CHAPTER XI.

THE HYPERELLIPTIC CASE OF RIEMANN'S THETA FUNCTIONS.

199. WE have seen (Chap. V.) that the hyperelliptic case* is a special one, characterised by the existence of a rational function of the second order. In virtue of this circumstance we are able to associate the theory with a simple algebraical relation, which we may take to be of the form

$$y^2 = 4(x-a_1)\dots(x-a_p)(x-c_1)\dots(x-c_{p+1}).$$

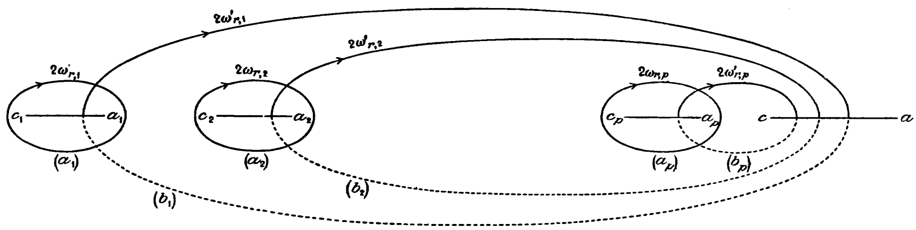
We have seen moreover (Chap. X. § 185) that in the hyperelliptic case, when p is greater than 2, there are always even theta functions which vanish for zero values of the argument. We may expect, therefore, that the investigation of the relations connecting the Riemann theta functions with the algebraical functions will be comparatively simple, and furnish interesting suggestions for the general case. It is also the fact that the grouping of the characteristics of the theta functions, upon which much of the ultimate theory of these functions depends, has been built up directly from the hyperelliptic case.

It must be understood that the present chapter is mainly intended to illustrate the general theory. For fuller information the reader is referred to the papers quoted in the chapter, and to the subsequent chapters of the present volume.

* For the subject-matter of this chapter, beside the memoirs of Rosenhain, Göpel, and Weierstrass, referred to in § 173, Chap. X., which deal with the hyperelliptic case, and general memoirs on the theta functions, the reader may consult, Prym, *Zur Theorie der Functionen in einer zweiblättrigen Fläche* (Zürich, 1866); Prym, *Neue Theorie der ultraellip. Funct.* (zweite Aus., Berlin, 1885); Schottky, *Abriss einer Theorie der Abel. Functionen von drei Variabeln* (Leipzig, 1880), pp. 147—162; Neumann, *Vorles. über Riem. Theorie* (Leipzig, 1884); Thomae, *Sammlung von Formeln welche bei Anwendung der . . Rosenhain'schen Functionen gebraucht werden* (Halle, 1876); Brioschi, *Ann. d. Mat.* t. x. (1880), and t. xiv. (1886); Thomae, *Crelle*, LXXI. (1870), p. 201; Krause, *Die Transformation der hyperellip. Funct. erster Ordnung* (Leipzig, 1886); Forsyth, "Memoir on the theta functions," *Phil. Trans.*, 1882; Forsyth, "On Abel's theorem," *Phil. Trans.*, 1883; Cayley, "Memoir on the . . theta functions," *Phil. Trans.*, 1880, and *Crelle*, Bd. 83, 84, 85, 87, 88; Bolza, *Göttinger Nachrichten* 1894, p. 268. The addition equation is considered in a dissertation by Hancock, Berlin, 1894 (Bernstein). For further references see the later chapters of this volume which deal with theta functions.

200. Throughout this chapter we suppose the relative positions of the branch places and period loops to be as in the annexed figure (4), the branch place a being at infinity.

Fig. 4.



In the general case, in considering the zeros of the function $\vartheta(u^{x,\,m}-e)$, we were led to associate with the place m, other p places $m_1, \ldots, m_p$, such that $\vartheta(u^{x,\,m})$ has $m_1, \ldots, m_p$ for its zeros (Chap. X. § 179). In this case we shall always take m at the branch place a, that is at infinity. It can be shewn that if b, b' denote any two of the branch places, the p integrals $u_1^{b,\,b'}, \ldots, u_p^{b,\,b'}$ are the p simultaneous constituents of a half-period, so that

$$u_r^{b,\,b'}=m_1\omega_{r,\,1}+\ldots\ldots+m_p\omega_{r,\,p}+m_1'\omega'_{r,\,1}+\ldots\ldots+m_p'\omega'_{r,\,p},\quad (r=1,\,2,\,\ldots,p),$$

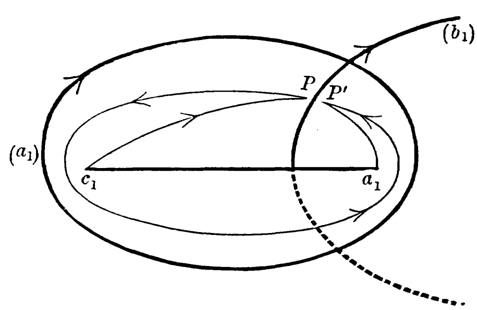
wherein $m_1, \ldots, m_p, m_1', \ldots, m_p'$ are integers, independent of r; this fact we shall often denote by putting $u^{b,\,b'}=\frac{1}{2}\Omega$. It can further be shewn that if, b remaining any branch place, b' is taken to be each of the other $2p+1$ branch places in turn, the $2p+1$ half-periods, $u^{b,\,b'}$, thus obtained, consist of p odd half-periods, and $p+1$ even half-periods. Thus if the branch places, b', for which $u^{b,\,b'}$ is an odd half-period be denoted by $b_1, \ldots, b_p$, we have, necessarily, $\vartheta(u^{b,\,b_1})=0, \ldots, \vartheta(u^{b,\,b_p})=0$, and we may take, for the places $m, m_1, \ldots, m_p$, the places $b, b_1, \ldots, b_p$. In particular it can be shewn that, when for b the branch place a is taken, and the branch places are situated as in the figure (4), each of $u^{a,\,a_1}, \ldots, u^{a,\,a_p}$ is an odd half-period. We have therefore the statement, which is here fundamental, *the function* $\vartheta(u^{x,\,a}-u^{x_1,\,a_1}-\ldots-u^{x_p,\,a_p})$ *has the places* $x_1, \ldots, x_p$ *as its zeros. It is assumed that the function* $\vartheta(u^{x,\,a})$ *does not vanish identically. This assumption will be seen to be justified.*

For our present purpose it is sufficient to prove (i) that each of the integrals $u^{b,\,b'}$ is a half-period, (ii) that each of the integrals $u^{a,\,a_1}, \ldots, u^{a,\,a_p}$ is an odd half-period. In regard to (i) the general statement is as follows: Let the period loops of the Riemann surface be projected on to the plane upon which the Riemann surface is constructed, forming such a network as that represented in the figure (4); denote the projection of the loop (a_r) by (A_r), and that of (b_r) by (B_r), and suppose (A_r), (B_r) affected with arrow heads, as in the figure, whereby to define the left-hand side, and the right-hand side; finally let a continuous curve be drawn on the plane of projection, starting from the projection of the branch place b' and ending in the projection of the branch place b; then if this curve cross the loop (A_r) m_r times from right to left, so that m_r is either $+1$ or -1, or 0, and cross the loop (B_r) m_r' times from right to left, we have

$$u_r^{b,\,b'} = m_1\omega_{r,\,1} + \dots\dots + m_p\omega_{r,\,p} + m_1'\omega'_{r,\,1} + \dots\dots + m_p'\omega'_{r,\,p}.$$

Thus, for instance, in accordance with this statement we should have $u_r^{a_1,\,c_1} = -\omega'_{r,\,1}$, and $u_r^{c_2,\,a_1} = \omega_{r,\,1} - \omega_{r,\,2}$, and it will be sufficient to prove the first of these results; the general proof is exactly similar. Now we can pass from c_1 to a_1, on the Riemann surface, by a curve lying in the upper

Fig. 5.



sheet which goes first to a point P on the left-hand side of the loop (b_1), and thence, following a course coinciding roughly with the right-hand side of the loop (a_1), goes to the point P', opposite to P on the right-hand side of (b_1), and thence, from P', goes to a_1. Thus we have

$$u_r^{a_1,\,c_1} = u_r^{P,\,c_1} - 2\omega'_{r,\,1} + u_r^{a_1,\,P'}.$$

On the other hand we can pass from c_1 to a_1 by a path lying entirely in the lower sheet, and consisting of two portions, from c_1 to P, and from P' to a_1, lying just below the paths from c_1 to P and from P' to a_1, which are in the upper sheet. Thus we have a result which we may write in the form

$$u_r^{a_1,\,c_1} = (u_r^{P,\,c_1})' + (u_r^{a_1,\,P'})'.$$

But, in fact, as the integral $u_r^{x,\,a}$ is of the form $\int \frac{(x,\,1)_{p-1}}{y}\,dx$, and y has different signs in the two sheets, we have

$$(u_r^{P,\,c_1})' = -\,u_r^{P,\,c_1}, \text{ and } (u_r^{a_1,\,P'})' = -\,u_r^{a_1,\,P'}.$$

Therefore, by addition of the equations we have

$$u_r^{a_1, c_1} = -\omega'_{r, 1},$$

which proves the statement made.

In regard now to the proof that $u^{a, a_1}, \ldots, u^{a, a_p}$ are all odd half-periods, we clearly have, in accordance with the results just obtained,

$$u_r^{a, a_i} = \omega_{r, i} - (\omega_{r, i+1} + \omega'_{r, i+1}) - \ldots\ldots - (\omega_{r, p} + \omega'_{r, p}) + (\omega'_{r, 1} + \ldots\ldots + \omega'_{r, p}),$$

which is equal to

$$(\omega'_{r, 1} + \omega'_{r, 2} + \ldots\ldots + \omega'_{r, i}) + (\omega_{r, i} - \omega_{r, i+1} - \ldots\ldots - \omega_{r, p}),$$

and if this be written in the form

$$m_1\omega_{r, 1} + \ldots\ldots + m_p\omega_{r, p} + m_1'\omega'_{r, 1} + \ldots\ldots + m'_p\omega'_{r, p}$$

we obviously have $m_1m_1' + \ldots\ldots + m_pm_p' = 1$.

Ex. i. We have stated that if b be any branch place there are p other branch places $b_1, b_2, \ldots, b_p$, such that $u^{b, b_1}, u^{b, b_2}, \ldots, u^{b, b_p}$ are odd half-periods, and that, if b' be any branch place other than $b, b_1, \ldots, b_p$, $u^{b, b'}$ is an even half-period. Verify this statement in case $p=2$, by calculating all the fifteen, $=\frac{1}{2}6 \,.\, 5$, integrals of the form $u^{b, b'}$, and prove that when b is in turn taken at a, c, c_1, c_2, a_1, a_2 the corresponding pairs b_1, b_2 are respectively

$$(a_1, a_2),\ (c_1, c_2),\ (c_2, c),\ (c_1, c),\ (a_2, a),\ (a_1, a).$$

Prove also that

$$u_r^{c, a} + u_r^{c_1, a_1} + u_r^{c_2, a_2} = 0.$$

Ex. ii. The reader will find it an advantage at this stage to calculate some of the results of the second and fifth columns in the tables given below (§ 204).

201. Consider now the $2p+1$ half-periods $u^{b, a}$ wherein b is any of the branch places other than a. From these we can form $\binom{2p+1}{2}$ half-periods, of the form $u^{b, a} + u^{b', a}$, wherein b, b' are any two different branch places, other than a, and $\binom{2p+1}{3}$ half-periods of the form $u^{b, a} + u^{b', a} + u^{b'', a}$, where b, b', b'' are any three different branch places other than a, and so on, and finally we can form $\binom{2p+1}{p}$ half-periods by adding any p of the half-periods $u^{b, a}$. The number

$$\binom{2p+1}{1} + \binom{2p+1}{2} + \ldots\ldots + \binom{2p+1}{p}$$

is equal to $-1 + \frac{1}{2}\left[(x+1)^{2p+1}\right]_{x=0}$, or to $2^{2p} - 1$, and therefore equal to the whole number of existent half-periods of which no two differ by a period, with the exclusion of the identically zero half-period ; we may say that this number is equal to the number of incongruent half-periods, omitting the identically zero half-period.

And in fact the $2^{2p}-1$ half-periods thus obtained are themselves incongruent. For otherwise we should have congruences of the form

$$u^{b_1, a} + u^{b_2, a} + \ldots\ldots + u^{b_r, a} \equiv u^{b_1', a} + u^{b_2', a} + \ldots\ldots + u^{b_s', a},$$

wherein any integral $u^{b_\kappa, a}$ that occurs on both sides of the congruence may be omitted. Since every one of these integrals is a half-period, and therefore $u^{b_\kappa, a} \equiv - u^{b_\kappa, a}$, we may put this congruence in the form

$$u^{b_1, a} + u^{b_2, a} + \ldots\ldots + u^{b_m, a} \equiv 0,$$

and here, since we are only considering the half-periods formed by sums of p, or less, different periods, m cannot be greater than $2p$. Now this congruence is equivalent with the statement that there exists a rational function having a for an m-fold pole and having $b_1, \ldots, b_m$ for zeros of the first order (Chap. VIII. § 158). Since a is at infinity, such a function can be expressed in the form (Chap. V. § 56)

$$(x, 1)_r + y\,(x, 1)_s,$$

and the number of its zeros is the greater of the integers $2r$, $2p+1+s$. Thus the function under consideration would necessarily be expressible in the form $(x, 1)_r$. But such a function, if zero at a branch place, would be zero to the second order. Thus no such function exists.

On the other hand the rational function y is zero to the first order at each of the branch places $a_1, \ldots, a_p, c_1, \ldots, c_p, c$, and is infinite at a to the $(2p+1)$th order ; hence we have the congruence

$$u^{a_1, a} + \ldots\ldots + u^{a_p, a} + u^{c_1, a} + \ldots\ldots + u^{c_p, a} + u^{c, a} \equiv 0.$$

202. With the half-period of which one element is expressed by

$$m_1 \omega_{r,1} + \ldots\ldots + m_p \omega_{r,p} + m_1' \omega'_{r,1} + \ldots\ldots + m_p' \omega'_{r,p},$$

we may associate the symbol

$$\begin{pmatrix} k_1', & k_2', & \ldots, & k_p' \\ k_1, & k_2, & \ldots, & k_p \end{pmatrix},$$

wherein k_s, equal to 0 or 1, is the remainder when m_s is divided by 2. The sum of two or more such symbols is then to be formed by adding the $2p$ elements separately, and replacing the sum by the remainder on division by 2. Thus for instance, when $p=2$, we should write $\binom{01}{11}+\binom{11}{01}=\binom{10}{10}$. If we call this symbol the characteristic-symbol, we have therefore proved, in the previous article, that *each of the* $2^{2p}-1$ *possible characteristic-symbols other than that one which has all its elements zero can be obtained as the sum of not more than* p *chosen from* $2p+1$ *fundamental characteristic-symbols, these* $2p+1$ *fundamental characteristic-symbols having as their sum the symbol of which all the elements are zero. In the method here adopted* p *of the fundamental symbols are associated with odd half-periods* (namely those given by $u^{a, a_1}, \ldots, u^{a, a_p}$), *and the other* $p+1$ *with even half-periods.* It is manifest that this theorem for characteristic-symbols, though derived by consideration of the hyperelliptic case, is true for all cases*. We may denote the fundamental symbols which correspond to the odd half-periods by the numbers $1, 3, 5, \ldots, 2p-1$, and those which correspond to the even half-periods by the numbers $0, 2, 4, 6, \ldots, 2p$, reserving the number $2p+1$ to represent the symbol of which all the elements are zero. Then a symbol which is formed by adding k of the fundamental symbols may be represented by placing their representative numbers in sequence.

Thus for instance, for $p=2$, Weierstrass has represented the symbols

$$\binom{10}{11}\ \binom{01}{01}\ \binom{00}{11}\ \binom{10}{01}\ \binom{01}{00}\ \binom{00}{00}$$

respectively by the numbers

$$1 \quad 3 \quad 0 \quad 2 \quad 4 \quad 5;$$

and, accordingly, represented the symbol $\binom{10}{10}$, which is equal to $\binom{00}{11}+\binom{10}{01}$, by the compound number 02. The $\binom{5}{2}=10$ combinations of the symbols 1, 3, 0, 2, 4 in pairs, represent the $2^{2p}-6$ symbols other than those here written. Further illustration is afforded by the table below (§ 204).

In case $p=3$, there will be seven fundamental symbols which may be represented by the numbers 0, 1, 2, 3, 4, 5, 6. All other symbols are represented either by a combination of two of these, or by a combination of three of them.

It may be mentioned that the fact that, for $p=3$, all the symbols are thus representable by seven fundamental symbols is in direct correlation with the fact that a plane quartic is determined when seven proper double tangents are given.

* The theorem is attributed to Weierstrass (Stahl, *Crelle*, LXXXVIII. pp. 119, 120). A further proof, and an extension of the theorem, are given in a subsequent chapter.

203. If in the half-period $\frac{1}{2}\Omega_{m,\,m'}$, of which an element is given by

$$\tfrac{1}{2}\Omega_{m,\,m'} = m_1\omega_{r,\,1} + \dots\dots + m_p\omega_{r,\,p} + m_1'\omega'_{r,\,1} + \dots\dots + m_p'\omega'_{r,\,p},$$

we write $\frac{1}{2}m_s = M_s + \frac{1}{2}k_s$, $\frac{1}{2}m_s' = M_s' + \frac{1}{2}k_s'$, where M_s, M_s' denote integers, and each of k_s, k_s' is either 0 or 1, we have (cf. the formulæ § 190, Chap. X.)

$$\vartheta(u + \tfrac{1}{2}\Omega_{m,\,m'}) = \vartheta(u\,;\; M + \tfrac{1}{2}k,\; M' + \tfrac{1}{2}k')\,e^{\lambda},$$

where

$$\lambda = [2\eta(M + \tfrac{1}{2}k) + 2\eta'(M' + \tfrac{1}{2}k')]\,[u + \omega(M + \tfrac{1}{2}k) + \omega'(M' + \tfrac{1}{2}k')] - \pi i(M + \tfrac{1}{2}k)(M' + \tfrac{1}{2}k'),$$

and therefore

$$\vartheta(u\,;\; \tfrac{1}{2}k,\, \tfrac{1}{2}k') = e^{-\lambda - \pi i M k'}\,\vartheta(u + \tfrac{1}{2}\Omega_{m,\,m'}).$$

The function represented by either side of this equation will sometimes be represented by $\vartheta(u \mid \frac{1}{2}\Omega_{m,\,m'})$; or if $\frac{1}{2}\Omega_{m,\,m'} = u^{b_1,\,a} + u^{b_2,\,a} + \dots\dots + u^{b_s,\,a}$, the function will sometimes be represented by $\vartheta(u \mid u^{b_1,\,a} + \dots\dots + u^{b_s,\,a})$, or by $\vartheta_{b_1 b_2 \dots b_s}(u)$.

We have proved in the last chapter (§§ 184, 185) that every odd half-period can be represented in the form

$$\tfrac{1}{2}\Omega \equiv u^{m_p,\,m} - u^{n_1,\,m_1} - \dots\dots - u^{n_{p-1},\,m_{p-1}},$$

and, when there are no even theta functions which vanish for zero values of the argument, that every even half-period can be represented in the form

$$\tfrac{1}{2}\Omega' \equiv u^{m_1',\,m_1} + \dots\dots + u^{m_p',\,m_p}\,;$$

in the hyperelliptic case every odd half-period can be represented in the form

$$\tfrac{1}{2}\Omega \equiv u^{a_p,\,a} - u^{n_1,\,a_1} - \dots\dots - u^{n_{p-1},\,a_{p-1}},$$

and every even half-period $\frac{1}{2}\Omega'$, for which $\vartheta(\frac{1}{2}\Omega')$ does not vanish, can be represented in the form

$$\tfrac{1}{2}\Omega' \equiv u^{b_1,\,a_1} + \dots\dots + u^{b_p,\,a_p},$$

and (§ 182, Chap. X.) the zeros of the function $\vartheta(u^{x,\,z} \mid \frac{1}{2}\Omega)$ consist of the place z and the places $n_1, \dots, n_p$, while the zeros of the function $\vartheta(u^{x,\,a} \mid \frac{1}{2}\Omega')$ are the places $b_1, \dots, b_p$. In case $p = 2$ there are no even theta functions vanishing for zero values of the argument; in case $p = 3$ there is one such function (§ 185, Chap. X.), and the corresponding even half-period $\frac{1}{2}\Omega''$ is such that we can put

$$\tfrac{1}{2}\Omega'' \equiv u^{a_3,\,a} - u^{x_1,\,a_1} - u^{x_2,\,a_2},$$

wherein x_1 is an arbitrary place and x_2 is the place conjugate to x_1. Since then $u^{x_2, a_2} \equiv -u^{x_1, a_2}$, this equation gives

$$\tfrac{1}{2}\Omega'' \equiv u^{a_3, a} - u^{a_2, a_1};$$

now, as in § 200, we easily find

$$u_r^{a_3, a} = -(\omega_{r,3} + \omega'_{r,1} + \omega'_{r,2} + \omega'_{r,3}), \quad u^{a_2, a_1} = \omega_{r,1} - \omega_{r,2} - \omega'_{r,2},$$

and therefore

$$\tfrac{1}{2}\Omega'' \equiv -\omega_{r,1} + \omega_{r,2} - \omega_{r,3} - (\omega'_{r,1} + \omega'_{r,3}).$$

Thus the even theta function which vanishes for zero values of the argument is that associated with the characteristic symbol $\begin{pmatrix}101\\111\end{pmatrix}$.

In the same way for $p = 4$, the 10 even theta functions which vanish for zero values of the argument are (§ 185, Chap. X.) associated with even half-periods given by

$$\tfrac{1}{2}\Omega'' = u^{a_4, a} - u^{b, a_3} - u^{a_1, a_2},$$

where b is in turn each of the ten branch places.

204. The following table gives the results for $p=2$. The reader is recommended to verify the second and fifth columns. The set of p equations represented by the equation $(\tfrac{1}{2}\Omega)_r = m_1\omega_{r,1} + m_2\omega_{r,2} + m_1'\omega'_{r,1} + m_2'\omega'_{r,2}$ is denoted by putting $\tfrac{1}{2}\Omega = \tfrac{1}{2}\begin{pmatrix}m_1' m_2'\\ m_1 m_2\end{pmatrix}$.

I. Six odd theta functions in the case $p=2$.

Function	We have	Weierstrass's number associated with this symbol		Putting the corresponding half-period $\equiv u^{a_2, a} - u^{n_1, a_1}$, we have for n_1 respectively
$\vartheta_{aa_1}(u)$	$u^{a, a_1} = \tfrac{1}{2}\begin{pmatrix}10\\10\end{pmatrix}$	02	(1)	a_2
$\vartheta_{aa_2}(u)$	$u^{a, a_2} = \tfrac{1}{2}\begin{pmatrix}11\\01\end{pmatrix}$	24	(3)	a_1
$\vartheta_{a_1a_2}(u)$	$u^{a_1, a_2} = \tfrac{1}{2}\begin{pmatrix}01\\-11\end{pmatrix}$	04	(13)	a
$\vartheta_{c_1c_2}(u)$	$u^{c_1, c_2} = \tfrac{1}{2}\begin{pmatrix}10\\-11\end{pmatrix}$	1	(24)	c
$\vartheta_{cc_1}(u)$	$u^{c_1, c} = \tfrac{1}{2}\begin{pmatrix}11\\-10\end{pmatrix}$	13	(02)	c_2
$\vartheta_{cc_2}(u)$	$u^{c_2, c} = \tfrac{1}{2}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}$	3	(04)	c_1

II. Ten even theta functions in the case $p=2$.

Function	We have	Weierstrass's number associated with this symbol		Putting the corresponding half-period $\equiv u^{b_1, a_1} + u^{b_2, a_2}$, we have for b_1, b_2
$\vartheta(u)$	$\frac{1}{2}\begin{pmatrix}00\\00\end{pmatrix}$	5		a_1, a_2
$\vartheta_{ac}(u)$	$u^{a,c} = \frac{1}{2}\begin{pmatrix}11\\00\end{pmatrix}$	23	(0)	c_1, c_2
$\vartheta_{ac_1}(u)$	$u^{a,c_1} = \frac{1}{2}\begin{pmatrix}00\\10\end{pmatrix}$	12	(2)	c, c_2
$\vartheta_{ac_2}(u)$	$u^{a,c_2} = \frac{1}{2}\begin{pmatrix}10\\01\end{pmatrix}$	2	(4)	c, c_1
$\vartheta_{a_1c_1}(u)$	$u^{c_1,a_1} = \frac{1}{2}\begin{pmatrix}10\\00\end{pmatrix}$	01	(12)	a_2, c_1
$\vartheta_{a_1c_2}(u)$	$u^{c_2,a_1} = \frac{1}{2}\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}$	0	(14)	a_2, c_2
$\vartheta_{a_2c_1}(u)$	$u^{c_1,a_2} = \frac{1}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}$	14	(23)	a_1, c_1
$\vartheta_{a_2c_2}(u)$	$u^{c_2,a_2} = \frac{1}{2}\begin{pmatrix}01\\00\end{pmatrix}$	4	(34)	a_1, c_2
$\vartheta_{ca_2}(u)$	$u^{c,a_2} = \frac{1}{2}\begin{pmatrix}00\\01\end{pmatrix}$	34	(03)	a_1, c
$\vartheta_{ca_1}(u)$	$u^{c,a_1} = \frac{1}{2}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	03	(01)	a_2, c

The numbers in brackets in the fourth column might be employed instead of the Weierstrass numbers; they are based on the branch places according to the correspondence

$$\begin{matrix} 1 & 3 & 0 & 2 & 4 \\ a_1 & a_2 & c & c_1 & c_2. \end{matrix}$$

But the Weierstrass notation is now so fully established that it will be employed here whenever any such notation is used.

It should be noticed that the letter notation for an odd function consists always of two a's or two c's; the letter notation for an even function contains one a and one c.

The expression of the half-period associated with any function as a sum of not more than two of the integrals $u^{b,a}$, which has been described in § 202, is of course immediately indicated by the letter notation employed for the functions.

Ex. Prove that if $\mathfrak{a} = \frac{1}{2}\begin{pmatrix}01\\11\end{pmatrix}$

$$u^{a,a_1} + \mathfrak{a} \equiv u^{a,a_2} \qquad u^{c_1,c_2} + \mathfrak{a} \equiv u^{a,c} \qquad u^{c,c_1} + \mathfrak{a} \equiv u^{a,c_2}$$

$$u^{a,a_2} + \mathfrak{a} \equiv u^{a,a_1} \qquad u^{c,c_2} + \mathfrak{a} \equiv u^{a,c_1}.$$

These equations effect a correspondence between five of the odd functions and the branch places.

205. Next we give the corresponding results for $p=3$. Each half-period can be formed as a sum of not more than 3 of the seven integrals $u^{b,\,a}$ (§ 202); the proper integrals are indicated by the suffix letters employed to represent the function. We may also associate the branch places with the numbers 0, 1, 2, 3, 4, 5, 6, say, in accordance with the scheme

$$\begin{array}{ccccccc} a_1, & a_2, & a_3, & c, & c_1, & c_2, & c_3 \\ 1, & 3, & 5, & 0, & 2, & 4, & 6\,; \end{array}$$

then the functions $\vartheta_1(u)$, $\vartheta_3(u)$, $\vartheta_5(u)$ will be odd, and the functions $\vartheta_0(u)$, $\vartheta_2(u)$, $\vartheta_4(u)$, $\vartheta_6(u)$ will be even; and every function will have a suffix formed of 1 or 2 or 3 of these numbers. There is however another way in which the 64 characteristics can be associated with the combinations of seven numbers, and one which has the advantage that all the seven numbers and their 21 combinations of two are associated with odd functions, while all the even functions except that in which the associated half-period is zero are associated with their 35 combinations of three. It will be seen in a later chapter in how many ways such a scheme is possible. One way is that in which the numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7$$

are associated respectively with the half-periods given by

$$u^{a_1,\,a},\ u^{a_2,\,a},\ u^{a_3,\,a},\quad u^{c,\,a}+u^{c_2,\,a}+u^{c_3,\,a},\quad u^{c,\,a}+u^{c_3,\,a}+u^{c_1,\,a},\quad u^{c,\,a}+u^{c_1,\,a}+u^{c_2,\,a},$$
$$u^{c_1,\,a}+u^{c_2,\,a}+u^{c_3,\,a}.$$

By § 201 the sum of these integrals is $\equiv 0$. The numbers thus obtained are given in the second column. Further every odd half-period can be represented by a sum $u^{a_3,\,a}-u^{n_1,\,a_1}-u^{n_2,\,a_2}$, and all the even half-periods except one as a sum $u^{b_1,\,a_1}+u^{b_2,\,a_2}+u^{b_3,\,a_3}$; the positions of n_1, n_2 or of b_1, b_2, b_3 are given in the fourth column.

I. 28 odd theta functions for $p=3$.

			$n_1, n_2=$
$\vartheta_{a_1}(u)$	1	$u^{a_1,\,a}\equiv\frac{1}{2}\begin{pmatrix}100\\100\end{pmatrix}$	$a_2,\ a_3$
$\vartheta_{a_2}(u)$	2	$u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}110\\010\end{pmatrix}$	$a_3,\ a_1$
$\vartheta_{a_3}(u)$	3	$u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}111\\001\end{pmatrix}$	$a_1,\ a_2$
$\vartheta_{a_1a_2}(u)$	12	$u^{a_1,\,a}+u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}010\\110\end{pmatrix}$	$a\ ,\ a_3$
$\vartheta_{a_1a_3}(u)$	13	$u^{a_1,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}011\\101\end{pmatrix}$	$a\ ,\ a_2$
$\vartheta_{a_2a_3}(u)$	23	$u^{a_2,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}001\\011\end{pmatrix}$	$a\ ,\ a_1$
$\vartheta_{cc_1}(u)$	74	$u^{c_1,\,a}+u^{c,\,a}\equiv\frac{1}{2}\begin{pmatrix}111\\100\end{pmatrix}$	$c_2,\ c_3$
$\vartheta_{cc_2}(u)$	75	$u^{c_2,\,a}+u^{c,\,a}\equiv\frac{1}{2}\begin{pmatrix}011\\010\end{pmatrix}$	$c_3,\ c_1$

Table I. (continued.)

			$n_1, n_2=$
$\vartheta_{cc_3}(u)$	76	$u^{c_3,\,a}+u^{c,\,a}\equiv\frac{1}{2}\begin{pmatrix}001\\001\end{pmatrix}$	c_1, c_2
$\vartheta_{c_2c_3}(u)$	56	$u^{c_2,\,a}+u^{c_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}010\\011\end{pmatrix}$	c, c_1
$\vartheta_{c_3c_1}(u)$	64	$u^{c_1,\,a}+u^{c_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}110\\101\end{pmatrix}$	c, c_2
$\vartheta_{c_1c_2}(u)$	45	$u^{c_1,\,a}+u^{c_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}100\\110\end{pmatrix}$	c, c_3
$\vartheta_{ca_1a_2}(u)$	37	$u^{c,\,a}+u^{a_1,\,a}+u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}101\\110\end{pmatrix}$	c, a_3
$\vartheta_{ca_1a_3}(u)$	27	$u^{c,\,a}+u^{a_1,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}100\\101\end{pmatrix}$	c, a_2
$\vartheta_{ca_2a_3}(u)$	17	$u^{c,\,a}+u^{a_2,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}110\\011\end{pmatrix}$	c, a_1
$\vartheta_{c_1a_2a_3}(u)$	14	$u^{c_1,\,a}+u^{a_2,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}001\\111\end{pmatrix}$	c_1, a_1
$\vartheta_{c_1a_3a_1}(u)$	24	$u^{c_1,\,a}+u^{a_3,\,a}+u^{a_1,\,a}\equiv\frac{1}{2}\begin{pmatrix}011\\001\end{pmatrix}$	c_1, a_2
$\vartheta_{c_1a_1a_2}(u)$	34	$u^{c_1,\,a}+u^{a_1,\,a}+u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}010\\010\end{pmatrix}$	c_1, a_3
$\vartheta_{c_2a_2a_3}(u)$	15	$u^{c_2,\,a}+u^{a_2,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}101\\001\end{pmatrix}$	c_2, a_1
$\vartheta_{c_2a_3a_1}(u)$	25	$u^{c_2,\,a}+u^{a_3,\,a}+u^{a_1,\,a}\equiv\frac{1}{2}\begin{pmatrix}111\\111\end{pmatrix}$	c_2, a_2
$\vartheta_{c_2a_1a_2}(u)$	35	$u^{c_2,\,a}+u^{a_1,\,a}+u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}110\\100\end{pmatrix}$	c_2, a_3
$\vartheta_{c_3a_2a_3}(u)$	16	$u^{c_3,\,a}+u^{a_2,\,a}+u^{a_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}111\\010\end{pmatrix}$	c_3, a_1
$\vartheta_{c_3a_3a_1}(u)$	26	$u^{c_3,\,a}+u^{a_3,\,a}+u^{a_1,\,a}\equiv\frac{1}{2}\begin{pmatrix}101\\100\end{pmatrix}$	c_3, a_2
$\vartheta_{c_3a_1a_2}(u)$	36	$u^{c_3,\,a}+u^{a_1,\,a}+u^{a_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}100\\111\end{pmatrix}$	c_3, a_3
$\vartheta_{cc_2c_3}(u)$	4	$u^{c,\,a}+u^{c_2,\,a}+u^{c_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}101\\011\end{pmatrix}$	a, c_1
$\vartheta_{cc_3c_1}(u)$	5	$u^{c,\,a}+u^{c_3,\,a}+u^{c_1,\,a}\equiv\frac{1}{2}\begin{pmatrix}001\\101\end{pmatrix}$	a, c_2
$\vartheta_{cc_1c_2}(u)$	6	$u^{c,\,a}+u^{c_1,\,a}+u^{c_2,\,a}\equiv\frac{1}{2}\begin{pmatrix}011\\110\end{pmatrix}$	a, c_3
$\vartheta_{c_1c_2c_3}(u)$	7	$u^{c_1,\,a}+u^{c_2,\,a}+u^{c_3,\,a}\equiv\frac{1}{2}\begin{pmatrix}010\\111\end{pmatrix}$	a, c

II. 36 *even characteristics for* $p=3$.

			b_1	b_2	b_3
$\vartheta(u)$		$\frac{1}{2}\begin{pmatrix}000\\000\end{pmatrix}$	a_1	a_2	a_3
$\vartheta_{a_1a_2a_3}(u)$	123	$u^{a_1,a}+u^{a_2,a}+u^{a_3,a}\equiv\frac{1}{2}\begin{pmatrix}101\\111\end{pmatrix}$	$*a,$	$x,$	$\bar{x}$
$\vartheta_c(u)$	456	$u^{c,a} \equiv\frac{1}{2}\begin{pmatrix}111\\000\end{pmatrix}$	c_1	c_2	c_3
$\vartheta_{c_1}(u)$	567	$u^{c_1,a} \equiv\frac{1}{2}\begin{pmatrix}000\\100\end{pmatrix}$	c	c_2	c_3
$\vartheta_{c_2}(u)$	647	$u^{c_2,a} \equiv\frac{1}{2}\begin{pmatrix}100\\010\end{pmatrix}$	c	c_3	c_1
$\vartheta_{c_3}(u)$	457	$u^{c_3,a} \equiv\frac{1}{2}\begin{pmatrix}110\\001\end{pmatrix}$	c	c_1	c_2
$\vartheta_{ca_1}(u)$	237	$u^{c,a}+u^{a_1,a} \equiv\frac{1}{2}\begin{pmatrix}011\\100\end{pmatrix}$	c	a_2	a_3
$\vartheta_{ca_2}(u)$	317	$u^{c,a}+u^{a_2,a} \equiv\frac{1}{2}\begin{pmatrix}001\\010\end{pmatrix}$	c	a_3	a_1
$\vartheta_{ca_3}(u)$	127	$u^{c,a}+u^{a_3,a} \equiv\frac{1}{2}\begin{pmatrix}000\\001\end{pmatrix}$	c	a_1	a_2
$\vartheta_{c_1a_1}(u)$	234	$u^{c_1,a}+u^{a_1,a} \equiv\frac{1}{2}\begin{pmatrix}100\\000\end{pmatrix}$	c_1	a_2	a_3
$\vartheta_{c_1a_2}(u)$	314	$u^{c_1,a}+u^{a_2,a} \equiv\frac{1}{2}\begin{pmatrix}110\\110\end{pmatrix}$	c_1	a_3	a_1
$\vartheta_{c_1a_3}(u)$	124	$u^{c_1,a}+u^{a_3,a} \equiv\frac{1}{2}\begin{pmatrix}111\\101\end{pmatrix}$	c_1	a_1	a_2
$\vartheta_{c_2a_1}(u)$	235	$u^{c_2,a}+u^{a_1,a} \equiv\frac{1}{2}\begin{pmatrix}000\\110\end{pmatrix}$	c_2	a_2	a_3
$\vartheta_{c_2a_2}(u)$	315	$u^{c_2,a}+u^{a_2,a} \equiv\frac{1}{2}\begin{pmatrix}010\\000\end{pmatrix}$	c_2	a_3	a_1
$\vartheta_{c_2a_3}(u)$	125	$u^{c_2,a}+u^{a_3,a} \equiv\frac{1}{2}\begin{pmatrix}011\\011\end{pmatrix}$	c_2	a_1	a_2
$\vartheta_{c_3a_1}(u)$	236	$u^{c_3,a}+u^{a_1,a} \equiv\frac{1}{2}\begin{pmatrix}010\\101\end{pmatrix}$	c_3	a_2	a_3
$\vartheta_{c_3a_2}(u)$	316	$u^{c_3,a}+u^{a_2,a} \equiv\frac{1}{2}\begin{pmatrix}000\\011\end{pmatrix}$	c_3	a_3	a_1
$\vartheta_{c_3a_3}(u)$	126	$u^{c_3,a}+u^{a_3,a} \equiv\frac{1}{2}\begin{pmatrix}001\\000\end{pmatrix}$	c_3	a_1	a_2
$\vartheta_{a_1c_2c_3}(u)$	156	$u^{a_1,a}+u^{c_2,a}+u^{c_3,a} \equiv\frac{1}{2}\begin{pmatrix}110\\111\end{pmatrix}$	a_1	c_1	c
$\vartheta_{a_1c_3c_1}(u)$	164	$u^{a_1,a}+u^{c_3,a}+u^{c_1,a} \equiv\frac{1}{2}\begin{pmatrix}010\\001\end{pmatrix}$	a_1	c_2	c
$\vartheta_{a_1c_1c_2}(u)$	145	$u^{a_1,a}+u^{c_1,a}+u^{c_2,a} \equiv\frac{1}{2}\begin{pmatrix}000\\010\end{pmatrix}$	a_1	c_3	c
$\vartheta_{a_1cc_1}(u)$	147	$u^{a_1,a}+u^{c,a}+u^{c_1,a}\equiv\frac{1}{2}\begin{pmatrix}011\\000\end{pmatrix}$	a_1	c_2	c_3

Table II. (continued).

			b_1	b_2	b_3
$\vartheta_{a_1cc_2}(u)$	157	$u^{a_1, a}+u^{c, a}+u^{c_2, a} \equiv \tfrac{1}{2}\begin{pmatrix}111\\110\end{pmatrix}$	a_1	c_3	c_1
$\vartheta_{a_1cc_3}(u)$	167	$u^{a_1, a}+u^{c, a}+u^{c_3, a} \equiv \tfrac{1}{2}\begin{pmatrix}101\\101\end{pmatrix}$	a_1	c_1	c_2
$\vartheta_{a_2c_2c_3}(u)$	256	$u^{a_2, a}+u^{c_2, a}+u^{c_3, a} \equiv \tfrac{1}{2}\begin{pmatrix}100\\001\end{pmatrix}$	a_2	c_1	c
$\vartheta_{a_2c_3c_1}(u)$	264	$u^{a_2, a}+u^{c_3, a}+u^{c_1, a} \equiv \tfrac{1}{2}\begin{pmatrix}000\\111\end{pmatrix}$	a_2	c_2	c
$\vartheta_{a_2c_1c_2}(u)$	245	$u^{a_2, a}+u^{c_1, a}+u^{c_2, a} \equiv \tfrac{1}{2}\begin{pmatrix}010\\100\end{pmatrix}$	a_2	c_3	c
$\vartheta_{a_2cc_1}(u)$	247	$u^{a_2, a}+u^{c, a}+u^{c_1, a} \equiv \tfrac{1}{2}\begin{pmatrix}001\\110\end{pmatrix}$	a_2	c_2	c_3
$\vartheta_{a_2cc_2}(u)$	257	$u^{a_2, a}+u^{c, a}+u^{c_2, a} \equiv \tfrac{1}{2}\begin{pmatrix}101\\000\end{pmatrix}$	a_2	c_3	c_1
$\vartheta_{a_2cc_3}(u)$	267	$u^{a_2, a}+u^{c, a}+u^{c_3, a} \equiv \tfrac{1}{2}\begin{pmatrix}111\\011\end{pmatrix}$	a_2	c_1	c_2
$\vartheta_{a_3c_2c_3}(u)$	356	$u^{a_3, a}+u^{c_2, a}+u^{c_3, a} \equiv \tfrac{1}{2}\begin{pmatrix}101\\010\end{pmatrix}$	a_3	c_1	c
$\vartheta_{a_3c_3c_1}(u)$	364	$u^{a_3, a}+u^{c_3, a}+u^{c_1, a} \equiv \tfrac{1}{2}\begin{pmatrix}001\\100\end{pmatrix}$	a_3	c_2	c
$\vartheta_{a_3c_1c_2}(u)$	345	$u^{a_3, a}+u^{c_1, a}+u^{c_2, a} \equiv \tfrac{1}{2}\begin{pmatrix}011\\111\end{pmatrix}$	a_3	c_3	c
$\vartheta_{a_3cc_1}(u$	347	$u^{a_3, a}+u^{c, a}+u^{c_1, a} \equiv \tfrac{1}{2}\begin{pmatrix}000\\101\end{pmatrix}$	a_3	c_2	c_3
$\vartheta_{a_3cc_2}(u)$	357	$u^{a_3, a}+u^{c, a}+u^{c_2, a} \equiv \tfrac{1}{2}\begin{pmatrix}100\\011\end{pmatrix}$	a_3	c_3	c_1
$\vartheta_{a_3cc_3}(u)$	367	$u^{a_3, a}+u^{c, a}+u^{c_3, a} \equiv \tfrac{1}{2}\begin{pmatrix}110\\000\end{pmatrix}$	a_3	c_1	c_2

It is to be noticed that every odd theta function is associated with either (i) any single one of a_1, a_2, a_3 or (ii) any pair of a_1, a_2, a_3 or any pair of c, c_1, c_2, c_3, or (iii) a triplet consisting of one of c, c_1, c_2, c_3 and two of a_1, a_2, a_3 or consisting of three from c, c_1, c_2, c_3. This may be stated by saying that *odd* suffixes are of one of the forms a, a^2, c^2, a^2c, c^3. Similarly an *even* suffix is of one of the forms c, ac, ac^2, a^3.

In the tables just given the fundamental characteristic-symbols, denoted by the numbers 1, 2, 3, 4, 5, 6, 7, are those associated with sums of integrals which may be denoted by

$$a_1,\ a_2,\ a_3,\ cc_2c_3,\ cc_3c_1,\ cc_1c_2,\ c_1c_2c_3.$$

We can equally well choose seven fundamental odd characteristic-symbols, associated with the integrals denoted by any one of the following sets:

$$\begin{array}{ccccccc}
c\,c_1, & c\,c_2, & c\,c_3, & c\,a_2a_3, & c\,a_3a_1, & c\,a_1a_2, & c_1c_2c_3 \\
c_1c\,, & c_1c_2, & c_1c_3, & c_1a_2a_3, & c_1a_3a_1, & c_1a_1a_2, & c\,c_2c_3 \\
c_2c\,, & c_2c_1, & c_2c_3, & c_2a_2a_3, & c_2a_3a_1, & c_2a_1a_2, & c\,c_3c_1 \\
c_3c\,, & c_3c_1, & c_3c_2, & c_3a_2a_3, & c_3a_3a_1, & c_3a_1a_2, & c\,c_1c_2 \\
a_1, & a_1a_2, & a_1a_3, & c_1a_2a_3, & c_2a_2a_3, & c_3a_2a_3, & ca_2a_3 \\
a_2, & a_2a_3, & a_2a_1, & c_1a_3a_1, & c_2a_3a_1, & c_3a_3a_1, & ca_3a_1 \\
a_3, & a_3a_1, & a_3a_2, & c_1a_1a_2, & c_2a_1a_2, & c_3a_1a_2, & ca_1a_2
\end{array}$$

The general theorem is—it is possible, corresponding to every even characteristic ϵ, to determine, in 8 ways, 7 odd characteristics $a, \beta, \gamma, \kappa, \lambda, \mu, \nu$, such that the combinations

$$a, \beta, \gamma, \kappa, \lambda, \mu, \nu, \epsilon a\beta, \epsilon a\kappa, \epsilon\lambda\mu$$

constitute all the 28 odd characteristics, and the combinations

$$\epsilon, a\beta\gamma, a\kappa\lambda, \beta\gamma\kappa$$

constitute all the 36 even characteristics. In the cases above $\epsilon=0$. The proof is given in a subsequent chapter.

206. Consider now what are the zeros of the functions

$$\vartheta(u), \ \vartheta(u|u^{b_1, a}+\dots\dots+u^{b_k, a}),$$

where $b_1, \dots, b_k$ denote any k of the branch places other than a ($k \ngtr p$), and u is given by

$$u_r = u_r^{x_1, a_1}+\dots\dots+u_r^{x_p, a_p}, \qquad (r=1, 2, \dots, p),$$

the functions being regarded as functions of x_1.

The zeros of $\vartheta(u)$ are the places $z_1, \dots, z_p$ determined by the congruence

$$u^{x_1, a_1}+\dots\dots+u^{x_p, a_p} \equiv u^{x_1, a}-u^{z_1, a_1}-\dots\dots-u^{z_p, a_p},$$

or, by*

$$u^{z_1, a}+u^{z_2, \bar{x}_2}+\dots\dots+u^{z_p, \bar{x}_p} \equiv 0.$$

Provided the places $a, \bar{x}_2, \dots, \bar{x}_p$ be not the zeros of a ϕ-polynomial, that is, provided none of the places $x_2, \dots, x_p$ be at a, and there be no coincidence expressible in the form $x_i = \bar{x}_j$, the places $z_1, z_2, \dots, z_p$ cannot be coresidual with any p other places (Chap. VI. § 98, and Chap. III.) and therefore (Chap. VIII. § 158) this congruence can only be satisfied when the places $z_1, \dots, z_p$ are the places

$$a, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_p;$$

these are then the zeros of $\vartheta(u)$, regarded as a function of x_1.

* The two places for which x has the same value, and y has the same value with opposite signs, are frequently denoted by x and $\bar{x}$.

The zeros of $\vartheta(u|u^{b_1, a}+\ldots\ldots+u^{b_k, a})$ are to be determined by the congruence

$$u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p}+u^{b_1, a}+\ldots\ldots+u^{b_k, a}\equiv u^{x_1, a}-u^{z_1, a_1}-\ldots\ldots-u^{z_p, a_p},$$

or, by

$$u^{z_1, b_1}+u^{z_2, \bar{x}_2}+\ldots\ldots+u^{z_p, \bar{x}_p}+u^{b_2, a}+\ldots\ldots+u^{b_k, a}\equiv 0,$$

which we may write also

$$(z_1, z_2, \ldots, z_p, a^{k-1})\equiv(b_1, \ldots, b_k, \bar{x}_2, \ldots, \bar{x}_p);$$

in particular the zeros of $\vartheta(u|u^{b, a})$ are the places $b, \bar{x}_2, \ldots, \bar{x}_p$.

207. Now, in fact, if the sum of the characteristics $q_1, \ldots, q_n$ differs from the sum of the characteristics $r_1, \ldots, r_n$ by a characteristic consisting wholly of integers, n being an integer not less than 2, then the quotient

$$f(u)=\frac{\vartheta(u;\, q_1)\,\vartheta(u;\, q_2)\ldots\ldots\vartheta(u;\, q_n)}{\vartheta(u;\, r_1)\,\vartheta(u;\, r_2)\ldots\ldots\vartheta(u;\, r_n)}$$

is a periodic function of u.

For, by the formula (§ 190, Chap. X.)

$$\vartheta(u+\Omega_m;\, q)=e^{\lambda_m(u)+2\pi i(mq'-m'q)}\,\vartheta(u;\, q),$$

where m denotes a row of integers, we have

$$\frac{f(u+\Omega_m)}{f(u)}=e^{2\pi i\,[m(\Sigma q'-\Sigma r')-m'(\Sigma q-\Sigma r)]},$$

and if $\Sigma q'-\Sigma r'$, $\Sigma q-\Sigma r$, each consist of a row of integers the right-hand side is equal to 1.

Hence, when the arguments, u, are as in § 206, the function $f(u)$ is a rational function of the places $x_1, \ldots, x_p$.

208. It follows therefore that the function

$$\frac{\vartheta^2(u|u^{b, a})}{\vartheta^2(u)}$$

is a rational function of the places $x_1, \ldots, x_p$. By what has been proved in regard to the zeros of the numerator and denominator it has, as a function of x_1, the zero b, of the second order, and is infinite at a, that is, at infinity, also to the second order. Thus it is equal to $M(b-x_1)$, where M does not depend on x_1. As the function is symmetrical in $x_1, x_2, \ldots, x_p$, it must therefore be equal to $K(b-x_1)\ldots(b-x_p)$, where K is an absolute constant. Therefore the function

$$\sqrt{(b-x_1)(b-x_2)\ldots(b-x_p)}=\frac{1}{\sqrt{K}}\,\frac{\vartheta(u|u^{b, a})}{\vartheta(u)}$$

may be interpreted as a single valued function of the places $x_1, \ldots, x_p$, on the Riemann surface, dissected by the $2p$ period loops. The values of the function on the two sides of any period loop have a quotient which is constant along that loop, and equal to ± 1.

The function has been considered by Rosenhain*, Weierstrass†, Riemann‡ and Brioschi§. We shall denote the quotient $\vartheta(u|u^{b,a})/\vartheta(u)$ by $q_b(u)$. There are $2p+1$ such functions, according to the position of b. Of these $q_{a_1}(u), \ldots, q_{a_p}(u)$ are odd functions, and $q_c(u), q_{c_1}(u), \ldots, q_{c_p}(u)$ are even functions. The functions are clearly generalisations of the functions $\sqrt{x} = \operatorname{sn} u$, $\sqrt{1-x} = \operatorname{cn} u$, $\sqrt{1-k^2x} = \operatorname{dn} u$, obtained from the consideration of the integral

$$u = \int_0^x \frac{dx}{\sqrt{4x(1-x)(1-k^2x)}}.$$

209. Consider next the function

$$F = \frac{\vartheta(u|u^{b_1,a} + \ldots\ldots + u^{b_k,a})\,\vartheta^{k-1}(u)}{\vartheta(u|u^{b_1,a})\ldots\ldots\vartheta(u|u^{b_k,a})},$$

wherein $b_1, \ldots, b_k$ are any k branch places other than a. We consider only the cases $k < p+1$. By what has been shewn, the function is rational in x_1, and if $z_1, \ldots, z_p$ denote the zeros of $\vartheta(u|u^{b_1,a} + \ldots\ldots + {}^{b_k,a})$ the zeros of the numerator, as here written, consist of the places

$$z_1, \ldots, z_p, a^{k-1}, \bar{x}_2^{k-1}, \ldots, \bar{x}_p^{k-1}$$

and the zeros of the denominator consist of the places

$$b_1, b_2, \ldots, b_k, \bar{x}_2^k, \ldots, \bar{x}_p^k.$$

Thus the rational function of x_1 has for zeros the places $z_1, \ldots, z_p, a^{k-1}$, and, for poles, the places $b_1, \ldots, b_k, \bar{x}_2, \ldots, \bar{x}_p$. It has already been otherwise shewn that these two sets of $p+k-1$ places are coresidual. Now any rational function, of the place x, which has these poles, can (Chap. VI. § 89) be written in the form

$$\frac{uy + v(x-b_1)\ldots(x-b_k)}{(x-b_1)\ldots(x-b_k)(x-x_2)\ldots(x-x_p)},$$

wherein u, v are suitable integral polynomials in x, so chosen that the numerator vanishes at the places $x_2, \ldots, x_p$. The denominator, as here written, vanishes to the second order at each of $b_1, \ldots, b_k$, and also vanishes at the places $x_2, \bar{x}_2, \ldots, x_p, \bar{x}_p$.

Let λ, μ be the highest powers of x respectively in u and v. Then, in order that this function may be zero at the place a, that is, at infinity, to the order $k-1$, it is necessary that the greater of the two numbers

$$2\lambda + 2p + 1 - 2(p+k-1),\ 2\mu + 2k - 2(p+k-1)$$

* *Mémoires par divers savants*, t. XI. (1851), pp. 361—468.

† By Weierstrass the function is multiplied by a certain constant factor and denoted by $al(u)$.

‡ In the general form enunciated, as a quotient of products of theta functions, *Werke* (Leipzig, 1876), p. 134 (§ 27).

§ *Annali di Mat.* t. X. (1880), t. XIV. (1886).

(wherein $2(p+k-1)$ is the order of infinity, at infinity, of the denominator) should be equal to $-(k-1)$. Since one of these numbers is odd and the other even, they cannot be both equal to $-(k-1)$. Further in order that the ratios of the $\lambda+\mu+2$ coefficients in u, v may be capable of being chosen so that the numerator vanishes in the places $x_2, \ldots, x_p$, it is necessary that $\lambda+\mu+1$ should not be less than $p-1$. And, since a rational function is entirely determined when its poles and all but p of its zeros are given, these conditions should entirely determine the function.

In fact we easily find from these conditions that the case $2\lambda+2p+1>2(\mu+k)$ can only occur when k is even, and then $\lambda=\frac{1}{2}k-1$, $\mu=p-1-\frac{1}{2}k$, and that the case $2\lambda+2p+1<2\mu+2k$ can only occur when k is odd, and then $\lambda=\frac{1}{2}(k-3)$, $\mu=p-\frac{1}{2}(k+1)$. In both cases $\lambda+\mu+2=p$.

By introducing the condition that the polynomial $uy+v(x-b_1)\ldots(x-b_k)$ should vanish in the places $x_2, \ldots, x_p$ we are able, save for a factor not depending on x, y, to express this polynomial as the product of $(x-b_1)\ldots(x-b_k)$ by a determinant of p rows and columns of which, for $r>1$, the rth row is formed with the elements

$$\frac{x_r^{\lambda}y_r}{\phi(x_r)}, \frac{x_r^{\lambda-1}y_r}{\phi(x_r)}, \ldots, \frac{y_r}{\phi(x_r)}, x_r^{\mu}, x_r^{\mu-1}, \ldots, 1,$$

wherein $\phi(x)$ denotes $(x-b_1)\ldots(x-b_k)$, the first row being of the same form with the omission of the suffixes.

Therefore, noticing that F is symmetrical in the places $x_1, \ldots, x_p$, we infer, denoting the product of the differences of $x_1, \ldots, x_p$ by $\Delta(x_1, \ldots, x_p)$, that

$$\frac{\vartheta(u|u^{b_1,a}+\ldots\ldots+u^{b_k,a})\,\vartheta^{k-1}(u)}{\vartheta(u|u^{b_1,a})\ldots\ldots\vartheta(u|u^{b_k,a})}=C\frac{\left|\frac{x_r^{\lambda}y_r}{\phi(x_r)}, \frac{x_r^{\lambda-1}y_r}{\phi(x_r)}, \ldots, \frac{y_r}{\phi(x_r)}, x_r^{\mu}, x_r^{\mu-1}, \ldots, 1\right|}{\Delta(x_1, \ldots, x_p)},$$

where C is an absolute constant, and the numerator denotes a determinant in which the first, second, ... rows contain, respectively, $x_1, x_2, \ldots$; and here

when k is even, $\qquad \lambda=\frac{1}{2}k-1, \quad \mu=p-1-\frac{1}{2}k$

and when k is odd, $\qquad \lambda=\frac{1}{2}(k-3), \quad \mu=p-\frac{1}{2}(k+1)$.

210. By means of the algebraic expression which we have already obtained for the quotients $\vartheta(u|u^{b,a})/\vartheta(u)$, we are now able to deduce an algebraic expression for the quotients

$$\vartheta(u|u^{b_1,a}+\ldots\ldots+u^{b_k,a})/\vartheta(u);$$

since it has already been shewn that by taking k in turn equal to $1, 2, \ldots, p$, and taking all possible sets $b_1, \ldots, b_k$ corresponding to any value of k, the half-periods represented by $u^{b_1,a}+\ldots\ldots+u^{b_k,a}$ consist of all possible half-periods except that one which is identically zero, it follows that, *in the* *hyperelliptic case, if u denote $u^{x_1, a_1} + \ldots\ldots + u^{x_p, a_p}$, and q denote in turn all possible half-integer characteristics except the identically zero characteristic, all the $2^{2p}-1$ ratios $\vartheta(u\,;\,q)/\vartheta(u)$ can be expressed algebraically in terms of $x_1, \ldots, x_p$, by the formulae which have been given.*

The simplest case is when $k=2$; then we have $\lambda=0$, $\mu=p-2$, and

$$\frac{\vartheta(u|u^{b_1, a}+u^{b_2, a})\,\vartheta(u)}{\vartheta(u|u^{b_1, a})\,\vartheta(u|u^{b_2, a})} = C\sum_{r=1}^{p} \frac{y_r}{(x_r-b_1)(x_r-b_2)}\,\frac{1}{R'(x_r)},$$

where $R(x)=(x-x_1)(x-x_2)\ldots(x-x_p)$, and C is an absolute constant. Denoting the quotient $\vartheta(u|u^{b_1, a}+u^{b_2, a})/\vartheta(u)$ by q_{b_1, b_2}, we have

$$q_{b_1, b_2} = A_{1, 2}\,q_{b_1}q_{b_2}\sum_{r=1}^{p}\frac{y_r}{(x_r-b_1)(x_r-b_2)}\,\frac{1}{R'(x_r)},$$

where $A_{1,2}$ is an absolute constant; and there are $p(2p+1)$ such functions.

When $k=3$, we have $\lambda=0$, $\mu=p-2$, and, if q_{b_1, b_2, b_3} denote the quotient $\vartheta(u|u^{b_1, a}+u^{b_2, a}+u^{b_3, a})/\vartheta(u)$, we obtain

$$q_{b_1, b_2, b_3} = B_{1, 2, 3}\,q_{b_1}q_{b_2}q_{b_3}\sum_{r=1}^{p}\frac{y_r}{(x_r-b_1)(x_r-b_2)(x_r-b_3)}\,\frac{1}{R'(x_r)},$$

where $B_{1, 2, 3}$ is an absolute constant. It is however clear that

$$\frac{q_{b_1, b_2}}{A_{12}q_{b_1}q_{b_2}} - \frac{q_{b_1, b_3}}{A_{13}q_{b_1}q_{b_3}} = (b_2-b_3)\,\frac{q_{b_1, b_2, b_3}}{B_{123}q_{b_1}q_{b_2}q_{b_3}},$$

so that the functions with three suffixes are immediately expressible by those with one and those with two suffixes.

More generally, the $2^{2p}-1$ quotients $\vartheta(u\,;\,q)/\vartheta(u)$, depending only on the p places $x_1, \ldots, x_p$, must be connected by $2^{2p}-p-1$ algebraical relations; and since (Chap. IX.) any argument can be expressed in the form $u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p}$, it follows that these may be regarded as relations connecting Riemann theta functions of arbitrary argument. This statement is true whether the surface be hyperelliptic or not.

Of such relations one simple and obvious one for the hyperelliptic case under consideration may be mentioned at once. We clearly have

$$\frac{q_{b_2, b_3}}{A_{23}q_{b_2}q_{b_3}}(b_2-b_3)+\frac{q_{b_3, b_1}}{A_{31}q_{b_3}q_{b_1}}(b_3-b_1)+\frac{q_{b_1, b_2}}{A_{12}q_{b_1}q_{b_2}}(b_1-b_2)=0,$$

and therefore

$$\frac{b_2-b_3}{A_{23}}\,\vartheta_{b_2b_3}(u)\,\vartheta_{b_1}(u)+\frac{b_3-b_1}{A_{31}}\,\vartheta_{b_3b_1}(u)\,\vartheta_{b_2}(u)+\frac{b_1-b_2}{A_{12}}\,\vartheta_{b_1b_2}(u)\,\vartheta_{b_3}(u)=0.$$

It is proved below (§ 213) that $A^2_{23} : A^2_{31} : A^2_{12}=(b_2\sim b_3) : (b_3\sim b_1) : (b_1\sim b_2)$.

Other relations will be given for the cases $p=2$, $p=3$. A set of relations connecting the q's of single and double suffixes, for any value of p, is given by Weierstrass (*Crelle* LII. *Werke* I. p. 336).

211. *Ex.* i. Prove that the rational function having the places $\bar{x}_1, \dots, \bar{x}_p, a$, as poles, and the branch place b as one zero, is given by

$$Z=(b-x)\dots\dots(b-x_p)\sum_0^p \frac{y_r}{x_r-b}\frac{1}{R'(x_r)},$$

where $R(\xi)=(\xi-x)(\xi-x_1)\dots\dots(\xi-x_p)$, and, in the summation, x_0, y_0 are to be replaced by x, y.

Prove that if u denote the argument

$$u=u^{x,\,a}+u^{x_1,\,a_1}+\dots\dots+u^{x_p,\,a_p},$$

then

$$\frac{\vartheta^2(u\,|\,u^{b,\,a})}{\vartheta^2(u)}=A\frac{Z^2}{(b-x)(b-x_1)\dots\dots(b-x_p)},$$

where A is an absolute constant.

Prove for example, in the elliptic case, with Weierstrass's notation, that

$$\frac{\sigma_i(u+v)}{\sigma\,(u+v)}=\sqrt{\wp(u+v)-e_i}=\tfrac{1}{2}\sqrt{(\wp u-e_i)(\wp v-e_i)}\left(\frac{\wp' u}{\wp u-e_i}-\frac{\wp' v}{\wp v-e_i}\right)\frac{1}{\wp u-\wp v}.$$

Ex. ii. If Z_r denote the function Z when the branch place b_r is put in place of b, and $R(b_r)$ denote $(b_r-x)(b_r-x_1)\dots\dots(b_r-x_p)$, and we put

$$\Phi=\frac{\vartheta(u\,|\,u^{b_1,\,a}+\dots\dots+u^{b_k,\,a})\,\vartheta^{k-1}(u)}{\vartheta(u\,|\,u^{b_1,\,a})\dots\dots\vartheta(u\,|\,u^{b_k,\,a})},$$

prove that

$$\Phi Z_1\dots\dots Z_k=BR(b_1)\dots\dots R(b_k)\left|\frac{y_r}{\phi(x_r)}x_r^{\lambda},\ \frac{y_r}{\phi(x_r)}x_r^{\lambda-1},\ \dots,\ \frac{y_r}{\phi(x_r)},\ x_r^{\mu},\ x_r^{\mu-1},\ \dots,\ 1\right|$$
$$\div\Delta(x,\,x_1,\,\dots,\,x_p),$$

where B is an absolute constant, $\Delta(x, x_1, \dots, x_p)$ denotes the product of all the differences of the $(p+1)$ quantities $x, x_1, \dots, x_p$, $\phi(x_r)=(x_r-b_1)\dots\dots(x_r-b_k)$, and the determinant is one of $p+1$ rows and columns in which, in the first row, x_0, y_0 are to be replaced by x, y.

Prove that, when k is even, $\lambda=\tfrac{1}{2}(k-2)$, $\mu=p-\tfrac{1}{2}k$, and, when k is odd, $\lambda=\tfrac{1}{2}(k-1)$, $\mu=p-\tfrac{1}{2}(k+1)$.

Ex. iii. Hence prove that the function $\dfrac{\vartheta(u\,|\,u^{b_1,\,a}+\dots\dots+u^{b_k,\,a})}{\vartheta(u)}$ is a constant multiple of

$$\frac{\sqrt{R(b_1)\dots\dots R(b_k)}\left|\frac{y_r}{\phi(x_r)}x_r^{\lambda},\ \frac{y_r}{\phi(x_r)}x_r^{\lambda-1},\ \dots,\ \frac{y_r}{\phi(x_r)},\ x_r^{\mu},\ x_r^{\mu-1},\ \dots,\ x_r,\ 1\right|}{\Delta(x,\,x_1,\,\dots,\,x_p)}.$$

This formula is true when $k=1$.

Ex. iv. A particular case is when $k=2$. Then the function $\vartheta(u\,|\,u^{b_1,\,a}+u^{b_2,\,a})/\vartheta(u)$ is a constant multiple of

$$\sqrt{(b_1-x)(b_1-x_1)\dots\dots(b_1-x_p)}\sqrt{(b_2-x)(b_2-x_1)\dots\dots(b_2-x_p)}\sum_0^p\frac{y_r}{(x_r-b_1)(x_r-b_2)}\frac{1}{R'(x_r)},$$

wherein $R(\xi)=(\xi-x)(\xi-x_1)\dots\dots(\xi-x_p)$.

Ex. v. Verify that the formula of Ex. iii. includes the formulae of the text (§ 210); shew that when x is put at infinity the values of λ, μ in the determinant of § 209 are properly obtained.

Ex. vi. Verify that the expression $\psi(x, b; a, \bar{x}_1, \ldots, \bar{x}_p)$ of § 130, Chap. VII., takes the form given for the function Z of Ex. i. when a is the place infinity.

Ex. vii. If $f(x)$ denote the polynomial

$$\lambda+\lambda_1 x+\lambda_2 x^2+\ldots\ldots+\lambda_{2p+2}x^{2p+2},$$

prove that any rational integral polynomial, $F(x, z)$, which is symmetric in the two variables x, z and of order $p+1$ in each of them, and satisfies the conditions

$$F(z, z)=2f(z), \left[\frac{d}{dx}F(x, z)\right]_{x=z}=\frac{d}{dz}f(z),$$

is of the form

$$F(x, z)=f(x, z)+(x-z)^2\psi(x, z),$$

where (cf. p. 195), with $\lambda_0=\lambda$, $\lambda_{2p+3}=0$,

$$f(x, z)=\sum_{i=0}^{p+1} x^i z^i\{2\lambda_{2i}+\lambda_{2i+1}(x+z)\},$$

and $\psi(x, z)$ is an integral polynomial, symmetric in x, z, of order $p-1$ in each*.

In case $p=2$, and $f(x)=(x-a_1)(x-a_2)(x-c)(x-c_1)(x-c_2)$, prove that a form of $F(x, z)$ is given by

$$F(x, z)=(x-a_1)(x-a_2)(z-c)(z-c_1)(z-c_2)+(z-a_1)(z-a_2)(x-c)(x-c_1)(x-c_2).$$

Ex. viii. If for purposes of operation we introduce homogeneous variables and write

$$f(x)=\lambda x_2^{2p+2}+\lambda_1 x_2^{2p+1}x_1+\ldots\ldots+\lambda_{2p+1}x_2 x_1^{2p+1}+\lambda_{2p+2}x_1^{2p+2},$$

prove that a form of $F(x, z)$ is given by

$$\bar{f}(x, z)=\frac{\lfloor p}{\lfloor 2p+1}\left(x_1\frac{\partial}{\partial z_1}+x_2\frac{\partial}{\partial z_2}\right)^{p+1}f(z),$$

where, after differentiation, x_1, x_2, z_1, z_2 are to be replaced by x, 1, z, 1 respectively.

This is the same as that which in the ordinary symbolical notation for binary forms is denoted by $\bar{f}(x, z)=2a_x^{p+1}a_z^{p+1}$, $f(x)$ being a_x^{2p+2}.

Ex. ix. Using the form of Ex. viii. for $F(x, z)$, prove that if $e_1, e_2, x, x_1, \ldots, x_p$ be any values of x, we have

$$\sum_{r=0}^{p}\frac{f(x_r)}{[G'(x_r)]^2}+\Sigma\Sigma\frac{\bar{f}(x_r, x_s)}{G'(x_r)G'(x_s)}=\frac{f(e_1)}{[G'(e_1)]^2}+\frac{f(e_2)}{[G'(e_2)]^2}+\frac{\bar{f}(e_1, e_2)}{G'(e_1)G'(e_2)},$$

where $G(\xi)=(\xi-e_1)(\xi-e_2)(\xi-x)(\xi-x_1)\ldots\ldots(\xi-x_p)$, and the double summation on the left refers to every one of the $\frac{1}{2}p(p+1)$ pairs of quantities chosen from $x, x_1, \ldots, x_p$.

Ex. x. Hence it follows†, when $y^2=f(x)$, $y_r^2=f(x_r)$, etc., and $R(\xi)=(\xi-x)(\xi-x_1)\ldots(\xi-x_p)$, that

$$R(e_1)R(e_2)\left[\sum_0^p\frac{y_r}{(e_1-x_r)(e_2-x_r)R'(x_r)}\right]^2-\frac{f(e_1)R(e_2)}{(e_1-e_2)^2R(e_1)}-\frac{f(e_2)R(e_1)}{(e_1-e_2)^2R(e_2)}+\frac{\bar{f}(e_1, e_2)}{(e_1-e_2)^2}$$

is equal to

$$R(e_1)R(e_2)\Sigma\Sigma\frac{2y_r y_s-f(x_r, x_s)}{G'(x_r)G'(x_s)},$$

* It follows that the hyperelliptic canonical integral of the third kind obtained on page 195 can be changed into the most general canonical integral, $R_{z, c}^{x, a}$ (p. 194), in which the matrix a has any value, by taking, instead of $f(x, z)$, a suitable polynomial $F(x, z)$ satisfying the conditions of Ex. vii.

† The result of this Example is given by Bolza, *Götting. Nachrichten*, 1894, p. 268.

where the summation refers to every pair from the $p+1$ quantities $x, x_1, \ldots, x_p$, and $\bar{f}(x, z)$ denotes the special value of $F(x, z)$ obtained in Ex. viii.

Ex. xi. It follows therefore by Ex. iv. that when b_1, b_2 are any branch places of the surface associated with the equation $y^2 - f(x) = 0$, there exists an equation of the form

$$C \frac{\vartheta^2 (u | u^{b_1, a} + u^{b_2, a})}{\vartheta^2 (u)} = R(b_1) R(b_2) \Sigma\Sigma \frac{2y_r y_s - f(x_r, x_s)}{G'(x_r) G'(x_s)} - \frac{\bar{f}(b_1, b_2)}{(b_1 - b_2)^2},$$

where C is an absolute constant, $G(\xi) = (\xi - b_1)(\xi - b_2)(\xi - x)(\xi - x_1) \ldots\ldots (\xi - x_p)$, and $u = u^{x, a} + u^{x_1, a_1} + \ldots\ldots + u^{x_p, a_p}$. The importance of this result will appear below.

212. The formulae of §§ 208, 210 furnish a solution of the inversion problem expressed by the p equations

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} \equiv u_i; \qquad (i = 1, 2, \ldots, p).$$

For instance the solution is given by the $2p + 1$ equations

$$\frac{\vartheta^2 (u | u^{b, a})}{\vartheta^2 (u)} = A (b - x_1)(b - x_2) \ldots (b - x_p);$$

from any p of these equations $x_1, \ldots, x_p$ can be expressed as single valued functions of the arbitrary arguments $u_1, \ldots, u_p$.

And it is easy to determine the value of A^2. For let $b_1, \ldots, b_p, b_1', \ldots, b_p'$ denote the finite branch places other than b. As already remarked (§ 201) we have

$$(c, c_1, \ldots, c_p) \equiv (a, a_1, \ldots, a_p)$$

and therefore

$$(b, b_1, \ldots, b_p) \equiv (a, b_1', \ldots, b_p').$$

Now we easily find by the formulae of § 190, Chap. X. that if P be a set of $2p$ integers, $P_1, \ldots, P_p, P_1', \ldots, P_p'$,

$$\frac{\vartheta^2 (u + \tfrac{1}{2}\Omega_P, \tfrac{1}{2}\Omega_P)}{\vartheta^2 (u + \tfrac{1}{2}\Omega_P)} = \frac{\vartheta^2 (u)}{\vartheta^2 (u; \tfrac{1}{2}P)} e^{-\pi i P P'};$$

hence, if $u^{b, a} = \tfrac{1}{2}\Omega_{P, P'}$, and $u_0 = u^{b_1, a} + \ldots\ldots + u^{b_p, a}$, we have, by the formula under consideration, writing $b_1, \ldots, b_p$ in place of $x_1, \ldots, x_p$, the equation

$$\frac{\vartheta^2 (u_0 | u^{b, a})}{\vartheta^2 (u)} = A (b - b_1) \ldots (b - b_p),$$

and, writing $b_1', \ldots, b_p'$ in place of $x_1, \ldots, x_p$, we have

$$\frac{\vartheta^2 (u_0 + u^{b, a} | u^{b, a})}{\vartheta^2 (u + u^{b, a})} = A (b - b_1') \ldots (b - b_p');$$

thus, by multiplication

$$e^{-\pi i P P'} = A^2 (b - b_1) \ldots (b - b_p)(b - b_1') \ldots (b - b_p'),$$

and hence

$$\frac{\vartheta^2(u|u^{b,a})}{\vartheta^2(u)} = \pm \frac{(b-x_1)(b-x_2)\dots(b-x_p)}{\sqrt{e^{\pi i PP'} f'(b)}},$$

where $f(x)$ denotes $(x-a_1)\dots(x-a_p)(x-c)(x-c_1)\dots(x-c_p)$, and $e^{\pi i PP'} = \pm 1$ according as $u^{b,a}$ is an odd or even half-period.

The reader should deduce this result from the equation (§ 171, Chap. IX.)

$$V(U_1, \dots, U_p;\ \xi_1, \gamma_1) \dots\dots V(U_1, \dots, U_p;\ \xi_k, \gamma_k) = \frac{(X - Z(x_1)) \dots\dots (X - Z(x_p))}{(X - Z(a_1)) \dots\dots (X - Z(a_p))}$$

by taking Z to be the rational function of the second order, x.

When $u = u^{x,a} + u^{x_1,a_1} + \dots\dots + u^{x_p,a_p}$, we deduce (see Ex. i. § 211)

$$\frac{\vartheta^2(u|u^{b,a})}{\vartheta^2(u)} = \pm \frac{(b-x)(b-x_1)\dots\dots(b-x_p)}{4\sqrt{e^{\pi i PP'}} f'(b)} \left[\sum_{r=0}^{p} \frac{y_r}{x_r - b} \frac{1}{R'(x_r)}\right]^2,$$

where $R(\xi) = (\xi - x)(\xi - x_1) \dots\dots (\xi - x_p)$.

If in particular we put b in turn at the places $a_1, \dots, a_p$, write $P(x) = (x-a_1)\dots(x-a_p)$ and $Q(x) = (x-c)(x-c_1)\dots(x-c_p)$, and use the equation

$$\frac{(x-x_1)\dots(x-x_p)}{P(x)} = 1 + \sum_1^p \frac{(a_i - x_1)\dots(a_i - x_p)}{(x-a_i)P'(a_i)},$$

we can infer that $x_1, \dots, x_p$ are the roots of the equation*

$$\sum_{i=1}^{p} \epsilon_i \sqrt{-\frac{Q(a_i)}{P'(a_i)}} \frac{\vartheta^2(u|u^{a_i,a})}{a_i - x} = \vartheta^2(u),$$

where ϵ_i is ± 1 and is such that we have

$$\frac{\vartheta^2(u|u^{a_i,a})}{\vartheta^2(u)} = \epsilon_i \frac{(a_i - x_1)\dots(a_i - x_p)}{\sqrt{-P'(a_i)Q(a_i)}}.$$

Another form of this equation for $x_1, \dots, x_p$ is given below (§ 216), where the equation determining y_i from x_i is also given.

213. We can also obtain the constant factor in the algebraic expression of the function $\vartheta(u|u^{b_1,a} + u^{b_2,a})\,\vartheta(u) \div \vartheta(u|u^{b_1,a})\,\vartheta(u|u^{b_2,a})$.

Let b_1, b_2 denote any branch places, and choose $z_1, \dots, z_p$ so that

$$u^{x_1,a_1} + \dots\dots + u^{x_p,a_p} + u^{b_1,a} \equiv u^{z_1,a_1} + \dots\dots + u^{z_p,a_p};$$

then $z_1, \dots, z_p, a$ are the zeros of a rational function which vanishes in $x_1, \dots, x_p, b_1$. Such a function can be expressed in the form

$$\frac{y + (x - b_1)(x, 1)^{p-1}}{(x-x_1)\dots\dots(x-x_p)},$$

* Cf. Weierstrass, *Math. Werke* (Berlin, 1894), vol. I. p. 328.

where $(x, 1)^{p-1}$ is an integral polynomial in x whose coefficients are to be chosen to satisfy the p equations

$$-y_i+(x_i-b_1)(x_i, 1)^{p-1}=0, \qquad (i=1, 2, \ldots, p);$$

thus the function is

$$\frac{y}{F(x)}+(x-b_1)\sum_{i=1}^{p}\frac{y_i}{x_i-b_1}\frac{1}{(x-x_i)F'(x_i)},$$

where $F(x)=(x-x_1)\ldots(x-x_p)$; and, if the coefficient of x^{2p+1} in the equation associated with the Riemann surface be taken to be 4, we have

$$y^2-(x-b_1)^2[F(x)]^2\left[\sum_{i=1}^{p}\frac{y_i}{x_i-b_1}\frac{1}{(x-x_i)F'(x_i)}\right]^2=4(x-x_1)\ldots(x-x_p)(x-z_1)\ldots(x-z_p)(x-b_1),$$

and therefore, putting b_2 for x,

$$\frac{(b_2-z_1)\ldots\ldots(b_2-z_p)}{(b_2-x_1)\ldots\ldots(b_2-x_p)}=(b_1-b_2)\left[\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)}\frac{1}{F'(x_i)}\right]^2.$$

Now we have found, denoting $u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p}$ by u, and $u^{z_1, a_1}+\ldots\ldots+u^{z_p, a_p}$ by v, the results

$$\frac{\vartheta^2(u|u^{b_2, a})}{\vartheta^2(u)}=\pm\frac{(b_2-x_1)\ldots\ldots(b_2-x_p)}{\sqrt{e^{\pi i PP'}}f'(b)}, \qquad \frac{\vartheta^2(v|u^{b_2, a})}{\vartheta^2(v)}=\pm\frac{(b_2-z_1)\ldots\ldots(b_2-z_p)}{\sqrt{e^{\pi i PP'}}f'(b)},$$

where $u^{b_2, a}=\frac{1}{2}\Omega_{P, P'}$; hence we have

$$\frac{\vartheta^2(v|u^{b_2, a})\,\vartheta^2(u)}{\vartheta^2(v)\,\vartheta^2(u|u^{b_2, a})}=\pm(b_1-b_2)\left[\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)}\frac{1}{F'(x_i)}\right]^2,$$

which, by the formulae of § 190, is the same as

$$\frac{\vartheta(u|u^{b_1, a}+u^{b_2, a})\,\vartheta(u)}{\vartheta(u|u^{b_1, a})\,\vartheta(u|u^{b_2, a})}=\epsilon\sqrt{b_1-b_2}\sum_{i=1}^{p}\frac{y_i}{2(x_i-b_1)(x_i-b_2)F'(x_i)},$$

where ϵ is a certain fourth root of unity.

Thus the method of this § not only reproduces the result of § 210, but determines the constant factor.

Ex. Determine the constant factors in the formulae of §§ 208, 210, 211.

214. Beside such formulae as those so far developed, which express products of theta functions algebraically, there are formulae which express differential coefficients of theta functions algebraically; as the second differential coefficients of $\vartheta(u)$ in regard to the arguments $u_1, \ldots, u_p$ are periodic functions of these arguments, this was to be expected.

We have (§ 193, Chap. X.) obtained* the formula

$$-\zeta_i(u^{x, m}-u^{x_1, m_1}-\ldots\ldots-u^{x_p, m_p})+\zeta_i(u^{\mu, m}-u^{x_1, m_1}-\ldots\ldots-u^{x_p, m_p})$$
$$=L_i^{x, \mu}+\sum_{k=1}^{p}\tilde{\nu}_{k, i}\,[(x_k, x)-(x_k, \mu)]\,\frac{dx_k}{dt};$$

* Cf. also Thomae, *Crelle*, LXXI., XCIV.

we denote by h_r the sum of the homogeneous products of $x_1, \ldots, x_p$, r together, without repetitions, and use the abbreviation

$$\chi_{p-i}(x;\ x_1, \ldots, x_p) = x^{p-i} - h_1 x^{p-i-1} + h_2 x^{p-i-2} - \ldots\ldots + (-)^{p-i} h_{p-i};$$

further, for the p fundamental integrals $u_1^{x,\mu}, \ldots, u_p^{x,\mu}$, we take the integrals

$$\int_\mu^x \frac{dx}{y}, \int_\mu^x \frac{x\,dx}{y}, \ldots, \int_\mu^x \frac{x^{p-1}\,dx}{y};$$

then it is immediately verified that

$$\tilde{\nu}_{k,i} = y_k \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)} \Big/ \frac{dx_k}{dt},$$

where $F(x)$ denotes $(x - x_1) \ldots (x - x_p)$.

Thus, if μ, ν denote the values of x and y at the place μ, we have, writing $a, a_1, \ldots, a_p$ for $m, m_1, \ldots, m_p$ (§ 200),

$$- \zeta_i(u^{x,a} - u^{x_1,a_1} - \ldots\ldots - u^{x_p,a_p}) + \zeta_i(u^{\mu,a} - u^{x_1,a_1} - \ldots\ldots - u^{x_p,a_p})$$

$$= L_i^{x,\mu} + \tfrac{1}{2}\sum_{k=1}^{p} \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)} \left[\frac{y + y_k}{x_k - x} - \frac{y_k + \nu}{x_k - \mu}\right];$$

therefore, also, the function

$$\zeta_i(u^{x,a} + u^{x_1,a_1} + \ldots\ldots + u^{x_p,a_p}) + L_i^{x,\mu} - \tfrac{1}{2}\sum_{k=1}^{p} \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)} \frac{y - y_k}{x - x_k}$$

is equal to

$$\zeta_i(u^{\mu,a} + u^{x_1,a_1} + \ldots\ldots + u^{x_p,a_p}) - \tfrac{1}{2}\sum_{k=1}^{p} \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)} \frac{\nu - y_k}{\mu - x_k},$$

which is independent of the place x.

Now let $R(t)$ denote $(t - x)(t - x_1) \ldots (t - x_p)$, and use the abbreviation given by the equation

$$\frac{y\chi_{p-i}(x; x_1, \ldots, x_p)}{R'(x)} + \frac{y_1\chi_{p-i}(x_1;\ x, x_2, \ldots, x_p)}{R'(x_1)} + \ldots + \frac{y_p\chi_{p-i}(x_p;\ x, x_1, \ldots, x_{p-1})}{R'(x_p)}$$
$$= f_{p-i}(x, x_1, \ldots, x_p);$$

then also

$$\frac{y_1\chi_{p-i-1}(x_1;\ x_2, \ldots, x_p)}{F'(x_1)} + \ldots\ldots + \frac{y_p\chi_{p-i-1}(x_p;\ x_1, \ldots, x_{p-1})}{F'(x_p)} = f_{p-i-1}(x_1, \ldots, x_p).$$

Now $\qquad \chi_{p-i}(x_1;\ x, x_2, \ldots, x_p) - \chi_{p-i}(x_1;\ x_1, x_2, \ldots, x_p)$

is equal to

$$[x_1^{p-i} - x_1^{p-i-1}(x + k_1) + x_1^{p-i-2}(xk_1 + k_2) - \ldots\ldots + (-1)^{p-i} x k_{p-i-1}]$$
$$- [x_1^{p-i} - x_1^{p-i-1}(x_1 + k_1) + x_1^{p-i-2}(x_1 k_1 + k_2) - \ldots\ldots + (-1)^{p-i} x_1 k_{p-i-1}],$$

wherein k_r denotes the sum of the homogeneous products of $x_2, \ldots, x_p$, without repetitions, r together, and is therefore equal to

$$(x_1 - x)\left[x_1^{p-i-1} - x_1^{p-i-2}k_1 + \ldots\ldots + (-)^{p-i-1}k_{p-i-1}\right]$$

or to

$$(x_1 - x)\,\chi_{p-i-1}(x_1;\ x_2, \ldots, x_p).$$

Hence

$$\frac{\chi_{p-i}(x_1;\ x, x_2, \ldots, x_p)}{R'(x_1)} = \frac{\chi_{p-i}(x_1;\ x_1, x_2, \ldots, x_p) + (x_1 - x)\,\chi_{p-i-1}(x_1;\ x_2, \ldots, x_p)}{(x_1 - x)\,F'(x_1)}$$

$$= -\frac{\chi_{p-i}(x_1;\ x_1, x_2, \ldots, x_p)}{F'(x_1)}\,\frac{1}{x - x_1} + \frac{\chi_{p-i-1}(x_1;\ x_2, \ldots, x_p)}{F'(x_1)}.$$

While, also,

$$\frac{\chi_{p-i}(x;\ x_1, \ldots, x_p)}{R'(x)} = \sum_{k=1}^{p} \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)}\,\frac{1}{x - x_k}.$$

Thus

$$f_{p-i}(x, x_1, \ldots, x_p) = \sum_{k=1}^{p} \frac{\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)}\,\frac{y - y_k}{x - x_k} + f_{p-i-1}(x_1, \ldots, x_p).$$

Therefore the expression

$$\zeta_i(u^{x,\,a} + u^{x_1,\,a_1} + \ldots + u^{x_p,\,a_p}) + L_i^{x,\,\mu} + L_i^{x_1,\,\mu} + \ldots + L_i^{x_p,\,\mu} - \tfrac{1}{2}f_{p-i}(x, x_1, \ldots, x_p)$$

is equal to

$$\zeta_i(u^{x,\,a} + u^{x_1,\,a_1} + \ldots + u^{x_p,\,a_p}) + L_i^{x_1,\,\mu} + \ldots + L_i^{x_p,\,\mu} - \tfrac{1}{2}f_{p-i}(a, x_1, \ldots, x_p).$$

In this equation the left-hand side is symmetrical in $x, x_1, \ldots, x_p$, and the right-hand side does not contain x. Hence the left-hand side is a constant in regard to x, and, therefore, also in regard to $x_1, \ldots, x_p$. That is, the left-hand side is an absolute constant, depending on the place μ. Denoting this constant by $-C$ we have

$$-\zeta_i(u^{x,\,a} + u^{x_1,\,a_1} + \ldots\ldots + u^{x_p,\,a_p}) = L_i^{x,\,\mu} + L_i^{x_1,\,\mu} + \ldots\ldots + L_i^{x_p,\,\mu}$$

$$-\frac{y\chi_{p-i}(x;\ x_1, \ldots, x_p)}{2R'(x)} - \ldots\ldots - \frac{y_p\chi_{p-i}(x_p;\ x, x_1, \ldots, x_{p-1})}{2R'(x_p)} + C.$$

215. From this equation another important result can be deduced. It is clear that the function

$$-\zeta_i(u^{x,\,a} + u^{x_1,\,a_1} + \ldots\ldots + u^{x_p,\,a_p}) - L_i^{x_1,\,a_1} - \ldots\ldots - L_i^{x_p,\,a_p}$$

does not become infinite when x approaches the place a, that is, the place infinity. If we express the value of this function by the equation just obtained, it is immediately seen that the limit of

$$\frac{-y_k\chi_{p-i}(x_k;\ x, x_1, \ldots, x_p)}{2R'(x_k)} \text{ is } -\frac{y_k\chi_{p-i-1}(x_k;\ x_1, \ldots, x_p)}{2F'(x_k)},$$

and that the expression

$$\frac{y\chi_{p-i}(x\,;\; x_1, \ldots, x_p)}{2R'(x)},$$

when expanded in powers of t by the substitutions $x = \frac{1}{t^2}$, $y = \frac{2}{t^{2p+1}}(1 + At^2 + \ldots)$, where A is a certain constant, contains *only odd powers of t.* Hence the limit when t is zero of the terms of the expansion of this expression other than those containing negative powers of t, is absolute zero, and therefore, *does not depend on the places* $x_1, \ldots, x_p$. The terms of the expansion which contain negative powers of t are cancelled by terms arising from the integral $L_i^{x,\,\mu}$. Since this integral does not contain $x_1, \ldots, x_p$ we infer that the difference

$$L_i^{x,\,\mu} - \frac{y\chi_{p-i}(x\,;\; x_1, \ldots, x_p)}{2R'(x)}$$

has a limit independent of $x_1, \ldots, x_p$, and, therefore, that

$$-\zeta_i(u^{x_1,\,a_1} + \ldots + u^{x_p,\,a_p}) = L_i^{x_1,\,a_1} + \ldots + L_i^{x_p,\,a_p} - \sum_{k=1}^{p} \frac{y_k\chi_{p-i-1}(x_k;\, x_1, \ldots, x_p)}{2F'(x_k)},$$

no additive constant being necessary because, as $\zeta_i(u)$ is an odd function, both sides of the equation vanish when $x_1, \ldots, x_p$ are respectively at the places $a_1, \ldots, a_p$. As any argument can be written, save for periods, in the form $u^{x_1,\,a_1} + \ldots + u^{x_p,\,a_p}$, this equation is theoretically sufficient to enable us to express $\zeta_i(u)$ for any value of u.

Ex. i. It can easily be shewn (§ 200) that

$$u^{c,\,a} + u^{c_1,\,a_1} + \ldots\ldots + u^{c_p,\,a_p} = 0.$$

Thus the final formula of § 214 immediately gives

$$-\zeta_i(u^{x_1,\,c_1} + \ldots\ldots + u^{x_p,\,c_p}) = L_i^{x_1,\,c_1} + \ldots\ldots + L_i^{x_p,\,c_p} - \sum_{k=1}^{p} \frac{y_k\chi_{p-i}(x_k;\; c,\; x_1, \ldots, x_p)}{2(x_k - c)F'(x_k)}.$$

Ex. ii. In case $p=1$ we infer from the formula just obtained, and from the final formula of § 214, respectively, the results

$$-\zeta_1(u^{x_1,\,a_1}) = L_1^{x_1,\,a_1}, \quad -\zeta_1(u^{x,\,a} + u^{x_1,\,a_1}) = L_1^{x,\,a_1} + L_1^{x_1,\,a_1} - \tfrac{1}{2}\frac{y - y_1}{x - x_1} + D,$$

where D is an absolute constant. Thus

$$\zeta_1(u^{x,\,a} + u^{x_1,\,a_1}) = \zeta_1(u^{x,\,a_1}) + \zeta_1(u^{x_1,\,a_1}) + \tfrac{1}{2}\frac{y - y_1}{x - x_1} - D.$$

This is practically equivalent with the well-known formula

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \tfrac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v}.$$

The identification can be made complete by means of the facts (i) The Weierstrass argument u is equal to $u^{a,\,x}$, in our notation, so that $y = -\wp'(u)$, (ii) $u^{x,\,a_1} = \omega + \omega' - u$, so that $\zeta_1(u^{x,\,a_1}) = \zeta_1(\omega + \omega' - u) = -L_1^{x,\,a_1} = -\int_{a_1}^{x} \frac{x\,dx}{y}$, as we easily find when $L_1^{x,\,\mu}$ is chosen as in § 138, Ex. i., (iii) $d\zeta u = \frac{x\,dx}{y}$, (iv) therefore $\zeta_1(u^{x,\,a_1}) = -\zeta u$, (v) the branch places c_1, a_1, c are chosen by Weierstrass (in accordance with the formula $e_1+e_2+e_3=0$) so that the limit of $\wp u - \frac{1}{u^2}$, when $u=0$, is 0. The effect of this is that the constant D is zero.

Ex. iii. For $p=2$ we have

$$-\zeta_1(u^{x,\,a}+u^{x_1,\,a_1}+u^{x_2,\,a_2}) = L_1^{x,\,\mu}+L_1^{x_1,\,\mu}+L_1^{x_2,\,\mu}$$
$$-\frac{y\,(x-x_1-x_2)}{2\,(x-x_1)\,(x-x_2)} - \frac{y_1\,(x_1-x-x_2)}{2\,(x_1-x)\,(x_1-x_2)} - \frac{y_2\,(x_2-x-x_1)}{2\,(x_2-x)\,(x_2-x_1)} + C_1$$

$$-\zeta_2(u^{x,\,a}+u^{x_1,\,a_1}+u^{x_2,\,a_2}) = L_2^{x,\,\mu}+L_2^{x_1,\,\mu}+L_2^{x_2,\,\mu}$$
$$-\frac{y}{2\,(x-x_1)\,(x-x_2)} - \frac{y_1}{2\,(x_1-x)\,(x_1-x_2)} - \frac{y_2}{2\,(x_2-x)\,(x_2-x_1)} + C_2$$

and

$$-\zeta_1(u^{x_1,\,a_1}+u^{x_2,\,a_2}) = L_1^{x_1,\,a_1}+L_2^{x_2,\,a_2} - \tfrac{1}{2}\frac{y_1-y_2}{x_1-x_2},\quad -\zeta_2(u^{x_1,\,a_1}+u^{x_2,\,a_2}) = L_2^{x_1,\,a_1}+L_2^{x_2,\,a_2},$$

where with a suitable determination of the matrix a which occurs in the definition of the integrals $L_i^{x,\,\mu}$ and in the function $\vartheta(u)$, we may take (§ 138, Ex. i. Chap. VII.)

$$L_1^{x,\,\mu} = \int_\mu^x \frac{dy}{4y}(\lambda_3 x + 2\lambda_4 x^2 + 3\lambda_5 x^3),\quad L_2^{x,\,\mu} = \int_\mu^x \frac{dy}{4y}\lambda_5 x^2.$$

For any values of p we obtain

$$-\zeta_p(u^{x_1,\,a_1}+\ldots\ldots+u^{x_p,\,a_p}) = L_p^{x_1,\,a_1}+\ldots\ldots+L_p^{x_p,\,a_p} = \frac{\lambda_{2p+1}}{4}\sum_{k=1}^{p}\int_{a_k}^{x_k}\frac{x^p dx}{y}.$$

Ex. iv. We have (§ 210) obtained $2^{2p}-1$ formulae of the form

$$\frac{\vartheta(u\,|\,u^{b_1,\,a}+\ldots\ldots+u^{b_k,\,a})}{\vartheta(u)} = Z,$$

where Z is an algebraical function, and the arguments $u_1, \ldots, u_p$ are given by

$$u = u^{x_1,\,a_1}+\ldots\ldots+u^{x_p,\,a_p};$$

the integrals being taken as in § 214, these equations lead to

$$\frac{\partial x_r}{\partial u_i} = \tilde{\nu}_{r,\,i}\frac{dx_r}{dt} = y_r\frac{\chi_{p-i}(x_r;\ x_1, \ldots, x_p)}{F'(x_r)}.$$

Hence we have

$$\zeta_i(u\,|\,u^{b_1,\,a}+\ldots\ldots+u^{b_k,\,a}) - \zeta_i(u) = \sum_{r=1}^{p} y_r\frac{\chi_{p-i}(x_r;\ x_1, \ldots, x_p)}{F'(x_r)}\frac{1}{Z}\frac{\partial Z}{\partial x_r}.$$

For instance, when $k=1$, and Z is a constant multiple of $\sqrt{(b_1-x_1)\ldots\ldots(b_1-x_p)}$, we obtain

$$\zeta_i(u\,|\,u^{b_1,\,a}) - \zeta_i(u) = \sum_{r=1}^{p} y_r\frac{\chi_{p-i}(x_r;\ x_1, \ldots, x_p)}{2F'(x_r)}\frac{1}{x_r-b_1},$$

so that

$$-\zeta_i(u\,|\,u^{b,\,a}) = L_i^{x_1,\,a_1}+\ldots\ldots+L_i^{x_p,\,a_p} - \sum_{r=1}^{p}\frac{y_r}{2F'(x_r)}\left[\chi_{p-i-1}(x_r;\ x_1, \ldots, x_p) + \frac{\chi_{p-i}(x_r;\ x_1, \ldots, x_p)}{x_r-b}\right]$$
$$= L_i^{x_1,\,a_1}+\ldots\ldots+L_i^{x_p,\,a_p} - \sum_{r=1}^{p}\frac{y_r}{2F'(x_r)}\frac{\chi_{p-i}(x_r;\ b,\ x_1, \ldots, x_p)}{x_r-b}.$$

By means of the formula

$$\zeta_i(u+\tfrac{1}{2}\Omega_{P,P'})=\eta_{i,1}P_1+\ldots\ldots+\eta_{i,p}P_p+\eta'_{i,1}P_1'\ldots\ldots+\eta'_{i,p}P_p'+\zeta_i(u|\tfrac{1}{2}\Omega_{P,P'}),$$

which is easily obtained from the formulae of § 190, we can infer that the formula just obtained is in accordance with the final formula of § 214.

Ex. v. We have seen (§ 185, Chap. X.) that in the hyperelliptic case there are $\binom{2p+1}{p}$ even theta functions which do not vanish ; and the corresponding half-periods are congruent to expressions of the form

$$u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p}.$$

It may be shewn in fact that these half-periods are obtained by taking for $x_1, \ldots, x_p$ the $\binom{2p+1}{p}$ possible sets of p branch places that can be chosen from $a_1, \ldots, a_p, c, c_1, \ldots, c_p$. Hence it follows from the formula of the text (p. 321) that if $\frac{1}{2}\Omega_k$ be any even half-period corresponding to a non-vanishing theta function, we have

$$\zeta_i(\tfrac{1}{2}\Omega_k)=(\tfrac{1}{2}H_k)_i.$$

This formula generalises the well-known elliptic function formula expressed by $\zeta\omega=\eta$. To explain the notation a particular case may be given; we have

$$\zeta_i(\omega_{1,r},\ \omega_{2,r},\ \ldots,\ \omega_{p,r})=\eta_{i,r}, \text{ or } \zeta_i(u^{c_{r+1}, a_r})=-L_i^{c_{r+1}, a_r}$$

and

$$\zeta_i(\omega'_{1,r},\ \omega'_{2,r},\ \ldots,\ \omega'_{p,r})=\eta'_{i,r}, \text{ or } \zeta_i(u^{c_r, a_r})\ \ =-L_i^{c_r, a_r}.$$

Thus each of the $2p^2$ quantities $\eta_{i,r}$, $\eta'_{i,r}$ can be expressed as ζ-functions of half-periods.

Ex. vi. The formula of the text (p. 321) is equivalent to

$$-\zeta_i(u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p})=L_i^{x_1, a_1}+\ldots\ldots+L_i^{x_p, a_p}-\tfrac{1}{2}\sum_{k=1}^{p}\frac{\partial x_k}{\partial u_{i+1}},$$

where

$$u_r=u_r^{x_1, a_1}+\ldots\ldots+u_r^{x_p, a_p}.$$

For example when $p=2$

$$-\zeta_1(u)+\tfrac{1}{2}\frac{\partial}{\partial u_2}(x_1+x_2)=L_1^{x_1, a_1}+L_1^{x_2, a_2}$$

$$-\zeta_2(u)\qquad\qquad\qquad =L_2^{x_1, a_1}+L_2^{x_2, a_2}.$$

216. It is easy to prove, as remarked in Ex. iii. § 215, that if

$$u=u^{x_1, a_1}+\ldots\ldots+u^{x_p, a_p},$$

and the matrix a (§ 138, Chap. VII.) be determined so that the integrals $L_i^{x,\mu}$ have the value found in § 138, Ex. i., then

$$-\zeta_p(u)=\tfrac{1}{4}\lambda_{2p+1}\sum_{k=1}^{p}\int_{a_k}^{x_k}\frac{x^p\,dx}{y}.$$

Therefore, if $-\frac{\partial}{\partial u_i}\zeta_r(u)$ be denoted by $\wp_{r,i}(u)$, we have

$$\wp_{p,i}(u)=-\frac{\partial\zeta_p(u)}{\partial u_i}=\tfrac{1}{4}\lambda_{2p+1}\sum_{k=1}^{p}\frac{x_k^p}{y_k}\frac{\partial x_k}{\partial u_i},$$

and thus, as follows from the definition of the arguments u,

$$\wp_{p,i}(u) = \tfrac{1}{4}\lambda_{2p+1} \sum_{k=1}^{p} \frac{x_k^p \chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)},$$

where $F(x)$ denotes $(x - x_1) \ldots (x - x_p)$.

Whence, if x be any argument whatever,

$$\sum_{i=1}^{p} x^{i-1} \wp_{p,i}(u), = \tfrac{1}{4}\lambda_{2p+1} \sum_{k=1}^{p} \frac{x_k^p \sum\limits_{i=1}^{p} x^{i-1}\chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)},$$

$$= \tfrac{1}{4}\lambda_{2p+1} \sum_{k=1}^{p} \frac{x_k^p F(x)}{(x - x_k) F'(x_k)};$$

but we have

$$\frac{\sum\limits_{i=1}^{p} x^{i-1} \wp_{p,i}(u)}{F(x)} = \sum_{k=1}^{p} \frac{\sum\limits_{i=1}^{p} x_k^{i-1} \wp_{p,i}(u)}{(x - x_k) F'(x_k)}.$$

Thus

$$\tfrac{1}{4}\lambda_{2p+1} x_k^p = \sum_{i=1}^{p} x_k^{i-1} \wp_{p,i}(u).$$

Thus, if we suppose $\lambda_{2p+1} = 4$, the values of $x_1, \ldots, x_p$ satisfying the inversion problem expressed by the equations

$$u \equiv u^{x_1, a_1} + \ldots\ldots + u^{x_p, a_p}$$

are the roots of the equation

$$F(x) = x^p - x^{p-1}\wp_{p,p}(u) - x^{p-2}\wp_{p,p-1}(u) - \ldots\ldots - \wp_{p,1}(u) = 0.$$

In other words, if the sum of the homogeneous products of r dimensions, without repetitions, of the quantities $x_1, \ldots, x_p$ be denoted by h_r, we have

$$h_r = (-)^{r-1}\wp_{p,p-r+1}(u).$$

Further, from the equation

$$\frac{\partial x_k}{\partial u_i} = \frac{y_k \chi_{p-i}(x_k;\ x_1, \ldots, x_p)}{F'(x_k)},$$

putting p for i, we infer that

$$y_k = F'(x_k)\frac{\partial x_k}{\partial u_p}, = -\left[\frac{\partial F(x)}{\partial u_p}\right]_{x=x_k},$$

because $F(x_k) = 0$. Thus, if we use the abbreviation

$$\psi(x) = -\frac{\partial F(x)}{\partial u_p} = x^{p-1}\wp_{p,p,p}(u) + x^{p-2}\wp_{p,p,p-1}(u) + \ldots\ldots + \wp_{p,p,1}(u)$$

we obtain

$$y_k = \psi(x_k).$$

These equations constitute a complete solution of the inversion problem. In the $\wp$-functions the matrix a is as in § 138, Ex. i., and the integrals of the first kind are as in § 214.

We have previously (§ 212) shewn that $x_1, \dots, x_p$ are determinable from p such equations as

$$\frac{\vartheta^2(u|u^{a_i, a})}{\vartheta^2(u)} = \pm \frac{(a_i - x_1) \dots (a_i - x_p)}{\sqrt{-P'(a_i)\, Q(a_i)}}, = \frac{(a_i - x_1) \dots (a_i - x_p)}{\mu_i}, \text{ say.}$$

Thus we have p equations of the form

$$\mu_i \frac{\vartheta^2(u|u^{a_i, a})}{\vartheta^2(u)} = a_i^p - a_i^{p-1} \wp_{p, p}(u) - a_i^{p-2} \wp_{p, p-1}(u) - \dots\dots - \wp_{p, 1}(u).$$

Ex. i. For $p=1$ we have

$$\mu_1 \frac{\vartheta^2(u|u^{a_1, a})}{\vartheta^2(u)} = a_1 - \wp_{1, 1}(u), \ = a_1 + \frac{\partial^2}{\partial u^2} \log \vartheta(u).$$

This is equivalent to the equation which is commonly written in the form

$$\wp u = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(u\sqrt{e_1 - e_3})}.$$

Ex. ii. For $p=2$ we have

$$\mu_1 \frac{\vartheta^2(u|u^{a_1, a})}{\vartheta^2(u)} = a_1^2 - a_1 \wp_{2, 2}(u) - \wp_{2, 1}(u),$$

$$\mu_2 \frac{\vartheta^2(u|u^{a_2, a})}{\vartheta^2(u)} = a_2^2 - a_2 \wp_{2, 2}(u) - \wp_{2, 1}(u).$$

We may denote the left-hand sides of these equations respectively by $\mu_1 q_1^2$, $\mu_2 q_2^2$.

Ex. iii. Prove that, with $\mu_1 q_1^2 = a_1^2 - a_1 \wp_{2, 2}(u) - \wp_{1, 2}(u)$, etc., $\mu_1 = \pm\sqrt{-f'(a_1)}$, we have

$$\frac{\mu_1 \mu_2}{a_1 - a_2}(q_1^2 q_2'^2 - q_2^2 q_1'^2)$$
$$= \wp_{22}(u)\, \wp_{12}(u') - \wp_{12}(u)\, \wp_{22}(u') + (a_1 + a_2)[\wp_{12}(u) - \wp_{12}(u')] + a_1 a_2 [\wp_{22}(u) - \wp_{22}(u')].$$

Ex. iv. Prove that

$$y_s = \frac{\partial x_s}{\partial u_1} + x_s \frac{\partial x_s}{\partial u_2} + \dots\dots + x_s^{p-1} \frac{\partial x_s}{\partial u_p}.$$

Ex. v. If, with $P(x)$ to denote $(x - a_1) \dots\dots (x - a_p)$, we put

$$V_r = \int_{a_1}^{x_1} \frac{P(x)}{x - a_r} \frac{dx}{2y} + \dots\dots + \int_{a_p}^{x_p} \frac{P(x)}{x - a_r} \frac{dx}{2y},$$

prove that

$$\frac{\partial}{\partial V_1} + \dots\dots + \frac{\partial}{\partial V_p} = 2 \frac{\partial}{\partial u_p}.$$

Ex. vi. With the same notation, shew that if

$$G = \int_{a_1}^{x_1} P(x) \frac{dx}{2y} + \dots\dots + \int_{a_p}^{x_p} P(x) \frac{dx}{2y},$$

then

$$\frac{\partial G}{\partial V_i} = -\frac{(a_i - x_1) \dots\dots (a_i - x_p)}{P'(a_i)}.$$

The arguments $V_1, \ldots, V_p$ are those used by Weierstrass (*Math. Werke*, Bd. I. Berlin, 1894, p. 297). The result of Ex. iv. is necessary to compare his results with those here obtained. The equation $y_r = \psi(x_r)$ is given by Weierstrass. The relation of Ex. vi. is given by Hancock (*Eine Form des Additionstheorem u. s. w. Diss.* Berlin, 1894, Bernstein).

With these arguments we have

$$\mu_i - \frac{\vartheta^2(u|u^{a_1,\,a})}{\vartheta^2(u)} = a_i^p - \tfrac{1}{2}P'(a_i)\frac{\partial}{\partial V_i}\zeta_p(u) = a_i^p - \tfrac{1}{4}P'(a_i)\frac{\partial}{\partial V_i}\left(\frac{\partial}{\partial V_1} + \ldots\ldots + \frac{\partial}{\partial V_p}\right)\log\vartheta(u).$$

Ex. vii. Prove from the formula

$$-\zeta_i(u^{x,\,a}+u) + \zeta_i(u^{\mu,\,a}+u) = L_i^{x,\,\mu} + \sum_{k=1}^{p}\tilde{\nu}_{k,\,i}\,[(x_k,\,x) - (x_k,\,\mu)]\frac{dx_k}{dt},$$

where

$$u = u^{x_1,\,a_1} + \ldots\ldots + u^{x_p,\,a_p},$$

that the function

$$\frac{\partial}{\partial u_i}\log\left[\frac{\vartheta^2(u^{x,\,a}+u)}{F(x)}\,e^{2\sum_{r=1}^{p} u_r L_r^{x,\,c}}\right] - \frac{y\chi_{p-i}(x;\,x_1,\,\ldots,\,x_p)}{F(x)}$$

is independent of the place x. Here c is an arbitrary place and $F(x) = (x-x_1)\ldots\ldots(x-x_p)$.

Ex. viii. If $R_{z,\,c}^{x,\,a}$ denote the integral $\Pi_{z,\,c}^{x,\,a} - 2\Sigma\Sigma a_{i,\,j}\,u_i^{z,\,c}\,u_j^{x,\,a}$, obtained in § 138, and $F_z^{x,\,a}$ denote $D_z R_{z,\,c}^{x,\,a}$, prove that in the hyperelliptic case, with the matrix a determined as in Ex. i. § 138, when the place a is at infinity,

$$F_a^{x,\,\mu} = -\frac{\sqrt{\lambda_{2p+1}}}{2}\int_\mu^x \frac{x^p dx}{y}.$$

Hence, when $\lambda_{2p+1} = 4$, shew that the equation obtained in § 215 (p. 321) is deducible from the equation (Chap. X. § 196)

$$F_m^{z_1,\,m_1} + \ldots\ldots + F_m^{z_p,\,m_p} = -\sum_{r=1}^{p}\mu_r(m)\,\zeta_r(u^{z_1,\,m_1} + \ldots\ldots + u^{z_p,\,m_p}).$$

Ex. ix. We can also express the function $\zeta_p(u+v) - \zeta_p(u) - \zeta_p(v)$, which is clearly a periodic function of the arguments u, v, in an algebraical form, and in a way which generalizes the formula of Jacobi's elliptic functions given by

$$Z(u) + Z(v) - Z(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v).$$

For if we take places $x_1, \ldots, \zeta_p$, such that

$$u \equiv u^{x_1,\,a_1} + \ldots\ldots + u^{x_p,\,a_p}$$

$$v \equiv u^{z_1,\,a_1} + \ldots\ldots + u^{z_p,\,a_p}$$

$$-u - v \equiv u^{\zeta_1,\,a_1} + \ldots\ldots + u^{\zeta_p,\,a_p},$$

these $3p$ places will be the zeros of a rational function which has $a_1, \ldots, a_p$ as poles, each to the third order. This function is expressible in the form $(My+NP)/P^2$, where P denotes $(x-a_1)\ldots\ldots(x-a_p)$, M is an integral polynomial in x of order $p-1$, and N is an integral polynomial in x of order p. Denoting this function by Z, we have

$$\zeta_p(u) + \zeta_p(v) - \zeta_p(u+v), \; = L_p^{x_1,\,a_1} + \ldots + L_p^{x_p,\,a_p} + L_p^{z_1,\,a_1} + \ldots + L_p^{z_p,\,a_p} + L_p^{\zeta_1,\,a_1} + \ldots + L_p^{\zeta_p,\,a_p},$$

$$= -\int_\infty^0 \left[\frac{dI}{dt}\,\frac{1}{Z-\mu}\right]_{t^{-1}} d\mu, \; = K \text{ say},$$

by § 154, Chap. VIII., where $I=L_p^{x,\mu}=\frac{1}{4}\lambda_{2p+1}\int_\mu^x \frac{x^p dx}{y}$. Writing Z in the form

$$\frac{(Ax^{p-1}+\ldots\ldots)\,y+(x^p+\ldots\ldots)\,P}{P^2},$$

and taking $\lambda_{2p+1}=4$, we find the value of the integral K to be $-2A$.

But from the equation

$$N^2P-4M^2Q=(x-x_1)\ldots\ldots(x-x_p)(x-z_1)\ldots\ldots(x-z_p)(x-\zeta_1)\ldots\ldots(x-\zeta_p),$$

where $Q=(x-c)(x-c_1)\ldots\ldots(x-c_p)$, we have, putting a_i for x,

$$p_i q_i \varpi_i = 2\sqrt{-Q(a_i)}\,(Aa_i^{p-1}+\ldots), \qquad (i=1,\,2,\,\ldots,\,p),$$

where $p_i=\sqrt{(a_i-x_1)\ldots\ldots(a_i-x_p)}$, $q_i=\sqrt{(a_i-z_1)\ldots\ldots(a_i-z_p)}$, $\varpi_i=\sqrt{(a_i-\zeta_1)\ldots\ldots(a_i-\zeta_p)}$; solving these equations for A we eventually have*

$$\zeta_p(u)+\zeta_p(v)-\zeta_p(u+v)=\sum_{i=1}^{p}\frac{p_i q_i \varpi_i}{P(a_i)\sqrt{-Q(a_i)}}.$$

Ex. x. Obtain, for $p=2$, the corresponding expression for $\zeta_1(u)+\zeta_1(v)-\zeta_1(u+v)$.

Ex. xi. Denoting $\frac{1}{P(a_i)\sqrt{-Q(a_i)}}$ by C_i, the equation

$$\zeta_p(u)+\zeta_p(v)-\zeta_p(u+v)=\sum_{i=1}^{p} C_i p_i q_i \varpi_i$$

gives

$$-\wp_{p,r}(u)+\wp_{p,r}(v)=\sum_{i=1}^{p} C_i\,[p_i^{(r)}q_i-p_iq_i^{(r)}]\,\varpi_i, \qquad (r=1,\,2,\,\ldots,\,p),$$

where $p_i^{(r)}$ denotes $\frac{\partial}{\partial u_r}\sqrt{(a_i-x_1)\ldots\ldots(a_i-x_p)}$. It has been shewn that p_i is a single valued function of u and it may be denoted by $p_i(u)$. Similarly ϖ_i is a single valued function of $u+v$, being equal to $p_i(-u-v)$. The equation here obtained enables us therefore to express $p_i(u+v)$ in terms of $p_i(u)$, $p_i(v)$, and the differential coefficients of these; for we have obtained sufficient equations to express $\wp_{p,r}(u)$, $\wp_{p,r}(v)$ in terms of the functions $p_i(u)$, $p_i(v)$. A developed result is obtained below in the case $p=2$, in a more elementary way.

217. We have obtained in the last chapter (§ 197) the equation

$$\sum_i\sum_j \wp_{i,j}(u^{x,m}-u^{x_1,m_1}-\ldots\ldots-u^{x_p,m_p})\,\mu_i(x)\,\mu_i(x_p)=D_xD_{x_p}R^{x,a}_{x_p,c}.$$

Hence, adopting that determination of the matrix a, occurring in the integrals $L_i^{x,\mu}$, and the function $\vartheta(u)$ (§ 192, Chap. X.), which gives the particular forms for $L_i^{x,\mu}$ obtained in § 138, Ex. i., we have in the hyperelliptic case

$$\sum_i\sum_j \wp_{i,j}(u^{x,a}+u^{x_1,a_1}+\ldots\ldots+u^{x_p,a_p})\,x^{i-1}x_r^{j-1}=\frac{f(x,x_r)-2yy_r}{4(x-x_r)^2},$$

where $f(x,z)=\sum_{i=0}^{p+1} x^iz^i\,[2\lambda_{2i}+\lambda_{2i+1}(x+z)]$. This equation is, however, in-

* This equation, with the integrals $L_p^{x,a}$ on the left-hand side, is given by Forsyth, *Phil. Trans.* 1883, Part I.

dependent of the particular matrix a adopted. For suppose, instead of the particular integral

$$L_i^{x,\mu}, = \int_\mu^x \frac{dx}{y} \sum_{k=i}^{2p+1-i} \lambda_{k+1+i}(k+1-i)\,x^k,$$

we take

$$L_i^{x,\mu} - \sum_{k=1}^{p} C_{i,k} u_k^{x,\mu},$$

where $C_{i,k} = C_{k,i}$; then (§ 138) this is equivalent to replacing the particular matrix a by $a + \frac{1}{2}C$, where C is an arbitrary symmetrical matrix, and we have the following resulting changes (p. 315)

$R_{z,c}^{x,a}$ (p. 194) becomes changed to $R_{z,c}^{x,a} - \Sigma\Sigma C_{i,k} u_i^{x,a} u_k^{z,c}$, so that,

$f(x, z)$ (p. 195) becomes changed to $f(x, z) - 4(x-z)^2 \Sigma\Sigma C_{i,k} x^{i-1} z^{k-1}$,

$\vartheta(u)$ (§ 189) becomes multiplied by $e^{\frac{1}{2}Cu^2}$,

and thus $\zeta_i(u)$ is increased by $C_{i,1}u_1 + \ldots\ldots + C_{i,p}u_p$, and instead of $\wp_{i,j}(u)$ we have $\wp_{i,j}(u) - C_{i,j}$.

Since now $u^{x,a} + u^{x_s,a_s} = u^{x_s,a} + u^{x,a_s}$, we have $\frac{1}{2}p(p+1)$ equations of the form

$$\sum_i\sum_j \wp_{i,j}(u)\, x_r^{i-1} x_s^{j-1} = \frac{f(x_r, x_s) - 2y_r y_s}{4(x_r - x_s)^2},$$

where $u = u^{x,a} + u^{x_1,a_1} + \ldots\ldots + u^{x_p,a_p}$, $r = 0, 1, \ldots, p$, and $s = 0, 1, \ldots, p$. Hence, if e_1, e_2 denote any quantities we obtain by calculation

$$\sum_i\sum_j \wp_{i,j}(u)\, e_1^{i-1} e_2^{j-1} = R(e_1)\,R(e_2) \sum_r\sum_s \frac{2y_r y_s - f(x_r, x_s)}{4G'(x_r)\,G'(x_s)};$$

here the matrix a is arbitrary, the polynomial $f(x_r, x_s)$ being correspondingly chosen, and

$$G(\xi) = (\xi - e_1)(\xi - e_2)(\xi - x)(\xi - x_1)\ldots(\xi - x_p),\quad R(\xi) = (\xi - x)(\xi - x_1)\ldots(\xi - x_p).$$

Suppose now that $f(x, z) = \bar{f}(x, z) + 4(x-z)^2 \sum_i\sum_j A_{i,j} x_r^{i-1} x_s^{j-1}$, where $\bar{f}(x, z)$ is the form obtained in Ex. viii. § 211; then we obtain

$$\sum_i\sum_j [\wp_{i,j}(u) - A_{i,j}]\, e_1^{i-1} e_2^{j-1} = R(e_1)\,R(e_2) \sum_r\sum_s \frac{2y_r y_s - \bar{f}(x_r, x_s)}{4G'(x_r)\,G'(x_s)},$$

and by Ex. x. § 211 this is equal to

$$\frac{1}{4} R(e_1)\,R(e_2) \left[\sum_0^p \frac{y_r}{(e_1 - x_r)(e_2 - x_r)\,R'(x_r)}\right]^2 - \frac{f(e_1)\,R(e_2)}{4(e_1 - e_2)^2\,R(e_1)}$$

$$- \frac{f(e_2)\,R(e_1)}{4(e_1 - e_2)^2\,R(e_2)} + \frac{\bar{f}(e_1, e_2)}{4(e_1 - e_2)^2},$$

and therefore

$$\sum_i\sum_j \wp_{i,j}(u)\, e_1^{i-1} e_2^{j-1} = \tfrac{1}{4} R(e_1) R(e_2) \left[\sum_0^p \frac{y_r}{(e_1 - x_r)(e_2 - x_r) R'(x_r)}\right]^2 - \frac{f(e_1) R(e_2)}{4(e_1 - e_2)^2 R(e_1)} - \frac{f(e_2) R(e_1)}{4(e_1 - e_2)^2 R(e_2)} + \frac{f(e_1, e_2)}{4(e_1 - e_2)^2}.$$

This is a very general formula*; in it the matrix a is arbitrary.

It follows from Ex. xi. § 211 that if b_1, b_2 be any branch places, we have

$$\sum_i\sum_j \wp_{i,j}(u)\, b_1^{i-1} b_2^{j-1} = \frac{f(b_1, b_2)}{4(b_1 - b_2)^2} + E \frac{\vartheta^2(u \mid u^{b_1, a} + u^{b_2, a})}{\vartheta^2(u)},$$

where E is a certain constant (cf. §§ 213, 212). This equation is also independent of the determination of the matrix a.

By solving $\frac{1}{2}p(p+1)$ equations of this form, wherein b_1, b_2 are in turn taken to be every pair chosen from any $p+1$ branch places, we can express $\sum_i\sum_j \wp_{i,j}(u)\, e_1^{i-1} e_2^{j-1}$ as a linear function of $\frac{1}{2}p(p+1)$ squared theta quotients, e_1, e_2 *being any quantities whatever.*

By putting b_2 at a, that is at infinity (first dividing by b_2^{p-1}), and putting x also at a, this becomes the formula already obtained (§ 216)

$$\mu_i \frac{\vartheta^2(u \mid u^{a_i, a})}{\vartheta^2(u)} = a_i^p - a_i^{p-1} \wp_{p,p}(u) - \dots\dots - \wp_{p,1}(u).$$

Ex. i. When $p=1$, taking the fundamental equation to be

$$y^2 = 4x^3 - g_2 x - g_3,$$

the expression

$$f(x, z), = \sum_0^{p+1} x^i z^i [2\lambda_{2i} + \lambda_{2i+1}(x+z)], = -2g_3 - g_2(x+z) + 4xz(x+z),$$

and

$$\frac{2ys - f(x, z)}{4(x-z)^2} = \frac{2ys - (y^2 + s^2) + 4(x^2 - z^2)(x-z)}{4(x-z)^2} = x + z - \tfrac{1}{4}\left(\frac{y-s}{x-z}\right)^2,$$

if $s^2 = 4z^3 - g_2 z - g_3$.

Therefore, by the formula at the middle of page 328, taking the matrix a to have the particular determination of § 138, Ex. i.,

$$\wp_{1,1}(u^{x, a} + u^{x_1, a_1}) = -(e_1 - x)(e_1 - x_1)(e_2 - x)(e_2 - x_1) \frac{x + x_1 - \frac{1}{4}\left(\frac{y - y_1}{x - x_1}\right)^2}{(x - e_1)(x - e_2)(x_1 - e_1)(x_1 - e_2)}$$

$$= -x - x_1 + \tfrac{1}{4}\left(\frac{y - y_1}{x - x_1}\right)^2;$$

this is a well-known result.

Ex. ii. When $p=2$, we easily find

$$\frac{R(e_1) R(e_2)}{G'(x_r) G'(x_s)} = -\frac{(x - e_1)(x - e_2)}{(x - x_r)(x - x_s)} \frac{1}{(x_r - x_s)^2}$$

* It is given by Bolza, *Göttinger Nachrichten*, 1894, p. 268.

and thus the expression

$$\wp_{1,1}(u)+(e_1+e_2)\,\wp_{1,2}(u)+e_1e_2\,\wp_{2,2}(u)$$

is equal to

$$-\frac{(x-e_1)(x-e_2)}{(x-x_1)(x-x_2)}\,\frac{2y_1y_2-f(x_1,x_2)}{4(x_1-x_2)^2}-\frac{(x_1-e_1)(x_1-e_2)}{(x_1-x)(x_1-x_2)}\,\frac{2yy_2-f(x,x_2)}{4(x-x_2)^2}$$
$$-\frac{(x_2-e_1)(x_2-e_2)}{(x_2-x)(x_2-x_1)}\,\frac{2yy_1-f(x,x_1)}{4(x-x_1)^2}.$$

Herein the matrix a is perfectly general. Adopting the particular determination of § 138, Ex. i., we have, since the term in $f(x,z)$ of highest degree in x is $\lambda_{2p+1}x^{p+1}z^p$, $=4x^3z^2$, say, by putting the place x at a, that is at infinity, the result

$$\wp_{1,1}(u)+(e_1+e_2)\,\wp_{1,2}(u)+e_1e_2\wp_{2,2}(u)=-\frac{2y_1y_2-f(x_1,x_2)}{4(x_1-x_2)^2}-x_1x_2(e_1+e_2)+e_1e_2(x_1+x_2),$$

where $u=u^{x_1,a_1}+u^{x_2,a_2}$.

Ex. iii. Prove, for $p=2$, when the matrix a is as in § 138, Ex. i., that

$$\wp_{11}(u)+\wp_{12}(u)\,.\,(e_1+e_2)+\wp_{22}(u)\,.\,e_1e_2=\frac{(a_1-e_1)(a_1-e_2)}{a_1-a_2}\,\mu_2q_2^2-\frac{(a_2-e_1)(a_2-e_2)}{a_1-a_2}\,\mu_1q_1^2$$
$$+\frac{\mu_1\mu_2}{a_1-a_2}\,q_{12}^2+\frac{f(a_1,a_2)}{4(a_1-a_2)^2}+e_1e_2(a_1+a_2)-(e_1+e_2)\,a_1a_2,$$

where e_1, e_2 are any quantities, $u=u^{x_1,a_1}+u^{x_2,a_2}$, and μ_1, μ_2 are as in § 216 (cf. § 213).

Ex. iv. From the formula, for $p=2$ (§§ 217, 216, 213),

$$\wp_{11}(u)+\wp_{12}(u)\,.\,(a_1+a_2)+\wp_{22}(u)\,.\,a_1a_2=\frac{\mu_1\mu_2}{a_1-a_2}\,q_{12}^2+\frac{f(a_1,a_2)}{4(a_1-a_2)^2},$$

where a_1, a_2 are the branch places as before denoted, infer (§ 216, Ex. iii.) that

$$\wp_{11}(u)-\wp_{11}(u')+\wp_{12}(u)\,\wp_{22}(u')-\wp_{12}(u')\,\wp_{22}(u)=\frac{\mu_1\mu_2}{a_1-a_2}\,[q_{12}^2-q'^2_{12}-q_1^2q_2'^2+q_2^2q_1'^2].$$

Prove also that, for any value of u, and any position of x,

$$\wp_{11}(u^{x,a}+u)-\wp_{11}(u)+\wp_{12}(u^{x,a}+u)\,\wp_{22}(u)-\wp_{22}(u^{x,a}+u)\,\wp_{12}(u)=0.$$

Ex. v. If $b_1, \dots, b_{p+1}$ be any $(p+1)$ branch places, and e_1, e_2 any quantities whatever, and $L(x)=(x-b_1)\dots\dots(x-b_{p+1})$, $M(x)=(x-e_1)(x-e_2)(x-b_1)\dots\dots(x-b_{p+1})$, prove that

$$\Sigma\Sigma\wp_{i,j}(u)\,e_1^{i-1}e_2^{j-1}=-L(e_1)\,L(e_2)\sum_r\sum_s\frac{(b_r-b_s)^2}{M'(b_r)\,M'(b_s)}\left[\frac{f(b_r,b_s)}{4(b_r-b_s)^2}+E_{r,s}\,\frac{\vartheta^2(u\,|\,u^{b_r,a}+u^{b_s,a})}{\vartheta^2(u)}\right],$$

where the matrix a has a perfectly general value, r, s consist of every pair of different numbers from the numbers $1, 2, \dots, (p+1)$, and $E_{r,s}$ are constants.

218. We conclude this chapter with some further details in regard to the case $p=2$, which will furnish a useful introduction to the problems of future chapters of the present volume. We have in case $p=1$ such a formula as that expressed by the equation

$$\frac{\sigma(u+u')\,\sigma(u-u')}{\sigma^2(u)\,\sigma^2(u')}=\wp(u')-\wp(u);$$

we investigate now, in case $p=2$, corresponding formulae for the functions

$$\frac{\vartheta(u+u')\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')},\qquad\frac{\vartheta(u+u'\,|\,u^{b,a})\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')};$$

by division of the results we obtain a formula expressing the theta quotient $\vartheta(u+u'|u^{b,a})\div\vartheta(u+u')$ by theta quotients of the arguments u, u'; this formula may be called the addition equation for the theta quotient $\vartheta(u\; u^{b,a})\div\vartheta(u)$. Though we shall in a future chapter obtain the result in another way, it will be found that a certain interest attaches to the mode of proof employed here.

Determine the places x_1, x_2, x_1', x_2' so that

$$u\equiv u^{x_1, a_1}+u^{x_2, a_2},\quad u'\equiv u^{x_1', a_1}+u^{x_2', a_2};$$

then, in order to find where the function $\vartheta(u^{x_1, a_1}+u^{x_2, a_2}+u^{x_1', a_1}+u^{x_2', a_2})$ vanishes, regarded as a function of x_1, we are to put

$$u^{x_1, a_1}+u^{x_2, a_2}+u^{x_1', a_1}+u^{x_2', a_2}\equiv u^{x_1, a}-u^{z_1, a_1}-u^{z_2, a_2},$$

or

$$(a, x_2, x_1', x_2', z_1, z_2)\equiv(a_1^3, a_2^3);$$

thus the places z_1, z_2 are positions of x_1 for which the determinant

$$\nabla=\begin{vmatrix} \dfrac{x_1y_1}{P(x_1)}, & \dfrac{y_1}{P(x_1)}, & x_1, & 1 \\ \dfrac{x_2y_2}{P(x_2)}, & \dfrac{y_2}{P(x_2)}, & x_2, & 1 \\ \dfrac{x_1'y_1'}{P(x_1')}, & \dfrac{y_1'}{P(x_1')}, & x_1', & 1 \\ \dfrac{x_2'y_2'}{P(x_2')}, & \dfrac{y_2'}{P(x_2')}, & x_2', & 1 \end{vmatrix}$$

wherein $P(x)$ denotes $(x-a_1)(x-a_2)$, vanishes. By considerations analogous to those of § 209 we therefore find, $\bar{\nabla}$ denoting the determinant derived from ∇ by changing the sign of y_1', y_2',

$$\frac{\vartheta(u+u')\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')}=A\frac{\nabla\bar{\nabla}P(x_1)\,P(x_2)\,P(x_1')\,P(x_2')}{(x_1-x_2)^2(x_1'-x_2')^2(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')},$$

where A is an absolute constant.

Now, if $\eta_1=y_1/P(x_1)$, etc., we find by expansion and multiplication,

$$\nabla\bar{\nabla}=(\eta_1\eta_2+\eta_1'\eta_2')^2(x_1-x_2)^2(x_1'-x_2')^2-[(\eta_1\eta_1'+\eta_2\eta_2')(x_1'-x_1)(x_2'-x_2) \\ -(\eta_1\eta_2'+\eta_2\eta_1')(x_1'-x_2)(x_2'-x_1)]^2,$$

and, if $a=(x_1'-x_1)(x_2'-x_2)$, $\beta=(x_1'-x_2)(x_2'-x_1)$, $a-\beta=(x_1'-x_2')(x_1-x_2)$, this leads to

$$\frac{\nabla\bar{\nabla}}{a-\beta}=(\eta_1^2-\eta_2'^2)(\eta_2^2-\eta_1'^2)\,a-(\eta_1^2-\eta_1'^2)(\eta_2^2-\eta_2'^2)\,\beta-\frac{a\beta}{a-\beta}(\eta_1-\eta_2)^2(\eta_1'-\eta_2')^2;$$

but, putting $y^2=4P(x)\,Q(x)$, $=4(x-a_1)(x-a_2)(x-c)(x-c_1)(x-c_2)$, we have

$$\frac{P(x_1)\,P(x_2)\,P(x_1')\,P(x_2')}{(a-\beta)\,a\beta}[(\eta_1^2-\eta_2'^2)(\eta_2^2-\eta_1'^2)\,a-(\eta_1^2-\eta_1'^2)(\eta_2^2-\eta_2'^2)\,\beta]$$

$$=\frac{16}{(x_1-x_2)(x_1'-x_2')}\left[\frac{Qx_1Px_2'-Qx_2'Px_1}{x_2'-x_1}\cdot\frac{Qx_2Px_1'-Qx_1'Px_2}{x_1'-x_2}\right. \\ \left.-\frac{Qx_1Px_1'-Qx_1'Px_1}{x_1'-x_1}\cdot\frac{Qx_2Px_2'-Qx_2'Px_2}{x_2'-x_2}\right],$$

and this expression is equal to

$$16\left[Qa_1 . Qa_2+\frac{Qa_2}{P'a_2}(a_1-x_1)(a_1-x_2)(a_1-x_1')(a_1-x_2')\right.$$
$$\left.+\frac{Qa_1}{P'a_1}(a_2-x_1)(a_2-x_2)(a_2-x_1')(a_2-x_2')\right],$$

as may be proved in various ways ; now we have proved (§§ 208, 212, 213) that

$$(a_1-x_1)(a_1-x_2)=\pm\sqrt{-P'(a_1)\,Q(a_1)}\,q_1^2,\quad (a_2-x_1)(a_2-x_2)=\pm\sqrt{-P'(a_2)\,Q(a_2)}\,q_2^2$$

and

$$\tfrac{1}{4}\left(\frac{\eta_1-\eta_2}{x_1-x_2}\right)^2=\pm\frac{1}{a_1-a_2}\frac{q_{12}^2}{q_1^2q_2^2},$$

where $q_1=\vartheta(u|u^{a_1,\,a})\div\vartheta(u)$, $q_2=\vartheta(u|u^{a_2,\,a})\div\vartheta(u)$, $q_{1,\,2}=\vartheta(u|u^{a_1,\,a}+u^{a_2,\,a})\div\vartheta(u)$; thus as $q_1^2q_2^2q_1'^2q_2'^2=\dfrac{P(x_1)\,P(x_2)\,P(x_1')\,P(x_2')}{P'(a_1)\,P'(a_2)\,Qa_1\,Qa_2}$, we have

$$\frac{1}{A}\frac{\vartheta(u+u')\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')},=\frac{\nabla\overline{\nabla}P(x_1)\,P(x_2)\,P(x_1')\,P(x_2')}{a\beta(a-\beta)^2},$$
$$=16\,Qa_1Qa_2\left[1-\frac{P'a_1}{P'a_2}q_1^2q_1'^2-\frac{P'a_2}{P'a_1}q_2^2q_2'^2\right]-16\frac{P'(a_1)\,P'(a_2)\,Qa_1Qa_2}{(a_1-a_2)^2}q_{12}^2q_{12}'^2,$$

where however we have assumed that the sign to be attached to the quotient

$$(a_1-x_1)(a_1-x_2)\div\sqrt{-P'(a_1)\,Q(a_1)}\,q_1^2$$

is the same for the places x_1', x_2' as for the places x_1, x_2. The product $\sqrt{-P'(a_1)\,Q(a_1)}$ $\sqrt{-P'(a_1)\,Q(a_1)}$ is, of course, here equal to $-P'(a_1)\,Q(a_1)$. Now,

$$P'(a_1)=(a_1-a_2)=-P'(a_2)\;;$$

thus we obtain

$$\frac{\vartheta(u+u')\,\vartheta(u-u')\,\vartheta^2}{\vartheta^2(u)\,\vartheta^2(u')}=1+q_1^2q_1'^2+q_2^2q_2'^2+q_{12}^2q_{12}'^2,$$

the value of the constant multiplier, ϑ^2, $=[\vartheta(0)]^2$, being determined by putting $u'=0$, in which case q_1', q_2', $q'_{1,\,2}$ all vanish.

If in this formula we write $v=u+u^{a_1,\,a}+u^{a_2,\,a}$ in place of u, we obtain, from the formulae

$$q_1^2(u+u^{a_1,\,a}+u^{a_2,\,a}),\ =q_1^2(v),\ =\frac{\vartheta^2(v|u^{a_1,\,a})}{\vartheta^2(v)},\ =\frac{\vartheta^2(u|u^{a_2,\,a})}{\vartheta^2(u|u^{a_1,\,a}+u^{a_2,\,a})}=\frac{q_2^2(u)}{q_{12}^2(u)},$$

$$q_2^2(u+u^{a_1,\,a}+u^{a_2,\,a})=-\frac{q_1^2(u)}{q_{12}^2(u)},\quad q_{12}^2(u+u^{a_1,\,a}+u^{a_2,\,a})=-\frac{1}{q_{12}^2(u)},$$

which are easy to verify from the formulae of § 190, Chap. X. and the table of characteristics given in this chapter, that

$$\frac{\vartheta^2.\,\vartheta(u+u'|u^{a_1,\,a}+u^{a_2,\,a})\,\vartheta(u-u'|u^{a_1,\,a}+u^{a_2,\,a})}{\vartheta^2(u|u^{a_1,\,a}+u^{a_2,\,a})\,\vartheta^2(u')}=1+\frac{q_2^2q_1'^2}{q_{12}^2}-\frac{q_2'^2q_1^2}{q_{12}^2}-\frac{q_{12}'^2}{q_{12}^2},$$

and therefore

$$\frac{\vartheta^2.\,\bar{\vartheta}(u+u')\,\bar{\vartheta}(u-u')}{\vartheta^2(u)\,\vartheta^2(u')}=q_{12}^2-q_{12}'^2-q_1^2q_2'^2+q_2^2q_1'^2,$$

where $\bar{\vartheta}(u)$ denotes $\vartheta(u \mid u^{a_1, a} + u^{a_2, a})$. But we can use the result of Ex. iv. § 217, to give the right-hand side a still further form, namely

$$\frac{a_1 - a_2}{\mu_1 \mu_2} [\wp_{11}(u) + \wp_{11}(u') + \wp_{12}(u) \wp_{22}(u') - \wp_{12}(u') \wp_{22}(u)].$$

Further if $u^{a_1, a} + u^{a_2, a} \equiv \frac{1}{2}\Omega_{m, m'}$, where m, m' consist of integers each either 0 or 1, we find, by adding $\frac{1}{2}\Omega_{m, m'}$ to u and u' and utilising the fact (§ 190) that

$$\lambda_m(u + u') = 2\lambda_{\frac{1}{2}m}(u) + 2\lambda_{\frac{1}{2}m}(u'),$$

that

$$\frac{\vartheta^2 \bar{\vartheta}(u + u') \bar{\vartheta}(u - u')}{\bar{\vartheta}^2(u) \bar{\vartheta}^2(u')} \frac{\mu_1 \mu_2}{a_1 - a_2} = \wp_{11}(v) - \wp_{11}(v') + \wp_{12}(v) \wp_{22}(v') - \wp_{12}(v') \wp_{22}(v),$$

where $v = u + \frac{1}{2}\Omega_{m, m'}$, $v' = u' + \frac{1}{2}\Omega_{m, m'}$. It should be noticed that

$$\wp_{i, j}(v) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \vartheta(u \,;\, \tfrac{1}{2}m, \tfrac{1}{2}m')\,; \text{ hence}$$

this formula can be expressed so as to involve only a single function in the form

$$\frac{\vartheta^2 \mu_1 \mu_2}{a_1 - a_2} \cdot \frac{\sigma(u + v) \sigma(u - v)}{\sigma^2(u) \sigma^2(v)} = \wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u) \wp_{22}(v) - \wp_{12}(v) \wp_{22}(u)$$

where $\sigma(u)$ denotes $\vartheta\left(u \,\middle|\, \frac{1}{2}\begin{pmatrix} 01 \\ 11 \end{pmatrix}\right)$, and $\wp_{i, j}(u) = -\dfrac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u)$. In Weierstrass's corresponding formula for $p = 1$, the function $\sigma(u)$ is determined so that $\sigma(u)/u = 1$ when $u = 0$. To introduce the corresponding conditions here would carry us further into detail. (See §§ 212, 213.)

Ex. Prove that if a_3 denote any one of the branch places c, c_1, c_2, $\alpha = (a_2 - a_3)$, $\beta = (a_3 - a_1)$, $\gamma = (a_1 - a_2)$, $P_1 = (a_1 - x_1)(a_1 - x_2)$, etc., $P_1' = (a_1 - x_1')(a_1 - x_2')$, etc., and

$$A = \left[\frac{y_1}{(x_1 - a_1)(x_1 - a_3)} - \frac{y_2}{(x_2 - a_1)(x_2 - a_3)}\right] \frac{1}{x_2 - x_1},$$

$$B = \left[\frac{y_1}{(x_1 - a_2)(x_1 - a_3)} - \frac{y_2}{(x_2 - a_2)(x_2 - a_3)}\right] \frac{1}{x_2 - x_1},$$

with similar notation for A', B', then the determinant Δ can be expressed in the form

$$\Delta = \frac{y_1 y_2}{P(x_1) P(x_2)} + \frac{y_1' y_2'}{P(x_1') P(x_2')} - X,$$

where

$$\gamma^2 X = AA'(P_1 P_3' + P_3 P_1') + BB'(P_2 P_3' + P_3 P_2') - AB'(\gamma\alpha P_3 + \gamma\beta P_3' + P_1 P_3' + P_2' P_3)$$
$$- A'B(\gamma\beta P_3 + \gamma\alpha P_3' + P_2 P_3' + P_1' P_3).$$

In this form Δ can be immediately expressed in terms of theta quotients.

219. Consider, nextly, the function

$$\frac{\vartheta(u + u' \mid u^{a_1, a}) \vartheta(u - u')}{\vartheta^2(u) \vartheta^2(u')}.$$

This is not a periodic function of u, u'. Thus we take in the first place the function

$$\frac{\vartheta(u+u'|u^{a_1, a})\,\vartheta(u-u')}{\vartheta(u)\,\vartheta(u|u^{a_1, a})\,\vartheta(u')\,\vartheta(u'|u^{a_1, a})}.$$

Put

$$u \equiv u^{x_1, a_1} + u^{x_2, a_2},\quad u' \equiv u^{x_1', a_1} + u^{x_2', a_2};$$

then, as functions of x_1, the zeros of $\vartheta(u)$, $\vartheta(u|u^{a_1, a})$ respectively are a, $\overline{x}_2$ and a_1, $\overline{x}_2$, the zeros of $\vartheta(u+u'|u^{a_1, a})$ are found in the usual way to be zeros of a rational function of the fifth order having a_1^2, a_2^3 as poles, and x_2, x_1', x_2' as zeros; such a function of x_1 is $\Delta_1/P(x_1)$, where $P(x_1)=(x_1-a_1)(x_1-a_2)$ and

$$\Delta_1 = \begin{vmatrix} \eta_1\,(x_1-a_1), & x_1^2, & x_1, & 1 \\ \eta_2\,(x_2-a_1), & x_2^2, & x_2, & 1 \\ \eta_1'\,(x_1'-a_1), & x_1'^2, & x_1', & 1 \\ \eta_2'\,(x_2'-a_1), & x_2'^2, & x_2', & 1 \end{vmatrix},$$

wherein $\eta_1 = y_1/P(x_1)$, etc.; the zeros of $\vartheta(u-u')$, as a function of x_1, are similarly zeros of a function of the sixth order having a_1^3, a_2^3 as poles and a, x_2, $\overline{x}_1'$, $\overline{x}_2'$ for its other zeros; such a function of x_1 is $\overline{\Delta}/P(x_1)$, where

$$\overline{\Delta} = \begin{vmatrix} \eta_1 x_1, & \eta_1, & x_1, & 1 \\ \eta_2 x_2, & \eta_2, & x_2, & 1 \\ -\eta_1' x_1', & -\eta_1', & x_1', & 1 \\ -\eta_2' x_2', & -\eta_2', & x_2', & 1 \end{vmatrix};$$

hence we find

$$\frac{\vartheta(u+u'|u^{a_1, a})\,\vartheta(u-u')}{\vartheta(u)\,\vartheta(u|u^{a_1, a})\,\vartheta(u')\,\vartheta(u'|u^{a_1, a})}$$
$$= C\,\frac{\Delta_1\overline{\Delta}\,(x_1-a_2)(x_2-a_2)(x_1'-a_2)(x_2'-a_2)}{(x_1-x_2)^2(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')(x_1'-x_2')^2},$$

wherein C is an absolute constant; for it is immediately seen that the two sides of this equation have the same poles and zeros.

We proceed to put the right-hand side into a particular form; for this purpose we introduce certain notations; denote the quantities c, c_1, c_2, which refer to the branch places other than a_1, a_2 by a_3, a_4, a_5 in any order; denote $(a_i-x_1)(a_i-x_2)$ by p_i, $(a_i-x_1')(a_i-x_2')$ by p_i'; denote by $\pi_{i,j}$ the expression

$$\tfrac{1}{2}\left[\frac{y_1}{(x_1-a_i)(x_1-a_j)} - \frac{y_2}{(x_2-a_i)(x_2-a_j)}\right]\frac{1}{x_2-x_1},$$

and write $p_{i,j}$ for $p_i p_j \pi_{i,j}$, with a similar notation $\pi'_{i,j}$, $p'_{i,j}$; also let $P(x)=(x-a_1)(x-a_2)$, $\eta_1=y_1/P(x_1)$, etc.

Then, by regarding the expression

$$(x_1'-x_2')^2(x_2-a_3)\,\frac{(a_2-a_4)(a_2-a_5)}{(a_2-x_2)(a_2-x_1')(a_2-x_2')}$$

as a function of a_2, and putting it into partial fractions in the ordinary way, we find that it is equal to

$$\frac{1}{a_2-x_2}(x_1'-x_2')^2(x_2-a_3)\frac{(x_2-a_4)(x_2-a_5)}{(x_2-x_1')(x_2-x_2')}+\frac{1}{a_2-x_1'}(x_1'-x_2')^2(x_2-a_3)\frac{(x_1'-a_4)(x_1'-a_5)}{(x_1'-x_2)(x_1'-x_2')}$$
$$+\frac{1}{a_2-x_2'}(x_1'-x_2')^2(x_2-a_3)\frac{(x_2'-a_4)(x_2'-a_5)}{(x_2'-x_2)(x_2'-x_1')};$$

using then the identities

$$-(x_2-a_3)(x_1'-x_2')=(x_2'-x_2)(x_1'-a_3)-(x_1'-x_2)(x_2'-a_3),$$
$$(x_2-a_3)(x_1'-x_2')=(x_1'-x_2)(x_2'-a_3)-(x_2'-x_2)(x_1'-a_3),$$

we are able to give the same expression the form

$$\tfrac{1}{4}\eta_2^2(x_2-a_1)\frac{(x_1'-x_2')^2}{(x_1'-x_2)(x_2'-x_2)}-\tfrac{1}{4}\eta_1'^2(x_1'-a_1)\frac{x_2'-x_2}{x_1'-x_2}-\tfrac{1}{4}\eta_2'^2(x_2'-a_1)\frac{x_1'-x_2}{x_1'-x_2}$$
$$+\frac{x_2'-a_3}{x_1'-a_2}(x_1'-a_4)(x_1'-a_5)+\frac{x_1'-a_3}{x_2'-a_2}(x_2'-a_4)(x_2'-a_5),$$

where $\tfrac{1}{4}\eta_1^2=(x_1-a_3)(x_1-a_4)(x_1-a_5)$, etc.; thus

$$\tfrac{1}{4}\eta_1'^2(x_1'-a_1)(x_2'-x_2)^2+\tfrac{1}{4}\eta_2'^2(x_2'-a_1)(x_1'-x_2)^2-\tfrac{1}{4}\eta_2^2(x_2-a_1)(x_1'-x_2')^2$$
$$=-(a_2-a_4)(a_2-a_5)(x_1'-x_2')^2(x_1'-x_2)(x_2'-x_2)(x_1-a_2)(x_2-a_3)\frac{1}{p_2^2p_2'^2}$$
$$+\frac{1}{p_2'^2}(x_1'-x_2)(x_2'-x_2)\{(x_2'-a_2)(x_2'-a_3)(x_1'-a_4)(x_1'-a_5)$$
$$+(x_1'-a_2)(x_1'-a_3)(x_2'-a_4)(x_2'-a_5)\}.$$

Now we have, by expansion,

$$\bar\Delta=(\eta_1\eta_2+\eta_1'\eta_2')(x_1-x_2)(x_1'-x_2')+(\eta_1\eta_1'+\eta_2\eta_2')(x_1'-x_1)(x_2'-x_2)$$
$$-(\eta_1\eta_2'+\eta_2\eta_1')(x_1'-x_2)(x_2'-x_1),$$

$$\Delta_1=\eta_1(x_1-a_1)(x_1'-x_2)(x_2'-x_2)(x_1'-x_2')-\eta_2(x_2-a_1)(x_1'-x_1)(x_2'-x_1)(x_1'-x_2')$$
$$+\eta_1'(x_1'-a_1)(x_2'-x_1)(x_2'-x_2)(x_1-x_2)-\eta_2'(x_2'-a_1)(x_1'-x_1)(x_1'-x_2)(x_1-x_2),$$

and in the product $\bar\Delta\Delta$ there will be two kinds of terms

(i) $$-\eta_1'\eta_2'(\eta_1-\eta_2)\gamma(x_1-x_2)(x_1'+x_2'-2a_1),$$

where γ denotes $(x_1'-x_1)(x_1'-x_2)(x_2'-x_1)(x_2'-x_2)$, there being four terms of this kind obtainable from this by the interchange of the suffixes 1 and 2, and the interchange of dashed and undashed letters,

(ii) $$\eta_1(x_2'-x_1)(x_1'-x_1)(x_1-x_2)\{\eta_1'^2(x_1'-a_1)(x_2'-x_2)^2+\eta_2'^2(x_2'-a_1)(x_1'-x_2)^2$$
$$-\eta_2^2(x_2-a_1)(x_1'-x_2')^2\},$$

there being three other terms similarly derivable from this one.

Consider now the expression

$$(a_2-a_4)(a_2-a_5)(p_{13}p_3p_1'+p'_{13}p_3'p_1)+p_{12}p_2p'_{23}p'_{45}+p'_{12}p_2'p_{23}p_{45},$$

and, of this, consider only the terms

$$(a_2-a_4)(a_2-a_5)p_{13}p_3p_1'+p_{12}p_2p'_{23}p'_{45};$$

by substitution of the values for p_{13} etc., and arrangement, we immediately find that these terms are equal to

$$-\tfrac{1}{8}p_2'^2p_2^2p_1'p_1\frac{\eta_1'\eta_2'(\eta_1-\eta_2)}{(x_1-x_2)(x_1'-x_2')^2}(x_1'+x_2'-2a_1)$$

$$-\tfrac{1}{2}(a_2-a_4)(a_2-a_5)p_1p_1'\frac{\eta_1(x_1-a_2)(x_2-a_3)-\eta_2(x_2-a_2)(x_1-a_3)}{x_1-x_2}$$

$$+\tfrac{1}{2}p_1p_1'p_2^2\frac{\eta_1-\eta_2}{(x_1-x_2)(x_1'-x_2')^2}\{(x_1'-a_2)(x_1'-a_3)(x_2'-a_4)(x_2'-a_5)$$

$$+(x_1'-a_4)(x_1'-a_5)(x_2'-a_2)(x_2'-a_3)\};$$

this expression, as we see by utilising an identity which was developed at the commencement of the investigation, is equal to

$$-\tfrac{1}{8}p_1p_1'p_2^2p_2'^2\frac{\eta_1'\eta_2'(\eta_1-\eta_2)}{(x_1-x_2)(x_1'-x_2')^2}(x_1'+x_2'-2a_1)+\tfrac{1}{8}\frac{p_2^2p_2'^2p_1p_1'}{(x_1-x_2)(x_1'-x_2')^2(x_1'-x_2)(x_2'-x_2)}K,$$

where K denotes

$$\eta_1[\eta_1'^2(x_1'-a_1)(x_2'-x_2)^2+\eta_2'^2(x_2'-a_1)(x_1'-x_2)-\eta_2^2(x_2-a_1)(x_1'-x_2')^2]$$

$$-\eta_2[\eta_2'^2(x_2'-a_1)(x_1'-x_1)^2+\eta_1'^2(x_1'-a_1)(x_2'-x_1)-\eta_1^2(x_1-a_1)(x_1'-x_2')^2].$$

Comparing this form with the terms occurring in the expansion for $\bar{\Delta}\Delta_1$, we obtain the result

$$\frac{1}{8}\frac{p_1p_1'p_2^2p_2'^2\bar{\Delta}\Delta_1}{(x_1-x_2)^2(x_1'-x_2')^2(x_1'-x_1)(x_1'-x_2)(x_2'-x_1)(x_2'-x_2)}$$

$$=(a_2-a_4)(a_2-a_5)(p_{13}p_3p_1'+p'_{13}p_3'p_1)+p_{12}p_2p'_{23}p'_{45}+p'_{12}p_2'p_{23}p_{45}.$$

Now we have (§§ 216, 213, 212) the formulae $p_i^2=\mu_iq_i^2$, $\frac{q^2_{i,j}}{q_i^2q_j^2}=\pm(a_i-a_j)\frac{p^2_{i,j}}{p_i^2p_j^2}$; we shall therefore put $p_i=M_iq_i$, $p_{i,j}=N_{i,j}q_{i,j}$; hence by the formula (p. 334) the quotient

$$\frac{\vartheta(u+u'|u^{a_1,a})\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')}$$

is a certain constant multiple of the function

$$(a_2-a_4)(a_2-a_5)M_1M_3N_{13}(q_{13}q_3q_1'+q'_{13}q_3'q_1)+N_{12}N_{23}N_{45}M_2(q_{12}q_2q'_{23}q'_{45}+q'_{12}q_2'q_{23}q_{45}).$$

Also we have $M_i^2=\mu_i$, $N^2_{i,j}=\pm\mu_i\mu_j/(a_i-a_j)$, where $\mu_i=\pm\sqrt{-f'(a_i)}$ when $i=1$ or 2, and $\mu_i=\pm\sqrt{f'(a_i)}$ when $i=3$, 4, 5. Hence it is easy to prove that the fourth powers of the quantities $(a_2-a_4)(a_2-a_5)M_1M_3N_{13}$, $N_{12}N_{23}N_{45}M_2$ are equal.

Hence we have

$$A\frac{\vartheta(u+u'|u^{a_1,a})\,\vartheta(u-u')}{\vartheta^2(u)\,\vartheta^2(u')}=\epsilon(q_{13}q_3q_1'+q'_{13}q_3'q_1)+q_{12}q_2q'_{23}q'_{45}+q'_{12}q_2'q_{23}q_{45},$$

where A is a certain constant, and ϵ a certain fourth root of unity. The value of ϵ is determined by a subsequent formula.

220. The equation just obtained (§ 219) taken with a previous formula gives the result

$$C\frac{\vartheta(u+u'|u^{a_1,a})}{\vartheta(u+u')}=\frac{\epsilon(q_{13}q_3q_1'+q'_{13}q_3'q_1)+q_{12}q_2q'_{23}q'_{45}+q'_{12}q_2'q_{23}q_{45}}{1+q_1^2q_1^{12}+q_2^2q_2^{12}+q_{12}^2q_{12}^{12}},$$

and limiting ourselves to one case, we may now take the places a_3, a_4, a_5 to be, respectively, c_1, c_2, c, and introduce Weierstrass's theta functions; *defining* the ten even functions* $\vartheta_5(u)$, $\vartheta_{23}(u), \dots, \vartheta_{03}(u)$ *to be respectively identical with the functions* $\vartheta(u)$, $\vartheta_{ac}(u), \dots, \vartheta_{ca_1}(u)$, *and the six odd functions* $\vartheta_{02}(u), \dots, \vartheta_3(u)$ *to be respectively the negatives of the functions* $\vartheta_{aa_1}(u), \dots, \vartheta_{cc_2}(u)$, the right-hand side of the equation is equivalent to

$$-\frac{\epsilon(\vartheta_5\vartheta_{02}\vartheta'_{01}\vartheta'_{12}+\vartheta_5'\vartheta'_{02}\vartheta_{01}\vartheta_{12})+\vartheta_{04}\vartheta_{24}\vartheta'_{14}\vartheta_3'+\vartheta'_{04}\vartheta'_{24}\vartheta_{14}\vartheta_3}{\vartheta_5^2\vartheta_5'^2+\vartheta_{02}^2\vartheta'^2_{02}+\vartheta_{24}^2\vartheta'^2_{24}+\vartheta_{04}^2\vartheta'^2_{04}};$$

here ϑ denotes $\vartheta(u)$, ϑ' denotes $\vartheta(u')$, and C is an absolute constant. This equation may be called the addition formula for the function q_1, and is one of a set which are the generalisation to the case $p=2$ of such formulae as that arising for $p=1$ in the form

$$\operatorname{sn}(u+u')=\frac{\operatorname{sn}u\operatorname{cn}u'\operatorname{dn}u'+\operatorname{sn}u'\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2u'}.$$

By interchanging the suffixes 1 and 2 we obtain an analogous expression for $\vartheta(u+u'|u^{a_2,a})\div\vartheta(u+u')$; if in this expression we add the half-period $u^{a_1,a}$ to u we obtain an expression for the function $\vartheta(u+u'|u^{a_1,a}+u^{a_2,a})\div\vartheta(u+u'|u^{a_1,a})$; and if this be multiplied by the expression just developed for the function $\vartheta(u+u'|u^{a_1,a})\div\vartheta(u+u')$ we obtain an expression for $\vartheta(u+u'|u^{a_1,a}+u^{a_2,a})\div\vartheta(u+u')$, and it can be shewn that the form obtained can be reduced to have the same denominator as in the expression here developed at length. The formulae are however particular cases of results obtained in subsequent chapters, and will not be further developed here. For that development such results as those contained in the following examples are necessary; these results are generalisations of such formulae as $\operatorname{sn}(u+K)=\operatorname{cn}u/\operatorname{dn}u$ which occur in the case $p=1$.

Ex. Prove, if $q_i(u)=\vartheta(u|u^{a_i,a})\div\vartheta(u)$, $q_{i,j}(u)=\vartheta(u|u^{a_i,a}+u^{a_j,a})\div\vartheta(u)$, etc., that (see the table § 204, and the formulae Chap. X. § 190)

$$q_1(u+u^{a_1,a})=-e^{\frac{1}{2}\pi i}/q_1(u),\quad q_2(u+u^{a_1,a})=-\frac{q_{1,2}(u)}{q_1(u)},$$

$$q_2(u+u^{a_1,a}+u^{a_2,a})=e^{\frac{1}{2}\pi i}\frac{q_1(u)}{q_{12}(u)},$$

and obtain the complete set of formulae.

221. In case $p=2$ there are five quotients of the form $\vartheta(u|u^{b,a})\div\vartheta(u)$, and ten of the form $\vartheta(u|u^{b_1,a}+u^{b_2,a})\div\vartheta(u)$, wherein b, b_1, b_2 denote any finite branch places. Since the arguments u may be written in the form $u^{x_1,a_1}+u^{x_2,a_2}$, the fifteen quotients are connected by thirteen algebraic relations. In virtue of the algebraic expression of these fifteen quotients, they may be studied independently of the theta functions. We therefore give below some examples of the equations connecting them.

* Königsberger, *Crelle*, LXIV. (1865), p. 22. In the letter notation (§ 204) the reduced characteristic symbols are such (§ 203) that each of k_s, k'_s is positive, or zero, and less than 2. In Weierstrass's notation the reduced symbols have the elements k'_s positive, or zero, and the elements k_s negative, or zero.

Ex. i. There is one relation, known as Göpel's biquadratic relation, which is of importance in itself, in view of developments that have arisen from it, and is of some historical interest.

Let $$q_c = \frac{\vartheta(u|u^{c,\,a})}{\vartheta(u)},\quad q_{a_1,\,a_2} = \frac{\vartheta(u|u^{a_1,\,a}+u^{a_2,\,a})}{\vartheta(u)},\quad q_{c_1,\,c_2} = \frac{\vartheta(u|u^{c_1,\,a}+u^{c_2,\,a})}{\vartheta(u)},$$

be three functions whose suffixes, together, involve all the five finite branch places. Then these three functions satisfy a biquadratic relation, which, if the functions be regarded as Cartesian coordinates in a space of three dimensions, represents a quartic surface with sixteen nodal points.

In fact, if p_a denote $\sqrt{(a-x_1)(a-x_2)}$, and $p_{b_1,\,b_2}$ denote the function

$$\tfrac{1}{2}p_{b_1}p_{b_2}\left[\frac{y_1}{(x_1-b_1)(x_1-b_2)} - \frac{y_2}{(x_2-b_1)(x_2-b_2)}\right]\frac{1}{x_2-x_1},$$

we have

$$p^2_{b_1,\,b_2} = \frac{4(x_2-b_1)(x_2-b_2)(x_1-e_1)(x_1-e_2)(x_1-e_3)+4(x_1-b_1)(x_1-b_2)(x_2-e_1)(x_2-e_2)(x_2-e_3)-2y_1y_2}{4(x_1-x_2)^2},$$

where b_1, b_2, e_1, e_2, e_3 are the finite branch places in any order; and if this be denoted by

$$\frac{\psi(x_1,\,x_2)-2y_1y_2}{4(x_1-x_2)^2},$$

it is immediately obvious that $\psi(x,\,x)=2y^2,\ =2f(x)$, say, and $\left[\frac{\partial}{\partial z}\psi(x,\,z)\right]_{z=x} = \frac{\partial f(x)}{\partial x}$; thus there is (§ 211, Ex. vii.) an equation of the form

$$p^2_{b_1,\,b_2} = \frac{f(x_1,\,x_2)-2y_1y_2}{4(x_1-x_2)^2} + Ax_1x_2 + B(x_1+x_2) + C,$$

where $f(x_1,\,x_2)$ is a certain symmetrical expression of frequent occurrence (cf. § 217), the same whatever branch places b_1, b_2 may be, and A, B, C are such that $\psi(x_1,\,x_2)$ vanishes when for x_1, x_2 are put any one of the four pairs of values $(b_1,\,b_2)$, $(e_2,\,e_3)$, $(e_3,\,e_1)$, $(e_1,\,e_2)$; therefore the difference between any two expressions such as $p^2_{b_1,\,b_2}$, formed for different pairs of finite branch places, is expressible in the form $Lx_1x_2+M(x_1+x_2)+N$; thus there must be an equation of the form

$$p^2_{a_1,\,c_1} = \lambda p^2_{a_1,\,a_2} + \mu p^2_{c_1,\,c_2} + \nu p^2_c + \rho,$$

where λ, μ, ν, ρ are independent of the places x_1, x_2.

Similarly

$$p^2_{a_2,\,c_2} = \lambda' p^2_{a_1,\,a_2} + \mu' p^2_{c_1,\,c_2} + \nu' p^2_c + \rho'.$$

But also it can be verified that

$$p_{a_1,\,a_2}p_{c_1,\,c_2} - p_{a_1,\,c_1}p_{a_2,\,c_2} = -(a_2-c_1)(a_1-c_2)p_c,\ =\kappa p_c,\ \text{say};$$

thus we have

$$[\lambda p^2_{a_1,\,a_2}+\mu p^2_{c_1,\,c_2}+\nu p^2_c+\rho][\lambda' p^2_{a_1,\,a_2}+\mu' p^2_{c_1,\,c_2}+\nu' p^2_c+\rho'] = [p_{a_1,\,a_2}p_{c_1,\,c_2}-\kappa p_c]^2,$$

and when the expressions $p_{a_1,\,a_2}$, etc., are replaced by the functions $q_{a_1,\,a_2}$, etc. (§ 210), this is the biquadratic relation in question. This proof is practically that given by Göpel (*Crelle*, xxxv. 1847, p. 291).

Ex. ii. Prove that

$$\frac{p^2_{a_1, a_2} - p^2_{a_1, c_1}}{a_2 - c_1} + p^2_{a_1} = (a_1 - a_2)(a_1 - c_1),$$

$$\frac{p^2_{c_1, c_2} - p^2_{a_1, c_1}}{c_2 - a_1} + p^2_{c_1} = (c_1 - c_2)(c_1 - a_1),$$

$$\frac{p^2_{a_1}}{(a_1 - c_1)(a_1 - c)} + \frac{p^2_{c_1}}{(c_1 - a_1)(c_1 - c)} + \frac{p^2_c}{(c - a_1)(c - c_1)} = 1,$$

and hence develop the method of *Ex.* i. in detail.

Ex. iii. For any value of p prove

(α) that the squares of any p of the theta quotients q_b, $= \vartheta(u \,|\, u^{b,\,a}) \div \vartheta(u)$, are connected by a linear relation,

(β) that the squares of any p of the theta quotients

$$q_b,\ q_{b, b_1},\ q_{b, b_2},\ q_{b, b_3},\ \ldots\ldots$$

are connected by a linear relation. (Weierstrass, *Math. Werke*, vol. I. p. 332.) These equations generalise the relations of *Ex.* ii.

Ex. iv. Another method of obtaining the biquadratic relations is as follows; if

$$\vartheta_q(v) \ = \Sigma e^{2\pi i v(n+q') + i\pi\tau(n+q')^2 + 2\pi i q(n+q')}$$

$$\Theta_q(V) = \Sigma e^{2\pi i V(n+q') + \frac{1}{2}i\pi\tau(n+q')^2 + 2\pi i q(n+q')},$$

$V = \frac{1}{2}v$, and, in Weierstrass's notation,

$$x = \vartheta_5(v),\ \ y = \vartheta_{01}(v),\ \ z = \vartheta_4(v),\ \ t = \vartheta_{23}(v),$$

so that $x : y : z : t = 1 : q_{a_1, c_1} : q_{a_2, c_2} : q_c$, and if a, b, c, d denote the values of x, y, z, t when $v = 0$, and the linear function $cx + dy - az - bt$ be denoted by $(c, d, -a, -b)$, etc., then it can be proved, by actual multiplication of the series, that

$$\Theta_3^2(V) = (c, d, -a, -b), \quad \Theta_{14}^2(V) = (d, -c, -b, a), \quad \Theta_{02}^2(V) = (b, -a, d, -c)$$

$$\Theta_5^2(V) = (a, b, c, d) \quad , \quad \Theta_1^2\ (V) = (b, -a, -d, c), \quad \Theta_{34}^2(V) = (a, b, -c, -d).$$

Relations of this character are actually obtained by Göpel, in this way. It will be sufficient, for the purpose of introducing the subject of a subsequent chapter, if the method of obtaining one of these relations be explained here. The general term of the series $\Theta_{02}(V)$ is (cf. the table § 204 and § 220)

$$-e^{\pi i v(n+q') + \frac{1}{2}i\pi\tau(n+q')^2 + 2\pi i q(n+q')},$$

where $q' = \frac{1}{2}(1, 0)$, $q = \frac{1}{2}(1, 0)$, namely is

$$-e^{\pi i[v_1(n_1+\frac{1}{2}) + v_2 n_2] + \frac{1}{2}\pi i[\tau_{11}(n_1+\frac{1}{2})^2 + 2\tau_{12}(n_1+\frac{1}{2})(n_2) + \tau_{22}n_2^2] + i\pi(n_1+\frac{1}{2})};$$

thus the exponent of the general term in the product $\Theta_{02}^2(V)$ is $\pi i L$, where L is equal to

$$v_1(n_1 + m_1 + 1) + v_2(n_2 + m_2) + \tfrac{1}{2}\tau_{11}[(n_1 + \tfrac{1}{2})^2 + (m_1 + \tfrac{1}{2})^2] + \tau_{12}[(n_1 + \tfrac{1}{2})n_2 + (m_1 + \tfrac{1}{2})m_2]$$
$$\tfrac{1}{2}\tau_{22}(n_2^2 + m_2^2) + n_1 + m_1 + 1;$$

there are therefore four kinds of terms in the product according to the evenness or oddness of the two integers n_1+m_1, n_2+m_2. Consider only one kind, namely when n_1+m_1, n_2+m_2 are both even, respectively equal to $2N_1$, $2N_2$, say; then L is equal to

$$2v_1(N_1+\tfrac{1}{2})+2v_2N_2+\tau_{11}(N_1+\tfrac{1}{2})^2+2\tau_{12}(N_1+\tfrac{1}{2})N_2+\tau_{22}N_2^2$$
$$+\tau_{11}\left(\frac{n_1-m_1}{2}\right)^2+2\tau_{12}\left(\frac{n_1-m_1}{2}\right)\left(\frac{n_2-m_2}{2}\right)+\tau_{22}\left(\frac{n_2-m_2}{2}\right)^2$$
$$+2N_1+1;$$

if now we put $\frac{n_1-m_1}{2}=M_1$, $\frac{n_2-m_2}{2}=M_2$, we have

$$n_1=N_1+M_1,\quad m_1=N_1-M_1,\quad n_2=N_2+M_2,\quad m_2=N_2-M_2;$$

thus, to any assigned values of the integers N_1, N_2, M_1, M_2 there correspond integers n_1, n_2, m_1, m_2 such that n_1+m_1, n_2+m_2 are both even; therefore, as

$$e^{2\pi i v_1(N_1+\frac{1}{2})+2\pi i v_2N_2+i\pi\tau_{11}(N_1+\frac{1}{2})^2+2i\pi\tau_{12}(N_1+\frac{1}{2})N_2+i\pi\tau_{22}N_2^2}$$

is a term of the series $\vartheta\left(v;\ \tfrac{1}{2}\begin{pmatrix}10\\00\end{pmatrix}\right)$, that is, of $\vartheta_{01}(v)$, and

$$e^{i\pi\tau_{11}M_1^2+2\pi i\tau_{12}M_1M_2+\tau_{22}M_2^2}$$

is a term of the series $\vartheta\left(0;\ \tfrac{1}{2}\begin{pmatrix}00\\00\end{pmatrix}\right)$, that is, of $\vartheta_5(v)$, and $e^{i\pi(2N_1+1)}=-1$, it follows that the terms of $\Theta_{02}^2(V)$ which are of the kind under consideration consist of all the terms of the product $-\vartheta_5.\vartheta_{01}(v)$, or $-ay$. It can similarly be seen that the three other sorts of terms, when n_1+m_1 is even and n_2+m_2 odd, when n_1+m_1 is odd and n_2+m_2 odd or even, are, in their aggregate the terms of the sum $bx+dz-ct$.

We can also, in a similar way, prove the equations

$$\Theta_{03}\Theta_{23}\Theta_3(V)\Theta_{14}(V)+\Theta_0\Theta_2\Theta_{02}(V)\Theta_5(V)=\Theta_{12}\Theta_{01}\Theta_1(V)\Theta_{34}(V),$$
$$\Theta_{03}^2=2(ac-bd),\ \Theta_{23}^2=2(ad+bc),\ \Theta_2^2=2(ab-cd),\ \Theta_{01}^2=2(ab+cd),$$
$$\Theta_0^2=a^2-b^2-c^2+d^2,\ \Theta_{12}^2=a^2-b^2+c^2-d^2,$$

Θ_{03} denoting $\Theta_{03}(0)$, etc.

Hence the equation of the quartic surface is obtainable in the form

$$\sqrt{2(ac-bd)(ad+bc)(c,\ d,\ -a,\ -b)(d,\ -c,\ -b,\ a)}$$
$$+\sqrt{(a^2-b^2-c^2+d^2)(ab-cd)(b,\ -a,\ d,\ -c)(a,\ b,\ c,\ d)}$$
$$=\sqrt{(a^2-b^2+c^2-d^2)(ab+cd)(b,\ -a,\ -d,\ c)(a,\ b,\ -c,\ -d)}.$$

A relation of this form is rationalised by Cayley in *Crelle's Journal*, LXXXIII. (1877), p. 215. The form obtained is shewn by Borchardt, *Crelle*, LXXXIII. (1877), p. 239, to be the same as that obtained by Göpel. See also Kummer, *Berlin. Monats.* 1864, p. 246, and *Berlin. Abhand.* 1866, p. 64; Cayley, *Crelle*, LXXXIV., XCIV.; and Humbert, *Liouville*, 4[me] Sér., t. IX. (1893); Schottky, *Crelle*, CV. pp. 233, 269; Wirtinger, *Untersuchungen über Thetafunctionen* (Leipzig, 1895).

The rationalised form of the equation, from which the presence of the sixteen nodes is obvious, is obtained in chapter XV. of the present volume.

Ex. v. Obtain the following relations, connecting the ratios of the values of the even theta functions for zero values of the arguments when $p=2$. They may be obtained from the relations (§ 212)

$$(b-x_1)(b-x_2)=\pm\sqrt{e^{\pi iPP'}f'(b)}\,\vartheta^2(u|u^{b,\,a})\div\vartheta^2(u)$$

by substituting special values for x_1 and x_2.

$$\vartheta^4 : \vartheta^4_c : \vartheta^4_{c_1} : \vartheta^4_{c_2} : \vartheta^4_{a_1c_1} : \vartheta^4_{a_1c_2} : \vartheta^4_{a_2c_1} : \vartheta^4_{a_2c_2} : \vartheta^4_{ca_2} : \vartheta^4_{ca_1}$$
$$=(c_1-c_2)(c_2-c)(c-c_1)\,.\,(a_1-a_2) : (a_1-a_2)(a_2-c)(c-a_1)\,.\,(c_1-c_2)$$
$$: (a_1-a_2)(a_2-c_1)(c_1-a_1)\,.\,(c_2-c) : (a_1-a_2)(a_2-c_2)(c_2-a_1)\,.\,(c_1-c)$$
$$: (c_2-c)(c-a_1)(a_1-c_2)\,.\,(c_1-a_2) : (c-c_1)(c_1-a_1)(a_1-c)\,.\,(c_2-a_2)$$
$$: (c-c_2)(c_2-a_2)(a_2-c)\,.\,(c_1-a_1) : (c-c_1)(c_1-a_2)(a_2-c)\,.\,(c_2-a_1)$$
$$: (c_1-c_2)(c_2-a_2)(a_2-c_1)\,.\,(a_1-c) : (c_1-c_2)(c_2-a_1)(a_1-c_1)\,.\,(a_2-c).$$

Infer that

$$\vartheta^4_{ca_2}\vartheta^4_{c_2a_2} : \vartheta^4_{a_1c_2}\vartheta^4_{ca_1} : \vartheta^4_c\vartheta^4_{c_2}=(a_2-c_1)^2 : (c_1-a_1)^2 : (a_1-a_2)^2.$$

We have proved (§§ 210, 213) that

$$\sqrt{a_2-c_1}\,\vartheta_{a_1}(u)\,\vartheta_{a_2c_1}(u)+\sqrt{c_1-a_1}\,\vartheta_{a_2}(u)\,\vartheta_{a_1c_1}(u)+\sqrt{a_1-a_2}\,\vartheta_{c_1}(u)\,\vartheta_{a_1a_2}(u)=0$$

and we have in fact, as follows from formulae developed subsequently, the equation

$$\vartheta_{ca_2}\vartheta_{c_2a_2}\vartheta_{a_1}(u)\,\vartheta_{a_2c_1}(u)+\vartheta_{a_1c_2}\vartheta_{ca_1}\vartheta_{a_2}(u)\,\vartheta_{a_1c_1}(u)=\vartheta_c\vartheta_{c_2}\vartheta_{c_1}(u)\,\vartheta_{a_1a_2}(u).$$

Ex. vi. Obtain formulae to express the ratios of the differential coefficients of the odd theta functions for zero values of the arguments.

Ex. vii. Prove that

$$\vartheta(u)\frac{\partial}{\partial u_2}\vartheta(u|u^{b_1,\,a}+u^{b_2,\,a})-\vartheta(u|u^{b_1,\,a}+u^{b_2,\,a})\frac{\partial}{\partial u_2}\vartheta(u)=\epsilon\sqrt{b_1-b_2}\,\vartheta(u|u^{b_1,\,a})\,\vartheta(u|u^{b_2,\,a}),$$

wherein b_1, b_2 are any two finite branch places, and ϵ is a certain fourth root of unity.

This result can be obtained in various ways; one way is as follows: Writing $u=u^{x_1,\,a_1}+u^{x_2,\,a_2}$, $u+u^{b_1,\,a}=v$, and $v=u^{z_1,\,b_1}+u^{z_2,\,b_2}$, we find, by the formula $\vartheta(u+\Omega_P)=e^{\lambda_P(u)}\vartheta(u;\,P)$, that

$$\frac{\partial}{\partial u_2}\log\frac{\vartheta(u|u^{b_1,\,a}+u^{b_2,\,a})}{\vartheta(u)}=L_2^{b_1,\,b_2}+\zeta_2(v-u^{b_2,\,a})-\zeta_2(v-u^{b_1,\,a}),$$

and, by the formula expressing $\zeta_i(u^{x,\,m}-u^{x_1,\,m_1}-\ldots\ldots-u^{x_p,\,m_p})-\zeta_i(u^{\mu,\,m}-u^{x_1,\,m_1}-\ldots\ldots-u^{x_p,\,m_p})$ by integrals and rational functions, the right-hand side is equal to

$$-\tfrac{1}{2}\frac{b_1-b_2}{z_1-z_2}\left[\frac{s_1}{(z_1-b_1)(z_1-b_2)}-\frac{s_2}{(z_2-b_1)(z_2-b_2)}\right],$$

where s_1, z_1 are the values of y, x respectively at the place z_1, and s_2, z_2 at the place z_2. This rational function of z_1, z_2 is however (§ 210) a certain constant multiple of $\vartheta(v|u^{b_1,\,a}+u^{b_2,\,a})/\vartheta(v)$, and hence the result can immediately be deduced.

One case of the relation, when b_1, b_2 are the places a_1, a_2, is expressible by Weierstrass's notation in the form

$$\vartheta_5(u)\frac{\partial}{\partial u_2}\vartheta_{04}(u)-\vartheta_{04}(u)\frac{\partial}{\partial u_2}\vartheta_5(u)=\epsilon\sqrt{a_1-a_2}\,\vartheta_{02}(u)\,\vartheta_{24}(u),$$

and it is interesting, using results which belong to the later part of this volume, to compare this with other methods of proof. We have*

$$\vartheta_4\vartheta_0\bar{\vartheta}_{04}(u+v)\,\vartheta_5(u-v)=\vartheta_5(u)\,\bar{\vartheta}_{04}(u)\,\vartheta_4(v)\,\vartheta_0(v)+\vartheta_2(u)\,\bar{\vartheta}_{13}(u)\,\bar{\vartheta}_{02}(v)\,\bar{\vartheta}_{24}(v)$$
$$+\vartheta_5(v)\,\bar{\vartheta}_{04}(v)\,\vartheta_4(u)\,\vartheta_0(u)+\vartheta_2(v)\,\bar{\vartheta}_{13}(v)\,\bar{\vartheta}_{02}(u)\,\bar{\vartheta}_{24}(u),$$

where ϑ_4, ϑ_0 denote $\vartheta_4(0)$, $\vartheta_0(0)$, and the bar denotes an odd function; if, herein, the arguments v_1, v_2 be taken very small, we may write $\vartheta(u+v)=\vartheta(u)+\left(v_1\frac{\partial}{\partial u_1}+v_2\frac{\partial}{\partial u_2}\right)\vartheta(u)$. Thus we obtain, eventually, remembering that the odd functions, and the first differential coefficients of the even functions, vanish for zero values of the arguments,

$$\vartheta_5(u)\,\vartheta'_{04}(u)-\vartheta_{04}(u)\,\vartheta'_5(u)=\frac{\vartheta_5\vartheta'_{04}}{\vartheta_4\vartheta_0}\vartheta_4(u)\,\vartheta_0(u)+\frac{\vartheta_2\vartheta'_{13}}{\vartheta_4\vartheta_0}\vartheta_{02}(u)\,\vartheta_{24}(u),$$

where $\vartheta'(u)=\frac{\partial}{\partial u_2}\vartheta(u)$, $\vartheta=\vartheta(0)$, $\vartheta'=\vartheta'(0)$.

Thus, by the formula of this example, putting $u=0$, we infer that

$$\left[\frac{\partial}{\partial u_2}\vartheta\left(u\,|\,u^{a_1,\,a}+u^{a_2,\,a}\right)\right]_{u=0}=0$$

or $\vartheta'_{04}=0$, and the result of the general formula agrees with the formula of this example.

In the cases $p>2$ we have even theta functions vanishing for zero values of the argument; here we have one of the differential coefficients of an odd function vanishing for zero values of the argument.

Note. Beside the references given in this chapter there is a paper by Bolza, *American Journal*, XVII. 11 (1895), "On the first and second derivatives of hyperelliptic σ-functions" (see *Acta Math.* XX. (Feb. 1896), p. 1: "Zur Lehre von den hyperelliptischen Integralen, von Paul Epstein"), which was overlooked till the chapter was completed. The fundamental formula of Klein, utilised by Bolza, is developed, in what appeared to be its proper place, in chapter XIV. of the present volume. See also Wiltheiss, *Crelle*, XCIX. p. 247, *Math. Annal.* XXXI. p. 417; Brioschi, *Rend. d. Acc. dei Lincei*, (Rome), 1886, p. 199; and further, Königsberger, *Crelle*, LXV. (1866), p. 342; Frobenius, *Crelle*, LXXXIX. (1880), p. 206.

To the note on p. 301 should be added the references; Prym, *Zur Theorie der Functnen. in einer zweiblätt. Fläche* (Zürich, 1866), p. 12; Königsberger, *Crelle*, LXIV. p. 20. To the note on p. 296 should be added; Harkness and Morley, *Theory of Functions*, chapter VIII., on double theta functions. In connection with § 205, notations for theta functions of three variables are given by Cayley and Borchardt, *Crelle*, LXXXVII. (1878).

* Krause, *Hyperelliptische Functionen*, p. 44; Königsberger, *Crelle*, LXIV. p. 28.