# CHAPTER XIV.

## FACTORIAL FUNCTIONS.

252. THE present chapter is concerned* with a generalisation of the theory of rational functions and their integrals. As in that case, it is convenient to consider the integrals and the functions together from the first. In order, therefore, that the reader may be better able to follow the course of the argument, it is desirable to explain, briefly, at starting, the results obtained. All the functions and integrals considered have certain fixed singularities, at places† denoted by $c_1, \dots, c_k$. A function or integral which has no infinities except at these fixed singularities is described as everywhere finite. The functions of this theory which replace the rational functions of the simpler theory have, beside the fixed singularities, no infinities except poles. But the functions differ from rational functions in that their values are not the same at the two sides of any period loop; these values have a ratio, described as the *factor*, which is constant along the loop; and a system of functions is characterised by the values of its factors. We consider two sets of factors, and, correspondingly, two sets of *factorial functions*, those of the *primary system* and those of the *associated system*; their relations are quite reciprocal. We have then a circumstance to which the theory of rational functions offers no parallel; *there may be everywhere finite factorial functions*‡. The number of such functions of the primary system which are linearly independent is denoted by $\sigma' + 1$; the number of the associated system by $\sigma + 1$. As in the case of algebraical integrals, we may have *everywhere finite factorial integrals*. The number of such integrals of the primary system which are linearly independent is denoted by $\varpi$, that of the associated system by $\varpi'$. The factorial integrals of the primary system are not integrals of factorial functions of that system; they are chosen so that the values $u$, $u'$

* The subject of the present chapter has been considered by Prym, *Crelle*, LXX. (1869), p. 354; Appell, *Acta Mathematica*, XIII. (1890); Ritter, *Math. Annal.* XLIV. (1894), pp. 261—374. In these papers other references will be found. See also Hurwitz, *Math. Annal.* XLI. (1893), p. 434, and, for a related theory, not considered in the present chapter, Hurwitz, *Math. Annal.* XXXIX. (1891), p. 1. For the latter part of the chapter see the references given in §§ 273, 274, 279.

† In particular the theory includes the case when $k=0$, and no such places enter.

‡ This statement is made in view of the comparison instituted between the development of the theory of rational functions and that of factorial functions. The factorial functions have (unless $k=0$) fixed infinities.

of such an integral on the two sides of a period loop are connected by an equation of the form $u' = Mu + \mu$, where $\mu$ is a constant and $M$ is the factor of the primary system of factorial functions which is associated with that period loop. The primary and associated systems are so related that if $F$ be a factorial function of either system, and $G'$ a factorial integral of the other system, $FdG'/dx$ is a rational function without assigned singularities. In the case of the rational functions, the smallest number of arbitrary assigned poles for which a function can always be constructed is $p+1$. In the present theory, as has been said, it may be possible to construct factorial functions of the primary system without poles; but when that is impossible, or $\sigma'+1=0$, the smallest number of arbitrary poles for which a factorial function of the primary system can always be constructed is $\varpi'+1$. Similarly when $\sigma+1=0$, the smallest number of arbitrary poles for which a factorial function of the associated system can always be constructed is $\varpi+1$. Of the two numbers $\sigma+1$, $\sigma'+1$, at least one is always zero, except in one case, when they are both unity. When $\sigma'+1$ is $>0$, the everywhere finite factorial functions of the primary system can be expressed linearly in terms of the everywhere finite factorial integrals of the same system. We can also construct factorial integrals of the primary system, which, beside the fixed singularities, have assigned poles; the least number of poles of arbitrary position for which this can be done is $\sigma+2$. And we can construct factorial integrals of the primary system which have arbitrary logarithmic infinities; the least number of such infinities of arbitrary position is $\sigma+2$. For the associated system of factors the corresponding numbers are $\sigma'+2$.

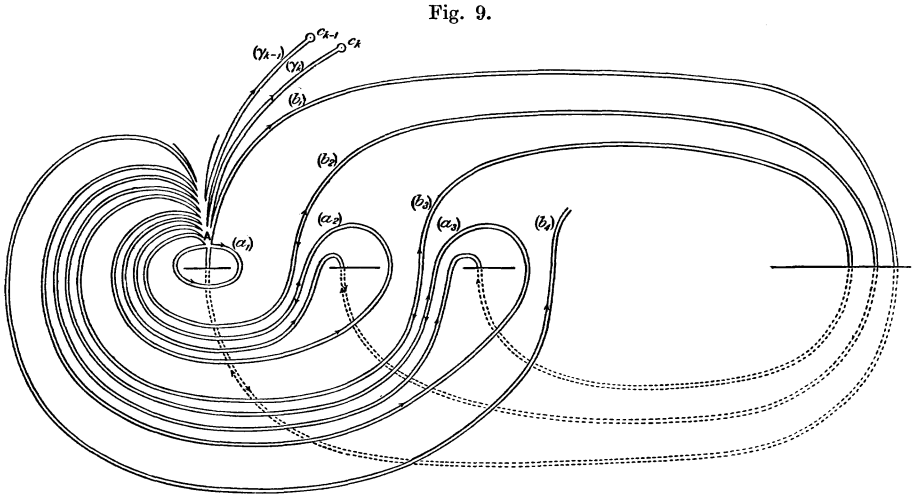
It will be found that all the formulae of the general theory are not immediately applicable to the ordinary theory of rational functions and their integrals. The exceptions, and the reasons for them, are pointed out in footnotes.

The deduction of these results occupies §§ 253—267 of this chapter. The section of the chapter which occupies §§ 271—278, deals, by examples, with the connection of the present theory with the theory of the Riemann theta functions. With a more detailed theory of factorial functions this section would be capable of very great development. The concluding section of the chapter deals very briefly with the identification of the present theory with the theory of automorphic functions.

253. Let $c_1, \ldots, c_k$ be arbitrary fixed places of the Riemann surface, which we suppose to be finite places and not branch places. In all the investigations of this chapter these places are to be the same. They may be called the essential singularities of the systems of factorial functions. We require the surface to be dissected so that the places $c_1, \ldots, c_k$ are excluded and the surface becomes simply connected. This may be effected in a manner analogous to that adopted in § 180, the places $c_1, \ldots, c_k$ occurring instead of

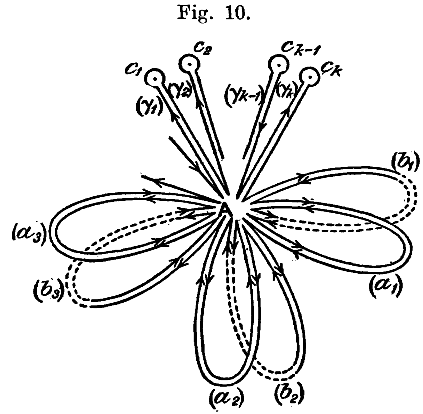
$z_1, \ldots, z_k$. But it is more convenient, in view of one development of the theory, to suppose the loops of § 180 to be deformed until the cuts* between the pairs of period loops become of infinitesimal length. Then the dissection will be such as that represented in figure 9; and this dissection is sufficiently

Fig. 9.



well represented by figure 10. We call the sides of the loops $(a_r)$, $(b_r)$, upon which the letters $a_r$, $b_r$ are placed, the left-hand sides of these loops, and by the left-hand sides of the cuts $(\gamma_1), \ldots, (\gamma_k)$, to the places $c_1, \ldots, c_k$, we mean

Fig. 10.



the sides which are on the left when we pass from $A$ to $c_1, \ldots, c_k$ respectively. The consideration of the effect of an alteration in these conventions is postponed till the theory of the transformation of the theta functions has been considered.

* These cuts are those generally denoted by $c_1, \ldots, c_{p-1}$. Cf. Forsyth, *Theory of Functions*, § 181.

254. In connection with the surface thus dissected we take now a series of $2p+k$ quantities

$$\lambda_1, \dots, \lambda_k, \quad h_1, \dots, h_p, \quad g_1, \dots, g_p,$$

which we call the fundamental constants; we suppose no one of $\lambda_1, \dots, \lambda_k$ to be a positive or negative integer, or zero; but we suppose $\lambda_1 + \dots + \lambda_k$ to be an integer, or zero; and we consider functions

(1) which are uniform on the surface thus dissected, and have, thereon, no infinities except poles,

(2) whose value on the left-hand side of the period loop $(a_i)$ is $e^{-2\pi i h_i}$ times the value on the right-hand side; whose value on the left-hand side of the period loop $(b_i)$ is $e^{2\pi i g_i}$ times the value on the right-hand side,

(3) which*, in the neighbourhood of the place $c_i$, are expressible in the form $t^{-\lambda_i}\phi_i$, where $t$ is the infinitesimal at $c_i$ and $\phi_i$ is uniform, finite, and not zero in the neighbourhood of the place $c_i$,

(4) which, therefore, have a value on the left-hand side of the cut $\gamma_i$ which is $e^{-2\pi i \lambda_i}$ times the value on the right-hand side.

Let $\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_N$ be any places; consider the expression

$$f = Ae^{\Pi_{\beta_1, m}^{x, a} + \dots + \Pi_{\beta_N, m}^{x, a} - \Pi_{\alpha_1, m}^{x, a} - \dots - \Pi_{\alpha_M, m}^{x, a} - 2\pi i [(h_1 + H_1) v_1^{x, a} + \dots + (h_p + H_p) v_p^{x, a}] - \sum_{i=1}^{k} \lambda_i \Pi_{c_i, m}^{x, a}},$$

wherein $A$ is independent of the place $x$,

$$N - M = \sum_{i=1}^{k} \lambda_i, \qquad \text{(i)},$$

$\Sigma\lambda$ being an integer (or zero), $m$ is an arbitrary place, and $H_1, \dots, H_p$ are integers. It is clear that this expression represents a function which is uniform on the dissected surface, which has poles at the places $\alpha_1, \dots, \alpha_M$, and zeros at the places $\beta_1, \dots, \beta_N$, and that in the neighbourhood of the place $c_i$ this function has the character required. For the period loop $(a_i)$ the function has the factor $e^{-2\pi i (h_i + H_i)} = e^{-2\pi i h_i}$, as desired; for the period loop $(b_i)$ the function has the factor $e^{2\pi i K}$, where

$$K = v_i^{\beta_1, m} + \dots + v_i^{\beta_N, m} - v_i^{\alpha_1, m} - \dots - v_i^{\alpha_M, m} - \sum_{r=1}^{p} (h_r + H_r) \tau_{r,i} - \sum_{r=1}^{k} \lambda_i v_i^{c_r, m},$$

and this factor is equal to $e^{2\pi i g_i}$ if only

$$v_i^{\beta_1, m} + \dots + v_i^{\beta_N, m} - v_i^{\alpha_1, m} - \dots - v_i^{\alpha_M, m} - \sum_{r=1}^{k} \lambda_i v_i^{c_r, m}$$
$$= g_i + G_i + \sum_{r=1}^{r=p} (h_r + H_r) \tau_{r,i}, \qquad \text{(ii)},$$

$G_i$ being an integer.

* It is intended, as already stated, that the places $c_1, \dots, c_k$ should be in the finite part of the surface and should not be branch places.

It follows therefore that, subject to the conditions (i) and (ii), such a function as has been described certainly exists.

Conversely it can be immediately proved that any such function must be capable of being expressed in the form here given, and that the conditions (i), (ii) are necessary.

Unless the contrary be expressly stated, we suppose the quantities $\lambda_1, \dots, \lambda_k, h_1, \dots, h_p, g_1, \dots, g_p$ always the same, and express this fact by calling the functions under consideration *factorial functions of the primary system.* The quantities $e^{-2\pi i\lambda_1}, \dots, e^{-2\pi i\lambda_k}, e^{-2\pi i h_1}, \dots, e^{-2\pi i h_p}, e^{2\pi i g_1}, \dots, e^{2\pi i g_p}$ are called *the factors.* It will be convenient to consider with these functions other functions of the same general character but with a different system of fundamental constants,

$$\lambda_1', \dots, \lambda_k', h_1', \dots, h_p', g_1', \dots, g_p',$$

connected with the original constants by the equations

$$\lambda_i + \lambda_i' + 1 = 0, \quad h_i + h_i' = 0, \quad g_i + g_i' = 0;$$

these functions will be said to be *functions of the associated system.* The factors associated therewith are the inverses of the factors of the primary system.

255. As has been remarked, the rational functions on the Riemann surface are a particular case of the factorial functions, arising when the factors are unity and no such places as $c_1, \dots, c_k$ are introduced. From this point of view the condition (i), which can be obtained as the condition that $\int d \log f$, taken round the complete boundary of the dissected surface, is zero, is a generalisation of the fact that the number of zeros and poles of a rational function is the same, and the condition (ii) expresses a theorem generalising Abel's theorem for integrals of the first kind.

Now Riemann's theory of rational functions is subsequent to the theory of the integrals; these arise as functions which are uniform on the dissected Riemann surface, but differ on the sides of a period loop by additive constants. In what follows we consider the theory in the same order, and enquire first of all as to the existence of functions whose differential coefficients are factorial functions. For the sake of clearness such functions will be called *factorial integrals;* and it will appear that just as rational functions are expressible by Riemann integrals of the second kind, so factorial functions are expressible by certain factorial integrals, provided the fundamental constants of these latter are suitably chosen. We define then a factorial integral of the primary system, $H$, as a function such that $dH/dx$ is a factorial function with the fundamental constants

$$\lambda_1 + 1, \dots, \lambda_k + 1, \; h_1, \dots, h_p, \; g_1, \dots, g_p;$$

thus $dH/dx$ has the same factors as the factorial functions of the primary system, but near the place $c_i$, $dH/dx$ is of the form $t^{-(\lambda_i+1)}\phi_i$, where $\phi_i$ is uniform, finite and not zero in the neighbourhood of $c_i$. Similarly we define a factorial integral of the associated system, $H'$, to be such that $dH'/dx$ is a factorial function with the fundamental constants

$$\lambda_1'+1, \ldots, \lambda_k'+1, \; h_1', \ldots, h_p', \; g_1', \ldots, g_p',$$

or

$$-\lambda_1, \ldots, -\lambda_k, \; -h_1, \ldots, -h_p, \; -g_1, \ldots, -g_p;$$

thus, if $f$ be any factorial function of the primary system, $f\,dH'/dx$ is a rational function on the Riemann surface, for which the places $c_1, \ldots, c_k$ are not in any way special. And similarly, if $f'$ be any factorial function of the associated system, and $H$ any factorial integral of the primary system, $f'\,dH/dx$ is a rational function.

The values of a factorial integral of the primary system, $H$, at the two sides of any period loop are connected by an equation of the form

$$\overline{H} = \mu H + \Omega,$$

where $\mu$ is one of the factors $e^{-2\pi i h_r}$, $e^{2\pi i g_r}$, and $\Omega$ is a quantity which is constant along the particular period loop. Near $c_i$, $H$ is of the form

$$A_i + t^{-\lambda_i}\phi_i + C_i \log t,$$

where $A_i$ is a constant, $\phi_i$ is uniform, finite, and, in general, not zero in the neighbourhood of $c_i$, and $C_i$ is a constant, which is zero unless $\lambda_i + 1$ be a positive integer (other than zero), and may be zero even when $\lambda_i + 1$ is a positive integer. After a circuit round $c_i$, $H$ will be changed into

$$\overline{H} = A_i + e^{-2\pi i\lambda_i} t^{-\lambda_i}\phi_i + 2\pi i C_i + C_i \log t;$$

thus, when $C_i = 0$,

$$\overline{H} = He^{-2\pi i\lambda_i} + A_i(1 - e^{-2\pi i\lambda_i}),$$

and when $C_i$ is not zero, and, therefore, $\lambda_i + 1$ is a positive integer,

$$\overline{H} = H + 2\pi i C_i;$$

in either case we have

$$\overline{H} = \gamma H + \Gamma,$$

where $\gamma = e^{-2\pi i\lambda_i}$, and $\Gamma$ is constant along the cut $(\gamma_i)$.

Thus, in addition to the fundamental factors of the system, there arise, for every factorial integral, $2p + k$ new constants, $2p$ of them such as that here denoted by $\Omega$ and $k$ of them such as that denoted by $\Gamma$. It will be seen subsequently that these are not all independent.

As has been stated we exclude from consideration the case in which any one of $\lambda_1, \dots, \lambda_k$ is an integer, or zero. Thus the constants $C_i$ will not enter; neither will the corresponding constants for the associated system.

256. Consider now the problem of finding *factorial integrals of the primary system which shall be everywhere finite.* Here, as elsewhere, when we speak of the infinities or zeros of a function, we mean those which are not at the places $c_1, \dots, c_k$, or which fall at these places in addition to the poles or zeros which are prescribed to fall there.

If $V$ be such a factorial integral, $dV/dx$ is only infinite when $dx$ is zero of the second order, namely $2p - 2 + 2n$ times, at the branch places of the surface. And $dV/dx$ is zero at $x = \infty$, $2n$ times*. Thus, if $N$ denote the number of zeros of $dV/dx$ which are not due to the denominator $dx$, or, as we may say (cf. § 21) the number of zeros of $dV$, we have by the condition (i) § 254,

$$N + 2n = 2p - 2 + 2n + \sum_{i=1}^{k} (\lambda_i + 1),$$

so that the number of zeros of $dV$ is $2p - 2 + \Sigma (\lambda_i + 1)$.

Now let $f_0$ denote a factorial function with the primary system of factors, but with behaviour at $c_i$ like $t^{-(\lambda_i+1)} \phi_i$, where $\phi_i$ is uniform, finite, and not zero at $c_i$. Then, if an everywhere finite factorial integral $V$ exists at all, $Z, = f_0^{-1} dV/dx$, will be a rational function on the Riemann surface, infinite at the (say $N_0$) zeros of $f_0$, and $2n + 2p - 2$ times at the branch places of the surface, and zero at the (say $M_0$) poles of $f_0$, and $2n$ times at $x = \infty$ (beside being zero at the zeros of $dV$). Conversely a rational function $Z$ satisfying these conditions will be such that $\int Z f_0 dx$ is a function $V$. Thus *the number of existent functions $V$ which are linearly independent is at least*

$$N_0 + 2n + 2p - 2 - (2n + M_0) - p + 1, \ = p - 1 + \sum_{i=1}^{k} (\lambda_i + 1),$$

*provided this be positive.* We are therefore sure, when this is the case, that functions $V$ do exist. To find the exact number, let $V_0$ be one such; then if $V$ be any other, $dV/dV_0$ is a rational function with poles in the $2p - 2 + \Sigma (\lambda + 1)$ zeros of $dV_0$; and conversely if $R$ be a rational function whose poles are the zeros of $dV_0$, the integral $\int R dV_0$ is a function $V$. Thus† *the number of functions $V$, when any exist,* is (§ 37, Chap. III.)

$$\varpi, \ = p - 1 + \Sigma (\lambda + 1) + \sigma + 1,$$

* These numbers may be modified by the existence of a branch place at infinity. But their difference remains the same.

† For the ordinary case of rational functions $\sigma + 1$, as here defined, is equal to unity, and, therefore, omitting the term $\Sigma (\lambda + 1)$, we have $\varpi = p$.

where $\sigma+1$ is the number of linearly independent differentials $dv$, of ordinary integrals of the first kind, which vanish in the $2p-2+\Sigma(\lambda+1)$ zeros of the differential $dV_0$ of any such function $V_0$. Since $dV/dV_0$ is a rational function, the number of differentials $dv$ vanishing in the zeros of $dV_0$ is the same as the number vanishing in the zeros of $dV$. Since $dv$ has $2p-2$ zeros, $\sigma+1$ vanishes when $\Sigma(\lambda+1)>0$.

*Ex.* For the hyperelliptic surface

$$y^2=(x-a)(x-b)(x, 1)_{2p},$$

the factorial integrals, $V$, having the same factors at the period loops as the root function $\sqrt{(x-a)(x-b)}$, and no other factors, are given by

$$\int\sqrt{(x-a)(x-b)}\,(x, 1)_{p-2}\frac{dx}{y}$$

and $\varpi=p-1$. Here $k=0$; there are no places $c_1, \dots, c_k$.

257. The number $\sigma+1$ is of great importance; when it is greater than zero, which requires $\Sigma(\lambda+1)$ to be negative or zero, *there are $\sigma+1$ factorial functions of the associated system which are nowhere infinite.*

For if $V$ be an everywhere finite factorial integral of the primary system, and $dv_1, \dots, dv_{\sigma+1}$ represent the linearly independent differentials of integrals of the first kind which vanish in the zeros of $dV$, the functions

$$\frac{dv_1}{dV}, \dots, \frac{dv_{\sigma+1}}{dV},$$

whose behaviour at a place $c_i$ is like that of $\frac{1}{t^{-(\lambda_i+1)}}\phi_i$, where $\phi_i$ is uniform, finite and not zero in the neighbourhood of $c_i$, namely of $t^{-\lambda_i'}\phi_i$, are clearly factorial functions of the associated system, without poles. Conversely if $K'$ denote an everywhere-finite factorial function of the associated system, the integral $\int K'dV$ is the integral of a rational function, and does not anywhere become infinite. Denoting it by $v$, $dv$ vanishes at the $2p-2+\Sigma(\lambda+1)$ zeros of $dV$ as well as at the $0+\sum_{i=1}^{k}\lambda_i'$, $=-\Sigma(\lambda+1)$, zeros of $K'$ (cf. the condition (i), § 254). Thus, to every factorial integral $V$ we obtain $\sigma+1$ functions $K'$; and since, when $\sigma+1>0$, the quotient of two differentials $dV$, $dV_0$ can* be expressed by the quotient of two differentials $dv$, $dv_0$, we cannot thus obtain more than $\sigma+1$ functions $K'$; while, conversely, to every function $K'$ we obtain a differential $dv$ which vanishes in the zeros of any assigned function $V$; and, as before, we cannot obtain any others by taking, instead of $V$, another factorial integral $V_0$.

* Cf. Chap. VI. § 98.

258. The existence of these everywhere finite factorial functions, $K'$, of the associated system can also be investigated *à priori* from the fundamental equations (i) and (ii) (§ 254). These give, in this case,

$$v_i^{\beta_1, m} + \ldots + v_i^{\beta_N, m} = -\sum_{r=1}^{k} (\lambda_r + 1) v_i^{c_r, m} - (g_i + G_i) - \tau_{i,1}(h_1 + H_1) - \ldots\ldots$$
$$- \tau_{i,p}(h_p + H_p), \qquad \text{(iii)}$$

and $$N = -\sum_{r=1}^{k} (\lambda_r + 1),$$

where $G_1, \ldots, G_p, H_1, \ldots, H_p$ are integers.

Hence no functions $K'$ exist unless $\Sigma(\lambda + 1)$ be a negative integer or be zero; we consider these possibilities separately.

When $\Sigma(\lambda + 1) = 0$, it is necessary, for the existence of such functions, that the fundamental constants satisfy the conditions

$$\sum_{r=1}^{k} (\lambda_r + 1) v_i^{c_r, m} + g_i + h_1\tau_{i,1} + \ldots\ldots + h_p\tau_{i,p} \equiv 0, \qquad (i = 1, 2, \ldots, p);$$

conversely, when these conditions are fulfilled, taking suitable integers $H_1, \ldots, H_p$, it is clear that the function

$$E_0 = Ae^{\sum_{r=1}^{k} (\lambda_r+1)\Pi_{c_r, m}^{x, a} + 2\pi i (h_1 + H_1) v_1^{x, a} + \ldots\ldots + 2\pi i (h_p + H_p) v_p^{x, a}},$$

wherein $A$ is an arbitrary constant, and $a$, $m$ are arbitrary places, is an everywhere finite factorial function of the associated system, and it can be immediately seen that every such function is a constant multiple of $E_0$. If then we denote the number of functions $K'$ by $\Sigma + 1$ (to be immediately identified with $\sigma + 1$), we have, in this case, $\Sigma + 1 = 1$; and there are $p$ functions $V$, given by $V = \int E_0^{-1} dv$, where $dv$ is in turn the differential of every one of the linearly independent integrals of the first kind; it is easy to see that every function $V$ can be thus expressed. Thus, in the zeros of a differential $dV$ there vanishes one differential $dv$, so that $\sigma + 1 = 1$. Hence $\sigma + 1 = \Sigma + 1$, and the formula $\varpi = p - 1 + \Sigma(\lambda + 1) + \sigma + 1$ is verified.

When $\Sigma(\lambda + 1)$ is negative and numerically greater than zero, and the equations (iii) have any solutions, let $t$ denote the number of linearly independent differentials $dv$ which vanish in the places of one and therefore of every set, $\beta_1, \ldots, \beta_N$, which satisfies these equations; then* the number of sets which satisfy these equations is $\infty^{s-p+t}$, where $s = -\Sigma(\lambda + 1)$; thus the quotient of two functions $K'$ is a rational function with $\Sigma + 1$, $= s - p + t + 1$ arbitrary constants, one of these being additive. This is then the number of linearly independent functions $K'$. If $K'$ be one of these functions, and

* Cf. § 158, Chap. VIII.; § 95, Chap. VI.

$dv_1, \ldots, dv_t$ denote the differentials vanishing in the zeros of $K'$, it is clear that the functions

$$\int \frac{dv_1}{K'}, \ldots, \int \frac{dv_t}{K'}$$

are finite factorial integrals of the primary system, that is, are functions $V$; conversely if $V$ be any finite factorial integral of the primary system, $\int K' dV$ is an integral, $v$, of the first kind such that $dv$ vanishes in the zeros of $K'$. Hence the number $t$, which expresses the number of differentials $dv$ which vanish in the zeros of $K'$, is equal to the number, $\varpi$, of functions $V$. But we have proved that $\varpi = p - 1 + \Sigma(\lambda + 1) + \sigma + 1$, and, above, that $t = p - 1 - s + \Sigma + 1$. Hence $\sigma + 1 = \Sigma + 1$.

Thus we have the results*: *The number, $\sigma + 1$, of everywhere finite factorial functions, $K'$, of the associated system is equal to the number of differentials $dv$ which vanish in the* $2p - 2 + \Sigma(\lambda + 1)$ *zeros of any differential $dV$;* hence (§ 21, Chap. II.) $\sigma + 1$ is less than $p$, unless $\Sigma(\lambda + 1) = -(2p - 2)$.

Also, when $\sigma + 1 > 0$, *the number, $\varpi$, of everywhere finite factorial integrals, $V$, of the primary system, is equal to the number of differentials $dv$ which vanish in the $s$, $= -\Sigma(\lambda + 1)$, zeros of any function $K'$.* The argument by which this last result is obtained does not hold when† $\sigma + 1 = 0$. When $\sigma + 1 > 0$, it follows that $\varpi$ is not greater than $p$.

Similarly when $s'$, $= -\Sigma(\lambda' + 1)$, $= \Sigma\lambda$, $= -s - k$, is $> 0$, we can prove, by considering the primary system, that there are $\sigma' + 1$ everywhere finite factorial functions $K$ of the primary system, where $\sigma' + 1$ is the number of differentials $dv$ vanishing in the $2p - 2 - \Sigma\lambda$, $= 2p - 2 + s + k$, zeros of any differential $dV$; and that, when $\sigma' + 1 > 0$, the number $\varpi'$, of everywhere finite factorial integrals, $V'$, of the associated system is equal to the number of differentials $dv$ vanishing in the $s'$ zeros of any function $K$. Hence $\sigma' + 1 = 0$ when $s > 0$, and, then, no functions $K$ exist. When $s = 0$ we have seen that there may or may not be functions $K'$; but there cannot be functions $K$ unless $k = 0$, since otherwise $2p - 2 + s + k > 2p - 2$. And then the existence of functions $K$ depends on the condition whether the fundamental constants be such that

$$\frac{1}{E_0}, \; = e^{-2\pi i [(h_1 + H_1) v_1^{x, a} + \ldots\ldots + (h_p + H_p) v_p^{x, a}]},$$

is a function of the primary system or not, $H_1, \ldots, H_p$ being suitable integers, namely whether there exist relations of the form

$$g_i + G_i + (h_1 + H_1) \tau_{i, 1} + \ldots\ldots + (h_p + H_p) \tau_{i, p} = 0, \qquad (i = 1, 2, \ldots, p),$$

* Which hold for the ordinary case of rational functions, $\sigma + 1$ being then unity.

† In the case of the factorial functions which are square roots of rational functions of which all the poles and zeros are of the second order, so that the places $c_1, \ldots, c_k$ are not present, and the numbers $g$, $h$ are half integers, we have $\varpi = p - 1$, $\sigma + 1 = 0$.

where $G_1, \dots, G_p$ are integers. In such case $E_0$ is a finite factorial function of the associated system.

On the whole then the theory breaks up into four cases (i) $\sigma+1=0$, $\sigma'+1=0$, (ii) $\sigma+1>0$, $\sigma'+1=0$, (iii) $\sigma+1=0$, $\sigma'+1>0$, (iv) $\sigma+1=1$, $\sigma'+1=1$. Of these the cases (ii) and (iii) are reciprocal.

259. One remark remains to be made in this connection. When $\sigma+1>0$ there are everywhere finite functions, $K'$, of the associated system, given (§ 257) by

$$\frac{dv_1}{dV'}, \frac{dv_2}{dV'}, \dots\dots, \frac{dv_{\sigma+1}}{dV'};$$

these have, at any one of the places $c_1, \dots, c_k$, a behaviour represented by that of $t^{-\lambda'}\phi$; hence the differential coefficients of these functions satisfy all the conditions whereby the differential coefficients, $dV'/dx$, of the everywhere finite factorial integrals of the associated system, are defined. Therefore* the functions $K'$ are expressible linearly in terms of the functions $V_1', \dots, V'_{\varpi'}$ by equations of the form

$$K_i', =\frac{dv_i}{dV'}, = \lambda_{i,1} V_1' + \dots\dots + \lambda_{i,\varpi'} V'_{\varpi'} + \lambda, \qquad (i=1, 2, \dots, (\sigma+1)),$$

where the coefficients, $\lambda_{i,j}$, $\lambda$, are constants.

Hence also the difference $\varpi'-(\sigma+1)$ is not negative. This is also obvious otherwise. For when $\sigma+1>0$, $-\Sigma(\lambda+1)$, $=s$, is zero or positive, and $\sigma+1 \not> p$ (§ 258), and, therefore, $\varpi'-\sigma$, $=p-(\sigma+1)+\sigma'+1+k+s$, can only be as small as zero when $k=0=s$, and $\sigma+1=p$; these are incompatible.

Similarly, when $\sigma'+1>0$, the everywhere finite factorial functions of the original system are linear functions of the factorial integrals $V_1, \dots, V_{\varpi}$.

It follows† therefore that of the $\varpi$ periods of the functions $V_1, \dots, V_{\varpi}$, at any definite period loop, only $\varpi-(\sigma'+1)$ can be regarded as linearly independent; in fact, $\sigma'+1$ of the functions $V_1, \dots, V_{\varpi}$ may be replaced by linear functions of the remaining $\varpi-(\sigma'+1)$, and of the functions $K_1, \dots, K_{\sigma'+1}$.

260. A factorial integral is such that its values at the two sides of a period loop of the first kind are connected by an equation of the form $u'=\mu_i u+\Omega_i$, its values at the two sides of a period loop of the second kind are connected by an equation of the form $u'=\mu'_i u+\Omega'_i$, and its values at the two sides of a loop $(\gamma_i)$ are connected by an equation of the form $u'=\gamma_i u+\Gamma_i$, where‡ $\Gamma_i=A_i(1-\gamma_i)$. Of the $2p+k$ periods $\Omega_i$, $\Omega'_i$, $\Gamma_i$ thus

* It is clearly assumed that $K'_i$ is not a constant; thus the reasoning does not apply to the ordinary case of rational functions.

† In the ordinary case of rational functions this number $\varpi-(\sigma+1)$ must be replaced by $p$. See the preceding note.

‡ § 255. The case where one of $\lambda_1, \dots, \lambda_k$ is zero or an integer is excluded.

arising, two at least can be immediately excluded. For it is possible, by subtracting one of the constants $A_1, \ldots, A_k$ from the factorial integral, to render one of the periods $\Gamma_1, \ldots, \Gamma_k$ zero; and by following the values of the factorial integral, which is single-valued on the dissected surface, once completely round the sides of the loops, we find, in virtue of $\gamma_1\gamma_2 \ldots \gamma_k = 1$, that

$$\sum_{i=1}^{p} [\Omega_i (1-\mu_i') - \Omega_i' (1-\mu_i)] = \Gamma_1 + \gamma_1\Gamma_2 + \gamma_1\gamma_2\Gamma_3 + \ldots + \gamma_1\gamma_2 \ldots \gamma_{k-1}\Gamma_k.$$

Thus there are certainly not more than $2p-2+k$ linearly independent periods of a factorial integral.

Suppose now that $V$ is any everywhere finite factorial integral of the original system, and $V_i'$ is any one of the corresponding integrals of the associated system. The integral $\int V dV_i'$, taken once completely round the boundary of the surface which is constituted by the sides of the period loops, is equal to zero. By expressing this fact we obtain an equation which is linear in the periods of $V$ and linear in the periods of $V_i'$. By taking $i$ in turn equal to $1, 2, \ldots, \varpi'$, we thus obtain $\varpi'$ linear equations for the periods of $V$, wherein the coefficients are the periods of $V_1', \ldots, V'_{\varpi'}$. As remarked above these coefficients are themselves connected by $\sigma+1$ linear equations; so that we thus obtain at most $\varpi' - (\sigma+1)$ linearly independent linear equations for the periods of $V$. If these are independent of one another and independent of the two reductions mentioned above, it follows that the $2p+k$ periods of $V$ are linearly expressible by only

$$2p-2+k-[\varpi'-(\sigma+1)]$$

periods, at most. Now we have

$$\varpi = p-1+\Sigma(\lambda+1)+\sigma+1,$$

$$\varpi' = p-1-\Sigma(\lambda)+\sigma'+1,$$

and therefore

$$\varpi+\varpi' = 2p-2+k+\sigma+1+\sigma'+1,$$

so that

$$2p-2+k-[\varpi'-(\sigma+1)] = \varpi-(\sigma'+1).$$

Thus $\varpi-(\sigma'+1)$ is the number of periods of a function $V$ which appear to be linearly independent; and, taking account of the existence of the functions $K_1, \ldots, K_{\sigma'+1}$, this is the same as the number of independent linear combinations of the functions $V_1, \ldots, V_{\varpi}$, which are periodic*. But the conclusions of this article require more careful consideration in particular cases; it is not shewn that the linear equations obtained are always independent, nor that they are the only equations obtainable.

*Ex.* i. Obtain the lineo-linear relation connecting the periods of the everywhere finite factorial integrals $V$, $V'$, of the primary and associated system, which is obtained by expressing that the contour integral $\int V dV'$ vanishes.

*Ex.* ii. In the case of the ordinary Riemann integrals of the first kind, the relation

$$\sum_{i=1}^{p} [\Omega_i (1-\mu_i') - \Omega_i' (1-\mu_i)] = \Gamma_1 + \gamma_1\Gamma_2 + \gamma_1\gamma_2\Gamma_3 + \ldots + \gamma_1\gamma_2 \ldots \gamma_{k-1}\Gamma_k$$

is identically satisfied, and further $k=0$. Thus the reasoning of the text does not hold†.

* We can therefore form linear combinations of the periodic functions $V$, for which the independent periods shall be 1, 0, ..., 0; 0, 1, ..., 0; etc., as in the ordinary case.

† In that case the numbers $\varpi'-(\sigma+1)$, $2p-2+k$, are to be replaced respectively by $p$ and $2p$. See the note † of § 259.

261. We enquire now how many arbitrary constants enter into the expression of a factorial function of the primary system which has $M$ poles of assigned position.

Supposing one such function to exist, denote it by $F_0$; then the ratio $F/F_0$, of any other such function to $F$, $F_0$, is a rational function with poles at the zeros of $F_0$; conversely if $R$ be any rational function with poles at the zeros of $F_0$, $F_0R$ is a factorial function of the primary system with poles at the assigned poles of $F_0$. The function $R$ contains

$$N - p + 1 + h + 1$$

arbitrary constants, one of them additive, where $N$ is the number of zeros of $F_0$, so that $N = M + \sum_{r=1}^{k} \lambda_r$, and $h+1$ is the number of differentials $dv$ vanishing in the zeros of $F_0$.

But in fact the number of differentials $dv$ vanishing in the zeros of $F_0$ is the same as the number of differentials $dV'$ vanishing in the poles of $F_0$, $V'$ being any everywhere finite factorial integral of the associated system.

For if $dv$ vanish in the zeros of $F_0$, the integral $\int dv/F_0$ is clearly a factorial integral, $V'$, of the associated system without infinities, and such that $dV'$ vanishes in the poles of $F_0$; conversely if $V'$ be any factorial integral of the associated system such that $dV'$ vanishes in the poles of $F_0$, the integral $\int F_0 dV'$ is an integral of the first kind, $v$, such that $dv$ vanishes in the zeros of $F_0$.

Thus, *the number of arbitrary constants in a factorial function of the primary system, with $M$ given arbitrary poles, is*

$$M + \sum_{r=1}^{k} \lambda_r - p + 1 + h + 1, \; = N - p + 1 + h + 1, \; = M - \varpi' + h + 1 + \sigma' + 1,$$

where $N$ is the number of zeros of the function, and $h+1$ the number of differentials $dV'$ vanishing in the $M$ poles*.

In particular, putting $M = 0$, $h + 1 = \varpi'$ (cf. § 258), we have the formula, already obtained,

$$\sigma' + 1 = \sum_{r=1}^{k} \lambda_r - p + 1 + \varpi'.$$

We can of course also obtain these results by considering the fundamental equations (i) and (ii), § 254.

262. Hence we can determine the smallest value of $M$ for which a factorial function of the primary system with $M$ given poles always exists.

* Counting the additive constant in the expression of a rational function, the last formula holds in the ordinary case.

When $M = \varpi' + 1$ it is not possible to determine a function $V'$, of the form

$$V' = A_1 V_1' + \dots\dots + A_{\varpi'} V'_{\varpi'},$$

wherein $A_1, \dots, A_{\varpi'}$ are constants, to vanish in $M$ arbitrary places; and therefore $h + 1 = 0$. Thus a factorial function of the primary system with $\varpi' + 1$ arbitrary poles will contain, in accordance with the formula of the last Article,

$$\varpi' + 1 + \sum_{r=1}^{k} \lambda_r - p + 1, \ = \sigma' + 2,$$

arbitrary constants.

When $\sigma' + 1 = 0$, this number is 1, and the factorial function is entirely determined save for an arbitrary constant multiplier. Hence we infer that when $\sigma' + 1 = 0$ the smallest value of $M$ is $\varpi' + 1$.

We consider in the next Article how to form the factorial function in question from other functions of the system. Of the existence of such a function we can be sure *à priori* by the formulae (i) (ii) of § 254. Such a function will have $N = \varpi' + 1 + \Sigma\lambda, = p$, zeros. They can be determined to satisfy the equations (ii). Then an expression of the function is given by the general formula of § 254.

When $\sigma' + 1 > 0$, there are $\sigma' + 1$ everywhere finite factorial functions $K_1, \dots, K_{\sigma'+1}$, of the primary system, and the general factorial function with $\varpi' + 1$ poles is of the form

$$F + \lambda_1 K_1 + \dots\dots + \lambda_{\sigma'+1} K_{\sigma'+1},$$

where $\lambda_1, \dots, \lambda_{\sigma'+1}$ are constants, and $F$ is any factorial function with the assigned poles. In this case also there exist no factorial functions with arbitrary poles less than $\varpi' + 1$ in number; the attempt to obtain such functions leads* always to a linear aggregate of $K_1, \dots, K_{\sigma'+1}$.

263. Suppose that $\sigma' + 1 = 0$; we consider the construction of the factorial function of the primary system with $\varpi' + 1$ arbitrary poles.

Firstly let $\sigma + 1 > 0$, so that there are $\sigma + 1$ everywhere finite functions, $K'$, of the associated system, and $\sigma + 1$ differentials $dv$ vanish in the $2p - 2 + \sum_{r=1}^{k} (\lambda_r + 1)$ zeros of any differential $dV$. Hence $s, = -\sum_{r=1}^{k} (\lambda_r + 1)$, is greater than zero or equal to zero. We take first the case when $s > 0$. Then $\varpi' = p - 1 - \sum_{r=1}^{k} \lambda_r = p - 1 + s + k$, and it is possible to determine a rational function with poles at $\varpi' + 1 = p + s + k$ arbitrary places. This function contains $s + k + 1$ arbitrary constants, one of these being additive. It can therefore be chosen to vanish at the places $c_1, \dots, c_k$, and will then

* For $M = \varpi' - r$, we shall have $h + 1 = r$, and, therefore, $M - \varpi' + h + 1 + \sigma' + 1 = \sigma' + 1$.

contain at least, and in general, $s+1$ arbitrary constants. Taking now any everywhere finite factorial function $K'$ of the associated system, let the rational function be further chosen to vanish in the $s$ zeros of $K'$; then the rational function is, in general, entirely determined save for an arbitrary constant multiplier. Denote the rational function thus obtained by $R$. Then $R/K'$ is a factorial function of the primary system with the $\varpi'+1$ assigned poles, and is the function we desired to construct. And since the ratio of two functions $K'$ is a rational function, it is immaterial what function $K'$ is utilised to construct the function required.

This reasoning applies also to the case in which $\sigma+1>0$, $s=0$, unless also $k=0$. Consider then the case in which $\sigma+1>0$, $s=0$ and $k=0$. There is (§ 258) only one function $K'$, of the form

$$E_0 = Ae^{2\pi i[(h_1+H_1)v_1^{x,a}+\dots\dots+(h_p+H_p)v_p^{x,a}]},$$

or $\sigma+1=1$; and $E_0^{-1}$ is a function of the primary system without poles. Thus $\sigma'+1=1$, and the case does not fall under that now being considered, for which $\sigma'+1=0$. The value of $\varpi'$ is $p$, and the factorial function with $\varpi'+1$ arbitrary poles is of the form $(F+C)E_0$, where $F+C$ is the general rational function with the given poles.

Nextly, let $\sigma+1=0$, as well as $\sigma'+1=0$. Then there exist no functions $K'$ and the previous argument is inapplicable. But, provided $\varpi'+1 \nless 2$, we can apply another method, which could equally have been applied when $\sigma+1>0$. For if $P$ be the factorial function of the primary system with $\varpi'+1$ assigned poles, and $V'$ be one of the $\varpi'$ factorial integrals of the associated system, and $v$ be any integral of the first kind, $P\dfrac{dV'}{dv}$ is a rational function whose poles are at the $\varpi'+1$ poles of $P$ and at the $2p-2$ zeros of $dv$. Conversely, if $R$ be any rational function with poles at these places (cf. § 37, Ex. ii. Chap. III.), and zeros at the $2p-2-\Sigma\lambda$ zeros of $dV'$, $R\bigg/\dfrac{dV'}{dv}$ is the factorial function required. It contains at least

$$\varpi'+1+2p-2-p+1-(2p-2-\Sigma\lambda), =1,$$

arbitrary constant multiplier.

In case $\varpi'+1<2$, so that $\varpi'=0$, $\Sigma\lambda=p-1$, there are no functions $V'$, and we may fall back upon the fundamental equations of § 254. In this case the least number of poles is 1.

264. Consider now the possibility of forming a factorial integral of the primary system whose only infinities are poles. We shew that it is possible to form such an integral with $\sigma+2$ arbitrary poles, and with no smaller number.

Suppose $G$ to be such a factorial integral, with $\sigma + 2$ poles, and, under the hypothesis $\varpi > 0$, let $V$ be an everywhere finite factorial integral, also of the primary system. Then $dG/dV$ is a rational function, with poles at the $2p - 2 + \Sigma(\lambda + 1)$ zeros of $dV$, and poles at the poles of $G$; near a pole of $G$, say $c$, the form of $dG/dV$ is given by

$$\frac{dG}{dV} = C\left(\frac{1}{t^2} + A + Bt + \dots\right) \div D_c V \,.\left[1 + t\frac{D_c^2 V}{D_c V} + \dots\right],$$

where $t$ is the infinitesimal for the neighbourhood of the place $c$, the quantities $C, A, B$ are constants, and $D_c V$ denotes a differentiation in regard to the infinitesimal; this is the same as

$$\frac{dG}{dV} = E\left[-\frac{1}{t^2} + \frac{1}{t}\frac{D_c^2 V}{D_c V} + \text{terms which are finite when } t = 0\right],$$

where $E = -C/D_c V$. Thus $dG/dV$ is infinite at a pole of $G$ like a constant multiple of

$$\psi = D_c \Gamma_c^{x,\,a} - \frac{D_c^2 V}{D_c V}\Gamma_c^{x,\,a},$$

$a$ being an arbitrary place.

Conversely if $R$ denote a rational function which is infinite to the first order at the zeros of $dV$, and infinite in the $\sigma + 2$ assigned poles of $G$ like functions of the form of $\psi$, $\int R\,dV$ will be such a factorial integral as desired.

Now $R$ is of the form (§ 20, Chap. II.)

$$A + A_1\Gamma_{e_1}^{x,\,a} + \dots\dots + A_r\Gamma_{e_r}^{x,\,a} + B_1\left[D_{x_1}\Gamma_{x_1}^{x,\,a} - \frac{D_{x_1}^2 V}{D_{x_1} V}\Gamma_{x_1}^{x,\,a}\right] + \dots\dots$$
$$+ B_{\sigma+2}\left[D_{x_{\sigma+2}}\Gamma_{x_{\sigma+2}}^{x,\,a} - \frac{D_{x_{\sigma+2}}^2 V}{D_{x_{\sigma+2}} V}\Gamma_{x_{\sigma+2}}^{x,\,a}\right],$$

wherein $a$ is an arbitrary place, $e_1, \dots, e_r$ denote the zeros of $dV$, $x_1, \dots, x_{\sigma+2}$ denote the assigned poles of $G$, and $A, A_1, \dots, A_r, B_1, \dots, B_{\sigma+2}$ are constants; the period of $R$, in this form, at a general period loop of the second kind, is given by

$$A_1\Omega_i(e_1) + \dots\dots + A_r\Omega_i(e_r) + B_1\left[D_{x_1}\Omega_i(x_1) - \frac{D_{x_1}^2 V}{D_{x_1} V}\Omega_i(x_1)\right] + \dots\dots$$
$$+ B_{\sigma+2}\left[D_{x_{\sigma+2}}\Omega_i(x_{\sigma+2}) - \frac{D_{x_{\sigma+2}}^2 V}{D_{x_{\sigma+2}} V}\Omega_i(x_{\sigma+2})\right],$$

where $\Omega_1(x), \dots, \Omega_p(x)$ are as in § 18, Chap. II., and this must vanish for $i = 1, 2, \dots, p$. Now (§ 258) in the places $e_1, \dots, e_r$ there vanish $\sigma + 1$ linear functions of $\Omega_1(x), \dots, \Omega_p(x)$. Thus, from the conditions expressing that the periods of $R$ are zero, we infer $\sigma + 1$ linear equations involving only the constants $B_1, \dots, B_{\sigma+2}$, which, since the places $x_1, \dots, x_{\sigma+2}$ are arbitrary, may be assumed to be independent. From these $\sigma + 1$ equations we can obtain the ratios $B_1 : B_2 : \ldots\ldots : B_{\sigma+2}$. There remain then, of the $p$ equations expressing that the periods of $R$ are zero, $p-(\sigma+1)$ independent equations containing effectively $r+1$ unknown constants. Thus the number of the constants $A_1, \ldots, A_r, B_1, \ldots, B_{\sigma+2}$ left arbitrary is $r+1-p+\sigma+1$, which is equal to $2p-2+\Sigma(\lambda+1)+1-p+\sigma+1$ or $\varpi$, and the total number of arbitrary constants in $R$ is $\varpi+1$. Thus we infer that, on the whole, $G$ is of the form*

$$[G]+C_1V_1+\ldots\ldots+C_{\varpi}V_{\varpi}+C,$$

where $[G]$ is a special function with the $\sigma+2$ assigned poles, multiplied by an arbitrary constant, and $C_1, \ldots, C_{\varpi}$, $C$ are arbitrary constants. And this result shews that $\sigma+2$ is the least number of poles that can be assigned for $G$. The argument applies to the case when $\sigma+1=0$ provided that $\varpi>0$.

The proof just given supposes $\varpi>0$; but this is not indispensable. Let $f_0$ be a factorial function with the primary system of multipliers but with a behaviour at the places $c_i$ like $t^{-(\lambda_i+1)}\phi_i$, where $\phi_i$ is uniform, finite and not zero in the neighbourhood of $c_i$. Then if, instead of $\int R\,dV$, we consider an integral $\int Rf_0dv$, wherein $dv$ is the differential of any Riemann integral of the first kind, and $R$ is a rational function which vanishes in the (say $M$) poles of $f_0$, and may become infinite in the zeros of $dv$ and the (say $N$) zeros of $f_0$, we shall obtain the same results. It is necessary to take $N>1$ (cf. § 37, Ex. ii. Chap. III.).

265. Another method, holding whether $\varpi=0$ or not, provided $\sigma+1>0$, may be indicated. Let $K'(x)$ be one of the everywhere finite factorial functions of the associated system. Consider the function of $x$,

$$\psi=\int\frac{1}{K'(x)}\,d\,[\Gamma_c^{x,a}+A\Pi_{c,\gamma}^{x,a}],$$

$a$, $c$, $\gamma$ being any places and $A$ a constant; when $x$ is in the neighbourhood of the place $c$ it is of the form

$$\int\frac{1}{K'(c)}\left[1-t\frac{DK'(c)}{K'(c)}\right]\left[\frac{1}{t^2}+\frac{A}{t}\right]dt,$$

where $t$ is the infinitesimal in the neighbourhood of the place $c$, and terms which will lead only to positive powers of $t$ under the integral sign are omitted; this is the same as

$$\frac{1}{K'(c)}\int\left\{\frac{1}{t^2}+\left[A-\frac{DK'(c)}{K'(c)}\right]\frac{1}{t}\right\}dt;$$

* In the ordinary case of rational functions, where $V$ is replaced by a Riemann normal integral $v$, the coefficients of $B_1, \ldots, B_{\sigma+2}$, in the expression for the general period of $R$, vanish for one value of $i$, namely when $V=v_i$. Thus $\sigma+1\,(=1)$ pole is sufficient to enable us to construct the factorial integral; it is the ordinary integral of the second kind.

hence if $A$ be $DK'(c)/K'(c)$, the function $\psi$ is infinite at $c$ like $-\frac{1}{t}\frac{1}{K'(c)}$. At the place $\gamma$ the function $\psi$ is infinite like $-\frac{A}{K'(\gamma)}\log t_\gamma$, where $t_\gamma$ is the infinitesimal in the neighbourhood of the place $\gamma$.

Putting now $M_{c,\gamma}^{x,a} = \Gamma_c^{x,a} + \frac{DK'(c)}{K'(c)}\Pi_{c,\gamma}^{x,a}$, consider the function

$$G(x) = \int \frac{1}{K'(x)} d\left\{A_1 M_{x_1,\gamma}^{x,a} + \ldots + A_{\sigma+2} M_{x_{\sigma+2},\gamma}^{x,a} + B_1 v_1^{x,a} + \ldots + B_p v_p^{x,a}\right\},$$

where $a$, $\gamma$ are arbitrary places and $A_1, \ldots, A_{\sigma+2}$, $B_1, \ldots, B_p$ are constants, subject to the conditions

(i) that

$$A_1 D_x M_{x_1,\gamma}^{x,a} + \ldots\ldots + A_{\sigma+2} D_x M_{x_{\sigma+2},\gamma}^{x,a} + B_1 \Omega_1(x) + \ldots\ldots + B_p \Omega_p(x)$$

vanishes at each of the $-\Sigma(\lambda+1)$ zeros of $K'(x)$,

(ii) that

$$A_1 \frac{DK'(x_1)}{K'(x_1)} + \ldots\ldots + A_{\sigma+2}\frac{DK'(x_{\sigma+2})}{K'(x_{\sigma+2})} = 0;$$

the first condition ensures that $G(x)$ is finite at the zeros of $K'(x)$, the second condition ensures that $G(x)$ is finite at the place $\gamma$. If we suppose* $v_1^{x,a}, \ldots, v_{\varpi}^{x,a}$ to be those integrals of the first kind whose differentials vanish at the zeros of $K'(x)$ (§ 258), the conditions (i) will involve only the constants $A_1, \ldots, A_{\sigma+2}$, $B_{\varpi+1}, \ldots, B_p$, and if these conditions be independent these $\sigma + 2 + (p - \varpi)$ coefficients will thereby be reduced to

$$\sigma + 2 + p - \varpi + \Sigma(\lambda+1), = 2;$$

thus, if the condition (ii) be independent of the conditions (i), the number of constants finally remaining is $\varpi + 2 - 1 = \varpi + 1$, and the form of $G(x)$ is

$$[G] + C_1 V_1 + \ldots\ldots + C_{\varpi} V_{\varpi} + C$$

as before.

*Ex.* Prove that, when $s$, $= -\Sigma(\lambda+1)$, is positive, we have

$$K'(x)\left[D_{x_1} M_{x,\gamma}^{x_1,a} + \ldots + D_{x_s} M_{x,\gamma}^{x_s,a}\right] = D_x\left\{K'(x)\left[\Gamma_{x_1}^{x,\gamma} + \ldots + \Gamma_{x_s}^{x,\gamma}\right]\right\}.$$

266. The factorial integral of the primary system with $\sigma + 2$ arbitrary poles can be simplified. If $x_1, \ldots, x_{\sigma+2}$ be the poles, its most general form may be represented by

$$EG(x_1, \ldots, x_{\sigma+2}) + E_1 V_1 + \ldots\ldots + E_{\varpi} V_{\varpi} + C,$$

* This is to simplify the explanation. In general it is $\varpi$ linear combinations of the normal integrals, whose differentials vanish in the zeros of $K'(x)$. The reduction corresponding to that of the text is then obtained by taking $\varpi$ linear combinations of the conditions (i).

where $E, E_1, \ldots, E_{\varpi}, C$ are arbitrary constants. Near a place $c_1$, one of the singular places of the factorial system, the integral will have a form represented by $A_1 + t^{-\lambda_1}\phi$; we may simplify the integral by subtracting from it the constant $A_1$; the consequence is that the additive period belonging to the loop $(\gamma_1)$ is zero; further there is one other linear relation connecting the additive periods of the integral, which is obtainable by following the value of the integral once round the boundary of the dissected surface (cf. § 260). Thus the number of periods of the integral is at most $2p - 2 + k$. We suppose the additive periods of the functions $G(x_1, \ldots, x_{\sigma+2})$, $V_1, \ldots, V_{\varpi}$, at the loop $(\gamma_1)$, to be similarly reduced to zero; then the constant $C$ is zero. The linear aggregate $E_1 V_1 + \ldots\ldots + E_{\varpi} V_{\varpi}$ may be replaced by an aggregate of the non-periodic functions $K_1, \ldots, K_{\sigma'+1}$, and $\varpi - (\sigma' + 1)$ of the integrals $V_1, \ldots, V_{\varpi}$, so that the integral under consideration takes the form

$$EG(x_1, \ldots, x_{\sigma+2}) + C_1 V_1 + \ldots + C_{\varpi-(\sigma'+1)} V_{\varpi-(\sigma'+1)} + F_1 K_1 + \ldots + F_{\sigma'+1} K_{\sigma'+1},$$

where $C_1, \ldots, C_{\varpi-(\sigma'+1)}, F_1, \ldots, F_{\sigma'+1}$ are constants. We can therefore, presumably, determine the constants $C_1, \ldots, C_{\varpi-(\sigma'+1)}$, so that $\varpi - (\sigma' + 1)$ of the additive periods of the integral vanish. The integral will then have $2p - 2 + k - (\varpi - \sigma' - 1), = \varpi' - (\sigma + 1)$, periods remaining, together with one period which is a linear function of them. A particular case* is that of Riemann's normal integral of the second kind, for which there are $p$ periods. *As in that case we suppose here that the period loops for which the additive periods of the factorial integral shall be reduced to zero are agreed upon beforehand.* We thus obtain a function

$$F \,.\, G_1(x_1, \ldots, x_{\sigma+2}) + F_1 K_1 + \ldots\ldots + F_{\sigma'+1} K_{\sigma'+1},$$

wherein $F, F_1, \ldots, F_{\sigma'+1}$ are arbitrary constants, and $G_1(x_1, \ldots, x_{\sigma+2})$ has additive periods only at $\varpi' - (\sigma + 1)$ prescribed period loops, beside a period which is a linear function of these. We may therefore further assign $\sigma' + 1$ zeros of the integral and choose $F$ so that the integral is infinite at $x_1$ like the negative inverse of the infinitesimal. When the integral is so determined we shall denote it by $\Gamma(x_1, x_2, \ldots, x_{\sigma+2})$. The assigned zeros are to be taken once for all, say at $a_1, \ldots, a_{\sigma'+1}$.

267. The factorial function of the primary system with $\varpi' + 1$ assigned arbitrary poles can be expressed in terms of the factorial integral of the primary system with $\sigma + 2$ assigned poles. Let $x_1, \ldots, x_{\varpi'+1}$ be the assigned poles of the factorial function. Then we may choose the constants $C_1, \ldots, C_{\varpi'-\sigma}$, so that the $\varpi' - (\sigma + 1)$ linearly independent periods of the aggregate

$$C_1 \Gamma(x_{\sigma+2}, x_1, \ldots, x_{\sigma+1}) + \ldots\ldots + C_{\varpi'-\sigma} \Gamma(x_{\varpi'+1}, x_1, \ldots, x_{\sigma+1})$$

are all zeros. The result is a factorial function with $x_1, \ldots, x_{\varpi'+1}$ as poles,

* Of the result. The reasoning must be amended by the substitution of $p$, $2p$ for $\varpi' - (\sigma + 1)$ and $2p - 2 + k$ respectively. Cf. the note † of § 260.

which vanishes in the places $a_1, \dots, a_{\sigma'+1}$. Or, taking arbitrary places $d_1, \dots, d_{\sigma+1}$ we may choose the constants $E_1, \dots, E_{\varpi'+1}$ so that the $\varpi' - (\sigma+1)$ linearly independent periods of the aggregate

$$E_1\Gamma(x_1, d_1, \dots, d_{\sigma+1}) + E_2\Gamma(x_2, d_1, \dots, d_{\sigma+1}) + \dots + E_{\varpi'+1}\Gamma(x_{\varpi'+1}, d_1, \dots, d_{\sigma+1})$$

are all zero, and at the same time the aggregate does not become infinite at $d_1, \dots, d_{\sigma+1}$. Then the addition, to the result, of an aggregate $F_1K_1 + \dots\dots + F_{\sigma'+1}K_{\sigma'+1}$, wherein $F_1, \dots, F_{\sigma'+1}$ are arbitrary constants, leads to the most general form of the factorial function with $x_1, \dots, x_{\varpi'+1}$ as poles. For the sake of definiteness we denote by $\psi(x;\, z, t_1, \dots, t_{\varpi'})$ the factorial function with poles of the first order at $z, t_1, \dots, t_{\varpi'}$, which is chosen so that it becomes infinite at $z$ like the negative inverse of the infinitesimal, and vanishes at the places $a_1, \dots, a_{\sigma'+1}$. A more precise notation would be* $\psi(x, a_1, \dots, a_{\sigma'+1};\, z, t_1, \dots, t_{\varpi'})$. This function contains no arbitrary constants.

Denoting this function now, temporarily, by $\psi$, and any everywhere finite factorial integral of the inverse system by $V'$, the value of the integral $\int \psi dV'$, taken round the boundary of the dissected surface formed by the sides of the period loops, is equal to the sum of its values round the poles of $\psi$. Since $\psi dV'/dx$ is a rational function the value of the integral taken round the boundary is zero. Near a pole of $\psi$, at which $t$ is the infinitesimal, the integral will have the form

$$\int \left[\frac{A}{t} + B + Ct + \dots\dots\right] [(DV') + t(D^2V') + \dots\dots]\, dt,$$

where $D$ denotes a differentiation. Thus the value obtained by taking the integral round this pole is $A(DV')$. If then the values of $A$ at the poles $\Gamma_1, \dots, \Gamma_{\varpi'}$ be denoted by $A_1, \dots, A_{\varpi'}$, we have, remembering that the value of $A$ at $z$ is $-1$, the $\varpi'$ equations

$$A_1(DV_1')_1 + \dots\dots + A_{\varpi'}(DV_1')_{\varpi'} = (DV_1')_z\ ,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_1(DV'_{\varpi'})_1 + \dots\dots + A_{\varpi'}(DV'_{\varpi'})_{\varpi'} = (DV'_{\varpi'})_z,$$

where $V_1', \dots, V'_{\varpi'}$ are the $\varpi'$ everywhere finite factorial integrals of the associated system, $(DV_i')_r$ denotes the differential coefficient of $V_i'$ at $t_r$, and $(DV_i')_z$ denotes the differential coefficient at $z$. Thus, if $\omega_r(x)$ denote, here, the linear aggregate of the form

$$E_1(DV_1')_x + \dots\dots + E_{\varpi'}(DV'_{\varpi'})_x,$$

wherein the constants $E_1, \dots, E_{\varpi'}$ are chosen so that $\omega_r(t_r) = 1$ and $\omega_r(t_s) = 0$ when $t_s$ is any one of the places $t_1, \dots, t_{\varpi'}$ other than $t_r$, we have $A_r = \omega_r(z)$. Hence we infer by the previous article (§ 266) that $\psi(x;\, z, t_1, \dots, t_{\varpi'})$ is equal to

$$\Gamma(z, d_1, \dots, d_{\sigma+1}) - \omega_1(z)\,\Gamma(t_1, d_1, \dots, d_{\sigma+1}) - \dots\dots - \omega_{\varpi'}(z)\,\Gamma(t_{\varpi'}, d_1, \dots, d_{\sigma+1}),$$

* Cf. § 122, Chap. VII. etc.

where $d_1, \ldots, d_{\sigma+1}$ are arbitrary places. For these two functions are infinite at the places $z, t_1, \ldots, t_{\varpi'}$ in the same way and both vanish at the places $a_1, \ldots, a_{\sigma'+1}$.

As in the case of the rational functions, the function $\psi(x; z, t_1, \ldots, t_{\varpi'})$ may be regarded as fundamental, and developments analogous to those given on pages 181, 189 of the present volume may be investigated. We limit ourselves to the expression of any factorial function of the primary system by means of it. The most general factorial function with poles of the first order at the places $z_1, \ldots, z_M$ may be expressed in the form

$$A_1\psi(x; z_1, t_1, \ldots, t_{\varpi'}) + \ldots\ldots + A_M\psi(x; z_M, t_1, \ldots, t_{\varpi'}) + B_1K_1 + \ldots\ldots + B_{\sigma'+1}K_{\sigma'+1},$$

where $A_1, \ldots, A_M, B_1, \ldots, B_{\sigma'+1}$ are constants. The condition that the function represented by this expression may not be infinite at $t_r$ is

$$A_1\omega_r(z_1) + \ldots\ldots + A_M\omega_r(z_M) = 0;$$

in case the $\varpi'$ equations of this form, for $r = 1, 2, \ldots, \varpi'$, be linearly independent, the factorial function contains $M + \sigma' + 1 - \varpi'$ arbitrary constants; but if there be $h + 1$ linearly independent aggregates of differentials, of the form

$$C_1dV_1' + \ldots\ldots + C_{\varpi'}dV'_{\varpi'},$$

which vanish in the $M$ assigned poles, then the equations of the form

$$A_1\omega_r(z_1) + \ldots\ldots + A_M\omega_r(z_M) = 0$$

are equivalent to only $\varpi' - (h + 1)$ equations, and the number of arbitrary constants in the expression of the factorial function is $M + \sigma' + 1 - \varpi' + h + 1$, in accordance with § 261.

*Ex.* i. Prove that a factorial integral of the primary system can be constructed with logarithmic infinities only in $\sigma+2$ places, but with no smaller number.

*Ex.* ii. If the factorial integral $G(x_1, x_2, \ldots, x_{\sigma+2})$ become infinite of the place $x_i$ like $\frac{R_i}{t}$, where $t$ is the infinitesimal at $x_i$, prove, by considering the contour integral $\int GdK_r'$, where $K_r'$ is one of the $\sigma+1$ everywhere finite factorial functions of the associated system, and $G$ denotes $G(x_1, x_2, \ldots, x_{\sigma+2})$, the $\sigma+1$ equations

$$\sum_{i=1}^{\sigma+2} R_iDK_r'(x_i) = 0,$$

$D$ denoting a differentiation. From these equations the ratio of the residues $R_1, R_2, \ldots, R_{\sigma+2}$ can be expressed.

268. The theory of this chapter covers so many cases that any detailed exhibition of examples of its application would occupy a great space. We limit ourselves to examining the case $p = 0$, for which explicit expressions can be given, and, very briefly, two other cases (§§ 268—270).

Consider the case $p=0$, $k=3$, there being three singular places such as have so far in this chapter been denoted by $c_1, c_2, \ldots$, but which we shall here denote by $\alpha, \beta, \gamma$, the associated numbers* being $\lambda_1=-3/2$, $\lambda_2=-3/2$, $\lambda_3=-2$. At these places the factorial functions of the associated system behave, respectively, like $t^{-\frac{1}{2}}\phi_1$, $t^{-\frac{1}{2}}\phi_2$, $t^{-1}\phi_3$, and the difference between the number of zeros and poles of such a function is $N'-M'=-\Sigma(\lambda+1)=2$. Thus there exist factorial functions of the associated system with no poles and two zeros. By the general formula of § 254, replacing $\Pi_{c,\gamma}^{x,a}$ by $\log\left(\dfrac{x-c}{x-\gamma}\Big/\dfrac{a-c}{a-\gamma}\right)$, the general form of such a function is found to be

$$K'(x)=\frac{Ax^2+Bx+C}{(x-\gamma)(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}}$$

and involves three arbitrary constants, so that $\sigma+1=3$. In what follows $K'(x)$ will be used to denote the special function $1/(x-\gamma)(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}$. The difference between the number of zeros and poles of factorial functions of the primary system is $N-M=-5$; hence $M=0$ is not possible, and $\sigma'+1=0$. Further

$$\varpi,\ =p-1+\Sigma(\lambda+1)+\sigma+1,\ =-1-2+3=0,$$
$$\varpi',\ =p-1-\Sigma\lambda+\sigma'+1\quad,\ =-1+5\quad=4,$$

and the factorial function of the primary system with fewest poles has $\varpi'+1=5$ poles, as also follows from the formula $N-M=-5$. This function is clearly given by

$$P(x)=\frac{(x-\alpha)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}(x-\gamma)^2}{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}.$$

Putting

$$\psi(x)=(x-\alpha)(x-\beta)(x-\gamma),\ f(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5),$$
$$\phi(x)=DK'(x)/K'(x)=-[(x-\gamma)^{-1}+\tfrac{1}{2}(x-\alpha)^{-1}+\tfrac{1}{2}(x-\beta)^{-1}],$$

and putting $\lambda_i=\psi(x_i)/f'(x_i)$, where $i$ is in turn equal to 1, 2, 3, 4, 5 and $f'(x)$ denotes the differential coefficient of $f(x)$, it is immediately clear that $P(x)$ is infinite at $x_1$ like $\lambda_1/(x-x_1)K'(x_1)$ It can be verified that

$$\sum_1^5\lambda_1=0,\ \sum_1^5 x_1\lambda_1=1,\ \sum_1^5 x_1\lambda_1\phi(x_1)=0,\ \sum_1^5 x_1^2\lambda_1\phi(x_1)=-2,\ \sum_1^5\lambda_1\phi(x_1)=0,$$

and these give

$$\sum_1^5\lambda_1[1+x_1\phi(x_1)]=0,\ \sum_1^5\lambda_1[2x_1+x_1^2\phi(x_1)]=0.$$

The factorial integral $G$, of the primary system, with $\sigma+2=4$ poles, $\tau, \xi, \eta, \zeta$, is (§ 265) given by

$$G(\tau,\xi,\eta,\zeta)=\int\frac{1}{K'(x)}\,d\left\{\sum_1^4 A_1\left[\frac{1}{x-\tau}-\phi(\tau)\log\frac{x-\tau}{x-c}\right]\right\},$$

* It was for convenience of exposition that, in the general theory, the case in which any of the numbers $\lambda_1, \ldots, \lambda_k$ are integers, was excluded.

where the sign of summation refers to $\tau$, $\xi$, $\eta$, $\zeta$ and the constants $A_1$, $A_2$, $A_3$, $A_4$ are to be chosen so that (i) the expression

$$A_1\phi(\tau) + A_2\phi(\xi) + A_3\phi(\eta) + A_4\phi(\zeta)$$

is zero, this being necessary in order that $G(\tau, \xi, \eta, \zeta)$ may not become infinite at the place $c$, and (ii) the expression

$$\sum_1^4 A_1\left[\frac{1}{(x-\tau)^2} + \phi(\tau)\left(\frac{1}{x-\tau} - \frac{1}{x-c}\right)\right]$$

vanishes to the fourth order when $x$ is infinite; the expression always vanishes to the second order when $x$ is infinite; the additional conditions are required because $K'(x)$ is zero to the second order when $x$ is infinite. Taking account of condition (i), we find, by expanding in powers of $\frac{1}{x}$, that the condition (ii) is equivalent to the two

$$\sum_1^4 A_1[1 + \tau\phi(\tau)] = 0, \quad \sum_1^4 A_1[2\tau + \tau^2\phi(\tau)] = 0.$$

Thus, introducing the values of $A_1, \ldots, A_4$ into the expression for $G(\tau, \xi, \eta, \zeta)$, we find, by proper choice of a multiplicative constant,

$$K'(x)\, DG(\tau, \xi, \eta, \zeta) = \begin{vmatrix} \frac{1}{(x-\tau)^2} + \frac{\phi(\tau)}{x-\tau}, & (\xi), & (\eta), & (\zeta) \\ \phi(\tau), & ., & ., & . \\ 1 + \tau\phi(\tau), & ., & ., & . \\ 2\tau + \tau^2\phi(\tau), & ., & ., & . \end{vmatrix} \quad \ldots(1),$$

in which the second, third and fourth columns differ from the first only in the substitution, respectively, of $\xi$, $\eta$, $\zeta$ in place of $\tau$.

The factorial integral $G(\tau, \xi, \eta, \zeta)$ thus determined can in fact be expressed without an integral sign. For we immediately verify that

$$\int dx\,(x-\gamma)\sqrt{(x-\alpha)(x-\beta)}\left[\frac{1}{(x-\tau)^2} + \frac{\phi(\tau)}{x-\tau}\right]$$

is equal, save for an additive constant, to

$$\sqrt{(x-\alpha)(x-\beta)}\left[\frac{\gamma-\tau}{x-\tau} + 1 + \tau\phi(\tau) + \tfrac{1}{2}\{x - \gamma - \tfrac{1}{2}(\alpha+\beta)\}\phi(\tau)\right]$$
$$+\left[2\tau + \tau^2\phi(\tau) - \{\gamma + \tfrac{1}{2}(\alpha+\beta)\}(1 + \tau\phi(\tau)) + \left\{\tfrac{1}{2}\gamma(\alpha+\beta) - \tfrac{1}{2}\left(\frac{\alpha-\beta}{2}\right)^2\right\}\phi(\tau)\right]$$
$$\times \log\left\{x - \frac{\alpha+\beta}{2} + \sqrt{(x-\alpha)(x-\beta)}\right\}$$
$$-\frac{2}{\sqrt{(\tau-\alpha)(\tau-\beta)}}\left[(\tau-\gamma)\phi(\tau) + 1 + \tfrac{1}{2}(\tau-\gamma)\left(\frac{1}{\tau-\alpha} + \frac{1}{\tau-\beta}\right)\right]$$
$$\times \log\frac{\sqrt{(x-\beta)(\tau-\alpha)} + \sqrt{(x-\alpha)(\tau-\beta)}}{\sqrt{x-\tau}},$$

and, by the definition of $\phi(x)$, the coefficient of the logarithm in the last line of this expression is zero; if we substitute these values in the expression found for $G(\tau, \xi, \eta, \zeta)$ we obviously have

$$G(\tau, \xi, \eta, \zeta) = \sqrt{(x-\alpha)(x-\beta)} \begin{vmatrix} \frac{\gamma-\tau}{x-\tau}, & (\xi), & (\eta), & (\zeta) \\ \phi(\tau), & ., & ., & . \\ 1+\tau\phi(\tau), & ., & ., & . \\ 2\tau+\tau^2\phi(\tau), & ., & ., & . \end{vmatrix} + \text{constant}, \ldots (2),$$

where the second, third and fourth columns of the determinant differ from the first only in the substitution, in place of $\tau$, respectively of $\xi, \eta, \zeta$. We proceed now to prove that this determinant is a certain constant multiple of $(x-\alpha)(x-\beta)(x-\mu)/(x-\tau)(x-\xi)(x-\eta)(x-\zeta)$, where $\mu$ is determined by

$$\frac{1}{\gamma-\mu} = \frac{1}{\gamma-\tau} + \frac{1}{\gamma-\xi} + \frac{1}{\gamma-\eta} + \frac{1}{\gamma-\zeta} - \frac{3}{2}\left(\frac{1}{\gamma-\alpha} + \frac{1}{\gamma-\beta}\right).$$

If we introduce constants, $A, B, C, A', B', C'$, depending only on $\alpha, \beta, \gamma$, defined by the identities

$$Cx^2 + Bx + A = \frac{2}{\alpha-\beta}(x-\beta)(x-\gamma),$$

$$C'x^2 + B'x + A' = \frac{4}{(\alpha-\beta)^2}(x-\gamma)\left(x - \frac{\alpha+\beta}{2}\right),$$

we can immediately verify that

$$A\phi(x) + B[1 + x\phi(x)] + C[2x + x^2\phi(x)] = -\frac{x-\gamma}{x-\alpha},$$

$$A'\phi(x) + B'[1 + x\phi(x)] + C'[2x + x^2\phi(x)] = -\frac{x-\gamma}{(x-\alpha)(x-\beta)},$$

and hence that

$$\frac{\gamma-\tau}{x-\tau} + [A + (x-\alpha)A']\phi(\tau) + [B + (x-\alpha)B'][1 + \tau\phi(\tau)] + [C + (x-\alpha)C'][2\tau + \tau^2\phi(\tau)]$$

$$= (x-\alpha)(x-\beta)\frac{\gamma-\tau}{(\alpha-\tau)(\beta-\tau)}\frac{1}{x-\tau};$$

thus

$$G(\tau, \xi, \eta, \zeta) = (x-\alpha)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}} \begin{vmatrix} \frac{\gamma-\tau}{(\alpha-\tau)(\beta-\tau)}\frac{1}{x-\tau}, & (\xi), & (\eta), & (\zeta) \\ \phi(\tau), & ., & ., & . \\ 1+\tau\phi(\tau), & ., & ., & . \\ 2\tau+\tau^2\phi(\tau), & ., & ., & . \end{vmatrix} + \text{constant}, \ldots\ldots (3)$$

now it is clear from the equation (2) that $G(\tau, \xi, \eta, \zeta)/\sqrt{(x-\alpha)(x-\beta)}$ is of the form $(x, 1)_3/(x-\tau)(x-\xi)(x-\eta)(x-\zeta)$, where $(x, 1)_3$ denotes an integral cubic polynomial; and since $1/K'(x)$ vanishes when $x=\gamma$, it follows from the equation (1) that the differential coefficient of $G(\tau, \xi, \eta, \zeta)$ vanishes when $x=\gamma$. Hence we have

$$G(\tau, \xi, \eta, \zeta) = L\frac{(x-\alpha)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}(x-\mu)}{(x-\tau)(x-\xi)(x-\eta)(x-\zeta)} + M,$$

where $\mu$ is such that the differential coefficient of this expression vanishes when $x=\gamma$, and has therefore the value already specified, $L$ is a constant whose value can be obtained from the equation (3) by calculation, and $M$ is a constant which we have not assigned. In the neighbourhood of the place $\alpha$, $G(\tau, \xi, \eta, \zeta)$ has the form $M + L(x-\alpha)^{\frac{3}{2}}[\lambda + \mu(x-\alpha) + \nu(x-\alpha)^2 + \ldots]$, and similarly in the neighbourhood of the place $\beta$. In the neighbourhood of the place $\gamma$, $G(\tau, \xi, \eta, \zeta)$ has the form

$$N + (x-\gamma)^2[\lambda' + \mu'(x-\gamma) + \nu'(x-\gamma)^2 + \ldots\ldots],$$

where $N$ is a constant, generally different from $M$.

In the general case of a factorial integral for $p=0$, $k=3$, the behaviour of the integral at $\alpha$, $\beta$, $\gamma$ is that of three expressions of the form

$$A+(x-\alpha)^{-\lambda}[P+Q(x-\alpha)+\ldots], \qquad B+(x-\beta)^{-\mu}[P'+Q'(x-\beta)+\ldots],$$
$$C+(x-\gamma)^{-\nu}[P''+Q''(x-\gamma)+\ldots],$$

provided no one of $\lambda+1$, $\mu+1$, $\nu+1$ be a positive integer; herein one of the constants $A$, $B$, $C$ may be taken arbitrarily and the others are thereby determined. The factorial integral becomes a factorial function only in the case when all of $A$, $B$, $C$ are zero.

We have seen that the factorial function of the primary system with fewest poles has 5 poles; let them be at $\tau, \tau_1, \xi, \eta, \zeta$; then, taking $G(\tau, \xi, \eta, \zeta)$ in the form just found, the factorial function can be expressed in the form

$$P(x) = CG(\tau, \xi, \eta, \zeta) + C_1G(\tau_1, \xi, \eta, \zeta) + D,$$

when the constants $C$, $C_1$, $D$ are suitably chosen.

For clearly $D$ can be chosen so that the function $P(x)$ divides identically by $(x-\alpha)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}$. It is then only necessary to choose the ratio $C : C_1$, if possible, so that the function $P(x)$ divides identically by $(x-\gamma)^2$. This requires only that

$$C\frac{x-\mu}{x-\tau} + C_1\frac{x-\mu_1}{x-\tau_1} = \rho\frac{(x-\gamma)^2}{(x-\tau)(x-\tau_1)},$$

where $\rho$ is a constant, or that the expression

$$C(x-\mu)(x-\tau_1) + C_1(x-\tau)(x-\mu_1)$$

divide by $(x-\gamma)^2$. Thus $C : C_1 = -(\gamma-\tau)(\gamma-\mu_1) : (\gamma-\mu)(\gamma-\tau_1)$, and

$$\frac{2\gamma-\mu-\tau_1}{(\gamma-\mu)(\gamma-\tau_1)} = \frac{2\gamma-\mu_1-\tau}{(\gamma-\mu_1)(\gamma-\tau)},$$

or

$$\frac{1}{\gamma-\mu} - \frac{1}{\gamma-\tau} = \frac{1}{\gamma-\mu_1} - \frac{1}{\gamma-\tau_1};$$

this condition is satisfied; both these expressions are by definition equal to

$$\frac{1}{\gamma-\xi} + \frac{1}{\gamma-\eta} + \frac{1}{\gamma-\zeta} - \tfrac{3}{2}\left(\frac{1}{\gamma-\alpha} + \frac{1}{\gamma-\beta}\right).$$

From the theoretical point of view it is however better to proceed as follows—Let the poles of $P(x)$ be at $x_1, \ldots, x_5$. Then $P(x)$ can be expressed in the form

$$P(x) = C_1 G(x_1, \xi, \eta, \zeta) + C_2 G(x_2, \xi, \eta, \zeta) + \ldots\ldots + C_5 G(x_5, \xi, \eta, \zeta) + C,$$

the constants $C, C_1, C_2, \ldots, C_5$ being suitably chosen. This equation requires, by equation (1),

$$K'(x)\,DP = \sum_1^5 C_r \left[\frac{1}{(x-x_r)^2} + \frac{\phi(x_r)}{(x-x_r)}\right] \Delta(\xi, \eta, \zeta)$$

$$+ \begin{vmatrix} 0, & E, & F, & G \\ \sum_1^5 C_r\phi(x_r), & \phi(\xi), & \phi(\eta), & \phi(\zeta) \\ \sum_1^5 C_r[1+x_r\phi(x_r)], & 1+\xi\phi(\xi), & 1+\eta\phi(\eta), & 1+\zeta\phi(\zeta) \\ \sum_1^5 C_r[2x_r+x_r^2\phi(x_r)], & 2\xi+\xi^2\phi(\xi), & 2\eta+\eta^2\phi(\eta), & 2\zeta+\zeta^2\phi(\zeta) \end{vmatrix}$$

wherein $\Delta(\xi, \eta, \zeta)$ is the minor, in the determinant occurring in equation (1), of the first element of the first row, and $E = (x-\xi)^{-2} + \phi(\xi)(x-\xi)^{-1}$, $F = (x-\eta)^{-2} + \phi(\eta)(x-\eta)^{-1}$, $G = (x-\zeta)^{-2} + \phi(\zeta)(x-\zeta)^{-1}$. If now we take $C_1, \ldots, C_5$ so that

$$\sum_1^5 C_r\phi(x_r) = 0, \quad \sum_1^5 C_r[1+x_r\phi(x_r)] = 0, \quad \sum_1^5 C_r[2x_r+x_r^2\phi(x_r)] = 0,$$

this leads to

$$\frac{\Delta(x_1, x_2, x_3)}{\Delta(\xi, \eta, \zeta)} DP = C_4 DG(x_1, x_2, x_3, x_4) + C_5 DG(x_1, x_2, x_3, x_5),$$

and the solution can be completed as before.

There are $\varpi' = 4$ everywhere finite factorial integrals of the associated system; if $V'$ be one of these, then by definition, $\frac{dV'}{dx}$ is a factorial function which has at $\alpha$ the form $(x-\alpha)^{-\frac{3}{2}}\phi$, and similarly at $\beta$, and has at $\gamma$ the form $(x-\gamma)^{-2}\phi$. Further $dV'/dx$ is zero to the second order at $x=\infty$. Hence we have

$$V' = \int \frac{(x, 1)_3\, dx}{(x-\alpha)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}(x-\gamma)^2},$$

and $dV'$ has $2p-2-\Sigma\lambda = -2+5 = 3$ zeros.

Thus $V'$ can be written in the form

$$V' = R\int \frac{dx}{(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}(x-\gamma)} + \frac{Lx^2+Mx+N}{(x-\gamma)(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}},$$
$$= NK'(x) + MK_1'(x) + LK_2'(x) + RV_0',$$

where $N$, $M$, $L$, $R$ are constants, $K'(x)$, $K_1'(x)$, $K_2'(x)$ are particular, linearly independent, everywhere finite factorial functions of the associated system, and $V_0'$ is a particular everywhere finite factorial integral of the associated system.

*Ex.* i. In case of a factorial system given by $p=0$, $k=2$, $\lambda_1=-\frac{3}{2}$, $\lambda_2=-\frac{3}{2}$, prove that $\sigma+1=2$, $\sigma'+1=0$, $\varpi=0$, $\varpi'=2$; prove that the factorial function of the primary system with fewest poles is $P(x)=(x-a)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}/(x-x_1)(x-x_2)(x-x_3)$; obtain the form of the factorial integral of the second kind of the primary system with fewest poles, and prove that it can be expressed in the form $AP(x)+B$; and shew that the everywhere finite factorial integrals of the associated system are expressible in the form $(Ax+B)/\sqrt{(x-a)(x-\beta)}$, their initial form being

$$V' = \int \frac{(Ax+B)\,dx}{(x-a)^{\frac{3}{2}}(x-\beta)^{\frac{3}{2}}}.$$

*Ex.* ii. When we take $p=0$ and $k$, $=2n+2$, places $c_1, \ldots, c_{2n+2}$, and each $\lambda=-\frac{1}{2}$, prove that the original and the associated systems coincide, that $\sigma+1=\sigma'+1=0$, $\varpi=\varpi'=n$, that the everywhere finite factorial integrals, and the integral with one pole are respectively

$$\int \frac{(x, 1)_{n-1}}{\sqrt{f(x)}}\,dx, \qquad \int \left[\frac{f(a)}{(x-a)^2} + \frac{1}{2}\frac{f'(a)}{x-a}\right]\frac{dx}{\sqrt{f(x)}},$$

where $f(x)=(x-c_1)\ldots\ldots(x-c_{2n+2})$. The factorial function with fewest poles is $\sqrt{f(x)}/(x, 1)_{n+1}$; express this in the form

$$\frac{\sqrt{f(x)}}{(x, 1)_{n+1}} = \sum_{i=1}^{n+1}\lambda_i \int \left[\frac{f(a_i)}{(x-a_i)^2} + \frac{1}{2}\frac{f'(a_i)}{x-a_i}\right]\frac{dx}{\sqrt{f(x)}} + \int \frac{(x, 1)_{n-1}}{\sqrt{f(x)}}\,dx + \text{constant},$$

$a_1, \ldots, a_{n+1}$ being the zeros of $(x, 1)_{n+1}$, and determine the $2n+1$ coefficients on the right-hand side.

269. One of the simplest applications of the theory of this chapter is to the case of the root functions already considered in the last chapter; such a function can be expressed in the form $e^{\psi}$, where

$$\psi = \Pi^{x,\gamma}_{\beta_1,\alpha_1} + \ldots\ldots + \Pi^{x,\gamma}_{\beta_N,\alpha_N} - 2\pi i \sum_1^p (h_i + H_i)\, v_i^{x,\gamma},$$

where $\beta_1, \ldots, \beta_N$ are the zeros, $\alpha_1, \ldots, \alpha_N$ the poles, $h_i$ is a rational numerical fraction, $H_i$ is an integer, and $\gamma$ is an arbitrary place. The singular places, $c_1, \ldots, c_k$ are entirely absent. The zeros and poles satisfy the equations expressed by

$$v^{\beta_1, \alpha_1} + \ldots\ldots + v^{\beta_N, \alpha_N} = g + G + \tau(h + H),$$

where $G_1, \ldots, G_p$ are integers; and since, if $m$ be the least common denominator of the $2p$ numbers $g$, $h$, the $m$th power of the function is a rational function, there is no function of the system which is everywhere finite, and the same is true of the associated system. Hence $\sigma + 1 = 0 = \sigma' + 1$, $\varpi = \varpi' = p - 1$; thus the function of the system with fewest poles has $p$ poles, and every function of the system can be expressed as a linear aggregate of such functions (§ 267. Cf. § 245, Chap. XIII.).

*Ex.* i. Prove that when the numbers $g$, $h$ are any half-integers, the everywhere finite integrals of the system are expressible in the form

$$V = \int \frac{dv}{\phi} \sum_1^{p-1} \lambda_i \sqrt{\Phi_i \Psi_i},$$

where $v$ is an arbitrary integral of the first kind, $\phi$ is the corresponding $\phi$-polynomial, and $\Phi_i$, $\Psi_i$ are $\phi$-polynomials with $p-1$ zeros each of the second order (cf. § 245, Chap. XIII.). It is in fact possible to represent *any* half-integer characteristic as the sum of two odd half-integer characteristics in $2^{p-2}(2^{p-1}-1)$ ways.

*Ex.* ii. In the hyperelliptic case, when the numbers $g$, $h$ are any half-integers, prove that the function of the system with $\varpi' + 1 = p$ poles is given by

$$\sqrt{u}\left\{\frac{y}{u\psi(x)} + \sum_1^p \frac{y_i}{u_i(x - x_i)\psi'(x_i)}\right\},$$

where the places $(x_1, y_1)$, ... are the poles in question,

$$\psi(x) = (x - x_1)\ldots(x - x_p),\ \psi'(x) = d\psi(x)/dx,\ u = (x - a)(x - b),$$

and $a$, $b$ are two suitably chosen branch places*, and $u_i = (x_i - a)(x_i - b)$. Shew that in the elliptic case this leads to the function $\dfrac{\sigma(u - v + w)}{\sigma(u - v)} e^{-\eta(u-v)}$.

270. In the case in which the factors at the period loops are any constants, the places $c_1, \ldots, c_k$ being still absent, it remains true that the number of zeros of any function of the system is equal to the number of poles; but here there may be an everywhere finite function of the system, and there will be such a function provided

$$g_i + \tau_{i,1} h_1 + \ldots\ldots + \tau_{i,p} h_p = -[G_i + \tau_{i,1} H_1 + \ldots\ldots + \tau_{i,p} H_p],\ (i = 1, 2, \ldots, p)$$

in which $G_1, \ldots, H_p$ are integers, the function being, in that case, expressed by

$$E = e^{-2\pi i \sum_1^p (h_i + H_i) v_i^{x, \gamma}};$$

* For the association of the proper pair of branch places $a$, $b$ with the given values of the numbers $g$, $h$, compare Chap. XI. § 208, Chap. XIII. § 245, and the remark at the conclusion of Ex. i.

then $E^{-1}$ is an everywhere finite function of the associated system, and $\sigma + 1 = \sigma' + 1 = 1$, $\varpi = \varpi' = p$. It is not necessary to consider this case, for it is clear that every function of the system is of the form $ER$, $R$ being a rational function.

When $\sigma + 1 = \sigma' + 1 = 0$ we have $\varpi = p - 1 = \varpi'$. Then every function of the system can be expressed linearly by means of functions of the system having $p$ poles. If $x_1, \dots, x_p$ be the poles of such a function and $z_1, \dots, z_p$ the zeros, and the relations connecting these be given by

$$v^{z_1, x_1} + \dots\dots + v^{z_p, x_p} = g + G + \tau(h + H).$$

There is beside the expression originally given, a very convenient way of expressing such a function, whose correctness is immediately verifiable, namely

$$\frac{\Theta(u - g - G - \tau h - \tau H)}{\Theta(u)} e^{-2\pi i (h+H) u},$$

wherein

$$u = v^{x, m} - v^{x_1, m_1} - \dots\dots - v^{x_p, m_p},$$

and $m, m_1, \dots, m_p$ are related as in § 179, Chap. X. Omitting a constant factor this is the same as

$$\frac{\Theta(u - g - \tau h)}{\Theta(u)} e^{-2\pi i h u}, = \phi(u), \text{ say};$$

since the difference between the values of the logarithm of $\phi(u)$ at the two sides of any period loop is independent of $u$, and of $x$, it follows that $\frac{\partial}{\partial x} \log \phi(u)$ is a rational function of $x$, and that $\frac{\partial}{\partial u_i} \log \phi(u)$ is a periodic function with $2p$ sets of simultaneous periods; thus the function $\phi(u)$ satisfies linear equations of the form

$$\frac{\partial^2 y}{\partial x^2} = Ry, \quad \frac{\partial^2 y}{\partial u_i \partial u_j} = R_{ij} y, \qquad (i, j = 1, 2, \dots, p),$$

where $R$, $R_{ij}$ are rational functions of $x$, and $2p$-ply periodic functions of $u$ given* by

$$R = \frac{\partial^2}{\partial x^2} \log \phi(u) + \left[\frac{\partial}{\partial x} \log \phi(u)\right]^2,$$

$$R_{ij} = \frac{\partial^2}{\partial u_i \partial u_j} \log \phi(u) + \left[\frac{\partial}{\partial u_i} \log \phi(u)\right]\left[\frac{\partial}{\partial u_j} \log \phi(u)\right].$$

*Ex.* The $2p$ constants $a$, $\lambda$ can be chosen so that

$$\phi(u) = \frac{\vartheta(u + a)}{\vartheta(u)} e^{\lambda u}$$

satisfies the equations $\phi(u + 2\omega) = A\phi(u)$, $\phi(u + 2\omega') = A'\phi(u)$, where $A$, $A'$ each represents $p$ given constants, and the notation is as in § 189, Chap. X.

* Cf. Halphen, *Fonct. Ellipt.*, Prem. Part. (Paris 1886), p. 235, and Forsyth, *Theory of Functions*, pp. 275, 285, for the case $p = 1$. By further development of the results given in Chap. XI. of this volume, and in the present chapter, it is clearly possible to formulate the corresponding analytical results for greater values of $p$.

271. We have seen (§ 261) that the number of arbitrary constants entering into the expression of a factorial function of the primary system with given poles is $N-p+1+h+1$, $=R$ say, where $N$ is the number of zeros of the function, and $h+1$ is the number of linearly independent differentials, $dv$, of integrals of the first kind, which vanish in the zeros of the function. When $h+1$ vanishes the assigning of the poles of the function, and of $R-1$ of the zeros determines the other $N-R+1$, $=p$, zeros; in any case the assigning of the poles and of $R-1$ of the zeros determines the other $N-R+1$, $=p-(h+1)$, of the zeros. Denote the poles by $\alpha_1, \dots, \alpha_M$ and the assigned zeros by $\beta_1, \dots, \beta_{R-1}$; then the remaining zeros $\beta_R, \dots, \beta_N$ are determined by the congruences

$$v_i^{\beta_1, a} + \dots + v_i^{\beta_{R-1}, a} - v_i^{\alpha_1, a} - \dots - v_i^{\alpha_M, a} - \sum_{r=1}^{k} \lambda_r v_i^{c_r, a} - (g_i + h_1 \tau_{i,1} + \dots + h_p \tau_{i,p})$$
$$\equiv -(v_i^{\beta_R, a} + \dots\dots + v_i^{\beta_N, a}),$$

$a$ being an arbitrary place. Now, let the form of the factorial function when the poles are given be

$$C_1 F_1(x) + \dots\dots + C_R F_R(x),$$

where $C_1, \dots, C_R$ are arbitrary constants, and $F_1(x), \dots, F_R(x)$ are linearly independent; then, when the zeros $\beta_1, \dots, \beta_{R-1}$ are assigned, the function is a constant multiple of the definite function

$$\Delta(x) = \begin{vmatrix} F_1(x), & \dots\dots, & F_R(x) \\ F_1(\beta_1), & \dots\dots, & F_R(\beta_1) \\ \dots & \dots & \dots \\ F_1(\beta_{R-1}), & \dots\dots, & F_R(\beta_{R-1}) \end{vmatrix};$$

the zeros of this function, other than $\beta_1, \dots, \beta_{R-1}$, are perfectly definite, and are determined by the congruences put down. Let $H$ denote the quantities given by

$$H_i = \sum_{r=1}^{k} \lambda_r v_i^{c_r, a} + g_i + h_1 \tau_{i,1} + \dots\dots + h_p \tau_{i,p};$$

take any places $\gamma_1, \dots, \gamma_{h+1}$, of assigned position, and take a place $m$ and $p$ dependent places $m_1, \dots, m_p$ defined as in § 179, Chap. X., and consider the function of $x$

$$\Theta(v^{x, m} - v^{\gamma_1, m_1} - \dots\dots - v^{\gamma_{h+1}, m_{h+1}} + v^{\beta_1, a} + \dots\dots + v^{\beta_{R-1}, a} - v^{a, m_{h+2}} - \dots\dots$$
$$- v^{a, m_p} - v^{\alpha_1, a} - \dots\dots - v^{\alpha_M, a} - H);$$

if the function does not vanish identically, its zeros, $x_1, \dots, x_p$, are (§ 179, Chap. X.) given by the congruences denoted by

$$-v^{\gamma_1, m_1} - \dots\dots - v^{\gamma_{h+1}, m_{h+1}} + v^{\beta_1, a} + \dots\dots + v^{\beta_{R-1}, a} - v^{a, m_{h+2}} - \dots\dots$$
$$- v^{a, m_p} - v^{\alpha_1, a} - \dots\dots - v^{\alpha_M, a} - H \equiv - v^{x_1, m_1} - \dots\dots - v^{x_p, m_p},$$

or, what is the same thing, by

$$v^{\beta_1, a} + \ldots\ldots + v^{\beta_{R-1}, a} - v^{a_1, a} - \ldots\ldots - v^{a_M, a} - H$$
$$\equiv v^{\gamma_1, x_1} + \ldots\ldots + v^{\gamma_{h+1}, x_{h+1}} - v^{x_{h+2}, a} - \ldots\ldots - v^{x_p, a};$$

now, from what has been said, it follows, comparing these congruences with those connecting the poles and zeros of $\Delta(x)$, that if $x_1, \ldots, x_{h+1}$ be taken at $\gamma_1, \ldots, \gamma_{h+1}$, these congruences determine $x_{h+2}, \ldots, x_p$ uniquely as the places $\beta_R, \ldots, \beta_N$. Thus the zeros of the theta function are the places $\gamma_1, \ldots, \gamma_{h+1}$ together with the zeros, other than $\beta_1, \ldots, \beta_{R-1}$, of the function $\Delta(x)$.

We suppose now $M$ to be as great as $p-1, = r+p-1$, say; as in § 184, p. 269, we take $n_1, \ldots, n_{p-1}$ to be the zeros of a $\phi$-polynomial of which all the zeros are of the second order, so that

$$v^{m_p, m} - v^{n_1, m_1} - \ldots\ldots - v^{n_{p-1}, m_{p-1}}$$

is an odd half-period, equal to $\frac{1}{2}\Omega_{s, s'}$ say; and we take the poles $a_{r+1}, \ldots, a_M$ at $n_1, \ldots, n_{p-1}$. Further*, in this article, we denote

$$\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'})\, e^{\pi i s' v^{x, z}} \text{ by } \lambda(x, z),$$

so that (§ 175, Chap. X.) $\lambda(x, z)$ is also equal to $e^{-\frac{1}{2}\pi i s'(s+\frac{1}{2}\tau s')}\,\Theta(v^{x, z}; \tfrac{1}{2}s, \tfrac{1}{2}s')$. The function $\lambda(x, z)$ must not be confounded with the function $\lambda(\zeta, \mu)$ of § 238.

Then in fact, denoting the arguments of the theta function by $V$, we have the following important formula,

$$e^{-2\pi i(h-\frac{1}{2}s')V}\,\Theta(V) = A\,\frac{\Delta(x)\prod_{j=1}^{r}\lambda(x, a_j)\prod_{j=1}^{h+1}\lambda(x, \gamma_j)\prod_{j=1}^{k}[\lambda(x, c_j)]^{\lambda_j}}{\prod_{j=1}^{R-1}\lambda(x, \beta_j)},$$

where $A$ is a quantity independent of $x$. In order to prove this it is sufficient to shew (i) that the right-hand side represents a single-valued function of $x$ on the Riemann surface dissected by the $2p$ period loops, (ii) that the right-hand side has no poles and has only the zeros of $\Theta(V)$, and (iii) that the two sides of the equation have the same factor for every one of the $2p$ period loops.

Now the function $\lambda(x, z)$ has no poles; its zeros are the place $z$, and the places $n_1, \ldots, n_{p-1}$. The places $n_1, \ldots, n_{p-1}$ occur on the right hand

($a$) as poles, each once in $\Delta(x)$, each $(R-1)$ times in the product $\prod_{j=1}^{R-1}\lambda(x, \beta_j)$;

($\beta$) as zeros, each $r$ times in $\prod_{j=1}^{r}\lambda(x, a_j)$, $h+1$ times in $\prod_{j=1}^{h+1}\lambda(x, \gamma_j)$, and

* For the introduction of the function $\lambda(x, z)$ see, beside the references given in chapter XIII. (§ 250), also Clebsch u. Gordan, *Abel. Functnen.* pp. 251—256, and Riemann, *Math. Werke* (1876), p. 134.

$\sum_{j=1}^{k} \lambda_j$ times in $\prod_{j=1}^{k} [\lambda(x, c_j)]^{\lambda_j}$; thus these places occur as zeros, on the right hand,

$$M-(p-1)+h+1+\Sigma\lambda_j-R, \;=N-p+1+h+1-R,$$

times, that is, not at all.

Thus the expression on the right hand may be interpreted as a single-valued function on the Riemann surface dissected by the $2p$ period loops—for we have seen that the places $n_1, \ldots, n_{p-1}$ do not really occur, and the multiplicity, at $c_j$, in the value of such a factor as $[\lambda(x, c_j)]^{\lambda_j}$ is cancelled by the assigned character of the factorial functions $F(x)$ occurring in $\Delta(x)$. Nextly, the zeros of the denominator of the right-hand side, other than at $n_1, \ldots, n_{p-1}$, are zeros of $\Delta(x)$, and the poles of $\Delta(x)$, other than $n_1, \ldots, n_{p-1}$, are zeros of the product $\prod_{j=1}^{r} \lambda(x, a_j)$, so that the right-hand side remains finite. The only remaining zeros of the right-hand side consist of $\gamma_1, \ldots, \gamma_{h+1}$ and the zeros of $\Delta(x)$ beside $\beta_1, \ldots, \beta_{R-1}$; and we have proved that these are the zeros of $\Theta(V)$. It remains then finally to examine the factors of the two sides of the equation at the period loops. The factors of the left-hand side at the $i$-th period loops respectively of the first and second kind are (see § 175, Chap. X.)

$$e^{-2\pi i(h_i-\frac{1}{2}s_i')} \text{ and } e^{-2\pi i\sum_{\mu=1}^{p}(h_\mu-\frac{1}{2}s'_\mu)\tau_{\mu,i}-2\pi i(V_i+\frac{1}{2}\tau_{i,i})};$$

the factor of the right-hand side at the $i$-th period loop of the first kind is $e^{\psi}$, where

$$\psi=-2\pi i h_i+r\pi i s_i'+(h+1)\pi i s_i'+\pi i s_i'\sum_{j=1}^{k}\lambda_j-(R-1)\pi i s_i';$$

now $R=N-p+1+h+1=r+\sum_{j=1}^{k}\lambda_j+h+1$; thus $\psi=-2\pi i h_i+\pi i s_i'$, and $e^{\psi}=e^{-2\pi i(h_i-\frac{1}{2}s_i')}$, or the factors of the two sides of the equation to be proved, at the $i$-th period loop of the first kind, are the same. Since the factor of $\lambda(x, z)$ at the $i$-th period loop of the second kind is $e^{\mu}$ where

$$\mu=-2\pi i[v_i^{x,z}+\tfrac{1}{2}s_i+\tfrac{1}{2}s_1'\tau_{i,1}+\ldots+s'_p\tau_{i,p}+\tfrac{1}{2}\tau_{i,i}]+\pi i(s_1'\tau_{i,1}+\ldots+s'_p\tau_{i,p}),$$
$$=-2\pi i(v_i^{x,z}+\tfrac{1}{2}s_i+\tfrac{1}{2}\tau_{i,i}),$$

it follows that the factor of the right-hand side at the $i$-th period loop of the second kind is $e^{\chi}$ where

$$\chi=2\pi i g_i-2\pi i\left[\sum_{j=1}^{r}v_i^{x,a_j}+\sum_{j=1}^{h+1}v_i^{x,\gamma_j}+\sum_{j=1}^{k}\lambda_j v_i^{x,c_j}-\sum_{j=1}^{R-1}v_i^{x,\beta_j}\right]$$
$$-\pi i\left[r+h+1+\sum_{j=1}^{h}\lambda_j-R+1\right](\tau_{i,i}+s_i),$$
$$=2\pi i g_i-\pi i(s_i+\tau_{i,i})-2\pi i\left[\sum_{j=1}^{r}v_i^{x,a_j}+\sum_{j=1}^{h+1}v_i^{x,\gamma_j}+\sum_{j=1}^{k}\lambda_j v_i^{x,c_j}-\sum_{j=1}^{R-1}v_i^{x,\beta_j}\right];$$

now we have

$$V_i = v_i^{x,\,m} - v_i^{\gamma_1,\,m_1} - \dots - v_i^{\gamma_{h+1},\,m_{h+1}} + v_i^{\beta_1,\,a} + \dots + v_i^{\beta_{R-1},\,a} - v_i^{a,\,m_{h+2}} - \dots - v_i^{a,\,m_p}$$

$$- v_i^{\alpha_1,\,a} - \dots - v_i^{\alpha_r,\,a} - v_i^{n_1,\,a} - \dots - v_i^{n_{p-1},\,a} - \sum_{j=1}^{k} \lambda_j v_i^{c_j,\,a} - g_i - h_1\tau_{i,1} - \dots - h_p\tau_{i,p},$$

and

$$\tfrac{1}{2}(s_i + s_1'\tau_{i,1} + \dots + s_p'\tau_{i,p}) = v^{m_p,\,m} - v^{n_1,\,m_1} - \dots - v^{n_{p-1},\,m_{p-1}};$$

thus

$$V_i - \tfrac{1}{2}(s_i + s_1'\tau_{i,1} + \dots + s_p'\tau_{i,p}) = v_i^{x,\,a} + v_i^{a,\,\gamma_1} + \dots + v_i^{a,\,\gamma_{h+1}} + v_i^{\beta_1,\,a} + \dots + v_i^{\beta_{R-1},\,a}$$

$$- v_i^{\alpha_1,\,a} - \dots - v_i^{\alpha_r,\,a} - \sum_{j=1}^{k} \lambda_j v_i^{c_j,\,a} - g_i - h_1\tau_{i,1} - \dots - h_p\tau_{i,p};$$

further

$$0 = -v_i^{x,\,a} + (h+1)\,v_i^{x,\,a} - (R-1)\,v_i^{x,\,a} + r v_i^{x,\,a} + \sum_{j=1}^{k} \lambda_j v_i^{x,\,a};$$

hence

$$V_i - \tfrac{1}{2}(s_i + s_1'\tau_{i,1} + \dots + s_p'\tau_{i,p}) = \sum_{j=1}^{h+1} v_i^{x,\,\gamma_j} - \sum_{j=1}^{R-1} v_i^{x,\,\beta_j} + \sum_{j=1}^{r} v_i^{x,\,\alpha_j}$$

$$+ \sum_{j=1}^{k} \lambda_j v_i^{x,\,c_j} - g_i - h_1\tau_{i,1} - \dots - h_p\tau_{i,p},$$

or

$$\sum_{\mu=1}^{p} (h_\mu - \tfrac{1}{2}s_\mu')\,\tau_{\mu,i} + V_i = -g_i + \tfrac{1}{2}s_i + \sum_{j=1}^{r} v_i^{x,\,\alpha_j} + \sum_{j=1}^{h+1} v_i^{x,\,\gamma_j} + \sum_{j=1}^{k} \lambda_j v_i^{x,\,c_j} - \sum_{j=1}^{R-1} v_i^{x,\,\beta_j},$$

and thence the identity of the factors taken by the two sides of the equation to be proved, at the $i$-th period loop of the second kind, is manifest.

And before passing on it is necessary to point out that if the functions $\lambda(x, z)$ be everywhere replaced by $\dfrac{\lambda(x, z)}{\psi}$, and $\Delta(x)$ be replaced by $\psi\Delta(x)$, $\psi$ being any quantity whatever, the value of the right-hand side of the equation is unaltered. For there are $R$ factors $\lambda(x, z)$ occurring in the numerator of the right-hand side of the equation beside $\Delta(x)$, and $R-1$ factors $\lambda(x, z)$ occurring in the denominator of the right-hand side of the equation. In particular $\psi$ may be a function of $x$.

272. We can now state the following result: Let $a, \alpha_1, \dots, \alpha_r$ be any assigned places; let $n_1, n_2, \dots, n_{p-1}$ be the zeros of a $\phi$-polynomial, or of a differential, $dv$, of the first kind, of which all the zeros are of the second order, and

$$v_i^{m_p,\,m} - v_i^{n_1,\,m_1} - \dots - v_i^{n_{p-1},\,m_{p-1}} = \tfrac{1}{2}(s_i + s_i'\tau_{i,1} + \dots + s_p'\tau_{i,p}), \quad (i = 1, 2, \dots, p),$$

$m, m_1, \dots, m_p$ being such places as in § 179, Chap. X.; let $h+1$ be the number of linearly independent differentials, $dv$, which vanish in the zeros of a factorial function of the primary system having $\alpha_1, \dots, \alpha_r, n_1, \dots, n_{p-1}$ as poles, or the number of differentials $dV'$, of everywhere finite factorial integrals of the associated system, which vanish in the places $n_1, \ldots, n_{p-1}$, $a_1, \ldots, a_r$; let $\gamma_1, \ldots, \gamma_{h+1}$ be any assigned places; denote $r+\sum_{j=1}^{k}\lambda_j+h+1$ by $R$, and let* $x_1, \ldots, x_R$ be any assigned places; let the general factorial function of the primary system having $a_1, \ldots, a_r, n_1, \ldots, n_{p-1}$ as poles be

$$C_1F_1(x)+\ldots\ldots+C_RF_R(x),$$

wherein $C_1, \ldots, C_R$ are constants, and let

$$\Delta(x_1, \ldots, x_R)=\left|\begin{matrix} F_1(x_1), & \ldots\ldots, & F_R(x_1) \\ F_1(x_2), & \ldots\ldots, & F_R(x_2) \\ \ldots & \ldots & \ldots \\ F_1(x_R), & \ldots\ldots, & F_R(x_R)\end{matrix}\right| \psi(x_1), \ldots, \psi(x_R),$$

where $\psi(x)$ denotes any function whatever; let

$$U_i=\sum_{j=1}^{R} v_i^{x_j, a}-\sum_{j=1}^{r} v_i^{a_j, a}-\sum_{j=1}^{h+1} v_i^{\gamma_j, a}-\sum_{j=1}^{k}\lambda_j v_i^{c_j, a},$$

which is independent of $a$, and let the row of $p$ quantities

$$g_i-\tfrac{1}{2}s_i+(h_1-\tfrac{1}{2}s_1')\tau_{i,1}+\ldots\ldots+(h_p-\tfrac{1}{2}s_p')\tau_{i,p}$$

be denoted† by $g-\frac{1}{2}s+\tau(h-\frac{1}{2}s')$; then if, modifying the definition of $\lambda(x, z)$, we put

$$\lambda(x, z)=\frac{\Theta(v^{x,z}+\frac{1}{2}\Omega_{s,s'})\,e^{\pi i s' v^{x,z}}}{\psi(x)\,\psi(z)},$$

we have

$$Ce^{-2\pi i(h-\frac{1}{2}s')[U-(g-\frac{1}{2}s)-\tau(h-\frac{1}{2}s')]}\,\Theta[U-(g-\tfrac{1}{2}s)-\tau(h-\tfrac{1}{2}s')]$$
$$=\frac{\Delta(x_1, x_2, \ldots, x_R)}{\prod\limits_{\substack{i,j=1,2,\ldots,R\\ i<j}}\prod\lambda(x_i, x_j)}\prod_{i=1}^{R}\left\{\prod_{j=1}^{r}\lambda(x_i, a_j)\prod_{j=1}^{h+1}\lambda(x_i, \gamma_j)\prod_{j=1}^{k}[\lambda(x_i, c_j)]^{\lambda_j}\right\},$$

wherein $C$ is a quantity independent of $x_1, \ldots, x_R$, which may depend on $c_1, \ldots, c_k, a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_{h+1}$.

273. The formula just obtained is of great generality; before passing to examples of its application it is desirable to explain the origin of a certain function which may be used in place of the unassigned function $\psi(x)$.

We have (§ 187, p. 274), in the notation of § 272,

$$\Pi_{x,z}^{x',z'}=\log\frac{\Theta(v^{x',x}+\frac{1}{2}\Omega_{s,s'})\,\Theta(v^{z',z}+\frac{1}{2}\Omega_{s,s'})}{\Theta(v^{x',z}+\frac{1}{2}\Omega_{s,s'})\,\Theta(v^{z',x}+\frac{1}{2}\Omega_{s,s'})};$$

if the zeros of the rational function of $x'$, $(x'-x)/(x'-z)$, be denoted by

* These replace the $x_1, \beta_1, \ldots, \beta_{R-1}$ of § 271.

† So that $V=U-(g-\frac{1}{2}s)-\tau(h-\frac{1}{2}s')$.

$x, x_1, \dots, x_{n-1}$, $n$ being the number of sheets of the fundamental Riemann surface, and the poles of the same function be denoted by $z, z_1, \dots, z_{n-1}$, we have, by Abel's theorem,

$$\Pi_{x_1, z_1}^{x', z'} + \dots\dots + \Pi_{x_{n-1}, z_{n-1}}^{x', z'} = -\Pi_{x, z}^{x', z'} + \log\left(\frac{x'-x}{x'-z}\Big/\frac{z'-x}{z'-z}\right),$$

$$= \log\frac{(x'-x)(z'-z)\,\Theta(v^{x', z} + \tfrac{1}{2}\Omega_{s, s'})\,\Theta(v^{z', x} + \tfrac{1}{2}\Omega_{s, s'})}{(x'-z)(z'-x)\,\Theta(v^{x', x} + \tfrac{1}{2}\Omega_{s, s'})\,\Theta(v^{z', z} + \tfrac{1}{2}\Omega_{s, s'})};$$

now let the places $x'$, $z'$ approach respectively indefinitely near to the places $x, z$ which, firstly, we suppose to be finite places and not branch places; then the right-hand side of the equation just obtained becomes

$$\log\left[\frac{1}{-(x-z)^2}\,\frac{\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'})\,\Theta(v^{x, z} - \tfrac{1}{2}\Omega_{s, s'})}{X(x)\,X(z)}\right],$$

where

$$X(x) = \sum_{i=1}^{p} \Theta_i(\tfrac{1}{2}\Omega_{s, s'})\,.\,Dv_i^{x, a}, \quad X(z) = \sum_{i=1}^{p} \Theta_i(\tfrac{1}{2}\Omega_{s, s'})\,.\,Dv_i^{z, a},$$

$D$ denoting a differentiation, and $a$ denoting an arbitrary place; but we have (Chap. X. § 175)

$$\Theta(v^{x, z} - \tfrac{1}{2}\Omega_{s, s'}) = e^{\pi i s s'} e^{2\pi i s' v^{x, z}}\,\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'}) = -\,e^{2\pi i s' v^{x, z}}\,\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'});$$

thus, on the whole, when the square roots are properly interpreted, we obtain

$$\lim._{x'=x,\, z'=z} \sqrt{-(x'-x)(z'-z)\,e^{-\Pi_{x, z}^{x', z'}}} = \frac{\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'})\,.\,e^{\pi i s' v^{x, z}}}{\sqrt{X(x)\,X(z)}}$$

$$= (x-z)\,e^{\tfrac{1}{2}\Pi_{x_1, z_1}^{x, z} + \dots\dots + \tfrac{1}{2}\Pi_{x_{n-1}, z_{n-1}}^{x, z}}. \qquad \text{(i)}$$

When the places $x, z$ are finite branch places we obtain a similar result. Denote the infinitesimals at these places by $t, t_1$, and, when $x', z'$ are near to $x, z$, respectively, suppose $x' = x + t^{w+1}$, $z' = z + t_1^{w_1+1}$; then from the equation given by Abel's theorem we obtain, if $\gamma$ denote an arbitrary place,

$$\sum_{r=1}^{w}\left[\Pi_{x_r, \gamma}^{x', z'} - \log t\right] + \sum_{r=w+1}^{n-1} \Pi_{x_r, \gamma}^{x', z'} + \sum_{r=1}^{w_1}\left[\Pi_{\gamma, z_r}^{x', z'} - \log t_1\right] + \sum_{r=w_1+1}^{n-1} \Pi_{\gamma, z_r}^{x', z'}$$

$$= -\log(t^w) - \log(t_1^{w_1}) + \log\left[\frac{t^{w+1} t_1^{w_1+1}}{-(x-z)^2}\,.\,\frac{-\Theta^2(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'})\,.\,e^{2\pi i s' v^{x, z}}}{t t_1 X(x)\,X(z)}\right],$$

where $X(x)$, $X(z)$ are of the same form as before, save that the differentiations $Dv_i^{x, a}$, $Dv_i^{z, a}$, are to be performed in regard to the infinitesimals $t, t_1$. If the limit of the first member of this equation, as $x', z'$ respectively approach to $x, z$, be denoted by $L$, we therefore have

$$\lim._{x'=x,\, z'=z} \sqrt{-t t_1 e^{-\Pi_{x, z}^{x', z'}}} = \frac{\Theta(v^{x, z} + \tfrac{1}{2}\Omega_{s, s'})\,.\,e^{\pi i s' v^{x, z}}}{\sqrt{X(x)\,X(z)}} = (x-z)\,e^{\frac{1}{2}L}. \qquad \text{(ii)}$$

The equations (i), (ii) are very noticeable; *there is no position of x for which the expression* $\Theta(v^{x,z}+\frac{1}{2}\Omega_{s,s'}) \cdot e^{\pi i s' v^{x,z}}/\sqrt{X(x)X(z)}$ *is infinite, and there is only one position of x, namely when x is at z, for which the expression vanishes;* for (§ 188, p. 281) the expression $\sqrt{X(x)}$ vanishes, to the first order, only when $x$ is at one of the places $n_1, \ldots, n_{p-1}$, and $\Theta(v^{x,z}+\frac{1}{2}\Omega_{s,s'})$ vanishes only when $x$ is at one of the places $z, n_1, \ldots, n_{p-1}$; there is no position of $x$ for which $\sqrt{X(x)}$ is infinite. Putting

$$\varpi_1(x, z) = \frac{\Theta(v^{x,z}+\frac{1}{2}\Omega_{s,s'})\, e^{\pi i s' v^{x,z}}}{\sqrt{X(x)X(z)}},$$

we have further $\varpi_1(x, z) = -\varpi_1(z, x)$, and if $t$ denote the infinitesimal near to $z$, we have, as $x$ approaches to $z$, $\text{limit}_{x=z}[\varpi_1(x, z)/t] = 1$. For every position of $x$ and $z$ on the dissected Riemann surface $\varpi_1(x, z)$ has a perfectly determinate value, *save for an ambiguity of sign*, and, as follows from the equations (i), (ii), this value is independent of the characteristic $(\frac{1}{2}s, \frac{1}{2}s')$.

There are various ways of dealing with the ambiguity in sign of the function $\varpi_1(x, z)$. For instance, let $\phi(x)$ be any $\phi$-polynomial vanishing in an arbitrary place $m$, and in the places $A_1, \ldots, A_{2p-3}$ (cf. § 244, Chap. XIII.), and let $Z(x)$ be that polynomial of the third degree in the $p$ fundamental linearly independent $\phi$-polynomials which vanishes to the second order in $A_1, \ldots, A_{2p-3}$ and in the places $m_1, \ldots, m_p$. Further let $\Phi(x)$ be that $\phi$-polynomial which vanishes to the second order in the places $n_1, \ldots, n_{p-1}$. Then we have shewn (§ 244) that the ratio $\sqrt{Z(x)}/\phi(x)\sqrt{\Phi(x)}$, save for an initial determination of sign for an arbitrary position of $x$, is single-valued on the dissected Riemann surface; hence instead of the function $\varpi_1(x, z)$ we may use the function

$$E_1(x, z) = \frac{\sqrt{Z(x)Z(z)}}{\phi(x)\phi(z)} \cdot \frac{\Theta(v^{x,z}+\frac{1}{2}\Omega_{s,s'})\, e^{\pi i s' v^{x,z}}}{\sqrt{\Phi(x)\Phi(z)}},$$

which has the properties; (i) on the dissected Riemann surface it is a single-valued function of $x$ and of $z$, (ii) $E_1(x, z) = -E_1(z, x)$, (iii) as a function of $x$ it has, beside the *fixed* zeros $m_1, \ldots, m_p$, only the zero given by $x = z$, and it has no infinities beside the *fixed* infinity given by $x = m$, where it is infinite to the first order. At the $r$-th period loops respectively of the first and second kind it has the factors

$$1,\ e^{-2\pi i (v_r^{x,z}+\frac{1}{2}\tau_{r,r})}.$$

But there can be no doubt, in view of the considerations advanced in chapter XII. of the present volume, as to the way in which the ambiguity of the sign of $\varpi_1(x, z)$ ought to be dealt with. Suppose that the Riemann surface now under consideration has arisen from the consideration of the functions there considered (§ 227) which are unaltered by the linear substitutions of the group. Let the places in the region $S$ of the $\zeta$ plane which correspond to the places $x$, $z$, $x'$, $z'$ of the Riemann surface be denoted by $\xi$, $\zeta$, $\xi'$, $\zeta'$. Then by comparing the equation obtained in chapter XII. (§ 234),

$$\lim._{\xi'=\xi,\ \zeta'=\zeta} \sqrt{-(\xi'-\xi)(\zeta'-\zeta)\,e^{-\Pi_{\xi,\zeta}^{\xi',\zeta'}}} = \frac{\Theta(v^{\xi,\zeta}+\tfrac{1}{2}\Omega_{s,s'})\,e^{\pi i s' v^{\xi,\zeta}}}{\sqrt{\dfrac{dv}{d\xi}\dfrac{dv}{d\zeta}}} = \varpi(\xi,\zeta),$$

with the equation here obtained,

$$\lim._{x'=x,\ z'=z} \sqrt{-tt_1 e^{-\Pi_{x,z}^{x',z'}}} = \frac{\Theta(v^{x,z}+\tfrac{1}{2}\Omega_{s,s'})\,e^{\pi i s' v^{x,z}}}{\sqrt{X(x)X(z)}} = \varpi_1(x,z),$$

and noticing that $X(x)$, $\dfrac{dv}{d\xi}$ agree in being differential coefficients of an integral of the first kind, which vanish to the second order at $n_1, \dots, n_{p-1}$, we deduce the equation

$$\varpi_1(x,z)\Big/\sqrt{\frac{dt}{d\xi}\cdot\frac{dt_1}{d\zeta}} = \varpi(\xi,\zeta);$$

now we have shewn that $\varpi(\xi,\zeta)$ is a single-valued function of $\xi$ and $\zeta$; and any one of the infinite number of values of $\xi$, which correspond to any value of $x$, has a continuous and definite variation as $x$ varies in a continuous way; hence it is possible, dividing $\varpi_1(x,z)$ by the factor $\sqrt{\dfrac{dt}{d\xi}\cdot\dfrac{dt_1}{d\zeta}}$, which by itself is of ambiguous sign, to destroy the original ambiguity while retaining the essential character of the function $\varpi_1(x,z)$. The modified function is infinitely many-valued, but each branch is separable from the others by a conformal representation. Thus the question of the ambiguity in the sign of $\varpi_1(x,z)$ is subsequent to the enquiry as to the function $\zeta$ which will conformably represent the Riemann surface upon a single $\zeta$ plane in a manner analogous to that contemplated in chapter XII. §§ 227, 230*.

In what follows however we do not need to enter into the question of the sign of $\varpi_1(x,z)$. It has been shewn in the preceding article that the final formula obtained is independent of the form taken for the function there denoted by $\psi(x)$. It is therefore permissible, for any position of $x$, to take for it the expression $\sqrt{X(x)}$, with any assigned sign, without attempting to give a law for the continuous variation of this expression. The advantage is in the greater simplicity of $\varpi_1(x,z)$; for example, when $x$ is at any one

* Klein has proposed to deal with the function $\varpi_1(x,z)$ by means of homogeneous variables. The reader may compare *Math. Annal.* XXXVI. (1890) p. 12, and Ritter, *Math. Annal.* XLIV. (1894) pp. 274—284. In the theory of automorphic functions the necessity for homogeneous variables is well established. Cf. § 279 of the present chapter. For the theory of the function $\varpi_1(x,z)$ in the hyperelliptic case see Klein, and Burkhardt, *Math. Annal.* XXXII. (1888).

of the places $n_1, \ldots, n_{p-1}$, the function $\lambda(x, z)$, as defined in § 271, vanishes independently of $z$; but this is not the case for $\varpi_1(x, z)$.

*Ex.* i. Prove that

$$\Pi_{a, c}^{x, z} = \log \frac{\varpi_1(x, a)\, \varpi_1(z, c)}{\varpi_1(x, c)\, \varpi_1(z, a)}.$$

*Ex.* ii. Prove that any rational function of which the poles are at $a_1, \ldots, a_M$ and the zeros at $\beta_1, \ldots, \beta_M$, can be put into the form

$$\frac{\varpi_1(x, \beta_1) \ldots\ldots \varpi_1(x, \beta_M)}{\varpi_1(x, a_1) \ldots\ldots \varpi_1(x, a_M)} e^{\lambda_1 v_1^{x, a} + \ldots\ldots + \lambda_p v_p^{x, a}}$$

where $\lambda_1, \ldots, \lambda_p$ are constants, and $a$ is a fixed place.

In what follows, as no misunderstanding is to be apprehended, we shall omit the suffix in the expression $\varpi_1(x, z)$, and denote it by $\varpi(x, z)$. The function $\varpi(\xi, \zeta)$ of chapter XII. does not recur in this chapter.

274. As an application of the formula of § 272 we take the case of the root form $\sqrt{X^{(3)}(x)}/\Phi(x)\sqrt{X(x)}$, where $X^{(3)}(x)$ is a cubic polynomial of the differential coefficients of the integrals of the first kind, having $3(p-1)$ zeros, each of the second order (cf. § 244, Chap. XIII.). Then the poles $\alpha_1, \ldots, \alpha_r$ are the $2p-2$ zeros of any given polynomial $\Phi(x)$, which is linear in the differential coefficients of integrals of the first kind. Thus $r = 2p-2$, $h+1=0$, $R, = r+h+1+\sum_1^k \lambda_j = 2p-2+0+0 = 2p-2$; $U = \sum_1^{2p-2} v^{x_j, \alpha_j}$, and, taking for the function $\psi(x)$, the expression $\sqrt{X(x)}$, the formula becomes

$$Ce^{-2\pi i[h-\frac{1}{2}s'][U-g+\frac{1}{2}s-\tau(h-\frac{1}{2}s')]}\,\Theta\Big[\sum_1^{2p-2} v^{x_j, \alpha_j} - (g-\tfrac{1}{2}s) - \tau(h-\tfrac{1}{2}s')\Big]$$

$$= \frac{\begin{vmatrix} \sqrt{X_1^{(3)}(x_1)} & \cdot & \sqrt{X_{2p-2}^{(3)}(x_1)} \\ \cdot & \cdot & \cdot \\ \sqrt{X_1^{(3)}(x_{2p-2})} & \cdot & \sqrt{X_{2p-2}^{(3)}(x_{2p-2})} \end{vmatrix} \prod_{i=1}^{2p-2}\prod_{j=1}^{2p-2} \varpi(x_i, \alpha_j)}{\prod_{i<j}^{i, j=1, 2, \ldots, 2p-2} \varpi(x_i, x_j) \prod_{i=1}^{2p-2} \Phi(x_1) \ldots\ldots \Phi(x_{2p-2})}.$$

Herein $\Phi(x)$ is a given polynomial with zeros at $\alpha_1, \ldots, \alpha_{2p-2}$, and the forms $\sqrt{X_1^{(3)}(x)}, \ldots, \sqrt{X_{2p-2}^{(3)}(x)}$ are any set of linearly independent forms, derived as in § 245, Chap. XIII., and having $(-g_1, \ldots, -h_1, \ldots, -h_p)$ for characteristic.

From this formula* that of § 250, Chap. XIII. is immediately obtainable. The result is clearly capable of extension to the case of a function

$$\sqrt{X^{(2r+1)}(x)}/\Phi_r(x)\sqrt{X(x)}.$$

* Cf. Weber, *Theorie der Abel'schen Functionen vom Geschlecht* 3, Berlin, 1876, § 24, p. 156; Noether, *Math. Annal.* XXVIII. (1887), p. 367; Klein, *Math. Annal.* XXXVI. (1890), p. 40. For the introduction of $\phi$-polynomials as homogeneous variables cf. §§ 110—114, Chap. VI. of the present volume. See also Stahl, *Crelle*, CXI. (1893), p. 106; Pick, *Math. Annal.* XXIX. "Zur Theorie der Abel'schen Functionen."

275. A general application of the formula of § 272 to the case of rational functions may be made by taking $\alpha_1, \ldots, \alpha_r$ to be any places whatever, $r$ being greater than $p-1$. Then $h+1=0$ and $R=r$; and if the general rational function with poles in $\alpha_1, \ldots, \alpha_r, n_1, \ldots, n_{p-1}$ be

$$A_1 F_1(x) + \ldots\ldots + A_{r-1}F_{r-1}(x) + A_r,$$

where $A_1, \ldots, A_r$ are constants, and we take for the function $\psi(x)$ the expression $\sqrt{X(x)}$, and modify the constant $C$ which depends in general upon $\alpha_1, \ldots, \alpha_r$, we obtain the result (cf. § 175, Chap. X.)

$$C\Theta[\sum_1^r v^{x_i, a_i}; \tfrac{1}{2}s, \tfrac{1}{2}s'], = Ce^{\pi i s'[U+\frac{1}{2}s+\frac{1}{4}\tau s']}\,\Theta[\sum_1^r v^{x_i, a_i} + \tfrac{1}{2}s + \tfrac{1}{2}\tau s'],$$

$$= \frac{\begin{vmatrix} F_1(x_1), & \ldots, & F_{r-1}(x_1), & 1 \\ \cdots & \cdots & \cdots & \cdots \\ F_1(x_r), & \ldots, & F_{r-1}(x_r), & 1 \end{vmatrix} \prod_{i=1}^{r}\prod_{j=1}^{r} \varpi(x_i, \alpha_j)}{\prod_{\substack{i,j=1,\ldots,r \\ i<j}} \varpi(x_i, x_j) \prod_{\substack{i,j=1,\ldots,r \\ i<j}} \varpi(\alpha_i, \alpha_j)} \sqrt{X(x_1)\ldots X(x_r)\,X(\alpha_1)\ldots X(\alpha_r)}.$$

276. This formula includes many particular cases*. We proceed to obtain a more special formula, deduced directly from the result of § 272. Let $\alpha_1, \ldots, \alpha_r = n_1, \ldots, n_{p-1}$. Then the everywhere finite factorial integrals of the associated system are the ordinary integrals of the first kind, and the number, $h+1$, of $dV'$ which vanish in the places $\alpha_1, \ldots, \alpha_r$, $n_1, \ldots, n_{p-1}$, that is, which vanish to the second order in the places $n_1, \ldots, n_{p-1}$, is 1. The number $R, = r + \sum_j \lambda_j + h + 1, = p \,.\, 1 + 0 + 1, = p$. The general function having the poles $n_1^2, \ldots, n_{p-1}^2$ is $F(x) = \Phi(x)/X(x)$, where $X(x)$ is the expression employed in § 273, and $\Phi(x)$ denotes the differential coefficient of the general integral of the first kind. Further

$$U = \sum_1^p v^{x_j, a} - \sum_1^{p-1} v^{n_j, a} - v^{\gamma, a}, = \sum_1^{p-1} v^{x_j, n_j} + v^{x_p, \gamma},$$

$\gamma$ being an arbitrary place. Hence

$$U - \tfrac{1}{2}s - \tfrac{1}{2}\tau s' = \sum_1^p v^{x_j, m_j} - v^{\gamma, m}, = V \text{ say},$$

and

$$e^{\pi i s'(U+\frac{1}{2}s+\frac{1}{2}\tau s')}\,\Theta(U + \tfrac{1}{2}s + \tfrac{1}{2}\tau s') = e^{\pi i s'(V+\Omega_{s,s'})}\,\Theta(V + \Omega_{s,s'})$$

is equal (§ 175, Chap. X.) to

$$e^{\pi i s'(V+\Omega_{s,s'})-2\pi i s'(V+\frac{1}{2}\tau s')}\,\Theta(V), = e^{-\pi i s'(V+s)}\,\Theta(V), = -e^{-\pi i V s'}\,\Theta(V),$$

since $ss'$ is an odd integer. Therefore taking for the function $\psi(x)$ the expression $\sqrt{X(x)}$, $\lambda(x, z)$ is $\varpi(x, z)$, and

$$\Delta(x_1, \ldots, x_R) = \begin{vmatrix} \Phi_1(x_1), & \cdot & , \Phi_p(x_1) \\ \cdot & , \cdot & , \cdot \\ \Phi_1(x_p), & \cdot & , \Phi_p(x_p) \end{vmatrix} \div \sqrt{X(x_1)\ldots X(x_p)},$$

* Cf. Klein, *Math. Annal.* XXXVI. p. 38.

where $\Phi(x), \ldots, \Phi_p(x)$ denote $dv_1^{x,a}/dt, \ldots, dv_p^{x,a}/dt$. Thus on the whole

$$Ce^{-\pi i V s'}\Theta(V) = \frac{\Delta(x_1, \ldots, x_p)}{\prod\limits_{i<j}^{i,j=1,\ldots,p}\prod \varpi(x_i, x_j)} \prod_{i=1}^{p} [\varpi(x_i, n_1), \ldots, \varpi(x_i, n_{p-1})\, \varpi(x_i, \gamma)],$$

where $C$ is a quantity which, beside the fixed constants of the surface, depends only on the place $\gamma$. Let us denote the expression

$$\frac{\varpi(x_i, n_1), \ldots\ldots, \varpi(x_i, n_{p-1})}{\sqrt{X(x_i)}},$$

which clearly has no zeros or poles, by $\mu(x_i)$; then we proceed to shew that in fact $C = A\mu(\gamma)$, where $A$ is a quantity depending only on the fixed constants of the surface, so that we shall have the formula

$$Ae^{-\pi i s' V}\Theta(V) = \frac{\begin{vmatrix} \Phi_1(x_1) & . & \Phi_p(x_1) \\ . & . & . \\ \Phi_1(x_p) & . & \Phi_p(x_p) \end{vmatrix} \mu(x_1), \ldots, \mu(x_p)\, \varpi(x_1, \gamma), \ldots, \varpi(x_p, \gamma)}{\prod\limits_{i<j}^{i,j=1,\ldots,p}\prod \varpi(x_i, x_j)\, \mu(\gamma)},$$

where

$$V = \sum_1^p v^{x_j, m_j} - v^{\gamma, m}.$$

In this formula $\gamma$ only occurs in the factors

$$\Psi = \frac{\varpi(x_1, \gamma), \ldots\ldots, \varpi(x_p, \gamma)}{\mu(\gamma)};$$

herein the factor $\sqrt{X(\gamma)}$ occurs once in the denominator of each of $\varpi(x_i, \gamma)$, and $p$ times as a denominator in $\mu(\gamma)$; thus this factor does not occur at all. In determining the factors of $\Psi$, as a function of $\gamma$, it will therefore be sufficient to omit this factor. Thus the factor of $\Psi$ at the $i$-th period loop of the first kind is $e^{\pi i s'(p-\overline{p-1})}$ or $e^{\pi i s'}$. At the $i$-th period loop of the second kind the factor of $\Theta(v^{x,z} + \frac{1}{2}\Omega_{s,s'})\, e^{\pi i s' v^{x,z}}$ is $e^{-2\pi i (v_i^{x,z} + \frac{1}{2}\tau_{i,i}) - \pi i s_i}$, and therefore the factor of $\Psi$ is

$$e^{-\pi i s_i - 2\pi i (v^{\gamma, x_p} + v^{n_1, x_1} + \ldots\ldots + v^{n_{p-1}, x_{p-1}} + \frac{1}{2}\tau_{i,i})}.$$

Consider now the expression

$$e^{-\pi i s' V}\Theta(V) = e^{\pi i s'(v^{\gamma, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p})}\, \Theta(v^{\gamma, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p});$$

at the $i$-th period loop of the first kind, this function, regarded as depending upon $\gamma$, has the factor $e^{\pi i s'}$; at the $i$-th period loop of the second kind it has the factor

$$e^{\pi i (\tau_{i,1}s'_1 + \ldots\ldots + \tau_{i,p}s'_p) - 2\pi i (v^{\gamma, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p} + \frac{1}{2}\tau_{i,i})};$$

but since

$$\pi i (s_i + \tau_{i,1} s'_1 + \ldots\ldots + \tau_{i,p} s'_p) = 2\pi i (v^{m_p, m} - v^{n_1, m_1} - \ldots\ldots - v^{n_{p-1}, m_{p-1}}),$$

it follows that

$$\pi i (\tau_{i,1} s'_1 + \ldots\ldots + \tau_{i,p} s'_p) - 2\pi i (v^{\gamma, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p})$$

is equal to

$$- \pi i s_i - 2\pi i (v^{\gamma, x_p} + v^{n_1, x_1} + \ldots\ldots + v^{n_{p-1}, x_{p-1}});$$

thus, changing $\gamma$ into $x$, we have proved that the function of $x$

$$e^{\pi i s' (v^{x, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p})} \Theta (v^{x, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_p, m_p})$$

has the same factors at the period loop as the function, of $x$, given by

$$\varpi (x, x_1) \ldots\ldots \varpi (x, x_p)/\mu (x);$$

it is clear that these functions have the same zeros, and no poles.

Hence the formula set down is completely established*.

277. We pass now to the particular case of the formula of § 272 which arises when the fundamental Riemann surface is hyperelliptic, and associated with the equation

$$y^2 = 4 (x^{2p+2} + \ldots\ldots).$$

Then the places $n_1, \ldots, n_{p-1}$ are branch places. We suppose also that $\mu + 1$ of the places $a_1, \ldots, a_r$ are branch places, say the place for which $x = d_1, \ldots, d_{\mu+1}$, and that $\mu + 1$ of the places $x_1, \ldots, x_r$ are branch places, say those at which $x = b_1, \ldots, b_{\mu+1}$. It is assumed that the branch places $n_1, \ldots, n_{p-1}, d_1, \ldots, d_{\mu+1}, b_1, \ldots, b_{\mu+1}$ are different from one another. We put $r - (\mu + 1) = \nu$; then the determinant of the functions $F_i(x_j)$, (§ 272), regarded as a function of $x_1$, is a rational function with poles in $n_1, \ldots, n_{p-1}, \alpha_1, \ldots, \alpha_\nu, d_1, \ldots, d_{\mu+1}$ and zero in $x_2, \ldots, x_\nu, b_1, \ldots, b_{\mu+1}$. Provided $\nu$ is not less than $\mu$, such a function is of the form

$$\frac{(x_1-n_1)\ldots(x_1-n_{p-1})(x_1-d_1)\ldots(x_1-d_{\mu+1})(x_1-b_1)\ldots(x_1-b_{\mu+1})(x_1, 1)_{\nu-1-\mu} + y_1(x_1, 1)_{\nu-1+\mu}}{(x_1-n_1)\ldots(x_1-n_{p-1})(x_1-d_1)\ldots(x_1-d_{\mu+1})(x_1-\alpha_1)\ldots(x_1-\alpha_\nu)}$$

where the degrees of $(x_1, 1)_{\nu-1-\mu}$, $(x_1, 1)_{\nu-1+\mu}$ are determined by the condition that the function is not to become infinite when $x_1$ is infinite. When $\nu = \mu$, the terms $(x_1, 1)_{\nu-1-\mu}$ are to be absent. When $\nu < \mu$, the conditions assigned do not determine the function; we shall suppose $\nu \geqq \mu$. The $2\nu - 1$ ratios of the coefficients in the numerator are to be determined by the conditions that the numerator vanishes in $x_2, \ldots, x_\nu$ and in the places conjugate†

* See the references given in the note *, § 274, and in particular Klein, *Math. Annal.* XXXVI. p. 39.

† The place conjugate to $(x, y)$ is $(x, -y)$

to $\alpha_1, \ldots, \alpha_\nu$. Hence, save for a factor independent of $x_1$, the determinant of the functions $F_i(x_j)$ is given by

$$\frac{\sqrt{\psi(x_1)}}{(x_1-n_1)\ldots(x_1-d_1)\ldots(x_1-\alpha_1)\ldots}$$

$$\begin{vmatrix} x_1^{\nu-1-\mu}\sqrt{\psi(x_1)}, & \ldots, & \sqrt{\psi(x_1)}, & x_1^{\nu-1+\mu}\sqrt{\phi(x_1)}, & \ldots, & \sqrt{\phi(x_1)} \\ x_2^{\nu-1-\mu}\sqrt{\psi(x_2)}, & \ldots, & \sqrt{\psi(x_2)}, & x_2^{\nu-1+\mu}\sqrt{\phi(x_2)}, & \ldots, & \sqrt{\phi(x_2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_1^{\nu-1-\mu}\sqrt{\psi(\alpha_1)}, & \ldots, & -\sqrt{\psi(\alpha_1)}, & \alpha_1^{\nu-1+\mu}\sqrt{\phi(\alpha_1)}, & \ldots, & \sqrt{\phi(\alpha_1)} \\ -\alpha_2^{\nu-1-\mu}\sqrt{\psi(\alpha_2)}, & \ldots, & -\sqrt{\psi(\alpha_2)}, & \alpha_2^{\nu-1+\mu}\sqrt{\phi(\alpha_2)}, & \ldots, & \sqrt{\phi(\alpha_2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

wherein $\psi(x) = (x-n_1)\ldots(x-n_{p-1})(x-d_1)\ldots(x-d_{\mu+1})(x-b_1)\ldots(x-b_{\mu+1})$, $\phi(x) = y^2/\psi(x)$, and the determinant has $2\nu$ rows and columns; denoting this determinant by $D_{\phi,\psi}$, the determinant of the functions $F_i(x_j)$ (§ 272) is therefore equal to

$$D_{\phi,\psi}\prod_{i=1}^{\nu}\frac{1}{(x_i-\alpha_1)\ldots(x_i-\alpha_\nu)\sqrt{(x_i-n_1)\ldots(x_i-n_{p-1})}}\sqrt{\frac{(x_i-b_1)\ldots(x_i-b_{\mu+1})}{(x_i-d_1)\ldots(x_i-d_{\mu+1})}}.$$

Hence, from § 272, taking $\psi(x) = \sqrt{(x-n_1)\ldots(x-n_{p-1})}$, so that $\varpi(x, z)$ will denote

$$\frac{\Theta(v^{x,z}+\tfrac{1}{2}\Omega_{s,s'})\,e^{\pi i s' v^{x,z}}}{\sqrt{(x-n_1)\ldots(x-n_{p-1})(z-n_1)\ldots(z-n_{p-1})}},$$

we have

$$C\Theta\left[\sum_1^{\nu} v^{x_i,\alpha_i} + \sum_1^{\mu+1} v^{b_i,d_i};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s'\right]$$

$$=\prod_{i=1}^{\nu}\prod_{j=1}^{\mu+1}\frac{\varpi(x_i,d_j)}{\varpi(x_i,b_j)}\sqrt{\frac{x_i-b_j}{x_i-d_j}}\;\frac{D_{\phi,\psi}\prod\limits_{i=1}^{\nu}\prod\limits_{j=1}^{\nu}\varpi(x_i,\alpha_j)}{\prod\limits_{\substack{i,j=1,\ldots,\nu\\ i<j}}\varpi(x_i,x_j)\prod\limits_{i=1}^{\nu}\prod\limits_{j=1}^{\nu}(x_i-\alpha_j)},$$

where $C$ is independent of $x_1, \ldots, x_\nu$.

Now, if $b$, $d$ be any two branch places, and $a$ an assigned branch place,

$$\frac{\varpi(x,d)}{\varpi(x,b)} = \frac{\Theta(v^{x,d};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s')}{\Theta(v^{x,b};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s')} = \frac{\Theta(v^{x,a}-v^{d,a};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s')}{\Theta(v^{x,a}-v^{b,a};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s')},$$

and hence, if

$$v_i^{d,a} = \tfrac{1}{2}(\delta_i + \delta_1'\tau_{i,1} + \ldots\ldots + \delta_p'\tau_{i,p}), \qquad (i = 1, 2, \ldots, p),$$

$$v_i^{b,a} = \tfrac{1}{2}(\beta_i + \beta_1'\tau_{i,1} + \ldots\ldots + \beta_p'\tau_{i,p}),$$

where $\beta_1, \ldots, \beta_p', \delta_1, \ldots, \delta_p'$ are integers, we have (§ 175, Chap. X.)

$$\frac{\varpi(x,d)}{\varpi(x,b)} = Ae^{\pi i(\delta'-\beta')v^{x,a}}\frac{\Theta[v^{x,a};\ \tfrac{1}{2}(s-\delta),\ \tfrac{1}{2}(s'-\delta')]}{\Theta[v^{x,a};\ \tfrac{1}{2}(s-\beta),\ \tfrac{1}{2}(s'-\beta')]},$$

where $A$ is independent of $x$. Thus the expression

$$e^{-\pi i(\delta'-\beta')v^{x,a}}\frac{\varpi(x,d)}{\varpi(x,b)}\sqrt{\frac{x-b}{x-d}},$$

which clearly has no poles or zeros, is such that its factors at the period loops are all $\pm 1$. The square of this function is therefore a constant, and the expression itself is a constant.

Therefore if

$$\sum_1^{\mu+1} v_i^{d_i, b_i} = \tfrac{1}{2}(\sigma_i + \sigma_1'\tau_{i,1} + \dots\dots + \sigma_p'\tau_{i,p}),$$

where $\sigma_1, \dots, \sigma_p'$ are integers, it follows that the function

$$e^{-\pi i\sigma'(v^{x_1, a_1}+\dots\dots+v^{x_\nu, a_\nu})}\prod_{i=1}^{\nu}\prod_{j=1}^{\mu+1}\frac{\varpi(x_i, d_j)}{\varpi(x_i, b_j)}\sqrt{\frac{x_i-b_j}{x_i-d_j}}$$

is independent of $x_1, \dots, x_\nu$. Further

$$\Theta(u-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau\sigma';\ \tfrac{1}{2}s, \tfrac{1}{2}s') = Be^{\pi i\sigma' u}\,\Theta[u;\ \tfrac{1}{2}(s-\sigma), \tfrac{1}{2}(s'-\sigma')]$$

by § 175, Chap. X. Thus on the whole we have

$$C\Theta\left[\sum_1^{\nu} v^{x_i, a_i};\ \tfrac{1}{2}(s-\sigma), \tfrac{1}{2}(s'-\sigma')\right]$$

$$= D_{\phi,\psi}\prod_{i=1}^{\nu}\prod_{j=1}^{\nu}\varpi(x_i, \alpha_j)\Big/\prod_{i<j}^{i,j=1,\dots,\nu}\varpi(x_i, x_j)\prod_{i=1}^{\nu}\prod_{j=1}^{\nu}(x_i-\alpha_j)\prod_{i<j}^{i,j=1,\dots,\nu}\varpi(\alpha_i, \alpha_j),$$

where $C$ is independent of $x_1, \dots, x_\nu$. Hence we can infer that $C$ is in fact independent also of $\alpha_1, \dots, \alpha_\nu$. For when the sets $x_1, \dots, x_\nu, \alpha_1, \dots, \alpha_\nu$ are interchanged, $D_{\phi,\psi}$ is multiplied by $(-)^{\nu^2+\nu-\mu} = (-1)^\mu$, and, since $\varpi(x, z) = -\varpi(z, x)$, this is also the factor by which the whole right-hand side is multiplied. The theta function on the left-hand side is also multiplied by $\pm 1$. Thus the square of the ratio of the right-hand side to the theta function on the left is unaltered by the interchange of the set $x_1, \dots, x_\nu$ with the set $\alpha_1, \dots, \alpha_\nu$. Thus $C^2$ is independent of $x_1, \dots, x_\nu$ and unaltered when $x_1, \dots, x_\nu$ are changed into $\alpha_1, \dots, \alpha_\nu$. Hence $C$ is an absolute constant.

It follows that the characteristic $\tfrac{1}{2}(s-\sigma)$, $\tfrac{1}{2}(s'-\sigma')$, and the theta functions, are even or odd according as $\mu$ is even or odd.

In the notation of § 200, Chap. XI., the half-periods $\tfrac{1}{2}\Omega_{s,s'}$ are given by

$$\tfrac{1}{2}\Omega_{s,s'} = v^{a_p, a} - v^{n_1, a_1} - \dots\dots - v^{n_{p-1}, a_{p-1}};$$

hence, if the half-periods given by

$$v^{a_1, a} + \dots\dots + v^{a_p, a}$$

be denoted by $\tfrac{1}{2}\Omega$, the half-periods associated with the characteristic $\tfrac{1}{2}(s-\sigma)$, $\tfrac{1}{2}(s'-\sigma')$ are congruent to expressions given by

$$\tfrac{1}{2}\Omega + v^{n_1, a} + \dots\dots + v^{n_{p-1}, a} + v^{b_1, a} + \dots\dots + v^{b_{\mu+1}, a} + v^{d_1, a} + \dots\dots + v^{d_{\mu+1}, a},$$

while $\psi$, which is of degree $p+1+2\mu$, is equal to

$$(x-n_1)\dots(x-n_{p-1})(x-b_1)\dots(x-b_{\mu+1})(x-d_1)\dots(x-d_{\mu+1});$$

by means of the formula (§ 201, Chap. XI.)

$$v^{a_1, a} + \dots\dots + v^{a_p, a} + v^{c_1, a} + \dots\dots + v^{c_p, a} + v^{c, a} \equiv 0,$$

the half-periods associated with the characteristic $\frac{1}{2}(s-\sigma)$, $\frac{1}{2}(s'-\sigma')$ can be reduced to be congruent to expressions denoted by

$$\tfrac{1}{2}\Omega + v^{e_1, a} + \dots\dots + v^{e_{p-2\mu}, a} + v^{e_{p+1-2\mu}, a},$$

where $e_1, \dots, e_{p-2\mu+1}$ are given by

$$\phi = 4(x-e_1)\dots\dots(x-e_{p+1-2\mu});$$

also, in taking all possible odd half-periods $\frac{1}{2}\Omega_{s, s'}$, all possible sets of $p-1$ of the branch places will arise for the set $n_1, \dots, n_{p-1}$. Hence it follows that the formula obtained includes as many results as there are ways of resolving $(x, 1)_{2p+2}$ into two factors $\phi_{p+1-2\mu}$, $\psi_{p+1+2\mu}$, of orders $p+1-2\mu$, $p+1+2\mu$, and (§ 201) that all possible half-integer characteristics arise, each associated with such a resolution. We have in fact, corresponding to $\mu = 0, 1, 2, \dots$, $E\left(\frac{p+1}{2}\right)$, a number of resolutions given by

$$\tfrac{1}{2}\binom{2p+2}{p+1} + \binom{2p+2}{p+3} + \binom{2p+2}{p+5} + \dots\dots, = 2^{2p}.$$

It has been shewn (§ 273) that the expression $\varpi(x, z)$ may be derived, by proceeding to a limit, from the integral $\Pi_{a, c}^{x, z}$. Hence the formula that has been obtained furnishes a definition of the theta function in terms of the algebraic functions and their integrals, and has been considered from this point of view by Klein, to whom it is due. After the investigation given above it is sufficient to refer* the reader, for further development, to Klein, *Math. Annal.* XXXII. (1888), p. 351, and to the papers there quoted.

*Ex.* i. Prove that the function $\Theta[u; \frac{1}{2}(s-\sigma), \frac{1}{2}(s'-\sigma')]$ vanishes to the $\mu$th order for zero values of the arguments.

*Ex.* ii. In the notation of § 200, Chap. XI., prove, from the result here obtained, that each of the sums

$$\sum_{i=1}^{4r+3} v^{c_i, a}, \quad \sum_{i=1}^{4r+2} v^{c_i, a}, \quad v^{a_j, a} + \sum_{i=1}^{4r+4} v^{c_i, a}, \quad v^{a_j, a} + \sum_{i=1}^{4r+3} v^{c_i, a}$$

represents an odd half-period; here $c_i$ is any one of the places $c, c_1, \dots, c_p$, $a_i$ is any one of the places $a_1, \dots, a_p$, $a_j$ is any one of the places $a_1, \dots, a_p$, and $r$ is an arbitrary integer

* See also Brill, *Crelle*, LXV. (1866), p. 273; and the paper of Bolza, *American Journal*, vol. XVII., referred to § 221, *note*, where Klein's formula is fundamental.

By means of the rule investigated on page 298, of the present volume, the characteristic $\frac{1}{2}(s-\sigma)$, $\frac{1}{2}(s'-\sigma')$ can be immediately calculated from the formula here (p. 436) given for it. Cf., also, Burkhardt, *Math. Annal.* XXXII., p. 426; Thompson, *American Journal*, XV. (1893), p. 91.

whose least value is zero, and whose greatest value is given by the condition that $i$ cannot be greater than $p+1$. Prove also that each of the sums

$$\sum_{i=1}^{4r+1} v^{c_i, a}, \quad \sum_{i=1}^{4r} v^{c_i, a}, \quad v^{a_j, a} + \sum_{i=1}^{4r+2} v^{c_i, a}, \quad v^{a_j, a} + \sum_{i=1}^{4r+1} v^{c_i, a}$$

represents an even half-period. For a more general result cf. the examples of § 303 (Chap. XVII.).

*Ex.* iii. By taking $\nu=p+1$, $\mu=0$, and the places $b$, $d$ so that $\frac{1}{2}\Omega_{s, s'} \equiv v^{b, d}$, finally putting $n_1, \ldots, n_{p-1}, b, d$ for $a_1, \ldots, a_p, a_{p+1}$, obtain, from the formula, the result

$$\frac{\Theta(v^{x, a} + v^{x_1, a_1} + \ldots\ldots + v^{x_p, a_p})}{\Theta(v^{z, a} + v^{x_1, a_1} + \ldots\ldots + v^{x_p, a_p})} = \frac{\sqrt{\psi(x)}}{\sqrt{\psi(z)}} \frac{\varpi(x, a)}{\varpi(z, a)} \frac{z-a}{x-a} \prod_{i=1}^{p} \frac{(x-x_i)(z-a_i)}{(x-a_i)(z-x_i)} e^{-\Pi_{x_i, a_i}^{x, z}},$$

where $\Pi_{x_i, a_i}^{x, z}$ replaces $\log \dfrac{\varpi(x, x_i)\, \varpi(z, a_i)}{\varpi(x, a_i)\, \varpi(z, x_i)}$, $\psi(x) = (x-a)\ldots(x-a_p)$, and the branch places $a, a_1, \ldots, a_p$ are, as in § 203, Chap. XI., such that the theta function in the numerator of the left-hand side vanishes as a function of $x$ at the places $\xi_1, \ldots, \xi_p$, conjugate to $x_1, \ldots, x_p$; and verify the result *à priori*. By the substitution

$$\frac{(x-x_i)(z-a_i)}{(x-a_i)(z-x_i)} e^{-\Pi_{x_i, a_i}^{x, z}} = e^{\Pi_{\xi_i, a_i}^{x, z}}$$

this formula can be further simplified. Deduce the results

$$\Pi_{x_1, z_1}^{x, z} + \ldots\ldots + \Pi_{x_p, z_p}^{x, z} = \log \frac{\Theta(v^{x, a} - v^{x_1, a_1} - \ldots\ldots - v^{x_p, a_p})\, \Theta(v^{z, a} - v^{z_1, a_1} - \ldots\ldots - v^{z_p, a_p})}{\Theta(v^{z, a} - v^{x_1, a_1} - \ldots\ldots - v^{z_p, a_p})\, \Theta(v^{x, a} - v^{z_1, a_1} - \ldots\ldots - v^{z_p, a_p})},$$

$$Z_i(u - v^{x, a}) - Z_i(u - v^{z, a}) = \Gamma_{x_1}^{x, z} \frac{\partial x_1}{\partial u_i} \Big/ \frac{dx_1}{dt} + \ldots\ldots + \Gamma_{x_p}^{x, z} \frac{\partial x_p}{\partial u_i} \Big/ \frac{dx_p}{dt},$$

where $u = v^{x_1, a_1} + \ldots\ldots + v^{x_p, a_p}$, $Z_i(u) = \dfrac{\partial}{\partial u_i} \log \Theta(u)$, and $\dfrac{dx_1}{dt}, \ldots$ are as in § 123, Chap. VII.

These results have already been given (Chap. X.).

278. It is immediately proved, by the formula (§ 187)

$$e^{\Pi_{z, a}^{x, \gamma}} = \frac{\Theta(v^{x, z} + \frac{1}{2}\Omega_{s, s'})\, \Theta(v^{\gamma, a} + \frac{1}{2}\Omega_{s, s'})}{\Theta(v^{x, a} + \frac{1}{2}\Omega_{s, s'})\, \Theta(v^{\gamma, z} + \frac{1}{2}\Omega_{s, s'})}$$

that the general expression of a factorial function given in § 254 can be written in the form

$$\prod_1^N \left[\Theta(v^{x, \beta_i} + \tfrac{1}{2}\Omega_{s, s'})\, e^{\pi i s' v^{x, \beta_i}}\right]$$

$$\div\, e^{2\pi i \sum_1^p (h_i + H_i) v_i^{x, \gamma}} \prod_1^M \left[\Theta(v^{x, a_i} + \tfrac{1}{2}\Omega_{s, s'})\, e^{\pi i s' v^{x, a_i}}\right] \prod_1^k \left[\Theta(v^{x, c_i} + \tfrac{1}{2}\Omega_{s, s'})\, e^{\pi i s' v^{x, c_i}}\right]^{\lambda_i}.$$

And, by the use of the expression $\varpi(x, z)$, this may be put into the form

$$e^{-2\pi i \sum_1^p (h_i + H_i) v_i^{x, \gamma}} \prod_1^N \varpi(x, \beta_i) \prod_1^M \left[\varpi(x, a_i)\right]^{-1} \prod_1^k \left[\varpi(x, c_i)\right]^{-\lambda_i}.$$

*Ex.* i. In the hyperelliptic case associated with an equation of the form

$$y^2=(x,\ 1)_{2p+2},$$

if $\bar{x}$ denote the place conjugate to the place $x$, it follows from the formula of § 273 that

$$\varpi(x,\ z)=(x-z)\,e^{\frac{1}{2}\Pi^{x,\ z}_{\bar{x},\ \bar{z}}},$$

unless $x$ or $z$ is a branch place.

*Ex.* ii. In the hyperelliptic case, if $k,\ k_1,\ \dots,\ k_p$ denote branch places, and

$$\phi(x)=(x-k)(x-k_1)\dots(x-k_p)$$

and the equation associated with the surface be $y^2=f(x)$, where $f(x)=\phi(x)\,\psi(x)$, and if we take places $x,\ x_1,\ \dots,\ x_p,\ z,\ z_1,\ \dots,\ z_p$, such that

$$v_i^{x_1,\ k_1}+\dots\dots+v_i^{x_p,\ k_p}\equiv v_i^{x,\ k},\quad v_i^{z_1,\ k_1}+\dots\dots+v_i^{z_p,\ k_p}\equiv v_i^{z,\ k},\qquad (i=1,\ 2,\ \dots,\ p),$$

then it is easily seen that the rational function having $\bar{x},\ x_1,\ \dots,\ x_p$ as zeros and $\bar{z},\ z_1,\ \dots,\ z_p$ as poles, can be put into the form $[y'\phi(x)+y\phi(x')]\div[y'\phi(z)+s\phi(x')]$, where $x',\ y'$ are the variables and $s$ is the value of $y'$ at the place $z$. Hence prove, by Abel's theorem, that

$$e^{\frac{1}{2}\Pi^{\bar{x},\ \bar{z}}_{x,\ z}}\ \frac{\sqrt{\phi(x)\,\psi(z)}+\sqrt{\phi(z)\,\psi(x)}}{2\sqrt[4]{f(x)f(z)}}=e^{-\frac{1}{2}\left(\Pi^{x,\ z}_{x_1,\ z_1}+\dots\dots+\Pi^{x,\ z}_{x_p,\ z_p}\right)}.$$

*Ex.* iii. Suppose now that $a,\ a_1,\ \dots,\ a_p$ are the branch places used in chapter XI. (§ 200), so that

$$e^{\Pi^{x,\ z}_{x_1,\ z_1}+\dots\dots+\Pi^{x,\ z}_{x_p,\ z_p}}=\frac{\Theta(v^{x,\ a}-v^{x_1,\ a_1}-\dots\dots-v^{x_p,\ a_p})\,\Theta(v^{z,\ a}-v^{z_1,\ a_1}-\dots\dots-v^{z_p,\ a_p})}{\Theta(v^{x,\ a}-v^{z_1,\ a_1}-\dots\dots-v^{z_p,\ a_p})\,\Theta(v^{z,\ a}-v^{x_1,\ a_1}-\dots\dots-v^{x_p,\ a_p})},$$

and suppose further that $\frac{1}{2}\Omega,=\frac{1}{2}(s+\tau s')$, is an *even* half-period such that

$$v^{k_1,\ a_1}+\dots\dots+v^{k_p,\ a_p}=v^{k,\ a}+\tfrac{1}{2}\Omega,\quad v^{x_1,\ a_1}+\dots\dots+v^{x_p,\ a_p}=v^{x,\ a}+\tfrac{1}{2}\Omega,$$

and
$$v^{z_1,\ a_1}+\dots\dots+v^{z_p,\ a_p}=v^{z,\ a}+\tfrac{1}{2}\Omega,$$

then deduce that

$$\frac{\Theta(v^{x,\ z}+\frac{1}{2}\Omega)\,e^{\pi i s' v^{x,\ z}}}{\Theta(\frac{1}{2}\Omega)}=\varpi(x,\ z)\,\frac{\sqrt{\phi(x)\,\psi(z)}+\sqrt{\phi(z)\,\psi(x)}}{2(x-z)\sqrt[4]{f(x)f(z)}}.$$

The results of examples i, ii, iii are given by Klein.

*Ex.* iv. Prove that, if $z,\ \zeta,\ c_1,\ \dots,\ c_p$ be arbitrary places, and $\gamma_1,\ \dots,\ \gamma_p$ be such that the places $\zeta,\ \gamma_1,\ \dots,\ \gamma_p$ are coresidual with the places $z,\ c_1,\ \dots,\ c_p$, then

$$\psi(x,\ \zeta\ ;\ z,\ c_1,\ \dots,\ c_p)=\frac{\varpi(x,\ \zeta)}{\varpi(x,\ z)\,\varpi(\zeta,\ z)}\,e^{\Pi^{x,\ z}_{\gamma_1,\ c_1}+\dots\dots+\Pi^{x,\ z}_{\gamma_p,\ c_p}};$$

hence deduce, by means of the result given in Ex. iv., page 174, that

$$\varpi(z,\ \zeta)=\frac{e^{\frac{1}{2}\left(\Pi^{z,\ \zeta}_{c_1,\ \gamma_1}+\dots\dots+\Pi^{z,\ \zeta}_{c_p,\ \gamma_p}\right)}}{\sqrt{D_\zeta\,\psi(\zeta,\ a\ ;\ z,\ c_1,\ \dots,\ c_p)}},$$

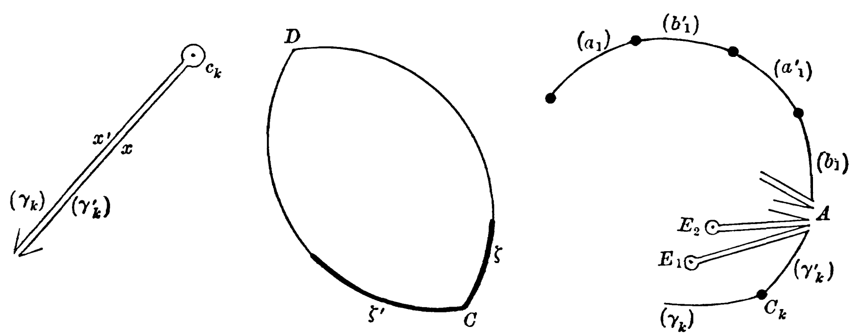
where $a$ is an arbitrary place.

279. The theory of the present chapter may be considered from another point of view. We have already seen, in chapter XII., that the theory of rational functions and their integrals may be derived with a fundamental surface consisting of a portion of a single plane bounded by circles, and the

change of independent variables involved justified itself by suggesting an important function, $\varpi(\zeta, \gamma)$. We explain now*, as briefly as possible, a more general case, in which the singular points, $c_1, \ldots, c_k$, of this chapter, are brought into evidence.

Suppose that a function $\zeta$ exists whereby the Riemann surface, dissected as in § 253, can be conformally represented upon the inside of a closed curvilinear polygon, in the plane of $\zeta$, whose sides are arcs of circles†; to the four sides, $(a_i)$, $(a_i')$, $(b_i)$, $(b_i')$, of a period-pair-loop are to correspond four sides of the polygon, to the two sides of a cut $(\gamma)$ are to correspond two sides of the polygon; the polygon will therefore have $2(2p+k)$ sides.

Fig. 11.




Then it is easily seen that if $C$ be the value of $\zeta$ at the angular point $C$ of the polygon, which corresponds to one of the singular points $c_1, \ldots, c_k$ on the Riemann surface, and $D$ be the value of $\zeta$ at the other intersection‡ of the circular arcs which contain the sides of the polygon meeting in $C$, we can pass from one of these sides to the other by a substitution of the form

$$\frac{\zeta' - C}{\zeta' - D} = e^{\frac{2\pi i}{l}} \frac{\zeta - C}{\zeta - D},$$

where $2\pi/l$ is the angle $C$ of the polygon, ($l$ being supposed an integer other than zero); as we pass from a point $\zeta$ of one of these sides to the corresponding point of the other side, the argument of the function $[(\zeta - C)/(\zeta - D)]^l$ increases by $2\pi$; if therefore $t$ be the infinitesimal at the corresponding singular point on the Riemann surface, we may write, for small values of $t$, $(\zeta - C)/(\zeta - D) = t^{\frac{1}{l}}$, so that $\zeta - C = t^{\frac{1}{l}}(C - D)(1 - t^{\frac{1}{l}})^{-1}$. Further if $\zeta$, $\zeta'$ be corresponding points

* Klein, *Math. Annal.* XXI. (1883), "Neue Beiträge zur Riemann'schen Functionentheorie"; Ritter, *Math. Annal.* XLI. (1893), p. 4; Ritter, *Math. Annal.* XLIV. (1894), p. 342.

† See Forsyth, *Theory of Functions*, chapter XXII., Poincaré, *Acta Math.* vols. I.—V. We may suppose that the polygon is such as gives rise to single-valued automorphic functions.

‡ Supposed to be outside the curvilinear polygon.

on the sides of the polygon which meet in $C$, we have for small values of $t$,

$$d\zeta = \frac{1}{l}(C-D)\,t^{\frac{1}{l}-1}\,dt,\ d\zeta' = \frac{1}{l}(C-D)\,t^{\frac{1}{l}-1}\,e^{2\pi i\left(\frac{1}{l}-1\right)}\,dt, \text{ or } d\zeta'/d\zeta = e^{2\pi i\left(\frac{1}{l}-1\right)}$$

ultimately, the factor omitted being a power series in $t^{\frac{1}{l}}$ or $(\zeta - C)/(\zeta - D)$, whose first term is unity.

We shall suppose now that the numbers $\lambda_1, \ldots, \lambda_k$ of this chapter are given by $\lambda_i = -m_i/l_i$, where $m_i$, $l_i$ are positive integers. Then a function whose behaviour near $c_i$ is that of an expression of the form $t^{-\lambda}\phi$, will, near $C_i$, behave like $(\zeta - C_i)^{m_i}\phi$, that is, will vanish a certain integral number of times. Further, for a purpose to be afterwards explained, we shall adjoin to the $k$ singular points $c_1, \ldots, c_k$, $m$ others, $e_1, \ldots, e_m$, for each of which the numbers $\lambda$ are the same and equal to $-\epsilon$, so that, if $t$ be the infinitesimal at any one of the places $e_1, \ldots, e_m$, the factorial functions considered behave like $t^{\epsilon}\phi$ at this place. These additional singular points, like the old, are supposed to be taken out from the surface by means of cuts $(\epsilon_1), \ldots, (\epsilon_m)$; and it is supposed that the corresponding curves in the curvilinear polygon of the $\zeta$-plane are also cuts passing to the interior of the polygon, as in the figure, so that at the point $E_1$ of the $\zeta$-plane which corresponds to the place $e_1$ of the Riemann surface, $\zeta$ is of the form $\zeta = E_1 + t\phi$, where $\phi$ is finite and not zero for small values of $t$, $t$ being the infinitesimal at $e_1$.

Factorial functions having these new singular points as well as the original singular points will be denoted by a bar placed over the top.

Let $dv$ denote the differential of an ordinary Riemann integral of the first kind which has $p-1$ zeros of the second order, at the places $n_1, \ldots, n_{p-1}$. Consider the function

$$Z_2 = \sqrt{\frac{dv}{d\zeta}}\,e^{-\frac{1}{2}\sum_{i=1}^{k}\left(1-\frac{1}{l_i}\right)\Pi^{x,a}_{c_i,c} - \sum_{1}^{p-1}\Pi^{x,a}_{n_i,c} + \frac{\rho}{2m}\sum_{1}^{m}\Pi^{x,a}_{e_i,c}};$$

where $a$, $c$ are arbitrary places, and $\rho$ is determined so that $Z_2$ is not infinite at the place $c$, or

$$\sum_{i=1}^{k}\left(1-\frac{1}{l_i}\right) + 2p - 2 = \rho;$$

this function is nowhere infinite on the Riemann surface; it vanishes to the first order only at $\zeta = \infty$; for each of the cuts $(\epsilon_1), \ldots, (\epsilon_m)$ it has a factor $e^{\frac{\pi i\rho}{m}}$; at a singular point $c_i$ it is expressible as a power series in $t^{\frac{1}{l}}$, or $(\zeta - C)/(\zeta - D)$, whose first term is unity. The values of $Z_2$ at the two sides of a period loop are such that $Z_2'/Z_2 = \sqrt{d\zeta/d\zeta'}$; but since these two sides correspond, on the $\zeta$-plane, to arcs of circles which can be transformed into one another by a substitution of the form $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$, wherein we suppose $\alpha\delta - \beta\gamma = 1$, it follows that $Z_2'/Z_2 = \gamma\zeta + \delta$. If then we also introduce the function $Z_1, = \zeta Z_2$, we have for the two sides of a period loop, equations of the form

$$Z_2' = \gamma Z_1 + \delta Z_2, \quad Z_1' = \alpha Z_1 + \beta Z_2.$$

Consider now a function

$$f = \overline{K}/Z_2^R,$$

where $\overline{K}$ is a factorial function with the $k+m$ singular points, and $R = 2m\epsilon/\rho$. At a singular point $c_i$, or $C_i$, its behaviour is that of a power series in $t^{\frac{1}{l}}$ or $(\zeta - C)/(\zeta - D)$, multiplied by $(\zeta - C_i)^{m_i}$; at a singular point $e_i$, or $E_i$, its behaviour is that of a power series in the infinitesimal $t$ multiplied by

$$t^\epsilon \Big/ \left(t^{\frac{\rho}{2m}}\right)^{\frac{2m\epsilon}{\rho}}$$

or unity; at a period loop it is multiplied by a factor of the form $\mu(\gamma\zeta + \delta)^{-R}$, where $\mu$ is the factor of $\overline{K}$. The function has therefore the properties of functions expressible by series of the form*

$$\Sigma R(\zeta_i)(\gamma_i \zeta + \delta_i)^R,$$

wherein the notation is, that $\zeta_i = (\alpha_i \zeta + \beta_i)/(\gamma_i \zeta + \delta_i)$ is one of the finite number of substitutions whereby the sides of the curvilinear polygon are related in pairs and $R(\zeta_i)$ is a rational function of $\zeta_i$. The equation connecting the values $f'$, $f$, of the function $f$, at the two sides of a period loop, may be put into the form

$$(\gamma Z_1 + \delta Z_2)^R f' = \mu Z_2^R f;$$

and *we may regard* $Z_2^R f$, *or* $\overline{K}$, *as a homogeneous form in the variables* $Z_1, Z_2$, *of dimension* $R$.

The difference between the number of zeros and poles of such a factorial function $\overline{K}$ is (§ 254)

$$\Sigma\lambda, = \Sigma\left(-\frac{m_i}{l_i}\right) - \epsilon m, = \Sigma\left(-\frac{m_i}{l_i}\right) - \tfrac{1}{2}\rho R, = \Sigma\left(-\frac{m_i}{l_i}\right) - R(p-1) - \tfrac{1}{2}R\Sigma\left(1 - \frac{1}{l_i}\right),$$
$$= \Sigma\left(-\frac{m_i}{l_i} + \frac{R}{2l_i}\right) - R(p-1) - \tfrac{1}{2}Rk;$$

adding the proper corrections for the zeros of the automorphic form $\overline{K}$ at the angular points $C_1, \ldots, C_k$ (Forsyth, *Theory of Functions*, p. 645) we have, for the excess of the number of zeros of the automorphic form over the number of poles

$$\Sigma\lambda + \Sigma\frac{m_i}{l_i} = -\frac{R}{2}\left[2p - 2 + k + m + 1 - \left(\Sigma\frac{1}{l_i} + m + 1\right)\right]$$
$$= -\frac{R}{2}\left[2p - 2 + q - \Sigma\frac{1}{\mu}\right],$$

where $q = k + m + 1$, $\Sigma\frac{1}{\mu} = \Sigma\frac{1}{l_i} + m + 1$.

We may identify this result with a known formula for automorphic

* Forsyth, *Theory of Functions*, p. 642. The quantity $R$ is, in Forsyth, taken equal to $-2m$.

functions $\Big[$Forsyth, *Theory of Functions*, p. 648; if in the formula $m\left(n-1-\Sigma\frac{1}{\mu}\right)$, there given, we substitute, by the formula of p. 608, § 293, $n=2N-1+q$, we obtain $m\left(2N-2+q-\Sigma\frac{1}{\mu}\right)\Big]$; for each of the angular points $C_1, \ldots, C_k$ is a cycle by itself, each of the points $E_1, \ldots, E_m$ is a cycle by itself, and the remaining angular points together constitute one cycle (cf. Forsyth, p. 596); the sum of the angles at the first $k$ cycles is $2\pi\Sigma\frac{1}{l_i}$, the sum of the angles at the second $m$ cycles is $2\pi m$, the sum of the angles at the other cycle is $2\pi$ *.

There is a way in which the adjoint system of singular points $e_1, \ldots, e_m$ may be eliminated from consideration. Imagine a continuously varying quantity, $x_2$, which is zero to the first order at $e_1, \ldots, e_m$ and is never infinite, and put $x_1 = xx_2$; the expression $\overline{K}x_2^{-\epsilon}$ may then be regarded as a homogeneous form in $x_1$, $x_2$ on the Riemann surface, without singular points at $e_1, \ldots, e_m$; and instead of the function $Z_2$ we may introduce the form $\zeta_2 = Z_2 x_2^{-\frac{\rho}{2m}}$, which is then without factor for the cuts $(\epsilon_1), \ldots, (\epsilon_m)$, or, as we may say, is *unbranched* at the places $e_1, \ldots, e_m$; and may also put $\zeta_1 = \zeta\zeta_2$.

Thus, (i), a factorial function, considered on the $\zeta$-plane, is a homogeneous automorphic form, (ii), introducing homogeneous variables on the Riemann surface, the consideration of factorial functions may be replaced by the consideration of homogeneous factorial forms.

*Ex.* Shew that the form

$$P(x, z) = x_2^{\frac{1}{m}} f(z)\, e^{\Pi^{x,a}_{z,c} - \frac{1}{m}(\Pi^{x,a}_{e_1,c} + \ldots\ldots + \Pi^{x,a}_{e_m,c}) + \sum_{i,j} \lambda_{i,j} v_i^{x,a} v_j^{z,c}},$$

where $a$, $c$ are arbitrary places and $\lambda_{i,j}$ are constants, is unbranched at $e_1, \ldots, e_m$, that it has no poles, and vanishes only at the place $z$. Here $f(z)$ is to be chosen so that, when $x$ approaches $z$, the ratio of $P(x, z)$ to the infinitesimal at $z$ is unity. At the $t$-th period loop of the second kind the function has a factor $(-)^M$ where

$$M = 2\pi i r + \frac{2\pi i}{m}(q'_2 - q) - \frac{2\pi i}{m}(v_t^{e_1,c} + \ldots\ldots + v_t^{e_m,c}) + \sum_{i,j} \lambda_{i,j} v_j^{z,c} \tau_{i,t},$$

$q'_2 - q$ denoting the number of circuits, made in passing from one side of the period loop to the other, of $x_2$ about $x_2 = 0$ other than those for which $x$ encloses places $e_1, \ldots, e_m$, and $r$ denoting the number of circuits † of $x$ about $z$.

* The formula is given by Ritter, *Math. Annal.* XLIV. p. 360 (at the top), the quantity there denoted by $q$ being here $-\frac{1}{2}\rho$. We do not enter into the conditions that the automorphic form be single-valued.

† The reader will compare the formula given by Ritter, *Math. Annal.* XLIV. p. 291. It may be desirable to call attention to the fact that the notation $\sigma+1$, $\sigma'+1$, as here used, does not coincide with that used by Ritter. The quantities denoted by him by $\sigma$, $\sigma'$ may, in a sense, be said to correspond respectively to those denoted here, for the factorial system including the singular points $e_1, \ldots, e_m$, by $\sigma'+1$ and $\varpi'$.