed by any transformation of odd order; let (AP) be the Göpel system formed by the transformed characteristics $\begin{pmatrix}K'\\K\end{pmatrix}$; then every one of the* *original functions is an integral polynomial of order r in the 2^p functions** $\vartheta(w;\ 2v, 2v', 2\zeta, 2\zeta' \mid AP)$: as follows from § 288, Chap. XV., the number of terms in the polynomial is at most, and in general, $\frac{1}{2}(r^p+1)$.

For the cases $p = 1, 2, 3$, and for any hyperelliptic case, it is not necessary to use the addition formula developed in Chap. XVIII. We may use instead the addition formula of § 286, Chap. XV. It is however then further to be shewn that only 2^p theta functions enter in the final formula. For the case $p = 3$ the reader may consult Weber, *Ann. d. Mat.* 2ª Ser., t. IX. (1878), p. 126.

369. We give an example of the application of the method here followed.

Suppose $p=1$, $r=3$, and that the transformation is that associated with the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$; then (§ 324, Chap. XVIII.) taking $M=3$, the function

$$\vartheta\left[u;\ 2\omega, 2\omega', 2\eta, 2\eta' \mid \tfrac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right],$$

or $\vartheta_{01}(u)$, is equal to $\vartheta_{01}(3w;\ 2v, 6v', 2\zeta/3, 2\zeta')$ or $\frac{1}{2}e^{\frac{1}{3}\eta\omega^{-1}u^2}\Psi_3(W;\ -\frac{1}{2}, 0)$. Now we have, with $a=(h+1)/3$,

$$C_0\Psi_3(W;\ -\tfrac{1}{2}, 0) = \sum_h \left[\Theta_{01}^3(W+a) + \Theta_{01}^3(W-a)\right];$$

also $\Theta_{01}^3(W+a)$ is equal to

$$\tfrac{1}{4}\sum_\epsilon \frac{\chi(a, a;\ 0, \epsilon)}{\chi(2a, 0;\ 0, \epsilon)} \sum_\alpha \epsilon_\alpha \Theta(W;\ A+P_\alpha) \sum_{\epsilon'} \frac{\chi(2a, a;\ P_\alpha, \epsilon')}{\chi(3a, 0;\ P_\alpha, \epsilon')} \sum_\beta \epsilon'_\beta e^{-\frac{1}{2}\pi i q'_\alpha q_\beta} \Theta(W+3a;\ A+P_\alpha+P_\beta)\,\Theta(W;\ A+P_\beta);$$

if we take the Göpel system to be $\frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so that $P_1=\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, this is equal to

$$\tfrac{1}{4}\sum_\epsilon \frac{\Theta_{01}^2(a)+\epsilon_1\Theta_{10}^2(a)}{\Theta_{01}(2a)\,\Theta_{01}+\epsilon_1\Theta_{10}(2a)\,\Theta_{10}}\,\Theta_{01}(W) \sum_{\epsilon'} \frac{\Theta_{01}(2a)\,\Theta_{01}(a)+\epsilon_1'\Theta_{10}(2a)\,\Theta_{10}(a)}{\Theta_{01}(3a)\,\Theta_{01}+\epsilon_1'\Theta_{10}(3a)\,\Theta_{10}}\,E_0$$

$$+\tfrac{1}{4}\sum_\epsilon \frac{\Theta_{01}^2(a)+\epsilon_1\Theta_{10}^2(a)}{\Theta_{01}(2a)\,\Theta_{01}+\epsilon_1\Theta_{10}(2a)\,\Theta_{10}}\,\epsilon_1\Theta_{10}(W) \sum_{\epsilon'} \frac{\Theta_{10}(2a)\,\Theta_{01}(a)-i\epsilon_1'\Theta_{01}(2a)\,\Theta_{10}(a)}{\Theta_{10}(3a)\,\Theta_{01}-i\epsilon_1'\Theta_{01}(3a)\,\Theta_{10}}\,E_1,$$

where Θ_{01} denotes $\Theta_{01}(0)$, etc., and

$$E_0 = \Theta_{01}(W+3a)\,\Theta_{01}(W) + \epsilon_1'\Theta_{10}(W+3a)\,\Theta_{10}(W),$$
$$E_1 = \Theta_{10}(W+3a)\,\Theta_{01}(W) - i\epsilon_1'\Theta_{01}(W+3a)\,\Theta_{10}(W).$$

Now, in accordance with the general rules, the denominator of the fraction

$$\frac{\Theta_{10}(2a)\,\Theta_{01}(a) - i\epsilon_1'\Theta_{01}(2a)\,\Theta_{10}(a)}{\Theta_{10}(3a)\,\Theta_{01} - i\epsilon_1'\Theta_{01}(3a)\,\Theta_{10}}$$

vanishes when $\epsilon_1' = -e^{\frac{1}{2}\pi i\binom{A}{P}} e^{\pi i(h-2K)q_1'+\pi i 2K'q_1}$, namely, as $\begin{pmatrix} K' \\ K \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = A$, when $\epsilon_1' = -ie^{\pi i(h+1)}$, and $a=(h+1)/3$; in fact, putting $a=(h+1+x)/3$,

$$\Theta_{10}(3a)\,\Theta_{01} - i\epsilon_1'\Theta_{01}(3a)\,\Theta_{10} = e^{\pi i(h+1)}\,\Theta_{10}(x)\,\Theta_{01} - i\epsilon_1'\Theta_{01}(x)\,\Theta_{10},$$
$$= \tfrac{1}{2}e^{\pi i(h+1)}\left[\Theta_{10}''\Theta_{01} - \Theta_{01}''\Theta_{10}\right]x^2,$$

* The expression of the transformed theta function in terms of $2^p=4$ theta functions is given by Hermite, *Compt. Rendus*, t. XL. (1855), for the case $p=2$. For the general hyperelliptic case cf. Königsberger, *Crelle*, LXIV. (1865), p. 32.

for small values of x, when $i\epsilon_1' = e^{\pi i(h+1)}$, because the differential coefficients of the even functions, being odd functions, vanish for zero argument; thus the denominator of the fraction vanishes to the second order. We find similarly, for $i\epsilon_1' = e^{\pi i(h+1)}$, $a = \frac{1}{3}(h+1+x)$, that the numerator of this fraction is equal to

$$e^{\pi i(h+1)}\left[\Theta'_{01}\left(\frac{h+1}{3}\right)\Theta_{10}\left(\frac{h+1}{3}\right) - \Theta'_{10}\left(\frac{h+1}{3}\right)\Theta_{01}\left(\frac{h+1}{3}\right)\right]x;$$

in the same case also we find that the expression E_1 is equal to

$$e^{\pi i(h+1)}\left[\Theta'_{10}(W)\Theta_{01}(W) - \Theta'_{01}(W)\Theta_{10}(W)\right]x,$$

while the expression $\Theta_{10}(W-3a)\Theta_{01}(W) - i\epsilon_1'\Theta_{01}(W-3a)\Theta_{10}(W)$ is equal to the negative of this. Thus the function $\Theta_{01}^3(W+a)$ can be expressed by the functions $\Theta_{10}(W)$, $\Theta_{01}(W)$, and their differential coefficients of the first order; but the function $\Theta_{01}^3(W+a) + \Theta_{01}^3(W-a)$ can be expressed by the functions $\Theta_{10}(W)$, $\Theta_{01}(W)$ only.

In the function $\Theta_{01}^3(W+a) + \Theta_{01}^3(W-a)$ the part

$$\sum_{\epsilon'} \frac{\Theta_{10}(2a)\Theta_{01}(a) - i\epsilon_1'\Theta_{01}(2a)\Theta_{10}(a)}{\Theta_{10}(3a)\Theta_{01} - i\epsilon_1'\Theta_{01}(3a)\Theta_{10}} E_1$$

furnishes only the single term for which $i\epsilon_1' = -e^{\pi i(h+1)}$, namely,

$$4e^{\pi i(h+1)} \frac{\Theta_{01}\left(\frac{h+1}{3}\right)\Theta_{10}\left(\frac{h+1}{3}\right)}{\Theta_{01}\Theta_{10}} \Theta_{01}(W)\Theta_{10}(W).$$

Ex. i. Prove that the final result is that $\frac{1}{2}C_0\vartheta_{01}(u)$ is equal to

$$\frac{\Theta_{01}^3(\frac{1}{3})\Theta_{01} - \Theta_{10}^3(\frac{1}{3})\Theta_{10}}{[\Theta_{01}^2(\frac{1}{3})\Theta_{01}^2 + \Theta_{10}^2(\frac{1}{3})\Theta_{10}^2][\Theta_{01}^4 + \Theta_{10}^4]}\{[\Theta_{01}^2(\tfrac{1}{3})\Theta_{01}^2 + \Theta_{10}^2(\tfrac{1}{3})\Theta_{10}^2]\vartheta_{01}^3(w)$$
$$- [\Theta_{10}^2(\tfrac{1}{3})\Theta_{01}^2 - \Theta_{01}^2(\tfrac{1}{3})\Theta_{10}^2]\vartheta_{01}(w)\vartheta_{10}^2(w)\}$$
$$+ \frac{\Theta_{10}^2(\frac{1}{3})\Theta_{01}^2(\frac{1}{3})[\Theta_{10}(\frac{1}{3})\Theta_{01} + \Theta_{01}(\frac{1}{3})\Theta_{10}]}{[\Theta_{01}^2(\frac{1}{3})\Theta_{01}^2 + \Theta_{10}^2(\frac{1}{3})\Theta_{10}^2]\Theta_{01}\Theta_{10}}\vartheta_{01}(w)\vartheta_{10}^2(w) + \tfrac{1}{2}\vartheta_{01}^3(w),$$

where Θ_{01}, Θ_{10} denote $\Theta_{01}(0)$ and $\Theta_{10}(0)$ respectively.

Ex. ii. Prove that

$$\Theta_{01}(W-\tfrac{1}{3})\Theta_{10}(W+\tfrac{1}{3}) - \Theta_{10}(W-\tfrac{1}{3})\Theta_{01}(W+\tfrac{1}{3})$$
$$= 2\frac{\Theta'_{10}(\frac{1}{3})\Theta_{01}(\frac{1}{3}) - \Theta'_{01}(\frac{1}{3})\Theta_{10}(\frac{1}{3})}{\Theta_{10}\Theta''_{01} - \Theta_{01}\Theta''_{10}}[\Theta'_{01}(W)\Theta_{10}(W) - \Theta'_{10}(W)\Theta_{01}(W)].$$

370. General formulae for the quadric transformation are also obtainable. The results are different, as has been seen, according as the characteristic (K, K') of the transformed function is zero (including integral) or not. The results are as follows:—

When (K, K') is zero, the transformed function can be expressed as a linear aggregate of the 2^p functions $\vartheta^2(w \mid A, P_i)$, whose characteristics are those of any Göpel system.

When (K, K') is not zero, the transformed function can be expressed as a linear aggregate of the 2^{p-1} products $\vartheta(w \mid A, P_i)\vartheta(w \mid A, K, P_i)$, in which the characteristics P_i are those of any Göpel group whereof the characteristic K, $=(K, K')$, is one constituent, and A is a characteristic such that $|A, K| \equiv |K|$, or $|A, K| \equiv |K| + 1$ (mod. 2), according as the function to be expressed is even or odd*.

When (K, K') is zero, the equation (I), § 367, putting $K = K' = \mu = 0$, and then increasing W by $\frac{1}{2}\mu\tau'$, where μ is a row of quantities each either 0 or 1, gives

$$C\Theta\left(2W;\ 2\tau' \,\middle|\, \begin{matrix}\mu/2\\0\end{matrix}\right) = \sum_h e^{-\pi i\mu h}\Theta^2\left(W + \tfrac{1}{2}h;\ \tau' \,\middle|\, \begin{matrix}\mu/2\\0\end{matrix}\right);$$

hence, from the fundamental formula of § 309 (p. 510), writing therein $v = 0$, $u = W + a$, $b = a = h/2$, $A = \frac{1}{2}\begin{pmatrix}\mu\\0\end{pmatrix}$, $P_i = \frac{1}{2}\begin{pmatrix}q_i'\\q_i\end{pmatrix}$, and $\begin{pmatrix}P_i\\A\end{pmatrix}e^{\pi i h q_i'}\epsilon_i = \zeta_i$, we obtain

$$2^p C\Theta\left(2W;\ 2\tau' \,\middle|\, \begin{matrix}\mu/2\\0\end{matrix}\right)$$

$$= \sum_h e^{-\pi i\mu h}\sum_\zeta \frac{\sum_i \begin{pmatrix}P_i\\A\end{pmatrix} e^{\pi i h q_i'}\zeta_i\Theta^2\left(0;\ \tau' \,\middle|\, \frac{1}{2}\begin{pmatrix}\mu\\h\end{pmatrix} + P_i\right)}{\sum_i \zeta_i\Theta^2(0;\ \tau' \mid AP_i)}\sum_i \zeta_i\Theta^2(W;\ \tau' \mid AP_i),$$

where C is independent of μ. It is assumed that the sum $\sum_i \zeta_i\Theta^2(0;\ \tau' \mid AP_i)$ is different from zero for each of the 2^p sets of values of the fourth roots ζ_i. This formula suffices to express any theta function of the second order with zero characteristic.

When (K, K') is other than zero, by putting in the equation (I), § 367, $r = 2$, $\mu = 0$, adding $\frac{1}{2}\tau' h'$ to W, where h' is a row of quantities each either 0 or 1, and then changing the sign of W, we obtain

$$Ce^{-\pi i\lambda'(K + \frac{1}{2}\tau'\lambda')}\Psi_2(W;\ K, K' + h') = \sum_h \left[e^{2\pi i\lambda' W}\Theta^2(W + a) + \epsilon e^{-2\pi i\lambda' W}\Theta^2(W - a)\right],$$

where $\lambda = K + h$, $\lambda' = K' + h'$, and C is the same constant as before, independent of W, K, K', h', and $a = \frac{1}{2}\lambda + \frac{1}{2}\tau'\lambda'$, the period τ' being omitted on the right side. Hence, taking the fundamental formula of § 309 (p. 510), putting therein $v = 0$, $u = W + a$, $b = a$, $A = 0$, $B = A$, and then writing $a = \frac{1}{2}\lambda + \frac{1}{2}\tau'\lambda' + \frac{1}{2}x$, where x is a row of p equal quantities, we find, provided $|K, P_i| \equiv 0$, (mod. 2),

* When (K, K') is zero, the function is necessarily even (§ 288, p. 463), and therefore $|K| \equiv |Q|$. We have seen (§ 327, Chap. XVIII.) that this is always true when r is odd. When r is 2, it is not always so, as is obvious by considering the transformation, for $p = 1$, in which $\alpha = 2$, $\beta = 0$, $\alpha' = 0$, $\beta' = 1$, and $(Q, Q') = (\frac{1}{2}, \frac{1}{2})$; then we find $(K, K') = (\frac{1}{2}, 1)$; thus $|Q| = 1$, $|K| = 2$.

and $\epsilon = e^{\pi i |K| + \pi i |A, K|}$, that $2^p C \Psi_2 (W; K, K' + h')$ is equal to the limit, when x vanishes, of the expression

$$e^{\pi i K'(K + \frac{1}{2}\tau' K')} \sum_h e^{-\pi i h h'} \sum_\zeta E_\zeta \sum_i \zeta_i e^{-\pi i K' q_i} \Theta (W; A, P_i) \{\Theta (W + x | A, K, P_i) \\ + \Theta (W - x | A, K, P_i)\},$$

where $\zeta_i = \binom{P_i}{A} e^{\pi i (h q'_i - h' q_i)} \epsilon_i$, and

$$E_\zeta = \frac{\sum_i \zeta_i e^{-\pi i (h - 2A) q_i' - \pi i h' x} \Theta^2 \left(\tfrac{1}{2} x + \tfrac{1}{2} K + \tfrac{1}{2} \tau' K' \,\middle|\, \tfrac{1}{2} \binom{h'}{h} + P_i\right)}{\sum_i \zeta_i e^{-\pi i K' q_i} \Theta (x; A, K, P_i) \Theta (0; A, P_i)}.$$

It can easily be proved (cf. § 308, p. 510) that the denominator of E_ζ vanishes, for $x = 0$, for the 2^{p-1} sets of values of the fourth roots ζ_i in which the fourth root corresponding to the characteristic K of the group (P) has the value $-\binom{A}{K} e^{\frac{1}{2}\pi i |K|}$, and that the corresponding expressions

$$U_\zeta = E_\zeta \sum_i \zeta_i e^{-\pi i K' q_i} \Theta (W; A, P_i) \{\Theta (W + x | A, K, P_i) + \Theta (W - x | A, K, P_i)\}$$

have the limit zero; the summation $\sum_\zeta$ is therefore to be taken only to extend to the 2^{p-1} sets of values in which this fourth root $= + \binom{A}{K} e^{\frac{1}{2}\pi i |K|}$. It may however happen that the denominator of E_ζ vanishes for other sets of values of the fourth roots ζ_i, when $x = 0$. We assume that for such sets of values the sum multiplying E_ζ in the expression U_ζ does not vanish for $x = 0$; by recurring to the proof of the formula of § 308, it is immediately seen that this is equivalent to assuming that the expression

$$\sum_i \epsilon_i \Theta^2 (U; P_i)$$

is not zero for general values of the arguments U for any set of values of the fourth roots ϵ_i (cf. (β), p. 514). That being so, the value of E_ζ when its denominator vanishes for $x = 0$, can always be obtained from the limiting expression given, by expanding its numerator and denominator in powers of x.

Ex. Applying the formula of this page for the case $p = 1$ to the function

$$\Theta_{11} (2W; 2\tau') = \tfrac{1}{2} \Psi_2 (W; -\tfrac{1}{2}, 1),$$

for which $(K, K') = (-\frac{1}{2}, 0)$ and $h' = 1$, we immediately find that the Göpel system in terms of which the function can be expressed is (A, AP_1), where $A = \frac{1}{2}\binom{1}{0}$, $P_1 = K = \frac{1}{2}\binom{0}{-1}$; we are to exclude the value of the expression U_ζ in which $\zeta_1 = -\binom{A}{K} = 1$; the value of E_ζ for $\zeta_1 = -1$ is easily found to be

$$E_\zeta = e^{-\pi i(x-h)}\,[\Theta_{10}^2(\tfrac{1}{2}x-\tfrac{1}{4})-\Theta_{10}^2(\tfrac{1}{2}x+\tfrac{1}{4})]\div\Theta_{11}(x)\,\Theta_{10}(0)$$

of which both numerator and denominator vanish for $x=0$. The final result of the formula is

$$C\Theta_{11}(2W;\ 2\tau')=-4\Theta_{10}(\tfrac{1}{4};\ \tau')\,\Theta_{10}'(\tfrac{1}{4};\ \tau')\,\Theta_{11}(W;\ \tau')\,\Theta_{10}(W;\ \tau')/\Theta_{11}'(0;\tau')\Theta_{10}(0;\ \tau').$$

Prove this result, and also

$$C\Theta_{01}(2W;\ 2\tau')=2\Theta_{00}^2(\tfrac{1}{4};\ \tau')\,\Theta_{00}(W;\ \tau')\,\Theta_{01}(W;\ \tau')/\Theta_{00}(0;\ \tau')\,\Theta_{01}(0;\ \tau'),$$

and (cf. § 365) obtain the formulae

$$\Theta_{10}(\tfrac{1}{4};\ \tau')\,\Theta_{10}'(\tfrac{1}{4};\ \tau')=-\frac{\pi}{2}\Theta_{10}^2(0;\ \tau')\,\Theta_{00}^2(\tfrac{1}{4};\ \tau'),$$

$$\Theta_{00}^4(\tfrac{1}{4};\ \tau')=\tfrac{1}{2}\Theta_{00}(0;\ \tau')\,\Theta_{01}(0;\ \tau')\,[\Theta_{00}^2(0;\ \tau')+\Theta_{01}^2(0;\ \tau')],$$

$$\Theta_{00}^2(0;\ 2\tau')=\tfrac{1}{2}[\Theta_{00}^2(0;\ \tau')+\Theta_{01}^2(0:\ \tau')],$$

$$C=\sqrt{2\,[\Theta_{00}^2(0;\ \tau')+\Theta_{01}^2(0;\ \tau')]}.$$

371. The preceding investigations of this chapter enable us to specify in all cases the form of the function $\vartheta\left(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta'\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ or $\vartheta\left(u\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ when expressed in terms of functions $\vartheta\left(w;\ 2v,\ 2v',\ 2\zeta,\ 2\zeta'\,\middle|\,\begin{matrix}K'\\K\end{matrix}\right)$ or $\vartheta\left(w\,\middle|\,\begin{matrix}K'\\K\end{matrix}\right)$. In many particular cases it is convenient to start from this form and determine the coefficients in the expression by particular methods. But it is proper to give a general method. For this purpose we should consider two stages, (i) the determination of the coefficients in the expression of the function $\vartheta\left(u\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ by means of functions $\psi_r(w;\ K,\ K'+\mu)$, (ii) the determination of the coefficients in the expression of the functions $\psi_r(w;\ K,\ K'+\mu)$ by means of functions $\vartheta\left(w\,\middle|\,\begin{matrix}K'\\K\end{matrix}\right)$. The preceding formulae of this chapter enable us to give a complete determination of the latter coefficients in a particular form, namely, in terms of theta functions whose arguments are fractional parts of the periods $2v,\ 2v'$; but this is by no means to be regarded as the final form.

372. Dealing first with the coefficients in the expression of the function $\vartheta\left(u\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ by functions $\psi_r(w;\ K,\ K'+\mu)$, there is one case in which no difficulty arises, namely, when the transformation is that associated with the matrix $\begin{pmatrix}r & 0\\0 & 1\end{pmatrix}$; then $\vartheta\left(u\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ is equal to $\vartheta\left(rw;\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta'\,\middle|\,\begin{matrix}K'/r\\K\end{matrix}\right)$, the row K' being in fact equal to rQ', namely $\vartheta\left(u\,\middle|\,\begin{matrix}Q'\\Q\end{matrix}\right)$ is $\tfrac{1}{2}\psi_r(w;\ K,\ K')$.

Now it can be shewn*, that if Ω_r be the matrix associated with any transformation of order r, and r be a prime number, or a number without square factors, then linear transformations, Ω, Ω', can be determined such that $\Omega_r = \Omega \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \Omega'$. Hence, in cases in which the matrices Ω, Ω' have been calculated, it is sufficient, first to carry out the transformation Ω upon the given function $\vartheta\left(u \,\middle|\, \begin{matrix} Q' \\ Q \end{matrix}\right)$; then to use the formulae for the transformation $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, whereby the original function appears as an integral polynomial of order r in 2^p theta functions; and finally to apply the transformation Ω' to these 2^p theta functions. All cases in which the order of transformation is not a prime number may be reduced to successive transformations of prime order (§ 332, Chap. XVIII.).

We can however make a statement of greater practical use, as follows. It is shewn in the Appendix II. (§§ 415, 416) that the matrix associated with any transformation of order r can be put into the form $\Omega \begin{pmatrix} A & B \\ 0 & B' \end{pmatrix}$, where Ω is the matrix of a linear transformation, and that, in whichever of the possible ways this is done, the determinant of the matrix B' is the same for all. In all cases in which this has been done the required coefficients are given by the equation

$$\frac{1}{\sqrt{|\omega|}} \vartheta\left(u;\ 2\omega, 2\omega', 2\eta, 2\eta' \,\middle|\, \begin{matrix} Q' \\ Q \end{matrix}\right)$$

$$= \frac{r^{\frac{1}{2}p}\epsilon}{\sqrt{|M|\,|v|\,|B'|}} e^{-\frac{\pi i}{r} dK' + \frac{\pi i}{r^2} \gamma K'^2} \sum_{\mu} e^{-\frac{\pi i}{r} d\mu - \frac{\pi i}{r^2} \gamma\mu^2} \vartheta\left[rw;\ 2v, 2rv', 2\zeta/r, 2\zeta' \,\middle|\, \begin{matrix} (K'+\mu)/r \\ K \end{matrix}\right],$$

wherein, (Q, Q') being a half-integer characteristic, ϵ is an eighth root of unity, $u = Mw$, $|M|$ is the determinant of the matrix M, etc., μ is in turn every row of integers each positive (or zero) and less than r, which satisfies the condition that the p quantities $\frac{1}{r} B'\mu$ are integral, and, finally, γ denotes the symmetrical matrix $\bar{B}B'$, while d denotes the row of integers formed by the diagonal elements of γ. It is shewn in the Appendix II., that the resulting range of values for μ is independent of how the original matrix is resolved into the form in question. For any specified form of the linear transformation Ω the value of ϵ can be calculated (as in Chap. XVIII., §§ 333—4); if ϵ_0

* Cf. Appendix II.; and for details in regard to the case $p=3$, Weber, *Ann. d. Mat.*, Ser. 2ª, t. IX. (1878—9). We have shewn (Chap. XVIII., § 324, Ex. i.) that the determinant of the matrix of transformation is $\pm r^p$. From the result quoted here it follows that that determinant is $+r^p$.

denote its value when the characteristic (Q, Q') is zero, its value for any other characteristic is given by

$$\epsilon/\epsilon_0 = e^{-\pi i [Q_1' + \frac{1}{2}d(\bar{\rho}\rho')][Q_1 + \frac{1}{2}d(\bar{\sigma}\sigma')] + \pi i QQ'},$$

where $\Omega = \begin{pmatrix} \rho & \sigma \\ \rho' & \sigma' \end{pmatrix}$, and $Q_1' = \bar{\rho}Q' - \bar{\rho}'Q - \frac{1}{2}d(\bar{\rho}\rho')$, $-Q_1 = \bar{\sigma}Q' - \bar{\sigma}'Q - \frac{1}{2}d(\sigma\sigma')$.

To prove this formula, we have first (§ 335, Chap. XVIII.), if $\Omega = \begin{pmatrix} \rho & \sigma \\ \rho' & \sigma' \end{pmatrix}$, the equation

$$\frac{1}{\sqrt{|\omega|}} \vartheta\left(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta' \,\middle|\, \begin{matrix} Q' \\ Q \end{matrix}\right) = \frac{\epsilon}{\sqrt{|M_1|\,|\omega_1|}} \vartheta\left(u_1;\ 2\omega_1,\ 2\omega_1',\ 2\eta_1,\ 2\eta_1' \,\middle|\, \begin{matrix} Q_1' \\ Q_1 \end{matrix}\right),$$

where $u = M_1 u_1$, $M_1\omega_1 = \omega\rho + \omega'\rho'$, etc. Writing $u_1 = 2\omega_1 U_1$, $\omega_1' = \omega_1\tau_1$, we have

$$\vartheta\left(u_1;\ 2\omega_1,\ 2\omega_1',\ 2\eta_1,\ 2\eta_1' \,\middle|\, \begin{matrix} Q_1' \\ Q_1 \end{matrix}\right) = e^{\frac{1}{2}\eta_1\omega_1^{-1}u_1^2}\, \Theta\left(U_1;\ \tau_1 \,\middle|\, \begin{matrix} Q_1' \\ Q_1 \end{matrix}\right),$$

and the equations $u_1 = M_2 w$, $M_2 v = \omega_1 A$, $M_2 v' = \omega_1 B + \omega_1' B'$, give, if $w = 2vW$, $v' = v\tau'$, and in virtue of $A\bar{B}' = r$, the equations $U_1 = AW$, $r\tau_1 = A\tau'\bar{A} - B\bar{A}$, while, by the equation $r\zeta = \bar{M}_2\eta_1 A$, we find $\eta_1\omega_1^{-1}u_1^2 = r\zeta v^{-1}w^2$. Now it is immediately seen that the exponent of the general term of $\Theta\left(U_1;\ \tau_1 \,\middle|\, \begin{matrix} Q_1' \\ Q_1 \end{matrix}\right)$ gives

$$2\pi i U_1 n + i\pi\tau_1 n^2 = 2\pi i r W\left(m + \frac{\mu}{r}\right) + \pi i r\tau'\left(m + \frac{\mu}{r}\right)^2 + \pi i d\left(m + \frac{\mu}{r}\right)$$
$$- i\pi(\gamma m^2 + dm) - 2\pi i \bar{B}\frac{B'\mu}{r} m - \frac{\pi i d\mu}{r} - \frac{\pi i}{r^2}\gamma\mu^2,$$

wherein $\gamma = \bar{B}B'$, and d denotes the row of diagonal elements of γ, and m, μ are obtained by putting $\bar{A}n = rm + \mu$, m being a row of integers, and μ a row of integers each less than r and positive (including zero); this equation is equivalent to $n - B'm = \frac{1}{r}B'\mu$; corresponding to every n it determines an unique m and an unique μ for which $\frac{B'\mu}{r}$ is integral; corresponding to any assigned μ for which $\frac{B'\mu}{r}$ is integral, and an assigned m, the equation determines an unique n. Since then $\gamma m^2 + dm$ is an even integer, and, for the terms which occur, $\bar{B}\frac{B'\mu}{r}m$ is an integer, we have

$$\Theta(U_1, \tau_1) = \sum_\mu e^{-\frac{\pi i d\mu}{r} - \frac{\pi i}{r^2}\gamma\mu^2}\, \Theta\left[rW;\ r\tau' \,\middle|\, \begin{matrix} \mu/r \\ d/2 \end{matrix}\right].$$

Increasing, in this equation, U_1 by $Q_1 + \tau_1 Q_1'$, we hence deduce

$$\Theta\left(U_1;\ \tau_1 \,\middle|\, \begin{matrix} Q_1' \\ Q_1 \end{matrix}\right) = e^{-\frac{\pi i}{r}dK' + \frac{\pi i}{r^2}\gamma K'^2} \sum_\mu e^{-\frac{\pi i d\mu}{r} - \frac{\pi i}{r^2}\gamma\mu^2}\, \Theta\left[rW;\ r\tau' \,\middle|\, \begin{matrix} (K' + \mu)/r \\ K \end{matrix}\right],$$

where $K' = \bar{A}Q_1'$, $-K = \bar{B}Q_1' - \bar{B}'Q_1 - \frac{1}{2}d(\bar{B}B')$, so that (K, K') is the characteristic of the final theta function of w. Since now the matrix $Mv\bar{B}' = M_1M_2v\bar{B}' = M_1\omega_1A\bar{B}' = rM_1\omega_1$, and therefore $|M|\,|v|\,|B'| = r^p\,|M_1|\,|\omega_1|$, we have, by multiplying the last obtained equation by $e^{\frac{1}{2}\eta_1\omega_1^{-1}u_1^2} = e^{\frac{1}{2}r\zeta v^{-1}w^2}$, the formula which was given above.

Ex. i. When $p=1$, the transformation associated with the matrix $\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ gives rise to the function $\Theta(W; \frac{1}{3}\tau')$; we have

$$\Theta(W; \tfrac{1}{3}\tau') = \Theta(3W; 3\tau') + \Theta\left(3W; 3\tau' \left| \begin{matrix}1/3\\ 0\end{matrix}\right.\right) + \Theta\left(3W; 3\tau' \left| \begin{matrix}-1/3\\ 0\end{matrix}\right.\right).$$

Other simple examples have already occurred for the quadric transformations (§ 365).

Ex. ii. Prove when $p=2$, by considering the transformation of order r (r odd) for which

$$\alpha = \begin{pmatrix}1, & -\mu\\ 0, & r\end{pmatrix}, \quad \beta = \begin{pmatrix}2\lambda, & 0\\ 0, & 0\end{pmatrix}, \quad \alpha' = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \quad \beta' = \begin{pmatrix}r & 0\\ \mu & 1\end{pmatrix},$$

that

$$\Theta\left[u_1 - \mu u_2, ru_2; \frac{1}{r}(\tau_{11} - 2\mu\tau_{12} + \mu^2\tau_{22} - 2\lambda), 2\tau_{12} - 2\mu\tau_{22}, r\tau_{22}\right]$$
$$= \tfrac{1}{2}\psi(0, 0) + \sum_{n=1}^{\frac{1}{2}(r-1)} e^{-\frac{2\pi i}{r}n^2\lambda}\,\psi(n, -n\mu),$$

where $\psi(n_1, n_2)$ denotes $\Theta\left(ru; r\tau \left| \begin{matrix}n_1/r, & n_2/r\\ 0 & \end{matrix}\right.\right) + \Theta\left(ru; r\tau \left| \begin{matrix}-n_1/r, & -n_2/r\\ 0 & \end{matrix}\right.\right)$. (Wiltheiss, *Crelle*, XCVI. (1884), pp. 21, 22.)

373. In regard now to the question of the coefficients which enter in the expression of the functions $\psi_r(w; K, K'+\mu)$ by means of functions $\vartheta\left(w \left| \begin{matrix}K'\\ K\end{matrix}\right.\right)$, the problem that arises is that of the determination of these coefficients in terms of given constants, as for instance the zero values of the original theta functions. The theory of this determination must be omitted from the present volume. In the case when the order of the transformation is odd these coefficients arise in this chapter expressed in terms of theta functions, $\vartheta\left(\frac{2vm + 2v'm'}{r}; 2v, 2v', 2\zeta, 2\zeta'\right)$, whose arguments are r-th parts of the periods $2v$, $2v'$. By means of two supplementary transformations, Δ, $r\Delta^{-1}$, (as indicated § 332, Chap. XVIII.), or by means of the formulae of Chap. XVII. (as indicated in Ex. vii., § 317, Chap. XVII.), we can obtain equations for functions $\vartheta(rw; 2v, 2v', 2\zeta, 2\zeta')$ as integral polynomials of degree r^2 in functions $\vartheta(w; 2v, 2v', 2\zeta, 2\zeta')$. By means of these equations the functions $\vartheta\left(\frac{2vm + 2v'm'}{r}; 2v, 2v', 2\zeta, 2\zeta'\right)$ are determined in terms of functions $\vartheta(0; 2v, 2v', 2\zeta, 2\zeta')$; or this determination may arise by elimination from the original equations of transformation, without use of the multiplication equations. There remains then further the theory of the relations connecting the functions $\vartheta(0; 2v, 2v', 2\zeta, 2\zeta')$ and the functions $\vartheta(0; 2\omega, 2\omega', 2\eta, 2\eta')$, which is itself a matter of complexity.

For the case $p=1$, the reader may consult, for instance, Weber, *Elliptische Functionen* (Braunschweig, 1891), Krause, *Theorie der doppeltperiodischen Functionen* (Erster Band, Leipzig, 1895). For the case $p=2$, Krause, *Hyperelliptische Functionen* (Leipzig, 1886), Königsberger, *Crelle*, LXIV., LXV., LXVII. For the form of the general results, the chapter, *Die Theilung*, of Clebsch u. Gordan, *Abel'sche Functionen* (Leipzig, 1866), which deals with the theta functions arising on a Riemann surface, may be consulted. For the hyperelliptic case, see also Jordan, *Traité des Substitutions* (Paris, 1870), p. 365, and Burkhardt, *Math. Annal.* XXXV., XXXVI., XXXVIII. (1890—1).

In particular cases, knowing the form of the expression of the functions

$$\vartheta\,(u\,;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta')$$

in terms of functions $\vartheta\,(w\,;\ 2v,\ 2v',\ 2\zeta,\ 2\zeta')$, we are able to determine the coefficients by the substitution of half-periods coupled with expansion of the functions in powers of the arguments. See, for instance, the book of Krause (*Hyperelliptische Functionen*) and Königsberger, *as above.*

Ex. i. In case $p=2$, $r=3$, the function $\Theta_5\,(3\,W,\ 3\tau')$ is a cubic polynomial of the functions $\Theta_5\,(W,\ \tau')$, $\Theta_{34}\,(W,\ \tau')$, $\Theta_1\,(W,\ \tau')$, $\Theta_{02}\,(W,\ \tau')$, of which the characteristics are respectively $\tfrac{1}{2}\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$, $\tfrac{1}{2}\begin{pmatrix}0, & 0\\ 0, & -1\end{pmatrix}$, $\tfrac{1}{2}\begin{pmatrix}1, & 0\\ -1, & -1\end{pmatrix}$, $\tfrac{1}{2}\begin{pmatrix}1, & 0\\ -1, & 0\end{pmatrix}$; these form a Göpel system. The only products of these functions which are theta functions of the third order and of zero characteristic are those contained in the equation

$$\Theta_5\,(3\,W,\ 3\tau')=A\phi_5^3+B\phi_5\phi_{34}^2+C\phi_5\phi_1^2+D\phi_5\phi_{02}^2+E\phi_{34}\phi_1\phi_{02},$$

where $\phi_5=\Theta_5\,(W,\ \tau')$, etc.; this equation contains the right number $\tfrac{1}{2}(r^p+1)=5$ of terms on the right side. Putting instead of the arguments W_1, W_2 respectively

$$W_1,\ W_2-\tfrac{1}{2}\,;\ \ W_1-\tfrac{1}{2}+\tfrac{1}{2}\tau_{11},\ W_2-\tfrac{1}{2}+\tfrac{1}{2}\tau_{21}\,;\ \ W_1-\tfrac{1}{2}+\tfrac{1}{2}\tau_{11},\ W_2+\tfrac{1}{2}\tau_{21},$$

we obtain in turn

$$\begin{aligned}
\Theta_{34}\,(3\,W,\ 3\tau') &= \ \ A\phi_{34}^3+B\phi_{34}\phi_5^2+C\phi_{34}\phi_{02}^2+D\phi_{34}\phi_1^2+E\phi_5\phi_1\phi_{02},\\
\Theta_1\,(3\,W,\ 3\tau') &= -A\phi_1^3-B\phi_1\phi_{02}^2+C\phi_1\phi_5^2+D\phi_1\phi_{34}^2+E\phi_5\phi_{02}\phi_{34},\\
\Theta_{02}\,(3\,W,\ 3\tau') &= -A\phi_{02}^3-B\phi_{02}\phi_1^2+C\phi_{02}\phi_{34}^2+D\phi_{02}\phi_5^2+E\phi_5\phi_1\phi_{34},
\end{aligned}$$

whereby the Göpel system of functions $\Theta_5\,(3\,W,\ 3\tau')$, $\Theta_{34}\,(3\,W,\ 3\tau')$, $\Theta_1\,(3\,W,\ 3\tau')$, $\Theta_{02}\,(3\,W,\ 3\tau')$ is expressed by means of the Göpel system ϕ_5, ϕ_{34}, ϕ_1, ϕ_{02}.

From the first two equations, by putting the arguments zero, we obtain

$$A=\frac{\bar{\Theta}_5\Theta_5-\bar{\Theta}_{34}\Theta_{34}}{\Theta_5^4-\Theta_{34}^4},\quad B=\frac{\bar{\Theta}_{34}\Theta_5^3-\bar{\Theta}_5\Theta_{34}^3}{\Theta_5\Theta_{34}\,(\Theta_5^4-\Theta_{34}^4)},$$

where $\bar{\Theta}_5=\Theta_5\,(0\,;\ 3\tau')$, etc., and $\Theta_5=\Theta_5\,(0\,;\ \tau')$, etc.; by the addition of other even half-periods to the arguments, for instance, those associated with the characteristics

$$\tfrac{1}{2}\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}0, & 1\\ 0, & 0\end{pmatrix},\ \tfrac{1}{2}\begin{pmatrix}0, & 0\\ -1, & 0\end{pmatrix},$$

we can obtain expressions for C, D, E; these substitutions give respectively

$$\begin{aligned}
\Theta_{23}\,(3\,W\,;\ 3\tau') &= A\phi_{23}^3-B\phi_{23}\phi_{24}^2+C\phi_{23}\phi_{04}^2-D\phi_{23}\phi_{03}^2+E\phi_{24}\phi_{04}\phi_{03},\\
\Theta_4\,(3\,W\,;\ 3\tau') &= A\phi_4^3-B\phi_4\phi_3^2-C\phi_4\phi_{14}^2+D\phi_4\phi_{13}^2-E\phi_3\phi_{14}\phi_{13},\\
\Theta_{12}\,(3\,W\,;\ 3\tau') &= A\phi_{12}^3+B\phi_{12}\phi_0^2+C\phi_{12}\phi_2^2+D\phi_{12}\phi_{01}^2+E\phi_0\phi_2\phi_{01}\,;
\end{aligned}$$

putting herein $W=0$ we obtain in succession the values of D, C and E, expressed in terms of the constants previously used, $\overline{\Theta}_5$, $\overline{\Theta}_{34}$, Θ_5, Θ_{34} and the constants $\overline{\Theta}_{23}$, $\overline{\Theta}_4$, $\overline{\Theta}_{12}$, Θ_{23}, Θ_{03}, Θ_4, Θ_{14}, Θ_{12}, Θ_0, Θ_2, Θ_{01}. Thus the zero values of each of the ten even functions $\Theta(W;\tau')$ enter in the expression of the coefficients A, B, C, D, E; there remains then the question of the expression of the zero values of the ten even functions in terms of four independent quantities (cf. Ex. iv., § 317, Chap. XVII.), and the question of the relations connecting the constants $\overline{\Theta}_5$, $\overline{\Theta}_{34}$, etc., and the constants Θ_5, Θ_{34}, etc. (cf. the following example).

Ex. ii. Denoting $\Theta_{01}(0;3\tau')\,\Theta_{01}(0;\tau')$ by C_{01}, etc., shew that when $p=2$ the result of Ex. iii., § 292 (p. 477) gives the equations

$$\begin{aligned} C_{01}+C_2 &= C_5+C_{34}-C_{12}-C_0,\\ C_4+C_{03} &= C_5-C_{34}+C_{12}-C_0,\\ C_{23}+C_{14} &= C_5-C_{34}-C_{12}+C_0, \end{aligned}$$

these being the only equations derivable from that result. By these equations, in virtue of the relations connecting the ten constants $\Theta(0;\tau')$, and the relations connecting the ten constants $\Theta(0;3\tau')$, (for the various even characteristics), the three ratios

$$\Theta_{34}(0;3\tau')/\Theta_5(0,3\tau'),\quad \Theta_{12}(0;3\tau')/\Theta_5(0;3\tau'),\quad \Theta_0(0;3\tau')/\Theta_5(0;3\tau')$$

are determinable in terms of the three

$$\Theta_{34}(0;\tau')/\Theta_5(0;\tau'),\quad \Theta_{12}(0,\tau')/\Theta_5(0;\tau'),\quad \Theta_0(0;\tau')/\Theta_5(0;\tau').$$

By addition of these equations we obtain

$$C_{01}+C_2+C_4+C_{03}+C_{23}+C_{14}+C_{34}+C_{12}+C_0=3C_5.$$

Obtain similarly from the result of Ex. iii., § 292, for any value of p, the equation

$$\Sigma\Theta\left[0;\,3\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}s'\\ s\end{pmatrix}\right]\Theta\left[0;\,\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}s'\\ s\end{pmatrix}\right]=(2^p-1)\,\Theta(0;3\tau')\,\Theta(0;\tau'),$$

where the summation on the left extends to all even characteristics except the zero characteristic; for instance, when $p=1$, this is the equation

$$\Theta_{01}(0;3\tau')\,\Theta_{01}(0;\tau')+\Theta_{10}(0;3\tau')\,\Theta_{10}(0;\tau')=\Theta_{00}(0;3\tau')\,\Theta_{00}(0;\tau'),$$

namely (cf. Ex. i., § 365 of this chapter) it is the modular equation for transformation of the third order which is generally written in the form (Cayley, *Elliptic Functions*, 1876, p. 188),

$$\sqrt{k\lambda}+\sqrt{k'\lambda'}=1.$$

As here in the case $p=2$, so for any value of p, we obtain, from the result of Ex. iii., § 292, 2^p-1 modular equations for the cubic transformation.

Ex. iii. From the formula of § 364 we obtain modular equations for the quadric transformation, in the form

$$2^p\,\Theta\left[0;\,2\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}k'\\ k\end{pmatrix}\right]\Theta\left[0;\,2\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}k'+s'\\ k\end{pmatrix}\right]=\sum_s e^{\pi isk'}\,\Theta\left[0;\,\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}s'\\ s\end{pmatrix}\right]\Theta\left[0;\,\tau'\,\middle|\,\tfrac{1}{2}\begin{pmatrix}s'\\ k+s\end{pmatrix}\right],$$

where s is a row of p quantities each either 0 or 1, so that the right side contains 2^p terms, and k, k', s' are any rows of p quantities each either 0 or 1.

374. In the fundamental equations of transformation we have considered only the case when the matrices α, α', β, β' are matrices of integers; the analytical theory can be formulated in a more general way, as follows; the argument is an application of the results of Chap. XIX.

Suppose we have the relations expressed (cf. Ex. ii., § 324, Chap. XVIII.) by

$$\begin{pmatrix} M, & 0 \\ 0, & r\bar{M}^{-1} \end{pmatrix} \begin{pmatrix} 2v, & 2v' \\ 2\zeta, & 2\zeta' \end{pmatrix} = \begin{pmatrix} 2\omega, & 2\omega' \\ 2\eta, & 2\eta' \end{pmatrix} \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix},$$

where r is a positive rational number, M is any matrix of p rows and columns, whose determinant does not vanish, α, β, α', β' are matrices of p rows and columns whose elements are *rational numbers not necessarily integers*, ω, ω', η, η' are matrices of p rows and columns satisfying the equations (B), § 140 (Chap. VII.), and v, v', ζ, ζ' are similar matrices satisfying similar conditions; then, as necessarily follows, the matrices α, β, α', β' satisfy the relation (viii) of § 324 (Chap. XVIII.).

If now x, y be any matrices of p rows and columns, the relations supposed are immediately seen to be equivalent to

$$\begin{pmatrix} M, & 0 \\ 0, & r\bar{M}^{-1} \end{pmatrix} \begin{pmatrix} 2vx, & 2v'y \\ 2\zeta x, & 2\zeta' y \end{pmatrix} = \begin{pmatrix} 2\omega, & 2\omega' \\ 2\eta, & 2\eta' \end{pmatrix} \begin{pmatrix} \alpha x, & \beta y \\ \alpha' x, & \beta' y \end{pmatrix};$$

we suppose that x, y *are such matrices of integers that* αx, βy, $\alpha' x$, $\beta' y$ *are matrices of integers, and, at the same time, such that* rx *is a matrix of integers*; such matrices x, y can be determined in an infinite number of ways.

Let u, w be two rows of p arguments connected by the equations $u = Mw$; when the arguments w are simultaneously increased by the elements of the row of quantities denoted by $2vxm + 2v'ym'$, in which m, m' are rows of p integers, the arguments u are increased by the elements of the row $2\omega n + 2\omega' n'$, where $n = \alpha xm + \beta ym'$, $n' = \alpha' xm + \beta' ym'$ are rows of integers. The resulting factor of the function $\vartheta(u;\ 2\omega, 2\omega', 2\eta, 2\eta')$ is e^R, where, if $H_\alpha = 2\eta\alpha + 2\eta'\alpha'$, etc., (cf. (v), § 324, Chap. XVIII.), R is given by

$$\begin{aligned} R &= H_n(u + \tfrac{1}{2}\Omega_n) - \pi i n n' \\ &= (H_\alpha xm + H_\beta ym')(Mw + Mvxm + Mv'ym') - \pi i n n' \\ &= (\bar{M}H_\alpha xm + \bar{M}H_\beta ym')(w + vxm + v'ym') - \pi i n n' \\ &= r(2\zeta xm + 2\zeta' ym')(w + vxm + v'ym') - \pi i n n'; \end{aligned}$$

now, since $\bar{\beta}'\alpha = r + \bar{\beta}\alpha'$, and because αx, βy, $\alpha' x$, $\beta' y$, rx are matrices of integers, we have

$$\begin{aligned} nn' &= \bar{x}\bar{\alpha}'\alpha xm^2 + (\bar{y}\bar{\beta}\alpha' x + \bar{y}\bar{\beta}'\alpha x)\, mm' + \bar{y}\bar{\beta}'\beta ym'^2 \\ &\equiv fm + f'm' + r\bar{y}xmm' \pmod{2}, \end{aligned}$$

where f, f' denote respectively the rows of integers formed by the diagonal elements of the symmetrical matrices $\bar{x}\bar{\alpha}'\alpha x$, $\bar{y}\bar{\beta}'\beta y$ (cf. § 327, Chap. XVIII.).

Thus, if we denote $\vartheta(u;\ 2\omega, 2\omega', 2\eta, 2\eta')$ by $\phi(w)$, we have

$$\phi(w + 2vxm + 2v'ym') = e^{r(2\zeta xm + 2\zeta' ym')(w + vxm + v'ym') + \pi i(fm + f'm') + \pi i(-r\bar{y}x)mm'}\, \phi(w).$$

Further if a, b denote the matrices of $2p$ columns and p rows, given respectively by

$$a=(2vx,\ 2v'y),\quad 2\pi i b=(2r\zeta x,\ 2r\zeta' y),$$

we have

$$\begin{aligned}\tfrac{1}{2}\frac{\pi i}{r}(\bar{a}b-\bar{b}a)&=\begin{pmatrix}\bar{x}\,\bar{v}\\ \bar{y}\,\bar{v}'\end{pmatrix}(\zeta x,\ \zeta' y)-\begin{pmatrix}\bar{x}\,\bar{\zeta}\\ \bar{y}\,\bar{\zeta}'\end{pmatrix}(vx,\ v'y)\\ &=\begin{pmatrix}\bar{x}(\bar{v}\zeta-\bar{\zeta}v)x, & \bar{x}(\bar{v}\zeta'-\bar{\zeta}v')y\\ \bar{y}(\bar{v}'\zeta-\bar{\zeta}'v)x, & \bar{y}(\bar{v}'\zeta'-\bar{\zeta}'v')y\end{pmatrix}\\ &=\tfrac{1}{2}\pi i\begin{pmatrix}0, & -\bar{x}y\\ \bar{y}x, & 0\end{pmatrix};\end{aligned}$$

so that $\bar{a}b-\bar{b}a=k$, say, is a skew symmetrical matrix of integers given by

$$\bar{a}b-\bar{b}a=k=\begin{pmatrix}0\ , & -r\bar{x}y\\ r\bar{y}x, & 0\end{pmatrix},$$

and we have

$$\sum^{\alpha<\beta} k_{\alpha,\beta}m_\alpha m_\beta'=\sum_{\alpha,\beta}(-r\bar{x}y)_{\alpha,\beta}m_\alpha m_\beta'=-r\bar{y}xmm',\qquad(\alpha,\beta=1,\ldots,p).$$

Finally, let λ, μ be rows of p quantities, the rows of conjugate complex quantities being denoted by λ_1, μ_1, and let λ, μ be taken so that the row of quantities $a(\lambda,\mu)$ consists of zeros, or

$$a(\lambda,\mu)=2vx\lambda+2v'y\mu=0,$$

so that $x\lambda=-\tau' y\mu$, where* $\tau'=v^{-1}v'$, is a symmetrical matrix, $=\rho'+i\sigma'$, say, ρ' and σ' being matrices of real quantities; then by

$$x\lambda_1=-\tau_1' y\mu_1=-(\rho'-i\sigma')y\mu_1,$$

we have

$$\begin{aligned}ik(\lambda,\mu)(\lambda_1,\mu_1)&=-ir(\bar{x}y\mu,\ -\bar{y}x\lambda)(\lambda_1,\mu_1)=-ir(\bar{y}x\lambda_1\mu-\bar{y}x\lambda\mu_1)\\ &=ir\bar{y}(\tau_1' y\mu_1\mu-\tau' y\mu\mu_1)=ir\bar{y}\left[(\rho'-i\sigma')-(\rho'+i\sigma')\right]y\mu\mu_1\\ &=2r\bar{y}\sigma' y\mu\mu_1=2r\sigma'\nu\nu_1,\end{aligned}$$

in which $\nu=y\mu$, $\nu_1=y\mu_1$; as in § 325, Chap. XVIII., since r is positive, the form $r\sigma'\nu\nu_1$ is necessarily positive except for zero values of μ.

On the whole, comparing formula (II), § 354, Chap. XIX., the function $\phi(w)$ satisfies the conditions of §§ 351—2, Chap. XIX., necessary for a Jacobian function of w in which the periods and characteristic are given † by

$$a=(2vx,\ 2v'y),\quad 2\pi i b=(2r\zeta x,\ 2r\zeta' y),\quad c=(\tfrac{1}{2}f,\ \tfrac{1}{2}f').$$

* The determinant of the matrix v is supposed other than zero, as in Chap. XVIII., § 324.

† In § 351, Chap. XIX., the row letters have σ elements; in the present case σ is equal to $2p$, and it is convenient to represent the corresponding row letters by two constituents, each of p elements; and similarly for the matrices of $2p$ columns and p rows.

To this function we now apply the result of § 359, Chap. XIX., in order to express it by theta functions of w. The condition for the matrix of integers there denoted by g, namely $\bar{g}\epsilon g = k$, is satisfied by $g = \begin{pmatrix} rx, & 0 \\ 0, & y \end{pmatrix}$, for

$$\begin{pmatrix} r\bar{x}, & 0 \\ 0, & \bar{y} \end{pmatrix}\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}\begin{pmatrix} rx, & 0 \\ 0, & y \end{pmatrix} = \begin{pmatrix} r\bar{x}, & 0 \\ 0, & \bar{y} \end{pmatrix}\begin{pmatrix} 0, & -y \\ rx, & 0 \end{pmatrix} = \begin{pmatrix} 0, & -r\bar{x}y \\ r\bar{y}x, & 0 \end{pmatrix};$$

hence, with the notation of § 358, Chap. XIX.,

$$K = ag^{-1} = (2vx,\ 2v'y)\begin{pmatrix} \frac{1}{r}x^{-1}, & 0 \\ 0, & y^{-1} \end{pmatrix} = (2v/r,\ 2v'),$$

$$2\pi iL = 2\pi ibg^{-1} = (2r\zeta x,\ 2r\zeta' y)\begin{pmatrix} \frac{1}{r}x^{-1}, & 0 \\ 0, & y^{-1} \end{pmatrix} = (2\zeta,\ 2r\zeta').$$

Hence, as our final result, by § 359, Chap. XIX., *the function* $\phi(w)$, *or* $\vartheta(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta')$, *can be expressed as a sum of constant multiples of functions** $\vartheta(w;\ 2v/r,\ 2v',\ 2\zeta,\ 2\zeta')$ *with different characteristics, the number of such terms being at most* $\sqrt{|K|} = r^p|x|\,|y|$, *where* $|x|$, $|y|$ *denote the determinants of the matrices* x, y. This is an extension of the result obtained when the matrices α, β, α', β' are formed with integers; as in that case there will be a reduction in the number of terms, from $r^p|x|\,|y|$, owing to the fact that the function $\phi(w)$ is even. A similar result holds whatever be the characteristic of the function $\vartheta(u;\ 2\omega,\ 2\omega',\ 2\eta,\ 2\eta')$. The generalisation is obtained quite differently by Prym and Krazer, *Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen* (Leipzig, 1892), *Zweiter Theil*, which should be consulted.

Ex. Denoting by E the matrix of p rows and columns of which the elements are zero, other than those in the diagonal, which are each unity, and taking for the matrices a, β, a', β' respectively $\frac{m}{n}E$, 0, 0, $\frac{n}{m}E$, where m, n are integers without common factor, we have the formula

$$m^p\,\Theta(u;\ \tau) = \sum_r\sum_s \Theta\left(\frac{n}{m}u;\ \frac{n^2}{m^2}\tau \,\middle|\, \begin{matrix} ms/n \\ nr/m \end{matrix}\right),$$

wherein r, s are rows of p positive integers, in which every element of r is 0 or numerically less than m, and every element of s is 0 or numerically less than n. This formula includes that of § 284, Ex. iii. (Chap. XV.); it is a particular case of a formula given by Prym and Krazer (*loc. cit.*, p. 77).

To obtain a verification—the general term of the right side is e^{ψ}, where

$$\psi = 2\pi iu\left(\frac{n}{m}N+s\right) + i\pi\tau\left(\frac{n}{m}N+s\right)^2 + 2\pi ir\left(\frac{n}{m}N+s\right);$$

* That is, functions $\vartheta(rw,\ 2v,\ 2rv',\ 2\zeta/r,\ 2\zeta')$; cf. § 284, p. 448.

hence $\sum_r e^{\psi}=0$ unless N/m is integral; when N/m is integral, $=M$, say, then $\sum_r e^{\psi}=m^p e^{\phi}$, where

$$\phi=2\pi iuK+i\pi\tau K^2,$$

K, $=nM+s$, obtaining all integral values when M takes all integral values and s takes all integral values (including zero) which are numerically less than n.

375. The theory of the transformation of theta functions may be said to have arisen in the problem of the algebraical transformation of the hyperelliptic theta quotients considered in Chap. XI. of this volume. To practically utilise the results of this chapter for that problem it is necessary to adopt conventions sufficient to determine the constant factors occurring in the algebraic expression of these theta quotients (cf. §§ 212, 213), and to define the arguments of the theta functions in an algebraical way. The reader is referred* to the forthcoming volumes of Weierstrass's lectures.

It has already (§ 174, p. 248) been remarked that when $p>3$ the most general theta function cannot be regarded as arising from a Riemann surface; for the algebraical problems then arising the reader is referred to the recent papers of Schottky and Frobenius (*Crelle*, CII. (1888), and following) and to the book of Wirtinger, *Untersuchungen über Thetafunctionen* (Leipzig, 1895).

* Cf. Rosenhain, *Mém. p. divers Savants*, XI. (1851), p. 416 ff.; Königsberger, *Crelle*, LXIV. (1865), etc.