CHAPTER XII.

A PARTICULAR FORM OF FUNDAMENTAL SURFACE.

222. JACOBI's inversion theorem, and the resulting theta functions, with which we have been concerned in the three preceding chapters, may be regarded as introducing a method for the change of the independent variables upon which the fundamental algebraic equation, and the functions associated therewith, depend. The theta functions, once obtained, may be considered independently of the fundamental algebraic equation, and as introductory to the general theory of multiply-periodic functions of several variables; the theory is resumed from this point of view in chapter XV., and the reader who wishes may pass at once to that chapter. But there are several further matters of which it is proper to give some account here. The present chapter deals with a particular case of a theory which is historically a development* of the theory of this volume; it is shewn that on a surface which is in many ways simpler than a Riemann surface, functions can be constructed entirely analogous to the functions existing on a Riemann surface. The suggestion is that there exists a conformal representation of a Riemann surface upon such a surface as that here considered, which would then furnish an effective change of the independent variables of the Riemann surface. We do not however at present undertake the justification of that suggestion, nor do we assume any familiarity with the general theory referred to. The present particular case has the historical interest that in it a function has arisen, which we may call the Schottky-Klein prime function, which is of great importance for any Riemann surface.

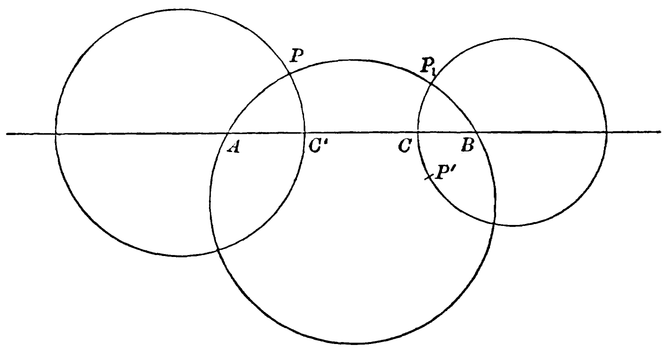
223. Let α, β, γ, δ be any quantities whatever, whereof three are definitely assigned, and the fourth thence determined by the relation $\alpha\delta-\beta\gamma=1$. Let ζ, ζ' be two corresponding complex variables associated together by the relation $\zeta'=(\alpha\zeta+\beta)/(\gamma\zeta+\delta)$. This relation can be put into the form

$$\frac{\zeta'-B}{\zeta'-A}=\mu e^{i\kappa}\frac{\zeta-B}{\zeta-A},$$

* Referred to by Riemann himself, *Ges. Werke* (Leipzig, 1876), p. 413.

wherein μ is real, and B, A are the roots of the quadratic equation $\zeta = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$, distinguished from one another by the condition that μ shall be less than unity. In all the linear substitutions which occur in this chapter it is assumed that B, A are not equal, and that μ is not equal to unity. We introduce now the ordinary representation of complex quantities by the points of a plane. Let the points A, B be marked as in the figure (6),

Fig. 6.



and a point C' be taken between A, B in such a way that $1 > AC'/C'B > \mu$, but otherwise arbitrarily; then the locus of a point P such that $AP/PB = AC'/C'B$ is a circle. Take now a point C also between A and B, such that $CB/AC = \mu C'B/AC'$, and mark the circle which is the locus of a point P' for which $P'B/AP' = CB/AC$; since $P'B/AP'$ is less than unity, this circle will lie entirely without the other circle. If now any circle through the points A, B cut the first circle, which we shall call the circle C', in the points P, Q, and cut the second circle, C, in P_1 and Q_1, P and P_1 being on the same side of AB, we have angle $AP_1B =$ angle APB, and $P_1B/AP_1 = \mu PB/AP$; therefore, if the point P be ζ, and the point P_1 be ζ_1, we have

$$\frac{\zeta_1 - B}{\zeta_1 - A} = \mu \frac{\zeta - B}{\zeta - A},$$

the argument of P vanishing when P is at the end of the diameter of the C' circle remote from C', and varying from 0 to 2π as P describes the circle C' in a clockwise direction; if then we pass along the circle C in a counter clockwise direction to a point P' such that the sum of the necessary positive rotation of the line BP_1 about B into the position BP', and the necessary negative rotation of the line AP_1 about A into the position AP', is κ, and ζ' be the point P', we have

$$\frac{\zeta' - B}{\zeta' - A} = e^{i\kappa} \frac{\zeta_1 - B}{\zeta_1 - A} = \mu e^{i\kappa} \frac{\zeta - B}{\zeta - A}.$$

Thus the transformation under consideration transforms any point ζ on the circle C' into a point on the circle C. If ζ denote any point within C'

the modulus of $(\zeta - B)/(\zeta - A)$ is greater than when ζ is on the circumference of C', and the transformed point ζ' is without the circle C, though not necessarily without the circle C'. If ζ denote any point without C' the transformed point is within the circle C.

224. Suppose * now we have given p such transformations as have been described, depending therefore on $3p$ given complex quantities, whereof 3 can be given arbitrary values by a suitable transformation $z' = (Pz + Q)/(Rz + S)$ applied to the whole plane; denote the general one by

$$\zeta' = \frac{\alpha_i \zeta + \beta_i}{\gamma_i \zeta + \delta_i}, \text{ wherein } \alpha_i \delta_i - \beta_i \gamma_i = 1, \qquad (i = 1, 2, \ldots, p),$$

or also by

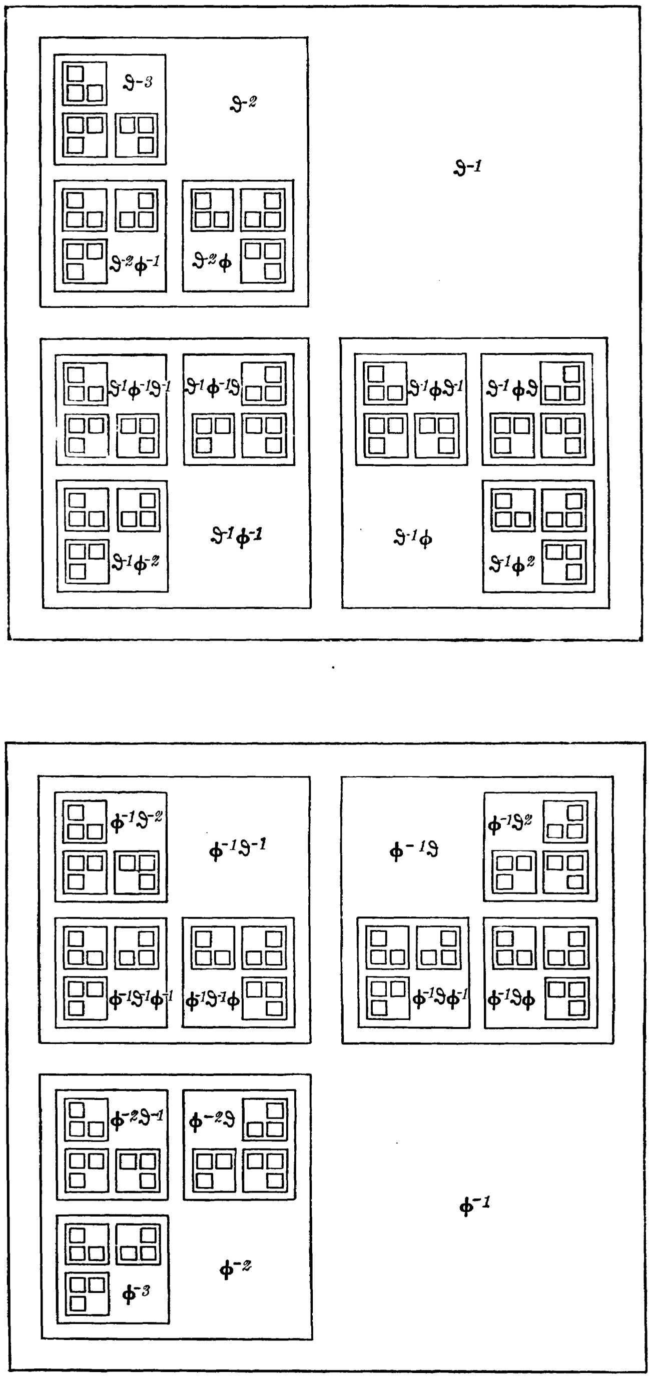
$$\zeta' = \vartheta_i \zeta, \ \zeta = \vartheta_i^{-1} \zeta',$$

the quantities corresponding to A, B, μ, α being denoted by A_i, B_i, μ_i, α_i; construct as here a pair of circles corresponding to each substitution, and *assume that the constants are such that, of the $2p$ circles obtained, each is exterior to all the others*; let the region exterior to all the circles be denoted by S, and the region derivable therefrom by the substitution ϑ_i be denoted by $\vartheta_i S$.

If the whole plane exterior to the circle C_i be subjected to the transformation ϑ_i, the circle C_i' will be transformed into C_i, the circle C_i itself will be transformed into a circle interior to C_i, which we denote by $\vartheta_i C_i$, and the other $2p-2$ circles which lie in a space bounded by C_i and C_i' will be transformed into circles lying in the region bounded by $\vartheta_i C_i$ and C_i, and, corresponding to the region S, exterior to all the $2p$ circles, we shall have a region $\vartheta_i S$ also bounded by $2p$ circles. But suppose that before we thus transform the whole plane by the transformation ϑ_i, we had transformed the whole plane by another transformation ϑ_j and so obtained, within C_j, a region $\vartheta_j S$ bounded by $2p$ circles, of which C_j is one. Then, in the subsequent transformation, ϑ_i, all the $2p-1$ circles lying within C_j will be transformed, along with C_j, into $2p-1$ other circles lying in a region, $\vartheta_i \vartheta_j S$, bounded by the circle $\vartheta_i C_j$. They will therefore be transformed into circles lying *within* $\vartheta_i C_j$—they cannot lie without this circle, namely in $\vartheta_i S$, because $\vartheta_i S$ is the picture of a space, S, whose only boundaries are the $2p$ fundamental circles $C_1, C_1', \ldots, C_p, C_p'$. Proceeding in the manner thus indicated we shall obtain by induction the result enunciated in the following statement, wherein ϑ_i^{-1} is the inverse transformation to ϑ_i, and transforms the circle C_i into C_i': *Let all possible multiples of powers of $\vartheta_1, \vartheta_1^{-1}, \ldots, \vartheta_p, \vartheta_p^{-1}$ be formed, and the corresponding regions, obtained by applying to S the transformations*

* The subject-matter of this section is given by Schottky, *Crelle*, CI. (1887), p. 227, and by Burnside, *Proc. London Math. Soc.* XXIII. (1891), p. 49.

Fig. 7.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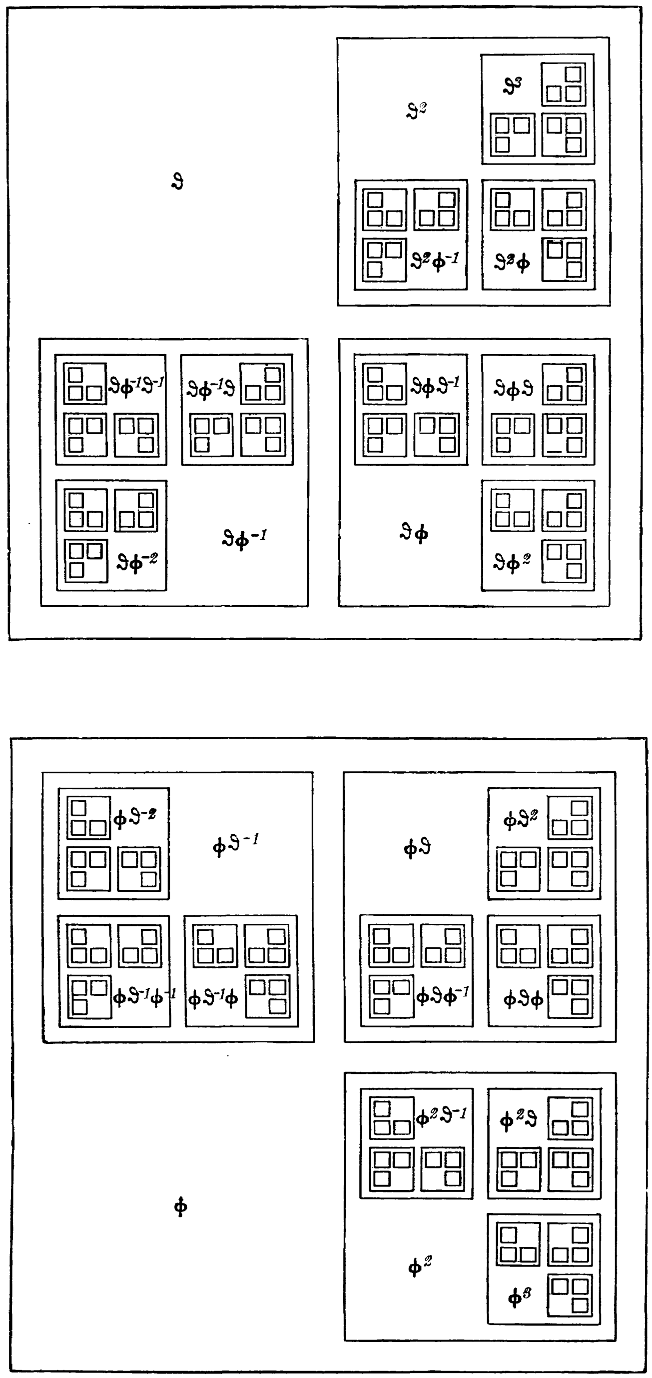
$\phi^{-1}\vartheta\phi^{-1}$
$\phi^{-1}\vartheta\phi$
$\phi^{-2}\vartheta^{-1}$
$\phi^{-2}\vartheta$
ϕ^{-1}
ϕ^{-2}
ϕ^{-3}

Fig. 7.


ϑ
ϑ^2
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$\phi\vartheta\phi^{-1}$
$\phi\vartheta\phi$
$\phi^2\vartheta^{-1}$
$\phi^2\vartheta$
ϕ
ϕ^2
ϕ^3

corresponding to all such products of powers, be marked out. In any such product the transformation first to be applied is that one which stands to the right. Let m be any one such product, of the form

$$m = \ldots\ldots \vartheta_i^{r_i} \vartheta_j^{r_j} \vartheta_k^{r_k},$$

formed by

$$\ldots\ldots + r_i + r_j + r_k, = h$$

factors, and let ϑ *be any transformation other than the inverse of* ϑ_k, *so that* $m\vartheta_k$ *is formed by the product of* $h+1$, *not* $h-1$, *factors. Then the region* mS *entirely surrounds the region* $m\vartheta S$.

Thus, the region $\vartheta_i S$ entirely surrounds the space $\vartheta_i\vartheta_j S$, and the latter surrounds $\vartheta_i\vartheta_j^2 S$, or $\vartheta_i\vartheta_j\vartheta_k S$; but $\vartheta_i S$ is surrounded by $\vartheta_i\vartheta_i^{-1} S$ or S. The reader may gain further clearness on this point by consulting the figure (7), wherein, for economy of space, rectangles are drawn in place of circles, and the case of only two fundamental substitutions, ϑ, ϕ, is taken.

The consequence of the previous result is—*The group of substitutions consisting of the products of positive and negative powers of* $\vartheta_1, \ldots, \vartheta_p$ *gives rise to a single covering of the whole plane, every point being as nearly reached as we desire, by taking a sufficient number of factors, and no point being reached by two substitutions.*

225. There are in fact certain points which are not reached as transformations of points of S, by taking the product of any *finite* number of substitutions. For instance the substitution ϑ_i^m is

$$\frac{\zeta' - B_i}{\zeta' - A_i} = \mu_i^m e^{im\kappa_i} \frac{\zeta - B_i}{\zeta - A_i},$$

and thus when m is increased indefinitely ζ' approaches indefinitely near to B_i, whatever be the position of ζ; but B_i is not reached for any finite value of m. In general the result of any infinite series of successive substitutions, $K = \alpha\beta\gamma\ldots$, applied to the region S, is, by what has been proved, a region lying within αS, in fact lying within $\alpha\beta S$, nay more, lying within $\alpha\beta\gamma S$, and so on—namely is a region which may be regarded as a point; denoting it by K, the substitution K transforms every point of the region S and in fact every other point of the plane into the same point K; and transforms the point K into itself. There will similarly be a point K' arising by the same infinite series of substitutions taken in the reverse order.

Such points are called the singular points of the group. There is an infinite number of them; but two of them for which the corresponding products of the symbols ϑ agree to a sufficient number of the left-hand factors are practically indistinguishable; none of them lie within regions that are obtained from S with a finite number of substitutions. The most important of these singular points are those for which the corresponding series of substitutions is periodic; of these the most obvious are those formed by indefinite repetition of one of the fundamental substitutions; we have already introduced the notation

$$\vartheta_i^{\infty} S = B_i,\ \vartheta_i^{-\infty} S = A_i,$$

to represent the results of such substitutions.

226. If ϑ, ϕ be any two substitutions given respectively by

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta},\quad \zeta' = \frac{A\zeta + B}{C\zeta + D},$$

wherein $\alpha\delta - \beta\gamma = 1 = AD - BC$, the compound substitution $\vartheta\phi$ is given by

$$\zeta' = \frac{\alpha\,(A\zeta + B) + \beta\,(C\zeta + D)}{\gamma\,(A\zeta + B) + \delta\,(C\zeta + D)} = \frac{(\alpha A + \beta C)\,\zeta + (\alpha B + \beta D)}{(\gamma A + \delta C)\,\zeta + (\gamma B + \delta D)},$$

and if this be represented by $\zeta' = (\alpha'\zeta + \beta')/(\gamma'\zeta + \delta')$, we have, in the ordinary notation of matrices

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and $\alpha'\delta' - \beta'\gamma' = (\alpha\delta - \beta\gamma)(AD - BC) = 1$. We suppose all possible substitutions arising by products of positive and negative powers of the fundamental substitutions $\vartheta_1, \ldots, \vartheta_p$ to be formed, and denote any general substitution by $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$, wherein, by the hypothesis in regard to the fundamental substitutions, $\alpha\delta - \beta\gamma = 1$. We may suppose all the substitutions thus arising to be arranged in order, there being first the identical substitution $\zeta' = (\zeta + 0)/(0\,.\,\zeta + 1)$, then the $2p$ substitutions whose products contain one factor, ϑ_i or ϑ_i^{-1}, then the $2p\,(2p-1)$ substitutions whose products are of one of the forms $\vartheta_i\vartheta_j$, $\vartheta_i\vartheta_j^{-1}$, $\vartheta_i^{-1}\vartheta_j$, $\vartheta_i^{-1}\vartheta_j^{-1}$, in which the two substitutions must not be inverse, containing two factors, then the $2p\,(2p-1)^2$ substitutions whose products contain three factors, and so on. So arranged consider the series

$$\Sigma\,(\text{mod}\,\gamma)^{-k},$$

wherein k is a real positive quantity, and the series extends to every substitution of the group except the identical substitution. Since the inverse substitution to $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$ is $\zeta = (\delta\zeta' - \beta)/(-\gamma\zeta' + \alpha)$, each set of $2p\,(2p-1)^{n-1}$ terms corresponding to products of n substitutions will contain each of its terms twice over.

Let now Θ_n denote a substitution formed by the product of n factors, and $\Theta_{n+1} = \Theta_n\vartheta_i$, where ϑ_i denotes any one of the primary $2p$ substitutions $\vartheta_1, \vartheta_1^{-1}, \ldots, \vartheta_p, \vartheta_p^{-1}$ other than the inverse of the substitution whose symbol stands at the right hand of the symbol Θ_n, so that Θ_{n+1} is formed with $n+1$

factors; then by the formula just set down $\gamma_{n+1} = \gamma_n \alpha_i + \delta_n \gamma_i$, where, if ϑ_i, or $\zeta' = (\alpha_i \zeta + \beta_i)/(\gamma_i \zeta + \delta_i)$, be put in the form $(\zeta' - B_i)/(\zeta' - A_i) = \rho_i (\zeta - B_i)/(\zeta - A_i)$, we have

$$\alpha_i, \qquad \beta_i, \qquad \gamma_i, \qquad \delta_i$$

respectively equal to

$$\frac{B_i \rho_i^{-\frac{1}{2}} - A_i \rho_i^{\frac{1}{2}}}{B_i - A_i}, \quad -\frac{A_i B_i (\rho_i^{-\frac{1}{2}} - \rho_i^{\frac{1}{2}})}{B_i - A_i}, \quad \frac{\rho_i^{-\frac{1}{2}} - \rho_i^{\frac{1}{2}}}{B_i - A_i}, \quad -\frac{A_i \rho_i^{-\frac{1}{2}} - B_i \rho_i^{\frac{1}{2}}}{B_i - A_i};$$

the signification of $\rho_i^{\frac{1}{2}}$ is not determined when the corresponding pair of circles is given; but we have supposed that the values of $\alpha_i, \beta_i, \gamma_i, \delta_i$ are given, and thereby the value of $\rho_i^{\frac{1}{2}}$. By these formulae we have

$$\frac{\gamma_{n+1}}{\gamma_n} = \rho_i^{-\frac{1}{2}} \frac{B_i + \delta_n/\gamma_n}{B_i - A_i} - \rho_i^{\frac{1}{2}} \frac{A_i + \delta_n/\gamma_n}{B_i - A_i}.$$

Herein the modulus of ρ_i may be either μ_i or μ_i^{-1}, according as ϑ_i is one of $\vartheta_1, \ldots, \vartheta_p$ or one of $\vartheta_1^{-1}, \ldots, \vartheta_p^{-1}$; the modulus of ρ_i is accordingly either less or greater than unity. If now $\Theta_n = \ldots \psi \phi \vartheta_r^{-1}$, where ϑ_r is one of the $2p$ fundamental substitutions $\vartheta_1, \ldots, \vartheta_p^{-1}$, and therefore $\Theta_n^{-1} = \vartheta_r \phi^{-1} \psi^{-1} \ldots$, the region $\Theta_n^{-1} S$ lies entirely within the region $\vartheta_r S$ (§ 224) or coincides with it; wherefore the point $\Theta_n^{-1}(\infty)$, or $-\delta_n/\gamma_n$, lies within the circle C_r when ϑ_r is one of $\vartheta_1, \ldots, \vartheta_p$, and lies within the circle C_r' when ϑ_r is one of $\vartheta_1^{-1}, \ldots, \vartheta_p^{-1}$; thus the points B_i and $-\delta_n/\gamma_n$ can only lie within the same one of the $2p$ fundamental circles $C_1, \ldots, C_p'$ when $r = i$ and ϑ_r is one of $\vartheta_1, \ldots, \vartheta_p$; and the points A_i and $-\delta_n/\gamma_n$ can only lie within the same one of the $2p$ fundamental circles $C_1, \ldots, C_p'$ when $r = i$ and ϑ_r is one of $\vartheta_1^{-1}, \ldots, \vartheta_p^{-1}$. Now, if the modulus of ρ_i be less than unity, and $r = i$, ϑ_r must be one of $\vartheta_1^{-1}, \ldots, \vartheta_p^{-1}$, namely must be ϑ_i^{-1}, since otherwise $\Theta_n \vartheta_i$ would consist of $n-1$ factors, and not $n+1$ factors; in that case therefore $B_i + \dfrac{\delta_n}{\gamma_n}$ is not of infinitely small modulus; if, however, the modulus of ρ_i be greater than unity, and $r = i$, ϑ_r must be ϑ_i, namely one of $\vartheta_1, \ldots, \vartheta_p$, and in that case the modulus of $A_i + \delta_n/\gamma_n$ is not infinitely small. Thus, according as $|\rho_i| \lessgtr 1$, we may put

$$| B_i + \delta_n/\gamma_n | > \lambda, \quad | A_i + \delta_n/\gamma_n | > \lambda,$$

where λ is a positive real quantity which is certainly not less than the distance of B_i, A_i, respectively, from the nearest point of the circle within which $-\delta_n/\gamma_n$ lies.

It follows from this that we have

$$\operatorname{mod}(\gamma_{n+1}/\gamma_n) > \sigma, \text{ or } \operatorname{mod}(\gamma_{n+1}^{-1}/\gamma_n^{-1}) < \frac{1}{\sigma},$$

where σ is a positive finite quantity, for which an arbitrary lower limit may be assigned independent of the substitutions of which Θ_n is compounded, and independent of n, *provided the moduli* $\mu_1, \ldots, \mu_p$ *be supposed sufficiently small, and the* p *pairs of circles be sufficiently distant from one another.*

Ex. Prove, in § 223, that if C' be chosen so that $C'C$ is as great as possible

$$\frac{1}{\sqrt{\mu}}\frac{C'C}{AB} = \frac{1-\sqrt{\mu}}{1+\sqrt{\mu}}\frac{1}{\sqrt{\mu}},$$

and the circles are both of radius $d\sqrt{\mu}/(1-\mu)$, where d is the length of AB.

We suppose the necessary conditions to be satisfied; then if γ_0 be the least of the p quantities $\operatorname{mod}[(\mu_i^{-\frac{1}{2}}e^{-\frac{1}{2}i\kappa_i} - \mu_i^{\frac{1}{2}}e^{\frac{1}{2}i\kappa_i})/(B_i - A_i)]$, and k be positive, the series $\Sigma \operatorname{mod} \gamma^{-k}$ is less than

$$\gamma_0^{-k}\left[2p + \frac{2p(2p-1)}{\sigma^k} + \frac{2p(2p-1)^2}{\sigma^{2k}} + \ldots\ldots\right],$$

and therefore certainly convergent if $\sigma^k > 2p-1$, which, as shewn above, may be supposed, $\mu_1, \ldots, \mu_p$ being sufficiently small.

227. Hence we can draw the following inference: Let $\sigma_1, \ldots, \sigma_p$ be assigned quantities, called multipliers, each of modulus unity, associated respectively with the p fundamental substitutions $\vartheta_1, \ldots, \vartheta_p$; with any compound substitution $\vartheta_1^{r_1}\vartheta_2^{r_2}\ldots$, let the compound quantity $\sigma_1^{r_1}\sigma_2^{r_2}\ldots$ be associated: let $f(x)$ denote any uniform function of x with only a finite number of separated infinities; let $\zeta' = (\alpha\zeta+\beta)/(\gamma\zeta+\delta)$ denote any substitution of the group, and σ be the multiplier associated with this substitution: then the series, extending to all the substitutions of the group,

$$\Sigma\sigma f\left(\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}\right)(\gamma\zeta+\delta)^{-k}$$

converges absolutely and uniformly * for all positions of ζ other than (i) the singular points of the group, and the points $\zeta = -\delta/\gamma$, namely the points derivable from $\zeta = \infty$ by the substitutions of the group, including the point $\zeta = \infty$ itself, (ii) the infinities of $f(\zeta)$ and the points thence derived by the substitutions of the group. The series represents therefore a well-defined continuous function of ζ for all the values of ζ other than the excepted ones. The function will have poles at the poles of $f(\zeta)$ and the points thence derived by the substitutions of the group; it may have essential singularities at the singular points of the group and at the essential singularities of

$$f((\alpha\zeta+\beta)/(\gamma\zeta+\delta)).$$

* In regard to ζ; for the convergence was obtained independently of the value of ζ.

Denote this function by $F(\zeta)$; if ϑ_0 denote any assigned substitution of the group, and ϑ denote all the substitutions of the group in turn, it is clear that $\vartheta\vartheta_0$ denotes all the substitutions of the group in turn including the identical substitution; recognising this fact, and denoting the multiplier associated with ϑ_0 by σ_0, we immediately find

$$F(\vartheta_0(\zeta)) = \sigma_0^{-1}(\gamma_0\zeta + \delta_0)^k F(\zeta),$$

or, the function is multiplied by the factor $\sigma_0^{-1}(\gamma_0\zeta+\delta_0)^k$ when the variable ζ is transformed by the substitution, ϑ_0, of the group. Thence also, if $G(\zeta)$ denote a similar function to $F(\zeta)$, formed with the same value of k and a different function $f(\zeta)$, the ratio $F(\zeta)/G(\zeta)$ remains entirely unaltered when the variable is transformed by the substitutions of the group. In order to point out the significance of this result we introduce a representation whereof the full justification is subsequent to the present investigation. Let a Riemann surface be taken, on which the $2p$ period loops are cut; let the circumference of the circle C_i of the ζ plane be associated with one side of the period loop (b_i) of the second kind, and the circumference of the circle C_i' with the other side of this loop; let an arbitrary curve which we shall call the i-th barrier be drawn in the ζ plane from an arbitrary point P of the circle C_i' to the corresponding point P' of the circle C_i, and let the two sides of this curve be associated with the two sides of the period loop (a_i) of the Riemann surface. Then the function $F(\zeta)/G(\zeta)$, which has the same value at any two near points on opposite sides of the barrier, and has the same value at any point Q of the circle C_i' as at the corresponding point Q' of the circle C_i, will correspond to a function uniform on the undissected Riemann surface. In this representation the whole of the Riemann surface corresponds to the region S; any region $\vartheta_i S$ corresponds to a repetition of the Riemann surface; thus if the only essential singularities of $F(\zeta)/G(\zeta)$ be at the singular points of the group, none of which are within S, $F(\zeta)/G(\zeta)$ corresponds to a rational function on the Riemann surface. It will appear that the correspondence thus indicated extends to the integrals of rational functions; of such integrals not all the values can be represented on the dissected Riemann surface, while on the undissected surface they are not uniform; for instance, of an integral of the first kind, u_i, the values u_i, $u_i+2\omega_{i,r}$, $u_i+2\omega'_{i,r}$, $u_i+2\omega_{i,r}+2\omega'_{i,r}$ may be represented, but in that case not the value $u_i+4\omega_{i,r}$; in view of this fact the repetition of the Riemann surface associated with the regions derived from S by the substitutions of the group is of especial interest—*we are able to represent more of the values of the integral in the ζ plane than on the Riemann surface.* These remarks will be clearer after what follows.

228. In what follows we consider only a simple case of the function $F(\zeta)$, that in which the multipliers $\sigma_1, \ldots, \sigma_p$ are all unity, $k=2$, and $f(\zeta)=1/(\zeta-a)$, a being a point which, for the sake of definiteness, we

suppose to be in the region S. We denote by $\zeta_i = \vartheta_i(\zeta) = (\alpha_i\zeta + \beta_i)/(\gamma_i\zeta + \delta_i)$ all the substitutions of the group, in turn, and call ζ_i the *analogue* of ζ by the substitution in question. The function

$$\Phi(\zeta, a) = \Sigma \frac{(\gamma_i\zeta + \delta_i)^{-2}}{\zeta_i - a},$$

has essential singularities at the singular points of the group, and has poles at the places $\zeta = a$, $\zeta = \infty$ and at the analogues of these places. Let the points ∞, a be joined by an arbitrary barrier lying in S, and the analogues of this barrier be drawn in the other regions. Then the integral of this uniformly convergent series, from an arbitrary point ξ, namely, the series

$$\Sigma \log \frac{\zeta_i - a}{\xi_i - a}, \ = \Pi_{a,\infty}^{\zeta,\xi}, \text{ say,}$$

is competent to represent a function of ζ which can only deviate from uniformity when ζ describes a contour enclosing more of the points a and its analogues than of the points ∞ and its analogues; this is prevented by the barriers. Thus the function is uniform over the whole ζ plane; it is infinite at $\zeta = a$ like $\log(\zeta - a)$, and at $\zeta = \infty$ like $-\log\left(\frac{1}{\zeta}\right)$, as we see by considering the term of the series corresponding to the identical substitution; its value on one side of the barrier $a\infty$ is $2\pi i$ greater than on the other side; it has analogous properties in the analogues of the points a, ∞, and the barrier $a\infty$; further, if $\zeta_n = \vartheta_n(\zeta)$ be any of the fundamental substitutions $\vartheta_1, \ldots, \vartheta_p$,

$$\Pi_{a,\infty}^{\zeta_n,\xi} - \Pi_{a,\infty}^{\zeta,\xi} = \sum_i \log \frac{\zeta_{in} - a}{\zeta_i - a} = \sum_i \log \frac{\zeta_{in} - a}{\xi_{in} - a} + \sum_i \log \frac{\xi_{in} - a}{\xi_i - a} - \sum_i \log \frac{\zeta_i - a}{\xi_i - a},$$

where ζ_{in} is obtained from ζ by the substitution $\vartheta_i\vartheta_n$; since the first and last of these sums contain the same terms, we have

$$\Pi_{a,\infty}^{\zeta_n,\xi} - \Pi_{a,\infty}^{\zeta,\xi} = \Pi_{a,\infty}^{\xi_n,\xi},$$

and the right-hand side is independent of ξ, being equal to $\Pi_{a,\infty}^{\zeta_n,\zeta}$; in order to prove this in another way, and obtain at the same time a result which will subsequently be useful, we introduce an abbreviated notation; denote the substitution ϑ_r simply by the letter r; then if j be in turn every substitution of the group whose product symbol has not a positive or negative power of the substitution n at its right-hand end, all the substitutions of the group have the symbol jn^h, h being in turn equal to all positive and negative integers (including zero); hence

$$\sum_i [\log(\xi_{in} - a) - \log(\xi_i - a)], \ = \sum_j \sum_h [\log(\xi_{jn^{h+1}} - a) - \log(\xi_{jn^h} - a)],$$

is equal to

$$\sum_j \log \frac{\vartheta_j(\xi_N) - a}{\vartheta_j(\xi_M) - a},$$

where $N=n^{\infty}$, $M=n^{-\infty}$; but, in fact, ξ_N is B_n, and ξ_M is A_n; thus $\Pi_{a,\infty}^{\xi_n,\xi}$ is independent of ξ; and if we introduce the definition

$$v_n^{\zeta}=\frac{1}{2\pi i}\sum_j \log\frac{\zeta-\vartheta_j(B_n)}{\zeta-\vartheta_j(A_n)},$$

where ϑ_n is one of the p fundamental substitutions, and, as before, j denotes all the substitutions whose product symbols have not a power of n at the right-hand end, we have

$$\Pi_{a,\infty}^{\zeta_n,\xi}-\Pi_{a,\infty}^{\zeta,\xi}=\Pi_{a,\infty}^{\zeta_n,\zeta}=2\pi i v_n^a.$$

Ex. If for abbreviation we put

$$P_{a,\infty}^{\zeta,\xi}=\sum_i \sigma_i\log\frac{\zeta_i-a}{\xi_i-a},$$

prove that

$$P_{a,\infty}^{\zeta_n,\xi}-\frac{1}{\sigma_n}P_{a,\infty}^{\zeta,\xi}=P_{a,\infty}^{c_n,c}+\frac{1-\sigma_n}{\sigma_n}P_{a,\infty}^{\xi,c},$$

c being an arbitrary point.

229. Introduce now the function $\Pi_{a,b}^{\zeta,\xi}$ defined by the equation

$$\Pi_{a,b}^{\zeta,\xi}=\Pi_{a,\infty}^{\zeta,\xi}-\Pi_{b,\infty}^{\zeta,\xi}=\sum_i\log\left(\frac{\zeta_i-a}{\xi_i-a}\Big/\frac{\zeta_i-b}{\xi_i-b}\right);$$

then, because a cross ratio of four quantities is unaltered by the same linear transformation applied to all the variables, we have also

$$\Pi_{a,b}^{\zeta,\xi}=\sum_i\log\left[\frac{\zeta-\vartheta_i^{-1}(a)}{\xi-\vartheta_i^{-1}(a)}\Big/\frac{\zeta-\vartheta_i^{-1}(b)}{\xi-\vartheta_i^{-1}(b)}\right]=\sum_r\log\left(\frac{a_r-\zeta}{b_r-\zeta}\Big/\frac{a_r-\xi}{b_r-\xi}\right),$$

where r, denoting ϑ_r, $=\vartheta_i^{-1}$, becomes in turn every substitution of the group. Thus we have

$$\Pi_{a,b}^{\zeta,\xi}=\Pi_{\zeta,\xi}^{a,b},\quad \Pi_{a,b}^{\zeta_n,\xi}-\Pi_{a,b}^{\zeta,\xi}=2\pi i v_n^{a,b},$$

where

$$v_n^{a,b},=v_n^a-v_n^b,=\frac{1}{2\pi i}\sum_j\log\left[\frac{a-\vartheta_j(B_n)}{a-\vartheta_j(A_n)}\Big/\frac{b-\vartheta_j(B_n)}{b-\vartheta_j(A_n)}\right],=\frac{1}{2\pi i}\Pi_{a,b}^{\xi_n,\xi},$$

j denoting as before every substitution whose product symbol has not a positive or negative power of n at the right-hand end and ξ being arbitrary; hence also

$$v_n^{\zeta,a}=\frac{1}{2\pi i}\Pi_{\zeta,a}^{\xi_n,\xi}=\frac{1}{2\pi i}\sum_i\log\left(\frac{\xi_{in}-\zeta}{\xi_i-\zeta}\Big/\frac{\xi_{in}-a}{\xi_i-a}\right)=\frac{1}{2\pi i}\sum_r\log\left(\frac{\zeta_r-\xi_n}{a_r-\xi_n}\Big/\frac{\zeta_r-\xi}{a_r-\xi}\right),$$

where r, $=i^{-1}$, denotes every substitution of the group.

There are essentially only p such functions $v_n^{\zeta, a}$, according as ϑ_n denotes $\vartheta_1, \vartheta_2, \ldots, \vartheta_p$; for, taking the expression given last but one, and putting $n = st$, that is, $\vartheta_n = \vartheta_s \vartheta_t$, we have

$$2\pi i v_{st}^{\zeta, a} = \Pi_{\zeta, a}^{\xi_{st}, \xi} = \Pi_{\zeta, a}^{\xi_{st}, \xi_t} + \Pi_{\zeta, a}^{\xi_t, \xi}$$

$$= \Pi_{\zeta, a}^{\eta_s, \eta} + \Pi_{\zeta, a}^{\xi_t, \xi},$$

where $\eta = \xi_t$, so that

$$v_{st}^{\zeta, a} = v_s^{\zeta, a} + v_t^{\zeta, a},$$

and in particular, when st is the identical substitution, as we see by the formula itself,

$$0 = v_s^{\zeta, a} + v_{s^{-1}}^{\zeta, a};$$

thus, if r denote $\vartheta_1^{\lambda_1} \vartheta_2^{\lambda_2} \ldots \vartheta_p^{\lambda_p} \ldots$, we obtain

$$v_r^{\zeta, a} = \lambda_1 v_1^{\zeta, a} + \ldots\ldots + \lambda_p v_p^{\zeta, a} + \ldots\ldots,$$

so that all the functions $v_r^{\zeta, a}$ are expressible as linear functions of $v_1^{\zeta, a}, \ldots, v_p^{\zeta, a}$.

230. It follows from the formula

$$v_n^{\zeta, a} = \frac{1}{2\pi i} \sum_j \log \left(\frac{\zeta - \vartheta_j (B_n)}{\zeta - \vartheta_j (A_n)} \Big/ \frac{a - \vartheta_j (B_n)}{a - \vartheta_j (A_n)} \right),$$

that the function $v_n^{\zeta, a}$ is never infinite save at the singular points of the group. But it is not an uniform function of ζ; for let ζ describe the circumference of the circle C_n in a counter clockwise direction; then, by the factor $\zeta - B_n$, $v_n^{\zeta, a}$ increases by unity; and no other increase arises; for, when the region within the circle C_n, constituted by $\vartheta_n S$ and regions of the* form $\vartheta_n \phi S$, contains a point $\vartheta_j (B_n)$, the product representing the substitution j has a positive power of ϑ_n as its left-hand factor, and in that case the region contains also the point $\vartheta_j (A_n)$. Similarly if ζ describe the circle C_n' in a clockwise direction, $v_n^{\zeta, a}$ increases by unity. But if ζ describe the circumference of any other of the $2p$ circles, no increase arises in the value of $v_n^{\zeta, a}$, for the existence of a point $\vartheta_j (B_n)$ in such a circle involves the existence also of a point $\vartheta_j (A_n)$.

It follows therefore that the function can be made uniform in the region S by drawing the barrier, before described, from an arbitrary point P of C_n' to the corresponding point P' of C_n. Then $v_n^{\zeta, a}$ is greater by unity on one side of this barrier than on the other side. Further if m denote any one of the substitutions $\vartheta_1, \ldots, \vartheta_p$, we have

$$v_n^{\zeta_m, b} - v_n^{\zeta, b} = v_n^{\zeta_m} - v_n^{\zeta} = v_n^{\zeta_m, \zeta} = \Pi_{\zeta_m, \zeta}^{\xi_n, \xi},$$

* Where ϕ denotes a product of substitutions in which ϑ_n^{-1} is not the left-hand factor.

where ξ is arbitrary; thus as $\Pi_{\xi_n, \zeta}^{\zeta_m, \zeta} = \Pi_{\zeta_m, \zeta}^{\xi_n, \xi}$, the difference is also independent of ζ, and we have, introducing a symbol for this constant difference,

$$v_n^{\zeta_m, b} - v_n^{\zeta, b} = \tau_{n, m} = \tau_{m, n}.$$

It follows therefore that if the p barriers, connecting the pairs of circles C_n', C_n, and their analogues for all the substitutions, be drawn in the interiors of the circles, the functions $v_1^{\zeta, a}, \ldots, v_p^{\zeta, a}$ are uniform in the region S, and in all the regions derivable therefrom by the substitutions of the group. The behaviour of the functions $v_1^{\zeta, a}, \ldots, v_p^{\zeta, a}$ in the region S is therefore entirely analogous to that of the Riemann normal integrals upon a Riemann surface, the correspondence of the pair of circumferences C_n, C_n' and the two sides of the barrier $P'P$, to the two sides of the period loops (b_n), (a_n), on the Riemann surface, being complete. And the regions within the circles $C_1, \ldots, C_p'$ enable us to represent, in an uniform manner, all the values of the integrals which would arise on the Riemann surface if the period loops (b_n) were not present. Thus the ζ plane has greater powers of representation than the Riemann surface. Further it follows, by what has preceded, that the integral $\Pi_{a, b}^{\zeta, \xi}$ is entirely analogous to the Riemann normal elementary integral of the third kind which has been denoted by the same symbol in considering the Riemann surface. On the Riemann surface the period loops (a_n) are not wanted for this function, which appears as a particular case of a more general canonical integral having symmetrical behaviour in regard to the first and second kinds of period loops; but the loops (b_n) are necessary; they render the function uniform by preventing the introduction of all the values of which the function is capable. In the ζ plane, however*, the function is uniform for all values of ζ, and the regions interior to the circles enable us to represent all the values of which the function is susceptible. Thus the introduction of Riemann's normal integrals appears a more natural process in the case of the ζ plane than in the case of the Riemann surface itself.

231. We may obtain a product expression for $\tau_{n, m}$ directly from the formula

$$\tau_{n, m} = \frac{1}{2\pi i} \sum_j \log \left[\frac{\zeta_m - \vartheta_j(B_n)}{\zeta - \vartheta_j(B_n)} \Big/ \frac{\zeta_m - \vartheta_j(A_n)}{\zeta - \vartheta_j(A_n)} \right];$$

let k denote in turn every substitution whose product symbol neither has a power of ϑ_m at its left-hand end nor a power of ϑ_n at its right-hand end; thus we may write $\vartheta_j = \vartheta_m^{-h} \vartheta_k$, or, for abbreviation, $j = m^{-h}k$; and for every substitution k, the substitution j has all the forms derivable by giving to h all positive and negative integral values including zero, except that, when k

* Barriers being drawn to connect the infinities of the function.

is the identical substitution, if $m=n$, h can only have the one value zero; then applying ϑ_j^{-1} to every quantity of the cross ratio under the logarithm sign, we have

$$\tau_{n,m}=\frac{1}{2\pi i}\sum_j \log\left(\frac{\zeta_{j^{-1}m}-B_n}{\zeta_{j^{-1}}-B_n}\Big/\frac{\zeta_{j^{-1}m}-A_n}{\zeta_{j^{-1}}-A_n}\right)$$

$$=\frac{1}{2\pi i}\sum_{k,h}\log\left(\frac{\zeta_{k^{-1}m^{h+1}}-B_n}{\zeta_{k^{-1}m^h}-B_n}\Big/\frac{\zeta_{k^{-1}m^{h+1}}-A_n}{\zeta_{k^{-1}m^h}-A_n}\right),$$

and therefore, if m be not equal to n,

$$\tau_{n,m}=\frac{1}{2\pi i}\sum_k \log\left(\frac{\vartheta_k^{-1}(B_m)-B_n}{\vartheta_k^{-1}(A_m)-B_n}\Big/\frac{\vartheta_k^{-1}(B_m)-A_n}{\vartheta_k^{-1}(A_m)-A_n}\right),$$

while when $m=n$, separating away the term for which k is the identical substitution,

$$\tau_{n,n}=\frac{1}{2\pi i}\log\left(\frac{\zeta_n-B_n}{\zeta-B_n}\Big/\frac{\zeta_n-A_n}{\zeta-A_n}\right)$$

$$+\frac{1}{2\pi i}\sum_k{}'\log\left(\frac{\vartheta_k^{-1}(B_n)-B_n}{\vartheta_k^{-1}(A_n)-B_n}\Big/\frac{\vartheta_k^{-1}(B_n)-A_n}{\vartheta_k^{-1}(A_n)-A_n}\right),$$

where Σ' denotes that the identical substitution, $\vartheta_k=1$, is not included; thus

$$\tau_{n,n}=\frac{1}{2\pi i}\log(\mu_n e^{i\kappa_n})+\frac{1}{2\pi i}\sum_s{}'\log\left[\frac{B_n-\vartheta_s(B_n)}{A_n-\vartheta_s(B_n)}\Big/\frac{B_n-\vartheta_s(A_n)}{A_n-\vartheta_s(A_n)}\right]^2,$$

where s denotes every substitution of the group other than the identical substitution, not beginning or ending with a power of ϑ_n, and excluding every substitution of which the inverse has already occurred.

These formulæ, like that for $v_n^{\zeta, a}$, are not definite unless the barriers (§ 227) are drawn.

232. *Ex.* i. If $v_n^{\zeta, a}=u_n+iw_n$, u_n, iw_n being the real and imaginary parts of $v_n^{\zeta, a}$, prove, as in the case of a Riemann surface, by taking the integral $\int u\,dw$ round the p closed curves each formed by the circumferences of a pair of circles and the two sides of the barrier joining them, that the imaginary part of $N_1{}^2\tau_{11}+\dots\dots+2N_1N_2\tau_{12}+\dots\dots$ is positive, $N_1, \dots, N_p$ being any real quantities and $u+iw=N_1v_1^{\zeta, a}+\dots\dots+N_pv_p^{\zeta, a}$. Prove also the result $\tau_{m,n}=\tau_{n,m}$ by contour integration.

Ex. ii. Prove that the function of ζ expressed by

$$\Gamma_a^{\zeta, \xi}=\frac{d}{da}\Pi_{a, b}^{\zeta, \xi}=\sum_r(\gamma_r a+\delta_r)^{-2}\left[\frac{1}{a_r-\zeta}-\frac{1}{a_r-\xi}\right],$$

has analogous properties to Riemann's normal elementary integral of the second kind.

Ex. iii. Prove that

$$\Gamma_{a_i}^{\zeta, \xi}=(\gamma_i a+\delta_i)^2\,\Gamma_a^{\zeta, \xi},$$

where $a_i=(\alpha_i a+\beta_i)/(\gamma_i a+\delta_i)$.

Ex. iv. With the notation

$$\Phi(z, \zeta) = \sum_r \frac{(\gamma_r z + \delta_r)^{-2}}{z_r - \zeta},$$

prove that

$$\Phi(z, \zeta_n) - \Phi(z, \zeta) = 2\pi i \frac{d}{dz} v_n^{z,\, a} = \Phi(z, \xi_n) - \Phi(z, \xi),$$

where ξ is an arbitrary point, and hence prove that if $z, c_1, \ldots, c_p, \xi$ be any arbitrary points, and $\xi_1 = \vartheta_1(\xi), \ldots, \xi_p = \vartheta_p(\xi)$, the function of ζ expressed by

$$\begin{vmatrix} \Phi(z, \zeta), & \Phi(z, \xi), & \Phi(z, \xi_1), \ldots, & \Phi(z, \xi_p) \\ \Phi(c_1, \zeta), & \Phi(c_1, \xi), & \Phi(c_1, \xi_1), \ldots, & \Phi(c_1, \xi_p) \\ \cdots & \cdots & \cdots & \cdots \\ \Phi(c_p, \zeta), & \Phi(c_p, \xi), & \Phi(c_p, \xi_1), \ldots, & \Phi(c_p, \xi_p) \\ 1, & 1, & 1, \ldots, & 1 \end{vmatrix},$$

is unchanged by the substitutions of the group, and has simple poles at $z, c_1, \ldots, c_p$, and their analogues, and a simple zero at ξ, and its analogues. Thus the function is similar to the function $\psi(x, a; z, c_1, \ldots, c_p)$ of § 122, and every function which is unchanged by the substitutions of the group can be expressed by means of it.

As a function of z, the function is infinite at $z = \xi$, $z = \zeta$, beside being infinite at $z = \infty$, and its analogues; when $(a_i z + \beta_i)/(\gamma_i z + \delta_i)$ is put for z, the function becomes multiplied by $(\gamma_i z + \delta_i)^2$. This last circumstance clearly corresponds with the fact (§ 123) that $\psi(x, a; z, c_1, \ldots, c_p)$ is not a rational function of z, but a rational function multiplied by $\frac{dz}{dt}$ (cf. Ex. iii.)

Ex. v. Prove that

$$\Gamma_a^{\zeta,\, \xi} = \sum_r \left(\frac{1}{a - \zeta_r} - \frac{1}{a - \xi_r}\right).$$

Ex. vi. In case $p = 1$, we have

$$v^{\zeta,\, a} = \frac{1}{2\pi i} \log\left(\frac{\zeta - B}{\zeta - A} \Big/ \frac{a - B}{a - A}\right), \quad \Pi_{a,b}^{\zeta,\, \xi} = \log \prod_{r=-\infty}^{\infty} \left(\frac{a_r - \zeta}{b_r - \zeta} \Big/ \frac{a_r - \xi}{b_r - \xi}\right), \quad \tau = \frac{1}{2\pi i} \log(\mu e^{i\kappa}),$$

where

$$(a_r - B)/(a_r - A) = (\mu e^{i\kappa})^r (a - B)/(a - A).$$

Putting, for abbreviation, $q = e^{i\pi\tau} = \sqrt{\mu e^{i\kappa}}$, and

$$\Theta(\zeta) = \sum_{n=-\infty}^{\infty} (-)^n q^{(n+\frac{1}{2})^2} \left(\frac{\zeta - B}{\zeta - A} \Big/ \frac{a - B}{a - A}\right)^{n+\frac{1}{2}},$$

prove, by applying the fundamental transformation once, that

$$\Theta(\zeta') = -\frac{1}{q} \frac{\zeta - A}{\zeta - B} \Big/ \frac{a - A}{a - B} \Theta(\zeta) = -e^{-2\pi i (v^{\zeta,\, a} + \frac{1}{2}\tau)} \Theta(\zeta),$$

and shew that $\Theta(\zeta)$ is a multiple of the Jacobian theta function $\Theta(v^{\zeta,\, a}, q; \tfrac{1}{2}, \tfrac{1}{2})$.

Ex. vii. Taking two circles as in figure 6 (§ 223), let $C'B/AC' = \sigma$ and $\frac{CB}{AC} \Big/ \frac{C'B}{AC'} = \mu$; take an arbitrary real quantity ω, and a pure imaginary quantity $\omega' = \frac{\omega}{i\pi} \log \mu$, and let

$\wp(u)$ denote Weierstrass's elliptic function of u with 2ω, $2\omega'$ as periods. Then prove, if a, c denote points outside both the circles, a' denote the inverse point of a in regard to either one of the circles, and P, Q be arbitrary real quantities,

(α) that the function

$$\left\{\wp\left[\frac{\omega}{i\pi}\log\left(\frac{\zeta-B}{\zeta-A}\right)^2\Big/\frac{a-B}{a-A}\frac{c-B}{c-A}\right]-\wp\left[\frac{\omega}{i\pi}\log\frac{a-B}{a-A}\Big/\frac{c-B}{c-A}\right]\right\}^{-1},$$

is unaltered by the substitution $(\zeta'-B)/(\zeta'-A)=\mu(\zeta-B)/(\zeta-A)$, and has poles of the first order, outside both the circles, only at the points $\zeta=a$, $\zeta=c$.

(β) that the function,

$$\frac{P+iQ}{\wp\left[\frac{\omega}{i\pi}\log\frac{1}{\sigma}\frac{\zeta-B}{\zeta-A}\right]-\wp\left[\frac{\omega}{i\pi}\log\frac{1}{\sigma}\frac{a-B}{a-A}\right]}+\frac{P-iQ}{\wp\left[\frac{\omega}{i\pi}\log\frac{1}{\sigma}\frac{\zeta-B}{\zeta-A}\right]-\wp\left[\frac{\omega}{i\pi}\log\frac{1}{\sigma}\frac{a'-B}{a'-A}\right]}$$

is real on the circumference of each circle, and, outside both the circles, has a pole of the first order only at the point $\zeta=a$. The arbitraries P, Q can be used to prescribe the residue at this pole.

Ex. viii. Prove that any two uniform functions of ζ having no discontinuities except poles, which are unaltered by the substitutions of the group, are connected by an algebraic relation (cf. § 235); and that, if these two be properly chosen, any other uniform function of ζ having no discontinuities except poles, which is unaltered by the substitutions of the group, can be expressed rationally in terms of them. The development of the theory on these lines is identical with the theory of rational functions on a Riemann surface, but is simpler on account of the absence of branch places. Thus for instance we have a theory of fundamental integral functions, an integral function being one which is only infinite in the poles of an arbitrarily chosen function x. And we can form a function such as $\bar{E}(x, z)$ (§ 124, Chap. VII.); but the essential part of that function is much more simply provided by the function, $\varpi(\zeta, \gamma)$, investigated in the following article.

233. The preceding investigations are sufficient to explain the analogy between the present theory and that of a Riemann surface. We come now to the result which is the main purpose of this chapter. In the equation

$$\Pi_{\zeta,\gamma}^{z,c}=\sum_i\log\left(\frac{z_i-\zeta}{c_i-\zeta}\Big/\frac{z_i-\gamma}{c_i-\gamma}\right)=\sum_i\log\{\zeta,\gamma/z_i,c_i\},$$

where $\{\zeta,\gamma/z_i,c_i\}$ denotes a cross ratio, let the point z approach indefinitely near to ζ, and the point c approach indefinitely near to γ; then separating away the term belonging to the identical substitution, and associating with the term belonging to any other substitution that belonging to the inverse substitution, we have, after applying a linear transformation to every element of the cross ratio arising from the inverse substitution

$$\Pi_{\zeta,\gamma}^{z,c}=\log\frac{(z-\zeta)(c-\gamma)}{(z-\gamma)(c-\zeta)}+\sum_i{}'\log\frac{(z_i-\zeta)(c_i-\gamma)}{(z_i-\gamma)(c_i-\zeta)}\cdot\frac{(z-\zeta_i)(c-\gamma_i)}{(z-\gamma_i)(c-\zeta_i)},$$

where Σ' denotes that, in the summation, of terms arising by a substitution and its inverse, only one is to be taken, and the identical substitution is excluded. Thus we have*

$$\lim_{z=\zeta,\, c=\gamma} [-(z-\zeta)(c-\gamma)e^{-\Pi^{z,\,c}_{\zeta,\,\gamma}}]^{\frac{1}{2}} = (\zeta-\gamma)\,\Pi'_i \frac{(\zeta_i-\gamma)(\gamma_i-\zeta)}{(\zeta_i-\zeta)(\gamma_i-\gamma)},$$
$$= (\zeta-\gamma)\,\Pi'_i \{\zeta, \gamma/\gamma_i, \zeta_i\},$$

where Π'_i has a similar signification to Σ'_i and $\{\zeta, \gamma/\gamma_i, \zeta_i\}$ denotes a cross ratio. Consider now the expression

$$\varpi(\zeta, \gamma), \;= (\zeta-\gamma)\,\Pi'_i \{\zeta, \gamma/\gamma_i, \zeta_i\};$$

it has clearly the following properties—it represents a perfectly definite function of ζ and γ, single-valued on the whole ζ-plane; it depends only on two variables, and $\varpi(\zeta, \gamma) = -\varpi(\gamma, \zeta)$; as a function of ζ it is infinite, save for the singular points of the group, only at $\zeta = \infty$, and not at the analogues of $\zeta = \infty$; it vanishes only at $\zeta = \gamma$ and the analogues of this point, and $\text{limit}_{\zeta=\gamma}\, \varpi(\zeta, \gamma)/(\zeta-\gamma) = 1$. Thus the function may be expected to generalise the irreducible factor of the form $x - a$, in the case of rational functions, and the factor $\sigma(u-a)$ in the case of elliptic functions, and to serve as a prime function for the functions of ζ now under consideration (cf. also Chap. VII. § 129 and Chaps. XIII. and XIV.). It should be noticed that the value of $\varpi(\zeta, \gamma)$ does not depend upon the choice we make in the product between any substitution and its inverse; this follows by applying the substitution ϑ_i^{-1} to every element of any factor.

234. We enquire now as to the behaviour of the function $\varpi(\zeta, \gamma)$ under the substitutions of the group. It will be proved that

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} = (-1)^{g_n+h_n} \frac{e^{-2\pi i (v_n^{\zeta,\,\gamma} + \frac{1}{2}\tau_{n,n})}}{\gamma_n \zeta + \delta_n},$$

where $(-1)^{g_n}$, $(-1)^{h_n}$ are certain $\pm$ signs to be explained.

This result can be obtained, save for a sign, from the definition of $\varpi(\zeta, \gamma)$, as a limit, from the function $\Pi^{\zeta,\,\gamma}_{z,\,c}$; but since, for our purpose, it is essential to avoid any such ambiguity, and because we wish to regard the function $\varpi(\zeta, \gamma)$ as fundamental, we adopt the longer method of dealing directly with the product $(\zeta-\gamma)\,\Pi'_i \{\zeta, \gamma/\gamma_i, \zeta_i\}$. We imagine the barriers, each connecting a pair of circles, which are necessary to render the functions $v_1^{\zeta,\,a}, \ldots, v_p^{\zeta,\,a}$

* This function occurs in Schottky, *Crelle*, CI. (1887), p. 242 (at the top of the page). See also p. 253, at the top. The function is modified, for a Riemann surface, by Klein, *Math. Annal.* XXXVI. (1890), p. 13. The modified function occurs also, in particular cases, in a paper by Pick, *Math. Annal.* XXIX., and in Klein, *Math. Annal.* XXXII. (1888), p. 367. For $p=1$, the theta function was of course expressed in factors by Jacobi. The function employed by Ritter, *Math. Annal.* XLIV. (p. 291), has a somewhat different character.

uniform, to be drawn; then the quantities $\tau_{n,m}$, $\tau_{n,n}$ given in § 231, and defined by $v_n^{\zeta_m,\zeta}$, $v_n^{\zeta_n,\zeta}$ are definite; so therefore is also $e^{\pi i v_n^{\zeta,\gamma}}$ and the quantity $e^{\pi i \tau_{n,n}}$, which is equal to

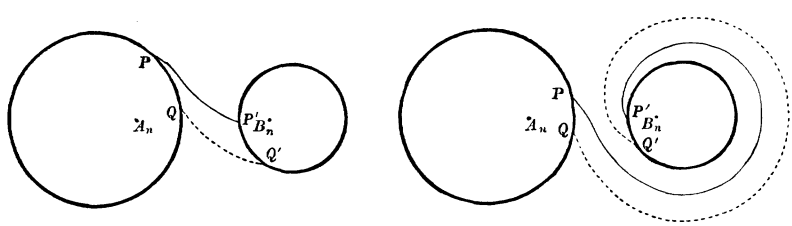
$$\mu_n^{\frac{1}{2}} e^{\frac{1}{2}i\kappa_n} \Pi'_s \left[\frac{B_n - (B_n)_s}{A_n - (B_n)_s} \Big/ \frac{B_n - (A_n)_s}{A_n - (A_n)_s} \right],$$

where s denotes a substitution, other than the identical substitution, not beginning or ending with a power of ϑ_n, and excluding the inverse of a substitution which has already occurred. This formula raises the question whether κ_n, which we take positive, is to be regarded as less than 2π or not, since otherwise the sign of $e^{\frac{1}{2}i\kappa_n}$ is not definite. But in fact, as it arises in this formula, from $v_n^{\zeta_n,\zeta}$, $\log \mu_n + i\kappa_n$ is the value of $\log \left(\frac{\zeta' - B_n}{\zeta' - A_n} \Big/ \frac{\zeta - B_n}{\zeta - A_n} \right)$ when ζ' has reached ζ_n from ζ *by a path which does not cross the barriers.* Thus κ_n is perfectly definite when the barriers are drawn, and the sign of the quantity

$$e^{-\pi i \tau_{n,n}} \mu_n^{\frac{1}{2}} e^{\frac{1}{2}i\kappa_n} \Pi'_s \left[\frac{B_n - (B_n)_s}{A_n - (B_n)_s} \Big/ \frac{B_n - (A_n)_s}{A_n - (A_n)_s} \right]$$

is perfectly definite and independent of the barriers. We denote it by $(-1)^{g_n - 1}$. The annexed figure illustrates two ways of drawing a barrier PP'. In the first case κ_n is less than 2π. In the second case ζ' must pass

Fig. 8.



once round the point B, and κ_n is greater than 2π. When κ_n is thus determined, the expression by means of κ_n of the $\rho_n^{\frac{1}{2}}$ which occurs in the formulae connecting α_n, β_n, γ_n, δ_n and A_n, B_n, ρ_n, for instance in the formula $\rho_n^{\frac{1}{2}} = (1 + \rho_n)/(\alpha_n + \delta_n)$, is also definite; it may be $\rho_n^{\frac{1}{2}} = \mu_n^{\frac{1}{2}} e^{\frac{1}{2}i\kappa_n}$ or $\rho_n^{\frac{1}{2}} = -\mu_n^{\frac{1}{2}} e^{\frac{1}{2}i\kappa_n}$. We shall put $\rho_n^{\frac{1}{2}} = (-1)^{h_n} \mu_n^{\frac{1}{2}} e^{\frac{1}{2}i\kappa_n}$. If the whole investigation had been commenced with a different sign for each of α_n, β_n, γ_n, δ_n, h_n would have become $h_n - 1$, but g_n, depending only on the circles and the barrier, would have the same value.

We have

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} = \frac{\zeta_n - \gamma}{\zeta - \gamma} \Pi'_i \frac{\zeta_{in} - \gamma}{\zeta_i - \gamma} \cdot \frac{\gamma_i - \zeta_n}{\gamma_i - \zeta} \cdot \frac{\zeta_i - \zeta}{\zeta_{in} - \zeta_n},$$

where i denotes in turn all substitutions which with their inverses give the whole group, except the identical substitution; thus i denotes all substitutions n^λ for $\lambda = 1, 2, 3, \ldots, \infty$, as well as all substitutions $n^h s n^k$, where s has the significance just explained and h, k take all positive and negative integer values including zero. Therefore

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} = \frac{\zeta_n - \gamma}{\zeta - \gamma} \prod_\lambda \frac{\zeta_{n^{\lambda+1}} - \gamma}{\zeta_{n^\lambda} - \gamma} \cdot \frac{\gamma_{n^\lambda} - \zeta_n}{\gamma_{n^\lambda} - \zeta} \cdot \frac{\zeta_{n^\lambda} - \zeta}{\zeta_{n^{\lambda+1}} - \zeta_n}$$

$$\prod_{h,s,k} \frac{\zeta_{n^h s n^{k+1}} - \gamma}{\zeta_{n^h s n^k} - \gamma} \cdot \frac{\gamma_{n^h s n^k} - \zeta_n}{\gamma_{n^h s n^k} - \zeta} \cdot \frac{\zeta_{n^h s n^k} - \zeta}{\zeta_{n^h s n^{k+1}} - \zeta_n}$$

$$= \frac{\zeta_n - \gamma}{\zeta - \gamma} \prod_\lambda \frac{\zeta_{n^{\lambda+1}} - \gamma}{\zeta_{n^\lambda} - \gamma} \prod_\lambda \frac{\zeta_{n^\lambda} - \zeta}{\zeta_{n^{\lambda+1}} - \zeta} \prod_\lambda \frac{\gamma_{n^\lambda} - \zeta_n}{\gamma_{n^\lambda} - \zeta} \cdot \frac{\zeta_{n^{\lambda+1}} - \zeta}{\zeta_{n^{\lambda+1}} - \zeta_n}$$

$$\prod_{h,s} \frac{(B_n)_{n^h s} - \gamma}{(A_n)_{n^h s} - \gamma} \cdot \frac{(A_n)_{n^h s} - \zeta}{(B_n)_{n^h s} - \zeta} \prod_{h,s,k} \frac{\gamma_{n^h s n^k} - \zeta_n}{\gamma_{n^h s n^k} - \zeta} \cdot \frac{\zeta_{n^h s n^{k+1}} - \zeta}{\zeta_{n^h s n^{k+1}} - \zeta_n},$$

the transformation of the second part of the product being precisely as in the first part,

$$= \frac{\zeta_n - \gamma}{\zeta - \gamma} \cdot \frac{B_n - \gamma}{\zeta_n - \gamma} \cdot \frac{\zeta_n - \zeta}{B_n - \zeta} \prod_\lambda \frac{\gamma - \zeta_{n^{1-\lambda}}}{\gamma - \zeta_{n^{-\lambda}}} \cdot \frac{\zeta_n - \zeta_{n^{-\lambda}}}{\zeta_n - \zeta_{n^{1-\lambda}}}$$

$$\prod_{h,s} \frac{(B_n)_{n^h s} - \gamma}{(A_n)_{n^h s} - \gamma} \cdot \frac{(A_n)_{n^h s} - \zeta}{(B_n)_{n^h s} - \zeta} \prod_{h,s,k} \frac{\gamma - \zeta_{n^{-k} s^{-1} n^{1-h}}}{\gamma - \zeta_{n^{-k} s n^{-h}}} \cdot \frac{\zeta_n - \zeta_{n^{-k} s^{-1} n^{-h}}}{\zeta_n - \zeta_{n^{-k} s^{-1} n^{1-h}}}$$

$$= \frac{B_n - \gamma}{B_n - \zeta} \cdot \frac{\zeta_n - \zeta}{\zeta - \gamma} \cdot \frac{\gamma - \zeta}{\gamma - A_n} \cdot \frac{\zeta_n - A_n}{\zeta_n - \zeta} \prod_{h,s} \frac{(B_n)_{n^h s} - \gamma}{(A_n)_{n^h s} - \gamma} \cdot \frac{(A_n)_{n^h s} - \zeta}{(B_n)_{n^h s} - \zeta}$$

$$\prod_{s,k} \frac{\gamma - (B_n)_{n^{-k} s^{-1}}}{\gamma - (A_n)_{n^{-k} s^{-1}}} \cdot \frac{\zeta_n - (A_n)_{n^{-k} s^{-1}}}{\zeta_n - (B_n)_{n^{-k} s^{-1}}};$$

since h and $-k$ have the same range of signification we may replace $-k$ by h, in the last form, and obtain, by a rearrangement of the second product,

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} = -\frac{B_n - \gamma}{B_n - \zeta} \cdot \frac{\zeta_n - A_n}{\gamma - A_n} \prod_{h,s} \frac{\zeta - (A_n)_{n^h s}}{\zeta - (B_n)_{n^h s}} \cdot \frac{\gamma - (B_n)_{n^h s}}{\gamma - (A_n)_{n^h s}}$$

$$\prod_{s,h} \frac{\gamma - (B_n)_{n^h s^{-1}}}{\gamma - (A_n)_{n^h s^{-1}}} \cdot \frac{\zeta_n - (A_n)_{n^h s^{-1}}}{\zeta_n - (B_n)_{n^h s^{-1}}};$$

but, from the formula

$$v_n^{\zeta, \gamma} = \frac{1}{2\pi i} \sum_j \log \frac{\zeta - \vartheta_j(B_n)}{\zeta - \vartheta_j(A_n)} \cdot \frac{\gamma - \vartheta_j(A_n)}{\gamma - \vartheta_j(B_n)},$$

where j can have the forms $n^h s$, $n^h s^{-1}$, or be the identical substitution, we have

$$e^{2\pi i v_n^{\zeta, \gamma}} = \frac{\zeta - B_n}{\zeta - A_n} \cdot \frac{\gamma - A_n}{\gamma - B_n} \prod_{h,s} \frac{\zeta - (B_n)_{n^h s}}{\zeta - (A_n)_{n^h s}} \cdot \frac{\gamma - (A_n)_{n^h s}}{\gamma - (B_n)_{n^h s}} \prod_{s,h} \frac{\zeta - (B_n)_{n^h s^{-1}}}{\zeta - (A_n)_{n^h s^{-1}}} \cdot \frac{\gamma - (A_n)_{n^h s^{-1}}}{\gamma - (B_n)_{n^h s^{-1}}};$$

therefore

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} e^{2\pi i v_n^{\zeta, \gamma}} = -\frac{\zeta_n - A_n}{\zeta - A_n} \prod_{s, h} \frac{\zeta - (B_n)_{n h_s - 1}}{\zeta_n - (B_n)_{n h_s - 1}} \cdot \frac{\zeta_n - (A_n)_{n h_s - 1}}{\zeta - (A_n)_{n h_s - 1}}$$

$$= -\frac{\zeta_n - A_n}{\zeta - A_n} \prod_{s, h} \frac{\zeta_{sn^{-h}} - B_n}{\zeta_{sn^{1-h}} - B_n} \cdot \frac{\zeta_{sn^{1-h}} - A_n}{\zeta_{sn^{-h}} - A_n}$$

$$= -\frac{\zeta_n - A_n}{\zeta - A_n} \prod_{s} \frac{(A_n)_s - B_n}{(B_n)_s - B_n} \cdot \frac{(B_n)_s - A_n}{(A_n)_s - A_n},$$

and hence

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} e^{2\pi i v_n^{\zeta, \gamma} + \pi i \tau_{n, n}} = \frac{\zeta_n - A_n}{\zeta - A_n} (-1)^{g_n} \mu_n^{\frac{1}{2}} e^{\frac{1}{2} i \kappa_n};$$

now from the formula $(\zeta_n - B_n)/(\zeta_n - A_n) = \rho_n (\zeta - B_n)/(\zeta - A_n)$, and the values of α_n, β_n, γ_n, δ_n given in § 226, we immediately find

$$(\zeta - A_n)/(\zeta_n - A_n) = [\zeta - A_n - \rho_n (\zeta - B_n)]/(B_n - A_n),$$

$$\gamma_n \zeta + \delta_n = [\rho_n^{-\frac{1}{2}} (\zeta - A_n) - \rho_n^{\frac{1}{2}} (\zeta - B_n)]/(B_n - A_n);$$

thus, as $\rho_n^{\frac{1}{2}} = (-1)^{h_n} \mu_n^{\frac{1}{2}} e^{\frac{1}{2} i \kappa_n}$, we have

$$(\zeta - A_n)/(\zeta_n - A_n) = (-1)^{h_n} \mu_n^{\frac{1}{2}} e^{\frac{1}{2} i \kappa_n} (\gamma_n \zeta + \delta_n);$$

hence, finally

$$\frac{\varpi(\zeta_n, \gamma)}{\varpi(\zeta, \gamma)} = (-1)^{g_n + h_n} \frac{e^{-2\pi i (v_n^{\zeta, \gamma} + \frac{1}{2}\tau_{n, n})}}{\gamma_n \zeta + \delta_n},$$

where $(-1)^{g_n} e^{-\pi i \tau_{n, n}} e^{\frac{1}{2} i \kappa_n}$ is independent of how the barriers are drawn, and $(-1)^{h_n} \gamma_n$, $(-1)^{h_n} \delta_n$ are independent of the signs attached to γ_n and δ_n.

235. The function $\varpi(\zeta, \gamma)$, whose properties have thus been deduced immediately from its expression as an infinite product, supposed to be convergent, may be regarded as fundamental. Thus, as can be immediately verified, the integral $\Pi_{\zeta, \gamma}^{z, c}$ is expressible by $\varpi(\zeta, \gamma)$, in the form

$$\Pi_{\zeta, \gamma}^{z, c} = \log \frac{\varpi(z, \zeta)\, \varpi(\gamma, c)}{\varpi(z, \gamma)\, \varpi(\zeta, c)},$$

and thence the integrals $v_n^{\zeta, \gamma}$ arise, by the definition $v_n^{\zeta, \gamma} = \frac{1}{2\pi i} \Pi_{\zeta, \gamma}^{z_n, z}$, and thence, also, integrals with algebraic infinities, by the definition

$$\Gamma_x^{\zeta, \gamma} = \frac{d}{dx} \Pi_{x, a}^{\zeta, \gamma}$$

(cf. Ex. ii, § 232). Further, if $F(\zeta)$ denote any uniform function of ζ whose value is unaltered by the substitutions of the group, which has no discontinuities except poles, it is easy to prove, by contour integration, as in the case of a Riemann surface, (i) That $F(\zeta)$ must be somewhere infinite in the region S, (ii) That $F(\zeta)$ takes any assigned value as many times within S as the sum of its orders of infinity within S, (iii) That if $\alpha_1, \ldots, \alpha_k$ be the poles and $\beta_1, \ldots, \beta_k$ the zeros of $F(\zeta)$ within S, and the barriers be supposed drawn,

$$v_i^{\beta_1, \alpha_1} + \ldots\ldots + v_i^{\beta_k, \alpha_k} = m_i + m_1'\tau_{i,1} + \ldots\ldots + m_p'\tau_{i,p}, \qquad (i = 1, \ldots, p),$$

where $m_1, \ldots, m_p, m_1', \ldots, m_p'$ are definite integers. Thence it is easy to shew that the ratio

$$F(\zeta) \Big/ \frac{\varpi(\zeta, \beta_1) \ldots\ldots \varpi(\zeta, \beta_k)}{\varpi(\zeta, \alpha_1) \ldots\ldots \varpi(\zeta, \alpha_k)} e^{-2\pi i (m_1' v_1^{\zeta, a} + \ldots + m'_p v_p^{\zeta, a})}$$

is a constant for all values of ζ. And replacing some of $\beta_1, \ldots, \alpha_k$ in this expression by suitable analogues, the exponential factor may be absorbed.

Ex. In the elliptic case where there is one fundamental substitution $(\zeta' - B)/(\zeta' - A) = \rho(\zeta - B)/(\zeta - A)$, we have $(\zeta_i - B)/(\zeta_i - A) = \rho^i(\zeta - B)/(\zeta - A)$, and thence putting u, v, respectively for the integrals v^ζ, v^γ, so that $e^{2\pi i u} = (\zeta - B)/(\zeta - A)$, $e^{2\pi i v} = (\gamma - B)/(\gamma - A)$, we immediately find

$$\frac{\zeta - \gamma_i}{\gamma - \gamma_i} \cdot \frac{\zeta - \zeta_i}{\gamma - \zeta_i} = \frac{1 - 2\rho^i \cos 2\pi(u - v) + \rho^{2i}}{(1 - \rho^i)^2}, \quad \zeta - \gamma = \frac{B - A}{2i} \frac{\sin \pi(u - v)}{\sin \pi u \sin \pi v},$$

and hence

$$\varpi(\zeta, \gamma) = \frac{B - A}{2i} \frac{\sin \pi(u - v)}{\sin \pi u \sin \pi v} \prod_{i=1}^{\infty} \frac{1 - 2\rho^i \cos 2\pi(u - v) + \rho^{2i}}{(1 - \rho^i)^2},$$

which*, putting $e^{\pi i \tau} = \rho^{\frac{1}{2}}$, is equal to

$$\frac{(B - A)\pi}{4i\omega} e^{-2\eta\omega(u - v)^2} \sigma[2\omega(u - v);\ 2\omega,\ 2\omega\tau] \div \sin \pi u \sin \pi v,$$

where ω is an arbitrary quantity, and

$$\eta\omega = \frac{\pi^2}{12} - 2\pi^2 \sum_1^\infty \frac{\rho^n}{(1 - \rho^n)^2}.$$

236. The further development of the theory of functions in the ζ plane may be carried out on the lines already followed in the case of the Riemann surface. We limit ourselves to some indications in regard to matters bearing on the main object of this chapter.

The excess of the number of zeros over the number of poles, in any region, of a function of ζ, $f(\zeta)$, which is uniform and without essential singularities within that region, is of course equal to the integral

$$\frac{1}{2\pi i} \int d \log f(\zeta),$$

* See, for instance, Halphen, *Fonct. Ellipt.* (Paris, 1886), vol. I. p. 400.

taken round the boundary of the region. If we consider, for example, the function $\Omega_n(\zeta), = dv_n^{\zeta, \gamma}/d\zeta$, which is nowhere infinite, in the region S, the number of its zeros within the region S is

$$\frac{1}{2\pi i}\sum_{r=1}^{p}\int\left[\frac{\Omega_n'(\zeta_r)}{\Omega_n(\zeta_r)} - \frac{\Omega_n'(\zeta)}{\Omega_n(\zeta)}\right]d\zeta,$$

where the dash denotes a differentiation in regard to ζ, and the sign of summation means that the integral is taken round the circles $C_1', \dots, C_p'$, in a counter-clockwise direction. Since $\Omega_n(\zeta_r) = (\gamma_r\zeta + \delta_r)^2\,\Omega_n(\zeta)$, the value is

$$\frac{1}{2\pi i}\sum_{r=1}^{p}\int\frac{2d\zeta}{\zeta - \underset{r}{\vartheta}^{-1}(\infty)}$$

or $2p$; thus as $\Omega_n(\zeta)$ vanishes to the second order at $\zeta = \infty$ in virtue of the denominator $d\zeta$, we may say that $dv_n^{\zeta, \gamma}$ has $2p-2$ zeros in the region S, in general distinct from $\zeta = \infty$. The function $\Omega_n(\zeta)$ vanishes in every analogue of these $2p-2$ places, but does not vanish in the analogues of $\zeta = \infty$.

The theory of the theta functions, constructed from the integrals $v_n^{\zeta, \gamma}$, and their periods $\tau_{n,m}$, will subsist, and, as in the case of the Riemann surface there will, corresponding to an arbitrary point m, which we take in the region S, be points $m_1, \dots, m_p$ in the region S, such that the zeros of the function $\Theta(v^{\zeta, m} - v^{\zeta_1, m_1} - \dots\dots - v^{\zeta_p, m_p})$ are the places $\zeta_1, \dots, \zeta_p$. And corresponding to any odd half period, $\frac{1}{2}\Omega_{s,s'}$, there will be places $n_1, \dots, n_{p-1}$, in the region S, which, repeated, constitute the zero of a differential $dv^{\zeta, \gamma}$, and satisfy the equations typified by

$$\tfrac{1}{2}\Omega_{s,s'} = v^{m_p, m} - v^{n_1, m_1} - \dots\dots - v^{n_{p-1}, m_{p-1}}.$$

The values of the quantities $e^{\pi i\tau_{n,n}}$ and the positions of $m_1, \dots, m_p$ may vary when the barriers which are necessary to define the periods $\tau_{n,m}$ are changed.

But it is one of the main results of the representation now under consideration that a particular theta function is derivable immediately from the function $\varpi(\zeta, \gamma)$; and hence, as is shewn in chapter XIV., that any theta function can be so derived. Let v denote the integral whose differential vanishes to the second order in each of the places $n_1, \dots, n_{p-1}$. Consider the expression $\sqrt{dv/d\zeta}$ in the region S. It has no infinities and it is single-valued in the neighbourhood of its zeros, as follows from the fact that the p zeros of $dv/d\zeta$ are all of the second order. Hence if the region S be made simply connected by drawing the p barriers, and joining the p pairs of circles by $p-1$ further barriers $(c_1), \dots, (c_{p-1})$, of which (c_r) joins the circumference C_r' to the circumference C_{r+1}, $\sqrt{dv/d\zeta}$ will be uniform in the region S so long as ζ does not cross any of the barriers. For the change in the value of $\sqrt{dv/d\zeta}$ when ζ is taken round any closed circuit may then be obtained by considering the equivalent circuits enclosing the zeros. But in fact the barriers $(c_1), \ldots, (c_{p-1})$ are unnecessary; to see this it is sufficient to see that any circuit in the region S which entirely surrounds a pair of circles, such as C_1', C_1, encloses an even number of the infinities of $dv/d\zeta$ which are at the singular points of the group. Since these infinities are among the logarithmic zeros and poles of $v_1^{\zeta, \gamma}, \ldots, v_p^{\zeta, \gamma}$, whereof v is a linear function, the proof required is included in the proof that any one of the functions $v_1^{\zeta, \gamma}, \ldots, v_p^{\zeta, \gamma}$ is unaltered when taken round a circuit entirely surrounding a pair of the circles, such as C_1', C_1. Thus when the barriers which render the functions $v_1^{\zeta, \gamma}, \ldots, v_p^{\zeta, \gamma}$ uniform are drawn, the function $\sqrt{dv/d\zeta}$ is entirely definite within the region S, save for an arbitrary constant multiplier, provided the sign of the function be given for some one point in the region S. And, this being done, if γ be any point, the function $\sqrt{\frac{dv}{d\zeta}}\sqrt{\frac{dv}{d\gamma}}$ is independent of this sign. This function, with a certain constant multiplier, which will be afterwards assigned, may be denoted by $\psi(\zeta)$.

237. We proceed now to prove the equation

$$\varpi(\zeta, \gamma) = A\,\frac{\Theta(v^{\zeta, \gamma} + \tfrac{1}{2}\Omega_{s, s'})e^{\pi i s' v^{\zeta, \gamma}}}{\psi(\zeta)} = Ae^{-\frac{1}{2}\pi i s'(s+\frac{1}{2}\tau s')}\,\frac{\Theta(v^{\zeta, \gamma};\ \tfrac{1}{2}s,\ \tfrac{1}{2}s')}{\psi(\zeta)},$$

where $s'v^{\zeta, \gamma} = s_1'v_1^{\zeta, \gamma} + \ldots\ldots + s_p'v_p^{\zeta, \gamma}$, and A is constant, independent of ζ and γ. It is clear first of all that the two sides of this equation have the same poles and zeros in the region S. For $\Theta(v^{\zeta, \gamma} + \frac{1}{2}\Omega_{s, s'})$ vanishes to the first order at the places $\gamma, n_1, \ldots, n_{p-1}$, and $\psi(\zeta)$ vanishes to the first order at $n_1, \ldots, n_{p-1}, \infty$, while $\varpi(\zeta, \gamma)$ vanishes to the first order at $\zeta = \gamma$, and is infinite to the first order at $\zeta = \infty$ *. Thus the quotient of the two sides of the equation has no infinities within the region S. Further the square of this quotient is uniform within the region S, independently of the barriers; for this statement holds of each of the factors

$$\varpi(\zeta, \gamma), \quad \psi^2(\zeta), \quad \Theta(v^{\zeta, \gamma} + \tfrac{1}{2}\Omega_{s, s'}), \quad e^{2\pi i s' v^{\zeta, \gamma}}.$$

And, if ζ be replaced by ζ_n, the square of the quotient of the two sides of the equation becomes (cf. § 175, Chap. X.) multiplied by the factor

$$\left[(-1)^{g_n + h_n}\,\frac{\psi(\zeta_n)/\psi(\zeta)}{\gamma_n\zeta + \delta_n}\right]^2,$$

which is equal to unity. Now† a function of ζ, which is unaltered by the substitutions of the group, and is uniform within the region S, and has no

* At the analogues of $\zeta = \infty$ neither $\varpi(\zeta, \gamma)$ nor $1/\psi(\zeta)$ becomes infinite.

† If $U + iV$ be the function, the integral $\int U dV$, taken round the $2p$ fundamental circles is expressible as a surface integral over S whose elements are positive or zero. In the case considered the former integral vanishes.

infinities, must, like a rational function on a Riemann surface, be a constant. Since the square root of a constant is also a constant the proof of the equation is complete.

From it we infer (i) that

$$\psi(\zeta_n)/\psi(\zeta) = (-1)^{g_n+h_n}(\gamma_n\zeta+\delta_n)(-1)^{s_n},$$

and (ii) that the values of $\psi(\zeta)$ on the two sides of a barrier have a quotient of the form $(-1)^{s'_n}$. The constant factor to be attached to $\psi(\zeta)$ may be chosen so that $A=1$. For this it is sufficient to take for the integral v the expression

$$v = \sum_{i=1}^{p} \Theta_i'(\tfrac{1}{2}\Omega_{s,s'})\, v_i^{\zeta,\gamma},$$

where $\Theta_i'(u) = \partial\Theta(u)/\partial u_i$. Then (cf. § 188, p. 281) the right-hand side, when ζ is near to γ, is equal to $A(\zeta-\gamma)+\dots$, while the left-hand side has the value $(\zeta-\gamma)+\dots$.

238. The developments of an equation analogous to that just obtained, which will be given in Chap. XIV. in connection with the functions there discussed, render it unnecessary for us to pursue the matter further here. The following forms an interesting example of theta functions, of another kind.

Suppose that the quantities $\mu_1, \dots, \mu_p$ are small enough to ensure (cf. § 226) the convergence of the series

$$\lambda(\zeta, \mu) = \sum_i \frac{[\gamma_i\zeta+\delta_i]^{-1}}{\zeta_i-\mu},$$

wherein μ denotes an arbitrary place within the region S, and i denotes a summation extending to every substitution of the group. It will appear that this function is definite in all cases in which the function $\varpi(\zeta, \mu)$ is definite. The function is immediately seen to verify the equations

$$\lambda(\zeta_n, \mu) = (\gamma_n\zeta+\delta_n)\lambda(\zeta, \mu), \quad \lambda(\zeta, \mu_n) = (\gamma_n\mu+\delta_n)\lambda(\zeta, \mu),$$

and

$$\begin{aligned}\lambda(\mu, \zeta) &= \sum_i \frac{1}{\alpha_i\mu+\beta_i-\zeta(\gamma_i\mu+\delta_i)} \\ &= -\sum_i \frac{1}{\delta_i\zeta-\beta_i-\mu(-\gamma_i\zeta+\alpha_i)} \\ &= -\sum \frac{(\gamma_r\zeta+\delta_r)^{-1}}{\zeta_r-m},\end{aligned}$$

where r denotes the substitution inverse to that denoted by i. Thus

$$\lambda(\zeta, \mu) = -\lambda(\mu, \zeta).$$

The function has one pole in the region S, namely at μ, and no other infinities, and if the series be uniformly convergent near $\zeta=\infty$, as we assume, the function vanishes to the first order at $\zeta=\infty$. The excess of the number of its zeros over the number of its poles in S, which is given by

$$\frac{1}{2\pi i}\sum_{n=1}^{p}\int\left[\frac{\lambda'(\zeta_n, \mu)}{\lambda(\zeta_n, \mu)}-\frac{\lambda'(\zeta, \mu)}{\lambda(\zeta, \mu)}\right]d\zeta,$$

where the dash denotes a differentiation in regard to ζ, and the integrals are taken counter-clockwise round the circles C_1', ..., C_p', namely by

$$\frac{1}{2\pi i}\sum_{n=1}^{p}\int\frac{d\zeta}{\zeta-\vartheta_n^{-1}\infty},$$

is equal to p. Thus the function has p zeros in S other than $\zeta=\infty$; denote these by $\mu_1, \dots, \mu_p$. Within any region $\vartheta_n S$ the function has the analogue of μ for a pole, and the analogues of $\mu_1, \dots, \mu_p$ for zeros; it does not vanish at the analogue of $\zeta=\infty$. This result may be verified also by investigating similarly the excess of the number of zeros over the number of poles in any such region; the result is found to be $p-1$.

Consider the ratio

$$f(\zeta)=[\lambda(\zeta, \mu)]^2\div\frac{dv}{d\zeta},$$

where v is any linear function of $v_1^{\zeta, \gamma}, \dots, v_p^{\zeta, \gamma}$; let $\zeta_1, \dots, \zeta_{2p-2}$ denote the zeros of dv. Then $f(\zeta)$ is uniform within the region S, and is unaltered by the substitutions of the group. It has poles $\mu^2, \zeta_1, \dots, \zeta_{2p-2}$, and no other infinities in S, and has zeros $\mu_1^2, \dots, \mu_p^2$, the square of a symbol being written to denote a zero or pole of the second order. Thus we have, precisely as for the case of rational functions on a Riemann surface,

$$2v_n^{\mu_p, \mu}+v_n^{\zeta_1, \mu_1}+v_n^{\zeta_2, \mu_1}+\dots\dots+v_n^{\zeta_{2p-3}, \mu_{p-1}}+v_n^{\zeta_{2p-2}, \mu_{p-1}}\equiv 0, \qquad (\mu=1, 2, \dots, p),$$

or (§ 179, p. 256),

$$(\mu^2, \zeta_1, \dots, \zeta_{2p-2})\equiv(\mu_1^2, \dots, \mu_p^2),$$

and therefore, if $m_1, \dots, m_p$ denote the points in S, derivable from μ (§ 236), such that $\Theta(v^{\zeta, \mu}-v^{x_1, m_1}-\dots\dots-v^{x_p, m_p})$ vanishes in $\zeta=x_1, \dots, \zeta=x_p$, we have (§ 182, p. 265).

$$(\mu_1^2, \dots, \mu_p^2)\equiv(m_1^2, \dots, m_p^2).$$

When the barriers are drawn, let

$$v_n^{\mu_1, m_1}+\dots\dots+v_n^{\mu_p, m_p}=\tfrac{1}{2}(k_i+k_1'\tau_{2,1}+\dots\dots+k_p'\tau_{i,p}), \qquad (i=1, 2, \dots, p),$$

$k_1, \dots, k_p, k_1', \dots, k_p'$ being integers.

Now consider the product $\lambda(\zeta, \mu)\varpi(\zeta, \mu)$. It has no poles, in S, and its zeros are $\mu_1, \dots, \mu_p$. It is an uniform function of ζ, and, subjected to one of the fundamental substitutions of the group it takes the factor

$$\frac{\lambda(\zeta_n, \mu)\varpi(\zeta_n, \mu)}{\lambda(\zeta, \mu)\varpi(\zeta, \mu)}, =(-1)^{g_n+h_n}e^{-2\pi i(v_n^{\zeta, \mu}+\frac{1}{2}\tau_{n,n})}.$$

Hence the function

$$F(\zeta) = \frac{\lambda(\zeta, \mu)\, \varpi(\zeta, \mu)}{\Theta(v^{\zeta, \mu} - \tfrac{1}{2}\Omega)} e^{\pi i k' v^{\zeta, \mu}},$$

wherein $k'v^{\zeta,\mu}$ denotes $k_1'v_1^{\zeta,\mu} + \dots\dots + k_p'v_p^{\zeta,\mu}$, and Ω denotes the p quantities $k_i + k_1'\tau_{i,1} + \dots\dots + k_p'\tau_{i,p}$, has, within S, no zeros or poles, and is such that, for a fundamental substitution,

$$F(\zeta_n)/F(\zeta) = (-1)^{g_n + h_n - k_n}$$

(cf. § 175, Chap. X.); thus, as in the previous article, $F(\zeta)$ is a constant thus, also, $g_n + h_n - k_n$ is an even integer, $= 2H_n$, say, and we have

$$\lambda(\zeta, \mu)\, \varpi(\zeta, \mu) = Ae^{-\pi i k' v^{\zeta, \mu}}\, \Theta(v^{\zeta, \mu} - \tfrac{1}{2}P),$$

where P denotes the p quantities $g_i + h_i + k_1'\tau_{i,1} + \dots\dots + k_p'\tau_{i,p}$, and A is independent of ζ. But, if ζ describe the circumference C_n, the left-hand side is unchanged, and the right-hand side obtains the factor $e^{-\pi i k'_n}$. Thus the integers $k_1', \dots, k_p'$ are all even; put $k_r' = 2H_r'$; then, as

$$\Theta\left(v^{\zeta, \mu} - \frac{g+h}{2} - \tau H'\right) = e^{2\pi i H'\left(v^{\zeta, \mu} - \frac{g+h}{2}\right) - \pi i \tau H'^2}\, \Theta\left(v^{\zeta, \mu} - \frac{g+h}{2}\right),$$

where the notation is that of § 175, Chap. X., we have

$$\lambda(\zeta, \mu)\, \varpi(\zeta, \mu) = B\Theta\left(v^{\zeta, \mu} - \frac{g+h}{2}\right),$$

wherein B is independent of ζ, and therefore, since the interchange of ζ, μ leaves both sides unaltered, B is also independent of μ. The value of B may be expressed by putting $\zeta = \mu$; thence we obtain, finally,

$$\lambda(\zeta, \mu)\, \varpi(\zeta, \mu) = \Theta(v^{\zeta, \mu} - \tfrac{1}{2}g - \tfrac{1}{2}h)/\Theta(\tfrac{1}{2}g + \tfrac{1}{2}h).$$

This equation may be regarded as equivalent to 2^p equations. For if in one of the p fundamental substitutions $\vartheta_r \zeta = (\alpha_r \zeta + \beta_r)/(\gamma_r \zeta + \delta_r)$, we consider the signs of α_r, β_r, γ_r, δ_r all reversed, the function $\lambda(\zeta, \mu)$, which involves the first powers of these quantities, will take a different value. The function $\varpi(\zeta, \mu)$, the p fundamental circles, and the integrals $v^{\zeta, \mu}$ and their periods $\tau_{n,m}$, and therefore the integers $g_1, \dots, g_p$, will remain unchanged, if the barriers remain unaltered. But the integer h_r will be increased by unity.

If, on the other hand, the coefficients α, β, γ, δ remaining unaltered, one of the barriers be drawn differently, the left-hand side of the equation remains unaltered; on the right-hand one of $h_1, \dots, h_p$ will be increased by an integer, say, for example, h_r increased by unity, and therefore each of $\tau_{1,r}, \dots, \tau_{p,r}$ also increased by unity. Putting u for $v^{\zeta, \mu} - \tfrac{1}{2}g - \tfrac{1}{2}h$, and neglecting integral increments of u, the exponent of the general term of the theta series is increased, save for integral multiples of $2\pi i$, by

$$2\pi i\left(-\tfrac{1}{2}\right) n_r + i\pi n_r^2,$$

which is an even multiple of πi, so that the general term is unchanged.

Ex. i. Prove that the function $\lambda(\zeta, \mu)$ can be written in the form

$$\lambda(\zeta, \mu) = \frac{1}{\zeta - \mu}\left[1 + \sum_i{}' (a_i + \delta_i)\{\zeta, \zeta_i \mid \mu, \mu_i\}\right],$$

where the sign of summation refers to all the substitutions of the group, other than the identical substitution, with the condition that when any substitution occurs its inverse must not occur, and $\{\zeta, \zeta_i \mid \mu, \mu_i\}$ denotes $\dfrac{\zeta-\mu}{\zeta-\mu_i}\Big/\dfrac{\zeta_i-\mu}{\zeta_i-\mu_i}$.

Ex. ii. In case $p=1$, where the fundamental substitution is

$$(\zeta' - B)/(\zeta' - A) = \rho\,(\zeta - B)/(\zeta - A),$$

putting $e^{2\pi i u} = (\zeta - B)/(\zeta - A)$, $e^{2\pi i v} = (\mu - B)/(\mu - A)$, prove that

$$\zeta - \mu = \frac{B-A}{2i}\,\frac{\sin\pi(u-v)}{\sin\pi u \sin\pi v}, \quad \{\zeta, \zeta_i \mid \mu, \mu_i\} = 4\rho^i\,\frac{\sin^2\pi(u-v)}{1 - 2\rho^i\cos 2\pi(u-v) + \rho^{2i}},$$

and hence

$$\lambda(\zeta, \mu) = \frac{2i\sin\pi u\sin\pi v}{(B-A)\sin\pi(u-v)}\left[1 + \sum_{i=1}^{\infty}\frac{4(-1)^{hi}\rho^{\frac{1}{2}i}(1+\rho^i)\sin^2\pi(u-v)}{1-2\rho^i\cos 2\pi(u-v)+\rho^{2i}}\right].$$

When $h=0$ this becomes*

$$\frac{4i\omega\sin\pi u\sin\pi v}{(B-A)\,\pi\sigma_3(0)}\,\frac{\sigma_3[2\omega(u-v)]}{\sigma[2\omega(u-v)]},$$

where the sigma functions are formed with 2ω, $2\omega\tau$ as periods, ω being an arbitrary quantity. Thus (§ 235, Ex.)

$$\varpi(\zeta, \mu)\,\lambda(\zeta, \mu) = e^{-2\eta\omega(u-v)^2}\frac{\sigma_3[2\omega(u-v)]}{\sigma_3(0)} = \frac{\vartheta_0(u-v)}{\vartheta_0(0)} = \frac{\Theta(u-v-\frac{1}{2})}{\Theta(\frac{1}{2})},$$

where the symbol ϑ_0 is as in Halphen, *Fonct. Ellip.* (Paris, 1886), Vol. I. pp. 260, 252. This agrees with the general result; in putting $\rho^{\frac{1}{2}} = e^{\pi i\tau}$ we have taken $g=1$; and, as stated, h is here taken zero.

When $h=1$ we similarly find

$$\lambda(\zeta, \mu) = \frac{4i\omega\sin\pi u\sin\pi v}{(B-A)\,\pi\sigma_3(\omega)}\,\frac{\sigma_3[2\omega(u-v+\frac{1}{2})]}{\sigma[2\omega(u-v)]}\,e^{-2\eta\omega(u-v)},$$

and hence

$$\varpi(\zeta, \mu)\,\lambda(\zeta, \mu) = e^{-2\eta\omega(u-v)^2 - 2\eta\omega(u-v)}\frac{\sigma_3[2\omega(u-v+\frac{1}{2})]}{\sigma_3(\omega)}, = \frac{\Theta(u-v)}{\Theta(0)},$$

also in agreement with the general formula. In these formulae $\Theta(u)$ denotes the series

$$\Sigma e^{2\pi i u n + i\pi\tau n^2} = 1 + 2q\cos(2\pi u) + 2q^4\cos(4\pi u) + 2q^9\cos(6\pi u) + \ldots\ldots,$$

where $q = e^{i\pi\tau}$.

* Cf. Halphen, *Fonct. Ellip.* (Paris, 1886), Vol. I. p. 422.

Ex. iii. Denoting

$$\sum_i{}' \frac{(\gamma_i\mu+\delta_i)^{-m}}{(\mu-\mu_i)^n}, \quad \sum_i{}' (a_i+\delta_i)^2 \frac{(\gamma_i\mu+\delta_i)^{-m}}{(\mu-\mu_i)^n},$$

where the summations include all substitutions of the group except the identical substitution, respectively by $u_{m,n}$, $v_{m,n}$, prove that, when ζ is near to μ,

$$\frac{\varpi(\zeta,\mu)}{\zeta-\mu} = 1-\tfrac{1}{2}(\zeta-\mu)^2 u_{2,2}+(\zeta-\mu)^3 u_{2,3}+\tfrac{1}{4}(\zeta-\mu)^4[u_{4,4}-6u_{2,4}+\tfrac{1}{2}u^2_{2,2}+v_{4,4}]+\ldots\ldots$$

Ex. iv. If z, s be two single-valued functions of ζ, without essential singularities, which are unaltered by the substitutions of the group, the algebraic * relation connecting z and s may be associated with a Riemann surface, whereon ζ is an infinitely valued function; and if z, s be properly chosen, any single-valued function of ζ without essential singularities, which is unaltered by the substitutions of the group, is a rational function on the Riemann surface. But if

$$\{\zeta, z\} = \frac{d^2}{dz^2}\log\frac{d\zeta}{dz} - \frac{1}{2}\left(\frac{d}{dz}\log\frac{d\zeta}{dz}\right)^2, \ = \frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2,$$

where $\zeta' = \frac{d\zeta}{dz}$, etc., we immediately find that the value $Z=(a\zeta+\beta)/(\gamma\zeta+\delta)$ gives

$$\{Z, z\} = \{\zeta, z\};$$

therefore, as $\{\zeta, z\}, \ = -\{z, \zeta\}\Big/\left(\frac{dz}{d\zeta}\right)^2$, is a single-valued function of ζ without essential singularities, and is unaltered by the substitutions of the group, we have

$$\{\zeta, z\} = 2I(z, s),$$

where I denotes a rational function. Therefore, if Y denote an arbitrary function, and $P = -\frac{d}{dz}\log\left(Y^2\frac{d\zeta}{dz}\right)$, Y and ζY are the solutions of the equation

$$\frac{d^2Y}{dz^2}+P\frac{dY}{dz}+\left[I+\tfrac{1}{4}P^2+\tfrac{1}{2}\frac{dP}{dz}\right]Y=0,$$

and if Y be chosen so that $Y^2\Big/\frac{dz}{d\zeta}$ is a rational function on the Riemann surface, the coefficients in this equation will also be rational functions. Thus for instance we may take for Y the function $\sqrt{\frac{dz}{d\zeta}}$, in which case $P=0$, or we may take for Y the function $\psi(\zeta), \ = \sqrt{\frac{dv}{d\zeta}\frac{dv}{dy}}$, considered in § 236, which is uniform on the ζ plane when the barriers are drawn, in which case $P = -\frac{d}{dz}\log\frac{dv}{dz}$, and the equation takes the form $\frac{d^2Y}{dv^2}+R.Y=0$, where R is a rational function, or again we may take for Y the uniform function of ζ, $\lambda(\zeta,\mu)$, considered in § 238†.

* Ex. viii. § 232.

† Cf. Riemann, *Ges. Werke* (Leipzig, 1876), p. 416, p. 415; Schottky, *Crelle*, LXXXIII. (1877), p. 336 ff.

Ex. v. If, as in Ex. iv., we suppose a Riemann surface constructed such that to every point ζ of the ζ plane there corresponds a place (z, s) of the Riemann surface, and in particular to the point $\zeta=\xi$ there corresponds the place (x, y), and if R, S be functions of ξ defined by the expansions

$$\frac{d}{dx}\log \varpi(\zeta, \xi)=-\frac{1}{z-x}+F+(z-x)R+\ldots\ldots, \quad \frac{\varpi(\zeta, \xi)}{\zeta-\xi}=1-\tfrac{1}{2}S(\zeta-\xi)^2+\ldots\ldots,$$

prove that

$$\tfrac{1}{6}\{\xi, x\}=R-\left(\frac{d\xi}{dx}\right)^2 S,$$

and that R, S are rational functions of x and y.

Ex. vi. The last two examples suggest a problem of capital importance—given any Riemann surface, to find a function ζ, which will effect a conformal representation of the surface to such a ζ-region as that here discussed. This problem may be regarded as that of finding a suitable form for the rational function $I(z, s)$. The reader may consult Schottky, *Crelle*, LXXXIII. (1877), p. 336, and *Crelle*, CI. (1887), p. 268, and Poincaré, *Acta Mathematica*, IV. (1884), p. 224, and *Bulletin de la Soc. Math. de France*, t. XI. (18 *May*, 1883), p. 112. In the elliptic case, taking

$$z=\wp\left(\frac{1}{2\pi i}\log\frac{\zeta-B}{\zeta-A}\right), \ =\wp(u),$$

where $\wp$ denotes Weierstrass's function with 1 and τ as periods, it is easy to prove that $\sqrt{\dfrac{du}{d\zeta}}$ and $\zeta\sqrt{\dfrac{du}{d\zeta}}$ are the solutions of the equation

$$(4z^3-g_2z-g_3)\frac{d^2Y}{dz^2}+(6z^2-\tfrac{1}{2}g_2)\frac{dY}{dz}+\pi^2Y=0.$$

239. There is one case of the theory which may be referred to in conclusion. Take p circles $C_1, \ldots, C_p$, exterior to one another, which are all cut at right angles by another circle O; take a further circle C cutting this orthogonal circle O at right angles; invert the circles $C_1, C_2, \ldots$ in regard to C. We shall obtain p circles $C_1', C_2', \ldots, C_p'$ also cutting the orthogonal circle O at right angles. The case referred to is that in which the circles $C_1, C_1', \ldots, C_p, C_p'$ are the fundamental circles and the angles $\kappa_1, \ldots, \kappa_p$ are all zero, so that, if ϑ_n denote one of the p fundamental substitutions, the corresponding points ζ, $\vartheta_n\zeta$ lie on a circle through A_n and B_n. We may suppose that the circles $C_1, \ldots, C_p$ are all interior to the circle C. It can be shewn by elementary geometry that A_n, B_n are inverse points in regard to the circle C as well as in regard to the circle C_n, and further that if ω denote the process of inversion in regard to the circle C and ω_n that of inversion in regard to C_n, the fundamental substitution ϑ_n is $\omega_n\omega$, so that $\omega\vartheta_n\omega=\vartheta_n^{-1}$, or $\omega\vartheta_n=\vartheta_n^{-1}\omega$. Hence if the points of intersection of the circles O, C_n be called a_n', b_n', the points of intersection of O, C_n' be called a_n, b_n, and the points of intersection of O, C be called a, b, it may be shewn without much difficulty that

$$v_n^{a_r,\, b_r}=P_{n,r}, \quad v_n^{a_n,\, b_n}=\tfrac{1}{2}+Q_n, \quad v_n^{a,\, b}=\tfrac{1}{2}+R, \quad (n, r=1, 2, \ldots, p;\ n\neq r),$$

where $P_{n,r}$, Q_n, R are integers, and the integrations are along the perimeters of the several circles. Hence it follows that the uniform functions of ζ expressed by $e^{2\Pi_{a_r, b_r}^{\zeta, c}}$, $e^{2\Pi_{a, b}^{\zeta, c}}$ are unaltered by the substitutions of the group. Denote them, respectively, by $x_r(\zeta)$ and $x(\zeta)$. Each of them has a single pole of the second order, and a single zero of the second order, and therefore, as in the case of rational functions on a hyperelliptic Riemann surface, we have, absorbing a constant factor in $x_r(\zeta)$, an equation of the form

$$x_r(\zeta) = \frac{x(\zeta) - x(a_r)}{x(\zeta) - x(b_r)}.$$

But it follows also that the function

$$y(\zeta), = e^{\Pi_{a, b}^{\zeta, c} + \Pi_{a_1, b_1}^{\zeta, c} + \dots\dots + \Pi_{a_p, b_p}^{\zeta, c}},$$

is unaltered by the substitutions of the group. Hence we have*, writing y, x for $y(\zeta)$, $x(\zeta)$, etc.,

$$y^2 = xx_1 \dots x_p = x \frac{[x - x(a_1)] \dots\dots [x - x(a_p)]}{[x - x(b_1)] \dots\dots [x - x(b_p)]}.$$

Thus the special case under consideration corresponds to a hyperelliptic Riemann surface; and, for example, the equations $v_n^{a_n, b_n} = \frac{1}{2} + Q_n$, etc., correspond to part of the results obtained in § 200, Chap. XI. It is manifest that the theory is capable of great development. The reader may consult Weber, *Göttinger Nachrichten*, 1886, "Ein Beitrag zu Poincaré's Theorie, u. s. w.," also, Burnside, *Proc. London Math. Soc.* XXIII. (1892), p. 283, and Poincaré, *Acta Math.* III. p. 80 and *Acta Math.* IV. p. 294 (1884); also Schottky, *Crelle*, CVI. (1890), p. 199. For the general theory of automorphic functions references are given by Forsyth, *Theory of Functions* (1893), p. 619. The particular case considered in this chapter is intended only to illustrate general ideas. From the point of view of the theory of this volume, Chapter XIV. may be regarded as an introduction to the theory of automorphic functions (cf. Klein, *Math. Annalen*, XXI. (1883), p. 141, and Ritter, *Math. Annalen*, XLIV. (1894), p. 261).

* The function x here employed is not identical in case $p=1$ with the z of Ex. vi. § 238.