CHAPTER X.

RIEMANN'S THETA FUNCTIONS. GENERAL THEORY.

173. THE theta functions, which are, certainly, the most important elements of the theory of this volume, were first introduced by Jacobi in the case of elliptic functions.* They enabled him to express his functions sn u, cn u, dn u, in the form of fractions having the same denominator, the zeros of this denominator being the common poles of the functions sn u, cn u, dn u. The ratios of the theta functions, expressed as infinite products, were also used by Abel †. For the case $p = 2$, similar functions were found by Göpel‡, who was led to his series by generalizing the form in which Hermite had written the general exponent of Jacobi's series, and by Rosenhain §, who first forms degenerate theta functions of two variables by multiplying together two theta functions of one variable, led thereto by the remark that two integrals of the first kind which exist for $p = 2$, become elliptic integrals respectively of the first and third kind, when two branch places of the surface for $p = 2$, coincide. Both Göpel and Rosenhain have in view the inversion problem enunciated by Jacobi; their memoirs contain a large number of the ideas that have since been applied to more general cases. In the form in which the theta functions are considered in this chapter they were first given, for any value of p, by Riemann ||. Functions which are quotients of theta functions had been previously considered by Weierstrass, without any mention of the theta series, for any hyperelliptic case ¶. These functions occur in the memoir of Rosenhain, for the case $p = 2$. It will be seen that

* *Fundamenta Nova* (1829); *Ges. Werke* (Berlin, 1881), Bd. I. See in particular, Dirichlet, Gedächtnissrede auf Jacobi, *loc. cit.* Bd. I., p. 14, and Zur Geschichte der Abelschen Transcendenten, *loc. cit.*, Bd. II., p. 516.

† *Œuvres* (Christiania, 1881), t. I. p. 343 (1827). See also Eisenstein, *Crelle*, XXXV. (1847), p. 153, etc. The equation (*b*) p. 225, of Eisenstein's memoir, is effectively the equation

$$\wp'^2(u) = 4\wp^3(u) - g_2\wp(u) - g_3.$$

‡ *Crelle*, XXXV. (1847), p. 277.

§ *Mém. sav. étrang.* XI. (1851), p. 361. The paper is dated 1846.

|| *Crelle*, LIV. (1857); *Ges. Werke*, p. 81.

¶ *Crelle*, XLVII. (1854); *Crelle*, LII. (1856); *Ges. Werke*, pp. 133, 297.

the Riemann theta functions are not the most general form possible. The subsequent development of the general theory is due largely to Weierstrass.

174. In the case $p=1$, the convergence of the series obtained by Jacobi depends upon the use of two periods 2ω, $2\omega'$, for the integral of the first kind, such that the ratio ω'/ω has its imaginary part positive. Then the quantity $q=e^{\pi i\frac{\omega'}{\omega}}$ is, in absolute value, less than unity.

Now it is proved by Riemann that if we choose normal integrals of the first kind $v_1^{x,a}, \dots, v_p^{x,a}$, so that $v_r^{x,a}$ has the periods $0\dots 0, 1, 0, \dots, \tau_{r,1}, \dots, \tau_{r,p}$, the imaginary part of the quadratic form

$$\phi = \tau_{11}n_1^2 + \dots\dots + \tau_{r,r}\,n_r^2 + \dots\dots + 2\tau_{1,2}\,n_1n_2 + \dots\dots + 2\tau_{r,s}\,n_rn_s + \dots\dots$$

is positive* for all real values of the p variables $n_1, \dots, n_p$. Hence for all rational integer values of $n_1, \dots, n_p$, positive or negative, the quantity $e^{i\pi\phi}$ has its modulus less than unity. Thus, if we write $\tau_{r,s}=\rho_{r,s}+i\kappa_{r,s}$, $\rho_{r,s}$ and $\kappa_{r,s}$ being real, and $a_1, =b_1+ic_1, \dots, a_p, =b_p+ic_p$, be any p constant quantities, the modulus of the general term of the p-fold series

$$\sum_{n_1=-\infty}^{n_1=\infty}\ \sum_{n_2=-\infty}^{n_2=\infty} \dots\dots \sum_{n_p=-\infty}^{n_p=\infty} e^{a_1n_1+\dots\dots+a_pn_p+i\pi\phi},$$

wherein each of the indices $n_1, \dots, n_p$ takes every real integer value independently of the other indices, is e^{-L}, where

$$L = -(b_1n_1+\dots\dots+b_pn_p)+\pi(\kappa_{11}n_1^2+\dots\dots+2\kappa_{1,2}\,n_1n_2+\dots\dots),$$
$$= -(b_1n_1+\dots\dots+b_pn_p)+\psi, \text{ say,}$$

where ψ is a real quadratic form in $n_1, \dots, n_p$, which is essentially positive for all the values of $n_1, \dots, n_p$ considered. When one (or more) of $n_1, \dots, n_p$ is large, L will have the same sign as ψ, and will be positive; and if μ be any positive integer $e^{L/\mu}$ is greater than $1+L/\mu$, and therefore $e^{-L}<\left(1+\frac{L}{\mu}\right)^{-\mu}$; now the series whose general term is $\left(1+\frac{L}{\mu}\right)^{-\mu}$ will be convergent or not according as the series whose general term is $\psi^{-\mu}$ is convergent or not, for the ratio $1+\frac{L}{\mu} : \psi$ has the finite limit $1/\mu$ for large values of $n_1, \dots, n_p$; and the series whose general term is $\psi^{-\mu}$ is convergent provided μ be taken

* The proof is given in Forsyth, *Theory of Functions*, § 235. If $w_1^{x,a}, \dots, w_p^{x,a}$ denote a set of integrals of the first kind such that $w_r^{x,a}$ has no periods at the b period loops except at b_r, and has there the period 1, and $\sigma_{r,1}, \dots, \sigma_{r,p}$ be the periods of $w_r^{x,a}$ at the a period loops, the quadratic function

$$\sigma_{11}n_1^2+\dots\dots+2\sigma_{12}n_1n_2+\dots\dots$$

has its imaginary part negative.

$>\frac{1}{2}p$. (Jordan, *Cours d'Analyse*, Paris, 1893, vol. I., § 318.) Hence the series whose general term is

$$e^{a_1 n_1+\dots\dots+a_p n_p+i\pi\phi},$$

is absolutely convergent.

In what follows we shall write $2\pi i u_r$ in place of a_r and speak of $u_1, \dots, u_p$ as the arguments; we shall denote by un the quantity $u_1 n_1 + \dots\dots + u_p n_p$, and by τn^2 the quadratic $\tau_{11} n_1^2 + \dots\dots + 2\tau_{12} n_1 n_2 + \dots\dots$. Then the Riemann theta function is defined by the equation

$$\Theta(u) = \Sigma e^{2\pi i u n + i\pi\tau n^2},$$

where the sign of summation indicates that each of the indices $n_1, \dots, n_p$ is to take all positive and negative integral values (including zero), independently of the others. By what has been proved it follows that $\Theta(u)$ is a single-valued, integral, analytical function of the arguments $u_1, \dots, u_p$.

The notation is borrowed from the theory of matrices (cf. Appendix ii.); τ is regarded as representing the symmetrical matrix whose (r, s)th element is $\tau_{r,s}$, n as representing a row, or column, letter, whose elements are $n_1, \dots, n_p$, and u, similarly, as representing such a letter with $u_1, \dots, u_p$ as its elements.

It is convenient, with $\Theta(u)$, to consider a slightly generalized function, given by

$$\Theta(u;\, q, q'), \text{ or } \Theta(u, q) = \Sigma e^{2\pi i u(n+q') + i\pi\tau(n+q')^2 + 2\pi i q(n+q')};$$

herein q denotes the set of p quantities $q_1, \dots, q_p$, and q' denotes the set of p quantities $q_1', \dots, q_p'$, and, for instance, $u(n+q')$ denotes the quantity $un + uq'$, namely

$$u_1 n_1 + \dots\dots + u_p n_p + u_1 q_1' + \dots\dots + u_p q_p',$$

and $\tau(n+q')^2$ denotes $\tau n^2 + 2\tau n q' + \tau q'^2$, namely

$$(\tau_{11} n_1^2 + \dots + 2\tau_{1,2} n_1 n_2 + \dots) + 2\sum_{s=1}^{p}\sum_{r=1}^{p} \tau_{r,s} n_r q_s' + (\tau_{11} q_1'^2 + \dots + 2\tau_{1,2} q_1' q_2' + \dots).$$

The quantities $q_1, \dots, q_p, q_1', \dots, q_p'$ constitute, in their aggregate, the *characteristic* of the function $\Theta(u;\, q)$; they may have any constant values whatever; in the most common case they are each either 0 or $\frac{1}{2}$.

The quantities $\tau_{i,j}$ are the periods of the Riemann normal integrals of the first kind at the second set of period loops. It is clear however that any symmetrical matrix, σ, which is such that for real values of $k_1, \dots k_p$ the quadratic form σk^2 has its imaginary part positive, may be equally used instead of τ, to form a convergent series of the same form as the Θ series. And it is worth while to make this remark in order to point out that the Riemann theta functions are not of as general a character as possible. For such a symmetrical matrix σ contains $\frac{1}{2}p(p+1)$ different quantities, while the periods $\tau_{r,s}$ are (Chap. I., § 7), functions of only $3p-3$ independent quantities. The difference $\frac{1}{2}p(p+1) -(3p-3) = \frac{1}{2}(p-2)(p-3)$, vanishes for $p=2$ or $p=3$; for $p=4$ it is equal to 1, and for greater values of p is still greater. We shall afterwards be concerned with the more general theta-function here suggested.

The function $\Theta(u)$ is obviously a generalization of the theta functions used in the theory of elliptic functions. One of these, for instance, is given by

$$\vartheta_1(u;\ \tfrac{1}{2},\tfrac{1}{2}) = \frac{\vartheta_1'(0)}{2\omega}e^{-2\eta\omega u^2}\sigma(2\omega u) = -\Sigma e^{2\pi i u(n+\frac{1}{2})+\pi i\tau(n+\frac{1}{2})^2+\pi i(n+\frac{1}{2})};$$

and the four elliptic theta functions are in fact obtained by putting respectively $q, q'=0, \frac{1}{2}$; $=\frac{1}{2}, \frac{1}{2}$; $=\frac{1}{2}, 0$; $=0, 0$.

175. There are some general properties of the theta functions, immediately deducible from the definition given above, which it is desirable to put down at once for purposes of reference. Unless the contrary is stated it is always assumed in this chapter that the characteristic consists of half integers; we may denote it by $\frac{1}{2}\beta_1, \ldots, \frac{1}{2}\beta_p, \frac{1}{2}\alpha_1, \ldots, \frac{1}{2}\alpha_p$, or shortly, by $\frac{1}{2}\beta, \frac{1}{2}\alpha$, where $\beta_1, \ldots, \beta_p, \alpha_1, \ldots, \alpha_p$ are integers, in the most common case either 0 or 1. Further we use the abbreviation $\Omega_{m,m'}$, or sometimes only Ω_m, to denote the set of p quantities

$$m_i + \tau_{i,1}m_1' + \ldots\ldots + \tau_{i,p}m_p', \qquad (i=1, 2, \ldots, p),$$

wherein $m_1, \ldots, m_p, m_1', \ldots, m_p'$ are $2p$ constants. When these constants are integers, the p quantities denoted by Ω_m are the periods of the p Riemann normal integrals of the first kind when the upper limit of the integrals is taken round a closed curve which is reducible to m_i circuits of the period loop b_i (or m_i crossings of the period loop a_i) and to m_i' circuits of the period loop a_i, i being equal to 1, 2, ..., p. (Cf. the diagram Chap. II. p. 21.) The general element of the set of p quantities denoted by Ω_m, will also sometimes be denoted by $m_i + \tau_i m'$, τ_i denoting the row of quantities formed by the ith row of the matrix τ. When $m_1, \ldots, m_p'$ are integers, the quantity $m_i + \tau_i m'$ is the period to be associated with the argument u_i.

Then we have the following formulae, (A), (B), (C), (D), (E):

$$\Theta(-u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{\pi i\beta\alpha}\,\Theta(u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha), \qquad \text{(A)}.$$

Thus $\Theta(u;\ \frac{1}{2}\beta, \frac{1}{2}\alpha)$ is an odd or even function of the variables $u_1, \ldots, u_p$ according as $\beta\alpha$, $=\beta_1\alpha_1 + \ldots\ldots + \beta_p\alpha_p$, is an odd or even integer; in the former case we say that the characteristic $\frac{1}{2}\beta, \frac{1}{2}\alpha$ is an odd characteristic, in the latter case that it is an even characteristic.

The behaviour of the function $\Theta(u)$ when proper simultaneous periods are added to the arguments, is given by the formulae immediately following, wherein r is any one of the numbers 1, 2, ..., p,

$$\Theta(u_1, \ldots, u_r+1, \ldots, u_p;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{\pi i\alpha_r}\,\Theta(u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha),$$

$$\Theta(u_1+\tau_{1,r}, u_2+\tau_{2,r}, \ldots, u_p+\tau_{p,r};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{-2\pi i(u_r+\frac{1}{2}\tau_{r,r})-\pi i\beta_r}\,\Theta(u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha).$$

Both these are included in the equation

$$\Theta(u+\Omega_m;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{-2\pi i m'(u+\frac{1}{2}\tau m')+\pi i(m\alpha-m'\beta)}\,\Theta(u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha), \qquad \text{(B)};$$

herein the quantities $m_1, \ldots, m_p, m_1', \ldots, m_p'$ are integers, $u + \Omega_m$ stands for the p quantities such as $u_r + m_r + m_1'\tau_{r,1} + \ldots\ldots + m_p'\tau_{r,p}$, and the notation in the exponent on the right hand is that of the theory of matrices; thus for instance $m'\tau m'$ denotes the expression

$$\sum_{r=1}^{p} m_r' (\tau_{r,1} m_1' + \ldots\ldots + \tau_{r,p} m_p'),$$

and is the same as the expression denoted by $\tau m'^2$.

Equation (B) shews that the partial differential coefficients, of the second order, of the logarithm of $\Theta(u; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$, in regard to $u_1, \ldots, u_p$, are functions of $u_1, \ldots, u_p$, with $2p$ sets of simultaneous periods.

Equation (B) is included in another equation; if each of β', α' denotes a row of p integers, we have

$$\Theta(u + \tfrac{1}{2}\Omega_{\beta', \alpha'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{-\pi i \alpha'(u + \frac{1}{2}\beta + \frac{1}{2}\beta' + \frac{1}{4}\tau\alpha')} \Theta(u; \tfrac{1}{2}\beta + \tfrac{1}{2}\beta', \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha'), \quad (C);$$

to obtain equation (B) we have only to put $\beta_r' = 2m_r$, $\alpha_r' = 2m_r'$ in equation (C). If, in the same equation, we put $\beta' = -\beta$, $\alpha' = -\alpha$, we obtain

$$\Theta(u - \tfrac{1}{2}\Omega_{\beta, \alpha}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{\pi i \alpha(u - \frac{1}{4}\tau\alpha)} \Theta(u; 0, 0) = e^{\pi i \alpha(u - \frac{1}{4}\tau\alpha)} \Theta(u);$$

from this we infer

$$\Theta(u; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha) = e^{\pi i \alpha(u + \frac{1}{2}\beta + \frac{1}{4}\tau\alpha)} \Theta(u + \tfrac{1}{2}\Omega_{\beta, \alpha}), \quad (D);$$

this is an important equation because it reduces a theta function with any half-integer characteristic to the theta function of zero characteristic.

Finally, when each of m, m' denotes a set of p integers, we have the equation

$$\Theta(u; \tfrac{1}{2}\beta + m, \tfrac{1}{2}\alpha + m') = e^{\pi i m \alpha} \Theta(u; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha), \quad (E);$$

thus the addition of integers to the quantities $\tfrac{1}{2}\alpha$ does not alter the theta function $\Theta(u; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$, and the addition of integers to the quantities $\tfrac{1}{2}\beta$ can at most change the sign of the function. Hence all the theta functions with half-integer characteristics are reducible to the 2^{2p} theta functions which arise when every element of the characteristic is either 0 or $\tfrac{1}{2}$.

176. We shall verify these equations in order in the most direct way. The method consists in transforming the exponent of the general term of the series, and arranging the terms in a new order. This process is legitimate, because, as we have proved, the series is absolutely convergent.

(A) If in the general term

$$e^{2\pi i u(n + \frac{1}{2}a) + i\pi\tau(n + \frac{1}{2}a)^2 + \pi i \beta(n + \frac{1}{2}a)},$$

we change the signs of $u_1, \ldots, u_p$, the exponent becomes

$$2\pi i u(-n - a + \tfrac{1}{2}a) + i\pi\tau(-n - a + \tfrac{1}{2}a) + \pi i \beta(-n - a + \tfrac{1}{2}a) + 2\pi i \beta \tau i + \pi i \beta a.$$

Since a consists of integers we may write m for $-n-a$, that is $m_r=-(n_r+a_r)$, for $r=1, 2, \ldots, p$; then, since β consists of integers, and therefore $e^{2\pi i\beta n}=1$, the general term becomes

$$e^{\pi i\beta a} \cdot e^{2\pi iu(m+\frac{1}{2}a)+i\pi\tau(m+\frac{1}{2}a)+\pi i\beta(m+\frac{1}{2}a)};$$

save for the factor $e^{\pi i\beta a}$, this is of the same form as the general term in the original series, the summation integers $m_1, \ldots, m_p$ replacing $n_1, \ldots, n_p$. Thus the result is obvious.

(B) The exponent

$$2\pi i(u+m+\tau m')(n+\tfrac{1}{2}a)+i\pi\tau(n+\tfrac{1}{2}a)^2+\pi i\beta(n+\tfrac{1}{2}a),$$

wherein $m+\tau m'$ stands for a row, or column, of p quantities of which the general one is

$$m_r+\tau_{r,1}m_1'+\ldots\ldots+\tau_{r,p}m_p',$$

is equal to

$$\begin{aligned}&2\pi iu(n+\tfrac{1}{2}a)+i\pi\tau(n+\tfrac{1}{2}a)^2+\pi i\beta(n+\tfrac{1}{2}a)+2\pi imn+\pi ima+2\pi i\tau m'n+\pi i\tau m'a\\ &=2\pi iu(n+m'+\tfrac{1}{2}a)+i\pi\tau(n+m'+\tfrac{1}{2}a)^2+\pi i\beta(n+m'+\tfrac{1}{2}a)-2\pi im'(u+\tfrac{1}{2}\tau m')\\ &\qquad+\pi i(ma-m'\beta)+2\pi imn.\end{aligned}$$

Replacing $e^{2\pi imn}$ by 1 and writing n for $n+m'$, the equation (B) is obtained.

(C) By the work in (B), replacing m, m' by $\frac{1}{2}\beta', \frac{1}{2}a'$ respectively, we obtain

$$\begin{aligned}&2\pi i(u+\tfrac{1}{2}\beta'+\tfrac{1}{2}\tau a')(n+\tfrac{1}{2}a)+i\pi\tau(n+\tfrac{1}{2}a)^2+\pi i\beta(n+\tfrac{1}{2}a)\\ &=2\pi iu(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)+i\pi\tau(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)+\pi i\beta(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)-\pi ia'(u+\tfrac{1}{4}\tau a')\\ &\qquad+\tfrac{1}{2}\pi i(\beta'a-a'\beta)+\pi i\beta'n,\end{aligned}$$

and this is immediately seen to be the same as

$$2\pi iu(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)+i\pi\tau(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)+\pi i(\beta+\beta')(n+\tfrac{1}{2}a'+\tfrac{1}{2}a)-\pi ia'(u+\tfrac{1}{2}\beta+\tfrac{1}{2}\beta'+\tfrac{1}{4}\tau a').$$

This proves the formula (C).

It is obvious that equations (D) are only particular cases of equation (C), and the equation (E) is immediately obvious.

It follows from the equation (A) that the number of odd theta functions contained in the formula $\Theta(u;\ \frac{1}{2}\beta,\ \frac{1}{2}a)$ is $2^{p-1}(2^p-1)$, and therefore that the number of even functions is $2^{2p}-2^{p-1}(2^p-1)$, or $2^{p-1}(2^p+1)$.

For the number of odd functions is the same as the number of sets of integers, $x_1, y_1, \ldots, x_p, y_p$, each either 0 or 1, for which

$$x_1y_1+\ldots\ldots+x_py_p=\text{an odd integer}.$$

These sets consist, (i), of the solutions of the equation

$$x_1y_1+\ldots\ldots+x_{p-1}y_{p-1}=\text{an odd integer},$$

in number, say, $f(p-1)$, each combined with each of the three sets

$$(x_p, y_p)=(0, 1),\ (1, 0),\ (0, 0),$$

together with, (ii), the solutions of the equation

$$x_1y_1+\ldots\ldots+x_{p-1}y_{p-1}=\text{an even integer},$$

in number $2^{2p-2}-f(p-1)$, each combined with the set

$$(x_p, y_p)=(1, 1).$$

Thus

$$\begin{aligned}f(p)&=3f(p-1)+2^{2p-2}-f(p-1)=2^{2p-2}+2f(p-1)\\ &=2^{2p-2}+2\{2^{2p-4}+2f(p-2)\}=\text{etc.}\\ &=2^{2p-2}+2^{2p-3}+2^{2p-4}+\ldots\ldots+2^p+2^{p-1}f(1)\\ &=2^{p-1}(2^p-1).\end{aligned}$$

Hence the number of even half periods is $2^{p-1}(2^p+1)$.

177. Suppose now that $e_1, \ldots, e_p$ are definite constants, that m denotes a fixed place of the Riemann surface, and x denotes a variable place of the surface. We consider p arguments given by $u_r = v_r^{x, m} + e_r$, where $v_1^{x, m}, \ldots, v_p^{x, m}$ are the Riemann normal integrals of the first kind. Then the function $\Theta(u)$ is a function of x. By equation (B) it satisfies the conditions

$$\Theta(u+k) = \Theta(u), \quad \Theta(u_r + \tau_r k') = e^{-2\pi i k'(u+\frac{1}{2}\tau k')}\,\Theta(u),$$

wherein k denotes a row, or column, of integers $k_1, \ldots, k_p$ and k' denotes a row or column * of integers $k_1', \ldots, k_p'$. As a function of x, the function $\Theta(v^{x, m} + e)$ cannot, clearly, become infinite, for the arguments $v_r^{x, m} + e_r$ are always finite; but the function does vanish; we proceed in fact to prove the fundamental theorem—*the function* $\Theta(v^{x, m} + e)$ *has always* p *zeros of the first order or zeros whose aggregate multiplicity is* p.

For brevity we denote $v_r^{x, m} + e_r$ by u_r. When the arguments $u_1, \ldots, u_p$ are nearly equal to any finite values $U_1, \ldots, U_p$, the function $\Theta(u)$ can be represented by a series of positive integral powers of the differences $u_1 - U_1, \ldots, u_p - U_p$. Hence the zeros of the function $\Theta(u)$, $= \Theta(v^{x, m} + e)$, are all of positive integral order. The sum of these orders of zero is therefore equal to the value of the integral

$$\frac{1}{2\pi i}\int d\log\Theta(u) = \frac{1}{2\pi i}\int \sum_{s=1}^{p} du_s \Theta_s'(u)/\Theta(u) = \frac{1}{2\pi i}\int dx \sum_{s=1}^{p} (du_s/dx)(\Theta_s'(u)/\Theta(u)),$$

wherein the dash denotes a partial differentiation in regard to the argument u_s, and the integral is to be taken round the complete boundary of the p-ply connected surface on which the function is single-valued, namely round the p closed curves formed by the sides of the period-pair-loops. (Cf. the diagram, p. 21.)

Now the values of $\dfrac{\Theta_s'(u)}{\Theta(u)}\dfrac{du_s}{dx}$ at two points which are opposite points on a period-loop a_r are equal, and in the contour integration the corresponding values of dx are equal and opposite. Hence the portions of the integral arising from the two sides of a period-loop a_r destroy one another. The values of $\dfrac{\Theta_s'(u)}{\Theta(u)}$ at two points which are opposite points on a period-loop b_r differ by $-2\pi i$, or 0, according as $s = r$ or not.

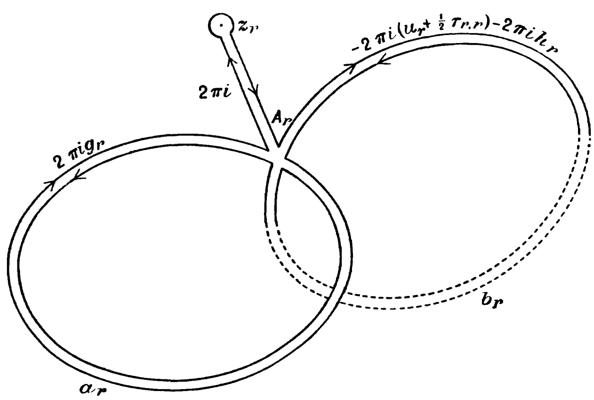
Hence the part of the integral which arises from the period-loop-pair (a_r, b_r) is equal to $-\int du_r$, taken once positively round the left-hand side of the loop b_r, namely equal to $-(-1) = 1$.

The whole value of the integral is, therefore, p; this is then the sum of the orders of zero of the function $\Theta(v^{x, m} + e)$.

* The notation $u_r + \tau_r k'$ denotes the p arguments $u_1 + \tau_1 k', \ldots, u_p + \tau_p k'$.

178. In regard to the position of the zeros of this function we are able to make some statement. We consider first the case when there are p distinct zeros, each of the first order. It is convenient to dissect the Riemann surface in such a way that the function $\log \Theta(v^{x,m} + e)$ may be regarded as single-valued on the dissected surface. Denoting the p zeros of $\Theta(v^{x,m} + e)$ by $z_1, \ldots, z_p$, we may suppose the dissection made by p closed curves such as the one represented in Figure [2], so that a zero of $\Theta(v^{x,m} + e)$ is associated with every one of the period-loop-pairs. Then the surface is still p-ply connected, and $\log \Theta(u)$ is single-valued on the surface bounded by the

Fig. 2.



p closed curves such as the one in the figure. For we proved that a complete circuit of the closed curve formed by the sides of the (a_r, b_r) period-loop-pair, gives an increment of $2\pi i$ for the function $\log \Theta(u)$; when the surface is dissected as in the figure this increment of $2\pi i$ is again destroyed in the circuit of the loop which encloses the point z_r. Any closed circuit on the surface as now dissected is equivalent to an aggregate of repetitions of such circuits as that in the figure; thus if x be taken round any closed circuit the value of $\log \Theta(u)$ at the conclusion of that circuit will be the same as at the beginning. From the formulae

$$\Theta(u_1, \ldots, u_r + 1, \ldots, u_p) = \Theta(u),$$

$$\Theta(u_1 + \tau_{r,1}, \ldots, u_r + \tau_{r,r}, \ldots, u_p + \tau_{r,p}) = e^{-2\pi i(u_r + \frac{1}{2}\tau_{r,r})} \Theta(u),$$

which we express by the statement that $\Theta(u)$ has the factors unity and $e^{-2\pi i(u_r + \frac{1}{2}\tau_{r,r})}$ for the period loops a_r and b_r respectively, it follows that $\log \Theta(u)$ can, at most, have, for opposite points of a_r, b_r, respectively, differences of the form $2\pi i g_r$, $-2\pi i(u_r + \frac{1}{2}\tau_{r,r}) - 2\pi i h_r$, wherein g_r and h_r are integers. The sides of the loops for which these increments occur are marked in the figure, u_r denoting the value of $v_r^{x,m} + e_r$ at the side opposite to that where the increment is marked; thus $u_r + \frac{1}{2}\tau_{r,r}$ is the mean of the values, u_r and $u_r + \tau_{r,r}$, which the integral u_r takes at the two sides of the loop b_r.

Since $\log \Theta(u)$ is now single-valued, the integral $\frac{1}{2\pi i}\int \log \Theta(u) \,.\, du_s$, taken round all the p closed curves constituting the boundary of the surface, will have the value zero. Consider the value of this integral taken round the single boundary in the figure. Let A_r denote the point where the loops a_r, b_r, and that round z_r, meet together. The contribution to the integral arising from the two sides of a_r will be $\int g_r dv_s^{x,m}$, *this* integral being taken once positively round the left side of a_r, from A_r back to A_r. This contribution is equal to $g_r \tau_{r,s}$. The contribution to the integral $\frac{1}{2\pi i}\int \log \Theta(u)\, du_s$ which arises from the two sides of the loop b_r is equal to

$$-\int [v_r^{x,m} + e_r + \tfrac{1}{2}\tau_{r,r} + h_r]\, dv_s^{x,m},$$

taken once positively round the left side of the curve b_r, from A_r back to A_r; this is equal to

$$-\int (v_r^{x,m} + \tfrac{1}{2}\tau_{r,r})\, dv_s^{x,m} + (e_r + h_r) f_{r,s},$$

where $f_{r,s}$ is equal to 1 when $r = s$, and is otherwise zero. Finally the part of the integral $\frac{1}{2\pi i}\int \log \Theta(u)\, du_s$, which arises by the circuit of the loop enclosing the point z_r, from A_r back to A_r, in the direction indicated by the arrow head in the figure, is $\int_{A_r}^{z_r} dv_s^{x,m}$ where A_r denotes now a definite point on the boundary of the loop b_r. If we are careful to retain this signification we may denote this integral by $v_s^{z_r, A_r}$. When we add the results thus obtained, for the p boundary curves, taking r in turn equal to $1, 2, \ldots, p$, we obtain

$$h_s + g_1\tau_{1,s} + \ldots\ldots + g_p\tau_{p,s} + e_s = \sum_{r=1}^{p} \left[-v_s^{z_r, A_r} + \int_{b_r} (v_r^{x,m} + \tfrac{1}{2}\tau_{r,r})\, dv_s^{x,m} \right],$$

wherein, on the right hand, the b_r attached to the integral sign indicates a circuit once positively round the left side of b_r from A_r back to A_r; and if k_s denote the quantity defined by the equation

$$k_s = \sum_{r=1}^{p} \int_{b_r} (v_r^{x,m} + \tfrac{1}{2}\tau_{r,r})\, dv_s^{x,m},$$

which, beside the constants of the surface, depends only on the place m, we have the result

$$h_s + g_1\tau_{1,s} + \ldots + g_p\tau_{p,s} + e_s = -v_s^{z_1, A_1} - \ldots - v_s^{z_p, A_p} + k_s \qquad (s = 1, 2, \ldots, p).$$

179. Suppose now that places $m_1, \ldots, m_p$ are chosen to satisfy the congruences

$$v_s^{m_1, A_1} + \ldots\ldots + v_s^{m_p, A_p} \equiv k_s; \qquad (s = 1, 2, \ldots, p);$$

this is always possible (Chap. IX. §§ 168, 169); it is not necessary for our purpose, to prove that only one set* of places $m_1, \ldots, m_p$, satisfies the conditions; these places, beside the fixed constants of the surface, depend only on the place m. Then, by the equations just obtained, we have

$$e_s \equiv -(v_s^{z_1, m_1} + \ldots\ldots + v_s^{z_p, m_p}); \qquad (s = 1, 2, \ldots, p).$$

Thus if we express the zero in the function $\Theta(v^{x, m} + e)$, it takes the form

$$\Theta(v_s^{x, m} - v_s^{z_1, m_1} - \ldots\ldots - v_s^{z_p, m_p} - h_s' - \tau_s g'),$$

where $g_1', \ldots, g_p', h_1', \ldots, h_p'$ are certain integers, and this, by the fundamental equation (B), § 175, is equal to

$$\Theta(v_s^{x, m} - v_s^{z_1, m_1} - \ldots\ldots - v_s^{z_p, m_p}),$$

save for the factor $e^{-2\pi i g'(v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p} - \frac{1}{2}\tau g')}$. This factor does not vanish or become infinite. Hence we have the result: *It is possible, corresponding to any place m, to choose p places, $m_1, \ldots, m_p$, whose position depends only on the position of m, such that the zeros of the function,*

$$\Theta(v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p}),$$

regarded as a function of x, are the places $z_1, \ldots, z_p$. This is a very fundamental result†.

It is to be noticed that the arguments expressed by $v^{x, m} - v^{z_1, m_1} - \ldots - v^{z_p, m_p}$ do not in fact depend on the place m. For the equations for $m_1, \ldots, m_p$, corresponding to any arbitrary position of m, were

$$v_s^{m_1, A_1} + \ldots\ldots + v_s^{m_p, A_p} \equiv k_s, \ = \sum_{r=1}^{p} \int_{b_r} (v_r^{x, m} + \tfrac{1}{2}\tau_{r, r})\, dv_s^{x, a},$$

a being an arbitrary place. If, instead of m, we take another place μ, we shall, similarly, be required to determine places $\mu_1, \ldots, \mu_p$ by the equations

$$v_s^{\mu_1, A_1} + \ldots\ldots + v_s^{\mu_p, A_p} \equiv k_s, \ = \sum_{r=1}^{p} \int_{b_r} (v_r^{x, \mu} + \tfrac{1}{2}\tau_{r, r})\, dv_s^{x, a}, \qquad (s = 1, 2, \ldots, p);$$

* If two sets satisfy the conditions, these sets will be coresidual (Chap. VIII., § 158).

† Cf. Riemann, *Ges. Werke* (1876), p. 125, (§ 22). The places $m_1, \ldots, m_p$ are used by Clebsch u. Gordan (*Abel. Functionen*, 1866), p. 195. In Riemann's arrangement the existence of the solution of the inversion problem is not proved before the theta functions are introduced.

thus

$$v_s^{\mu_1, m_1} + \ldots\ldots + v_s^{\mu_p, m_p} \equiv \sum_{r=1}^{p} \int_{b_r} v_r^{m, \mu}\, dv_s^{x, a}, = \sum_{r=1}^{p} f_{s, r}\, v_r^{\mu, m}, \qquad (s = 1, 2, \ldots, p),$$

wherein $f_{s, r} = 1$ when $r = s$, and is otherwise zero, as we see by recalling the significance of the b_r attached to the integral sign. Thus (Chap. VIII., § 158), the places $\mu_1, \ldots, \mu_p, m$ are coresidual with the places $m_1, \ldots, m_p, \mu$, and the arguments

$$v_s^{x, m} - v_s^{z_1, m_1} - \ldots\ldots - v_s^{z_p, m_p},$$

are congruent to arguments of the form

$$v_s^{x, \mu} - v_s^{z_1, \mu_1} - \ldots\ldots - v_s^{z_p, \mu_p}.$$

The fact that the places $\mu_1, \ldots, \mu_p, m$ are coresidual with the places $m_1, \ldots, m_p, \mu$, which is expressed by the equations

$$v_s^{\mu_1, m_1} + \ldots\ldots + v_s^{\mu_p, m_p} + v_s^{m, \mu} \equiv 0, \qquad (s = 1, 2, \ldots, p),$$

will also, in future, be often represented in the form

$$(\mu_1, \ldots, \mu_p, m) \equiv (m_1, \ldots, m_p, \mu).$$

If the places $m_1, \ldots, m_p$ are not zeros of a ϕ-polynomial, this relation determines $\mu_1, \ldots, \mu_p$ uniquely from the place μ.

Ex. In case $p = 1$, prove that the relation determining $m_1, \ldots, m_p$ leads to

$$v^{m_1, m} \equiv \tfrac{1}{2}(1 + \tau).$$

Hence the function $\Theta(v^{x, z} + \frac{1}{2} + \frac{1}{2}\tau)$ vanishes for $x = z$, as is otherwise obvious.

180. The deductions so far made, on the supposition that the p zeros of the function $\Theta(v^{z, m} + e)$ are distinct, are not essentially modified when this is not so. Suppose the zeros to consist of a p_1-tuple zero at z_1, a p_2-tuple zero at $z_2, \ldots$, and a p_k-tuple zero at z_k, so that $p_1 + \ldots\ldots + p_k = p$. The surface may be dissected into a simply connected surface as in Figure 3. The function $\log \Theta(v^{x, m} + e)$ becomes a single-valued function of x on the dissected surface; and its differences, for the two sides of the various cuts, are those given in the figure. To obtain these differences we remember that $\log \Theta(v^{x, m} + e)$ increases by $2\pi i$ when x is taken completely round the four sides of a pair of loops (a_r, b_r). The mode of dissection of Fig. 3, may of course also be used in the previous case when the zeros of $\Theta(v^{x, m} + e)$ are all of the first order.

The integral $\dfrac{1}{2\pi i}\displaystyle\int \log \Theta(v^{x, m} + e)\, dv_s^{x, m}$, taken along the single closed boundary constituted by the sides of all the cuts, has the value zero. Its value is, however, in the case of Figure 3,

$$p_1 v_s^{z_1, A_1} + \dots\dots + p_k v_s^{z_k, A_1}$$

$$+ g_1 \int_{a_1} dv_s^{x, m} - h_1 \int_{b_1} dv_s^{x, m} - \int_{b_1} (v_1^{x, m} + e_1 + \tfrac{1}{2}\tau_{1,1})\, dv_s^{x, m} - (p-1)\, v_s^{A_2, A_1}$$

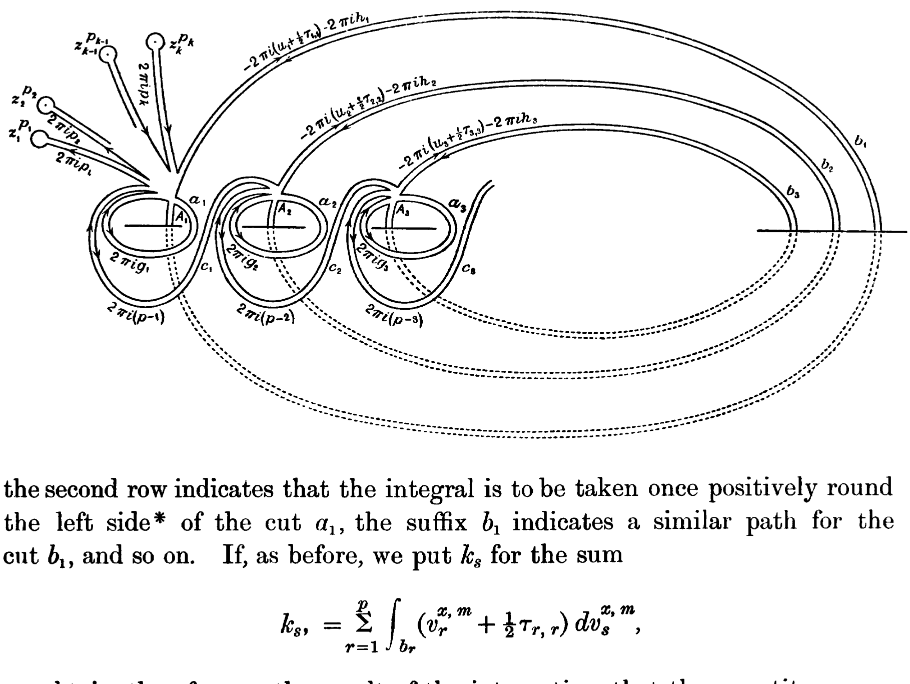
$$+ g_2 \int_{a_2} dv_s^{x, m} - h_2 \int_{b_2} dv_s^{x, m} - \int_{b_2} (v_2^{x, m} + e_2 + \tfrac{1}{2}\tau_{2,2})\, dv_s^{x, m} - (p-2)\, v_s^{A_3, A_1}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ g_p \int_{a_p} dv_s^{x, m} - h_p \int_{b_p} dv_s^{x, m} - \int_{b_p} (v_p^{x, m} + e_p + \tfrac{1}{2}\tau_{p,p})\, dv_s^{x, m},$$

wherein the first row is that obtained by the sides of the cuts, from A_1, excluding the zeros $z_1, \dots, z_k$, and the second row is that obtained from the cuts a_1, b_1, c_1, and so on. The suffix a_1 to the first integral sign in

Fig. 3.



the second row indicates that the integral is to be taken once positively round the left side* of the cut a_1, the suffix b_1 indicates a similar path for the cut b_1, and so on. If, as before, we put k_s for the sum

$$k_s, \; = \sum_{r=1}^{p} \int_{b_r} (v_r^{x, m} + \tfrac{1}{2}\tau_{r, r})\, dv_s^{x, m},$$

we obtain, therefore, as the result of the integration, that the quantity

$$h_s + g_1 \tau_{s, 1} + \dots\dots + g_p \tau_{s, p} + e_s$$

* By the left side of a cut a_1, or b_1, is meant the side upon which the increments of $\log \Theta(u)$ are marked in the figure. The general question of the effect of variation in the period cuts is most conveniently postponed until the transformation of the theta functions has been considered.

is equal to

$$k_s - p_1 v_s^{z_1, A_1} - \dots\dots - p_k v_s^{z_k, A_1} + (p-1)\, v_s^{A_2, A_1} + (p-2)\, v_s^{A_3, A_2} + \dots\dots + v_s^{A_p, A_{p-1}}$$

and this is immediately seen to be the same as

$$k_s - v_s^{z_1, A_1} - \dots\dots - v_s^{z_1, A_{p_1}} - v_s^{z_2, A_{p_1+1}} - \dots\dots - v_s^{z_2, A_{p_1+p_2}} - \dots\dots - v_s^{z_k, A_p}.$$

We thus obtain, of course, the same equations as before (§ 179), save that z_1 is here repeated p_1 times, ..., and z_k is repeated p_k times. And we can draw the inference that $\Theta(v^{x, m} + e)$ can be written in the form $\Theta(v_s^{x, m} - v_s^{z_1, m_1} - \dots\dots - v_s^{z_p, m_p} - h_s - \tau_s g)$, which, save for a finite non-vanishing factor, is the same as $\Theta(v_s^{x, m} - v_s^{z_1, m_1} - \dots\dots - v_s^{z_p, m_p})$; the argument $v_s^{x, m} - v_s^{z_1, m_1} - \dots\dots - v_s^{z_p, m_p}$ does not depend on the place m.

181. From the results of §§ 179, 180, we can draw an inference which leads to most important developments in the theory of the theta functions.

For, from what is there obtained it follows that if $z_1, \dots, z_p$ be any places whatever, the function $\Theta(v^{x, m} - v^{z_1, m_1} - \dots\dots - v^{z_p, m_p})$ has $z_1, \dots, z_p$ for zeros. Hence, putting z_p for x we infer that *the function*

$$\Theta(v^{m_p, m} - v^{z_1, m_1} - \dots\dots - v^{z_{p-1}, m_{p-1}}) \qquad \text{(F)}$$

vanishes identically for all positions of $z_1, \dots, z_{p-1}$. Putting

$$f_s = v_s^{z_1, m_1} + \dots\dots + v_s^{z_{p-2}, m_{p-2}} - v_s^{m_p, m},$$

for $s = 1, 2, \dots, p$, this is the same as the statement that the function $\Theta(v^{x, m_{p-1}} + f)$ vanishes identically for all positions of x and for all values of $f_1, \dots, f_p$ which can be expressed in the form arising here. When $f_1, \dots, f_p$ are arbitrary quantities it is not in general possible to determine places $z_1, \dots, z_{p-2}$ to express $f_1, \dots, f_p$ in the form in question. Nevertheless the case which presents itself reminds us that in the investigation of the zeros of $\Theta(v^{x, m} + e)$ we have assumed that the function does not vanish identically, and it is essential to observe that this is so for general values of $e_1, \dots, e_p$. If, for a given position of x, the function $\Theta(v^{x, m} + e)$ vanished identically for all values of $e_1, \dots, e_p$, the function $\Theta(r)$ would vanish for all values of the arguments $r_1, \dots, r_p$. We assume * from the original definition of the theta function, by means of a series, that this is not the case.

Further the function $\Theta(v^{x, m} + e)$ is by definition an analytical function of each of the quantities $e_1, \dots, e_p$; and if an analytical function do not vanish

* The series is a series of integral powers of the quantities $e^{2\pi i r_1}, \dots, e^{2\pi i r_p}$.

for all values of its argument, there must exist a continuum of values of the argument, of finite extent in two dimensions, within which the function does not vanish *. Hence, for each of the quantities $e_1, \ldots, e_p$ there is a continuum of values of two dimensions, within which the function $\Theta(v^{x,m}+e)$ does not vanish identically. And, by equation (B), § 175, this statement remains true when the quantities $e_1, \ldots, e_p$ are increased by any simultaneous periods. Restricting ourselves then, first of all, to values of $e_1, \ldots, e_p$ lying within these regions, there exist (Chap. IX. § 168) positions of $z_1, \ldots, z_p$ to satisfy the congruences

$$e_s \equiv v_s^{z_1, m_1} + \ldots\ldots + v_s^{z_p, m_p}, \qquad (s=1, 2, \ldots, p);$$

and, since to each set of positions of $z_1, \ldots, z_p$, there corresponds only one set of values for $e_1, \ldots, e_p$, the places $z_1, \ldots, z_p$ are also, each of them, variable within a certain two-dimensionality. Hence, within certain two-dimensional limits, there certainly exist arbitrary values of $z_1, \ldots, z_p$ such that the function $\Theta(v^{x,m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p})$ does not vanish identically. For such values, and the corresponding values of $e_1, \ldots, e_p$, the investigation so far given holds good. And therefore, for such values, the function $\Theta(v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}})$ vanishes identically. Since this function is an analytical function of the places† $z_1, \ldots, z_{p-1}$, and vanishes identically for all positions of each of these places within a certain continuum of two dimensions, it must vanish identically for all positions of these places.

Hence the theorem (F) holds without limitation, notwithstanding the fact that for certain special forms of the quantities $e_1, \ldots, e_p$, the function $\Theta(v^{x,m}+e)$ vanishes identically. The important part played by the theorem (F) will be seen to justify this enquiry.

182. It is convenient now to deduce in order a series of propositions in regard to the theta functions (§§ 182—188); and for purposes of reference it is desirable to number them.

(I.) If $\zeta_1, \ldots, \zeta_p$ be p places which are zeros of one or more linearly independent ϕ-polynomials, that is, of linearly independent linear aggregates of the form $\lambda_1\Omega_1(x) + \ldots\ldots + \lambda_p\Omega_p(x)$ (Chap. II. § 18, Chap. VI. § 101), then the function

$$\Theta(v^{x,m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_p, m_p})$$

vanishes identically for all positions of x.

For then, if $\tau+1$ be the number of linearly independent ϕ-polynomials which vanish in the places $\zeta_1, \ldots, \zeta_p$, we can, taking $\tau+1$ arbitrary places

* E.g. a single-valued analytical function of an argument z, $=x+iy$, cannot vanish for all rational values of x and y without vanishing identically.

† By an analytical function of a place z on a Riemann surface, is meant a function whose values can be expressed by series of integral powers of the infinitesimal at the place.

$z_1, \ldots, z_{\tau+1}$, determine $p-\tau-1$ places $z_{\tau+2}, \ldots, z_p$, such that $(z_1, \ldots, z_p) \equiv (\zeta_1, \ldots, \zeta_p)$ (see Chap. VI. § 93, etc., and for the notation, § 179). Then the argument

$$v_s^{x, m} - v_s^{\zeta_1, m_1} - \ldots\ldots - v_s^{\zeta_p, m_p}, \qquad (s = 1, 2, \ldots, p),$$

can be put in the form

$$v_s^{x, m} - v_s^{z_1, m_1} - \ldots\ldots - v_s^{z_p, m_p},$$

save for integral multiples of the periods; thus (§§ 179, 180) the theta function vanishes when x is at any one of the perfectly arbitrary places $z_1, \ldots, z_{\tau+1}$. Thus, since by hypothesis $\tau+1$ is at least equal to 1, the theta function vanishes identically.

It follows from this proposition that if $z_2', \ldots, z_p'$ be the remaining zeros of a ϕ-polynomial determined to vanish in each of $z_2, \ldots, z_p$, and neither x nor z_1 be among $z_2', \ldots, z_p'$, then the zeros of the function

$$\Theta\left(v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p}\right),$$

regarded as a function of z_1, are the places $x, z_2', \ldots, z_p'$.

From this Proposition and the results previously obtained, we can infer that *the function* $\Theta\left(v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p}\right)$ *vanishes only* (i) *when* x *coincides with one of the places* $z_1, \ldots, z_p$, or (ii) *when* $z_1, \ldots, z_p$ *are zeros of a* ϕ*-polynomial.*

(II.) Suppose a rational function exists, of order, Q, not greater than p, and let $\tau+1$ be the number of ϕ-polynomials vanishing in the poles of this function. Take $\tau+1$ arbitrary places

$$\zeta_1, \ldots, \zeta_q, x_1, \ldots, x_{\tau+1-q},$$

wherein $q = Q - p + \tau + 1$, and suppose $z_1, \ldots, z_Q$ to be a set of places coresidual with the poles of the rational function, of which, therefore, q are arbitrary. Then the function

$$\Theta\left(v^{m_p, m} + v^{\zeta_1, z_1} + \ldots\ldots + v^{\zeta_q, z_q} - v^{x_1, m_1} - \ldots\ldots \right.$$
$$\left. - v^{x_{\tau+1-q}, m_{\tau+1-q}} - v^{z_{q+1}, m_{\tau+2-q}} - \ldots\ldots - v^{z_Q, m_{p-q}}\right)$$

vanishes identically.

For if we choose $\zeta_{q+1}, \ldots, \zeta_Q$ such that $(\zeta_1, \ldots, \zeta_Q) \equiv (z_1, \ldots, z_Q)$, the general argument of the theta function under consideration is congruent to the argument

$$v^{m_p, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_{\tau+1-q}, m_{\tau+1-q}} - v^{\zeta_{q+1}, m_{\tau+2-q}} - \ldots\ldots - v^{\zeta_Q, m_{p-q}}.$$

This value of the argument is a particular case of that occurring in (F), § 181, the last $q-1$ of the upper limits in (F) being put equal to the lower limits. Hence the proposition follows from (F).

(III.) If r denote such a set of arguments $r_1, \ldots, r_p$ that $\Theta(r) = 0$, and, for the positions of z under consideration, the function $\Theta(v^{x, z} + r)$ does not vanish for all positions of x, then there are unique places $z_1, \ldots, z_{p-1}$, such that

$$r \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}}.$$

In this statement of the proposition a further abbreviation is introduced which will be constantly employed. The suffix indicating that the equation stands as the representative of p equations is omitted.

Before proceeding to the proof it may be remarked that if $m', m_1', \ldots, m_p'$ be places such that (cf. § 179)

$$(m', m_1, \ldots, m_p) \equiv (m, m_1', \ldots, m_p')$$

and therefore, also,

$$v^{m', m} - v^{m_1', m_1} - \ldots\ldots - v^{m_p', m_p} \equiv 0,$$

then the equation

$$r \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}}$$

is the same as the equation

$$r \equiv v^{m_p', m'} - v^{z_1, m_1'} - \ldots\ldots - v^{z_{p-1}, m'_{p-1}}.$$

This proposition (III.) is in the nature of a converse to equation (F). Since the function $\Theta(v^{x, z} + r)$ does not vanish identically, its zeros, $z_1, \ldots, z_p$, are such that

$$v^{x, z} + r \equiv v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p};$$

now we have

$$v^{z_1, m_1} + v^{z_p, m_p} \equiv v^{z_p, m_1} + v^{z_1, m_p},$$

so that the zeros $z_1, \ldots, z_p$ may be taken in any order; since $\Theta(r)$ vanishes, z is one of the zeros of $\Theta(v^{x, z} + r)$; hence, we may put $z_p = z$, and obtain

$$\begin{aligned} r &\equiv v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p} - v^{x, z_p}, \\ &\equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}}, \end{aligned}$$

which is the form in question.

If the places $z_1, \ldots, z_{p-1}$ in this equation are not unique, but, on the contrary, there exists also an equation of the form

$$r \equiv v^{m_p, m} - v^{z_1', m_1} - \ldots\ldots - v^{z'_{p-1}, m_{p-1}},$$

then, from the resulting equation

$$v^{z_1', z_1} + \ldots\ldots + v^{z'_{p-1}, z_{p-1}} \equiv 0,$$

we can (Chap. VIII. § 158) infer that there is an infinite number of sets of places $z_1', \ldots, z'_{p-1}$, all coresidual with the set $z_1, \ldots, z_{p-1}$; hence we can put

$$v^{x, z} + r \equiv v^{x, m} - v^{z_1', m_1} - \ldots\ldots - v^{z'_{p-1}, m_{p-1}} - v^{z, m_p},$$

wherein at least one of the places $z_1', \ldots, z'_{p-1}$ is entirely arbitrary. Then the function $\Theta(v^{x, z} + r)$ vanishes for an arbitrary position of x, that is, it vanishes identically; this is contrary to the hypothesis made.

It follows also that whenever it is possible to find places $z_1, \ldots, z_{p-1}$ to satisfy the inversion problem expressed by the p equations

$$v^{z_1, m_1} + \ldots\ldots + v^{z_{p-1}, m_{p-1}} = u,$$

the function $\Theta(v^{m_p, m} - u)$ vanishes; conversely, when u is such that this function vanishes we can solve the inversion problem referred to.

(IV.) When r is such that $\Theta(r)$ vanishes, and $\Theta(v^{x, z} + r)$ does not, for the values of z considered, vanish identically for all positions of x, the zeros of $\Theta(v^{x, z} + r)$, other than z, are independent of z and depend only on the argument r.

This is an immediate corollary from Proposition (III.); but it is of sufficient importance to be stated separately.

(V.) If $\Theta(r) = 0$, and $\Theta(v^{x, z} + r)$ vanish identically for all positions of x and z, but $\Theta(v^{x, z} + v^{\xi, \zeta} + r)$ do not vanish identically, in regard to x, for the positions of z, ξ, ζ considered, then it is possible to find places $z_1, \ldots, z_{p-2}$ such that

$$r \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-2}, m_{p-2}} - v^{\xi, m_{p-1}},$$

and these places $z_1, \ldots, z_{p-2}$ are definite.

Under the hypotheses made, we can put

$$v^{x, z} + v^{\xi, \zeta} + r \equiv v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_p, m_p},$$

wherein $z_1, \ldots, z_p$ are the zeros of $\Theta(v^{x, z} + v^{\xi, \zeta} + r)$; now z is clearly a zero; for the function $\Theta(v^{\xi, \zeta} + r)$ is of the same form as $\Theta(v^{x, z} + r)$, and vanishes identically; and ζ is also a zero; for, putting ζ for x, the function $\Theta(v^{x, z} + v^{\xi, \zeta} + r)$ becomes $\Theta(v^{\xi, z} + r)$, which also vanishes identically. Putting, therefore, ζ, z for z_{p-1} and z_p respectively, the result enunciated is obtained, the uniqueness of the places $z_1, \ldots, z_{p-2}$ being inferred as in Proposition (III.).

We may state the theorem differently thus: If $\Theta(v^{x, z} + r)$ vanish for all positions of x and z, and $\Theta(v^{x, z} + v^{\xi, \zeta} + r)$ do not in general vanish identically, the equations

$$r \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-2}, m_{p-2}} - v^{z_{p-1}, m_{p-1}}$$

can be solved, and in the solution one of $z_1, \ldots, z_{p-1}$ may be taken arbitrarily, and the others are thereby determined. Hence also we can find places $z_1', \ldots, z'_{p-1}$, other than $z_1, \ldots, z_{p-1}$, such that

$$v^{z_1', z_1} + \ldots\ldots + v^{z'_{p-1}, z_{p-1}} = 0,$$

one of the places $z_1', \ldots, z'_{p-1}$ being arbitrary. Hence by the formula $Q - q = p - \tau - 1$, putting $Q = p - 1$, $q = 1$, we infer $\tau + 1 = 2$, so that a ϕ-polynomial vanishing in $z_1, \ldots, z_{p-1}$ can be made to vanish in the further arbitrary place z. Thus, when $\Theta\,(v^{x, z} + r)$ vanishes identically, we can write

$$v^{x, z} + r \equiv v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}} - v^{z, m_p},$$

wherein the places $z_1, \ldots, z_{p-1}, z$ are zeros of a ϕ-polynomial (cf. Prop. I.).

(VI.) The propositions (III.) and (V.) can be generalized thus: If $\Theta\,(v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ be identically zero for all positions of the places $x_1, z_1, \ldots, x_q, z_q$, and the function $\Theta\,(v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ do not vanish identically in regard to x, then places $\zeta_1, \ldots, \zeta_{p-1}$ can be found to satisfy the equations

$$r \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

and, of these places, q are arbitrary, the others being thereby determined.

These arbitrary places, $\zeta_1, \ldots, \zeta_q$, say, must be such that the function $\Theta\,(v^{x, z} + v^{\zeta_1, z_1} + \ldots\ldots + v^{\zeta_q, z_q} + r)$ does not vanish identically.

For as before we can put

$$v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r \equiv v^{x, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_p, m_p},$$

wherein $\zeta_1, \ldots, \zeta_p$ are the zeros of the function $\Theta\,(v^{x, z} + v^{x_1, z_1} + \ldots + v^{x_q, z_q} + r)$. It is clear that z is one zero of this function; also putting z_1 for x the function becomes $\Theta\,(v^{x_1, z} + v^{x_2, z_2} + \ldots\ldots + v^{x_q, z_q} + r)$, which vanishes, by the hypothesis. Thus the places $z, z_1, \ldots, z_q$ are all zeros of the function

$$\Theta\,(v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r).$$

Putting then $z_1, \ldots, z_q, z$ respectively for $\zeta_1, \ldots, \zeta_q, \zeta_p$ in the congruence just written, it becomes

$$v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + v^{z_1, m_1} + \ldots\ldots + v^{z_q, m_q} + v^{\zeta_{q+1}, m_{q+1}} + \ldots\ldots$$
$$+ v^{\zeta_{p-1}, m_{p-1}} + v^{z, m_p} + r \equiv v^{x, m},$$

and this is the same as

$$r \equiv v^{m_p, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_q, m_q} - v^{\zeta_{q+1}, m_{q+1}} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}};$$

replacing $x_1, \ldots, x_q$ by $\zeta_1, \ldots, \zeta_q$ we have the result stated.

Hence also, we can find places $\zeta_1', \ldots, \zeta'_{p-1}$, other than $\zeta_1, \ldots, \zeta_{p-1}$, such that

$$v^{\zeta_1', \zeta_1} + \ldots\ldots + v^{\zeta'_{p-1}, \zeta_{p-1}} \equiv 0,$$

q of the places $\zeta_1', \ldots, \zeta'_{p-1}$ being arbitrary. Therefore a ϕ-polynomial can be chosen to vanish in $\zeta_1, \ldots, \zeta_{p-1}$ and in $q\ (= p-1-(Q-q)$, when $Q = p-1)$ other arbitrary places. Thus the argument

$$v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_{q-1}, z_{q-1}} + r,$$

for which the theta function vanishes identically, can be written in the form

$$v^{x, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{q-1}, m_{q-1}} - v^{\zeta_q, m_q} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}} - v^{z, m_p},$$

wherein $z_1, \ldots, z_{q-1}, \zeta_q, \ldots, \zeta_{p-1}, z$ are zeros of $q+1$ linearly independent ϕ-polynomials.

(VII.) If the function $\Theta\,(v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ be identically zero for all positions of the places $x_1, z_1, x_2, z_2, \ldots, x_q, z_q$, and, for general positions of $x_1, z_1, \ldots, x_q, z_q$, the function $\Theta\,(v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ be not identically zero, as a function of x, for proper positions of z, and be not identically zero, as a function of z, for proper positions of x, then we can find places $\zeta_1, \ldots, \zeta_{p-1}$, of which q places are arbitrary, such that

$$r \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

and can also find places $\xi_1, \ldots, \xi_{p-1}$, of which q places are arbitrary, such that

$$-r \equiv v^{m_p, m} - v^{\xi_1, m_1} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}}.$$

This is obvious from the last proposition, if we notice that

$$\Theta\,(v^{z, x} + v^{z_1, x_1} + \ldots\ldots + v^{z_q, x_q} - r) = \Theta\,(v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r).$$

We can hence infer that

$$2v^{m_p, m} + v^{m_1, \zeta_1} + v^{m_1, \xi_1} + \ldots\ldots + v^{m_{p-1}, \zeta_{p-1}} + v^{m_{p-1}, \xi_{p-1}} \equiv 0,$$

and this is the same (Chap. VIII. § 158) as the statement that the set of $2p$ places constituted by $\xi_1, \ldots, \xi_{p-1}, \zeta_1, \ldots, \zeta_{p-1}$ and the place m, repeated, is coresidual with the set of $2p$ places constituted by the places $m_1, \ldots, m_p$, each repeated. This result we write (cf. § 179) in the form

$$(m^2, \xi_1, \ldots, \xi_{p-1}, \zeta_1, \ldots, \zeta_{p-1}) \equiv (m_1^2, m_2^2, \ldots, m_p^2).$$

(VIII.) We can now prove that if $\zeta_1, \ldots, \zeta_{p-1}$ be arbitrary places, places $\xi_1, \ldots, \xi_{p-1}$ can be found such that

$$(m^2, \xi_1, \ldots, \xi_{p-1}, \zeta_1, \ldots, \zeta_{p-1}) \equiv (m_1^2, m_2^2, \ldots, m_p^2).$$

Let r denote the set of p arguments given by

$$r \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

$\zeta_1, \ldots, \zeta_{p-1}$ being quite arbitrary. Then, by theorem (F), (§ 181), the function $\Theta(r)$ certainly vanishes. It may happen that also the function $\Theta(v^{x,z}+r)$ vanishes identically for all positions of x and z. It may further happen that also the function $\Theta(v^{x,z}+v^{x_1,z_1}+r)$ vanishes identically for all positions of x, z, x_1, z_1. We assume* however that there is a finite value of q such that the function $\Theta(v^{x,z}+v^{x_1,z_1}+\ldots\ldots+v^{x_q,z_q}+r)$ does not vanish identically for all positions of $x, z, x_1, z_1, \ldots, x_q, z_q$. Then by Proposition VII. it follows that we can find places $\xi_1, \ldots, \xi_{p-1}$, such that

$$-r \equiv v^{m_p, m} - v^{\xi_1, m_1} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}};$$

comparing this with the equations defining the argument r, we can, as in Proposition (VII.) infer that the congruence stated at the beginning of this Proposition also holds.

(IX.) Hence follows a very important corollary. Taking any other arbitrary places $\zeta_1', \ldots, \zeta'_{p-1}$, we can find places $\xi_1', \ldots, \xi'_{p-1}$ such that

$$(m^2, \xi_1', \ldots, \xi'_{p-1}, \zeta_1', \ldots, \zeta'_{p-1}) \equiv (m_1^2, m_2^2, \ldots, m_p^2);$$

therefore the set $\xi_1, \ldots, \xi_{p-1}, \zeta_1, \ldots, \zeta_{p-1}$ is coresidual with the set $\xi_1', \ldots, \xi'_{p-1}$, $\zeta_1', \ldots, \zeta'_{p-1}$. Now, of a set of $2p-2$ places coresidual with a given set we can in general take only $p-2$ arbitrarily; when, as here, we can take $p-1$ arbitrarily, each of the sets must be the zeros of a ϕ-polynomial (Chap. VI. § 93). Thus the places $\xi_1, \ldots, \xi_{p-1}, \zeta_1, \ldots, \zeta_{p-1}$ are zeros of a ϕ-polynomial.

Therefore, if $a_1, \ldots, a_{2p-2}$ be the zeros of any ϕ-polynomial whatever, that is, the zeros of the differential of any integral of the first kind, *the places $m_1, \ldots, m_p$ are so derived from the place m that we have*

$$(m^2, a_1, \ldots, a_{2p-2}) \equiv (m_1^2, m_2^2, \ldots, m_p^2), \qquad (G);$$

in other words, if $c_1, \ldots, c_p$ denote any independent places, the places $m_1, \ldots, m_p$ satisfy the equations

$$2\left[v_s^{m_1, c_1} + \ldots\ldots + v_s^{m_p, c_p}\right] \equiv 2v_s^{m, c_p} + v_s^{a_1, c_1} + v_s^{a_2, c_1} + \ldots\ldots + v_s^{a_{2p-3}, c_p} + v_s^{a_{2p-2}, c_p},$$

for $s = 1, 2, \ldots, p$. Denoting the right hand, whose value is perfectly definite, by A_s, and supposing $g_1, \ldots, g_p, h_1, \ldots, h_p$ to denote proper integers, these equations are the same as

$$v_s^{m_1, c_1} + \ldots\ldots + v_s^{m_p, c_p} \equiv \tfrac{1}{2}A_s + \tfrac{1}{2}(h_s + g_1\tau_{s,1} + \ldots\ldots + g_p\tau_{s,p}), \qquad (G'),$$

where $s = 1, 2, \ldots, p$.

* It will be seen in Proposition XIV. that if $\Theta(v^{x,z}+v^{x_1,z_1}+\ldots\ldots+v^{x_q,z_q}+r)$ vanishes identically, then all the partial differential coefficients of $\Theta(u)$, in regard to $u_1, \ldots, u_p$, up to and including those of the $(q+1)$th order, also vanish for $u=r$.

There are however 2^{2p} sets of places $m_1, \ldots, m_p$, corresponding to any position of the place m, which satisfy the equation* (G). For in equations (G′) there are 2^{2p} values possible for the right-hand side in which each of $g_1, \ldots, g_p, h_1, \ldots, h_p$ is either 0 or 1, and any two sets of values $g_1, \ldots, g_p, h_1, \ldots, h_p$ and $g_1', \ldots, g_p', h_1', \ldots, h_p'$, such that g_i, g_i' differ by an even integer, and h_i, h_i' differ by an even integer, for $i = 1, 2, \ldots, p$, lead to the same positions for the places $m_1, \ldots, m_p$. (Chap. VIII. § 158.)

We have seen (§ 179) that the places $m_1, \ldots, m_p$ depend only on the place m and on the mode of dissection of the Riemann surface. We are to see, in what follows, that the 2^{2p} solutions of the equation (G) are to be associated, in an unique way, each with one of the 2^{2p} essentially distinct theta functions with half integer characteristics.

183. The equation (G) can be interpreted geometrically. Take a non-adjoint polynomial, Δ, of any grade μ, which has a zero of the second order at the place m; it will have $n\mu - 2$ other zeros. Take an adjoint polynomial ψ, of grade $(n-1)\sigma + n - 3 + \mu$, which vanishes in these other $n\mu - 2$ zeros of Δ. Then (Chap. VI. § 92, Ex. ix.) ψ will be of the form $\lambda\psi_0 + \Delta\phi$, where ψ_0 is a special form of ψ, λ is an arbitrary constant, and ϕ is a general ϕ-polynomial. The polynomial ψ will have $2p$ zeros other than those prescribed; denote them by $k_1, \ldots, k_{2p}$. If ϕ' be any ϕ-polynomial, with $a_1, \ldots, a_{2p-2}$ as zeros, we can form a rational function, given by $(\lambda\psi_0 + \Delta\phi)/\Delta\phi'$, whose poles are the places $a_1, \ldots, a_{2p-2}$, together with the place m repeated, its zeros being the places $k_1, \ldots, k_{2p}$. Hence (Chap. VI. § 96) we have

$$(m^2, a_1, \ldots, a_{2p-2}) \equiv (k_1, k_2, \ldots, k_{2p-1}, k_{2p}),$$

and therefore, by equation (G),

$$(m_1^2, \ldots, m_p^2) \equiv (k_1, k_2, \ldots, k_{2p-1}, k_{2p}) \qquad (\text{G}'');$$

hence (Chap. VI. § 90) it is possible to take the polynomial ψ so that its zeros $k_1, \ldots, k_{2p}$ consist of p zeros each of the second order, and *the places $m_1, \ldots, m_p$ are one of the sets of p places thus obtained.*

There are 2^{2p} possible polynomials ψ which have the necessary character, as we have already seen by considering the equation (G′); but, in fact, a certain number of these are composite polynomials formed by the product of the polynomial Δ and a ϕ-polynomial of which the $2p-2$ zeros consist of $p-1$ zeros each repeated. To prove this it is sufficient to prove that there exist such ϕ-polynomials having only $p-1$ zeros, each of the second order; for it is clear that if Φ denote such a polynomial, the product $\Delta\Phi$ is of grade

* If for any set of values for $g_1, \ldots, g_p, h_1, \ldots, h_p$ the equations (G′) are capable of an infinity of (coresidual) sets of solutions, the correct statement will be that there are 2^{2p} lots of coresidual sets, belonging to the place m, which satisfy the equation (G). The corresponding modification may be made in what follows.

$(n-1)\sigma+n-3+\mu$ and satisfies the conditions imposed on the polynomial ψ. That there are such ϕ-polynomials Φ is immediately obvious algebraically. If we form the equation giving the values of x at the zeros of the general ϕ-polynomial,

$$\lambda_1\phi_1+\dots\dots+\lambda_p\phi_p,$$

the $p-1$ conditions that the left-hand side should be a perfect square, will determine the necessary ratios $\lambda_1 : \lambda_2 : \dots : \lambda_p$, and, in general, in only a finite number of ways. (Cf. also Prop. XI. below.)

It is immediately seen, from equation (G″), that if $m_1, \dots, m_p$ be the double zeros of one such polynomial ψ as described, and $m_1', \dots, m_p'$ of another, both sets being derived from the same place m, then

$$v^{m_1', m_1}+\dots\dots+v^{m_p', m_p}=\tfrac{1}{2}\Omega_{\beta, \alpha}, \qquad \text{(H)}$$

where $\Omega_{\beta, \alpha}$ stands for p quantities such as

$$\beta_s+\alpha_1\tau_{s, 1}+\dots\dots+\alpha_p\tau_{s, p},$$

$\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p$ being integers.

We may give an example of the geometrical relation thus introduced, which is of great importance. It will be sufficient to use only the usual geometrical phraseology.

Suppose the fundamental equation is of the form

$$C+(x, y)_1+(x, y)_2+(x, y)_3+(x, y)_4=0,$$

representing a plane quartic curve ($p=3$). Then if a straight line be drawn touching the curve at a point m, it will intersect it again in 2 points A, B. Through these 2 points A, B, ∞^3 conics can be drawn; of these conics there are a certain number which touch the fundamental quartic in three points P, Q, R other than A and B. There are $2^{2p}=64$ sets of three such points P, Q, R; but of these some consist of the two points of contact of double tangents of the quartic taken with the point m itself.

In fact there are (Salmon, *Higher Plane Curves*, Dublin, 1879, p. 213) 28, $=2^{p-1}(2^p-1)$, double tangents; these do not depend at all on the point m; there are therefore 36, $=2^{p-1}(2^p+1)$, proper sets of three points P, Q, R in which conics passing through A and B touch the curve. One of these sets of three points is formed by the points m_1, m_2, m_3. It has been proved that the numbers $2^{p-1}(2^p-1)$, $2^{p-1}(2^p+1)$ are respectively the numbers of odd and even theta functions of half integer characteristics (§ 176).

184. (X.) We have seen in Proposition (VIII.) (§ 182) that the places $m_1, \dots, m_p$ are one set from 2^{2p} sets of p places all satisfying the same equivalence (G). We are now to see the interpretation of the other $2^{2p}-1$ solutions of this equation.

Let $m_1', \dots, m_p'$ be any set, other than $m_1, \dots, m_p$, which satisfies the congruence (G). Then, by equations (G′), we have

$$2\left(v_s^{m_1', m_1}+\dots\dots+v_s^{m_p', m_p}\right)\equiv 0, \qquad (s=1, 2, \dots, p),$$

and therefore, if $\Omega_{\beta, \alpha}$ denote the set of p quantities of which a general one is given by

$$\beta_s + \alpha_1\tau_{s,1} + \ldots\ldots + \alpha_p\tau_{s,p}, \qquad (s = 1, 2, \ldots, p),$$

where $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p$ are certain integers, we have

$$v_s^{m_1', m_1} + \ldots\ldots + v_s^{m_p', m_p} = \tfrac{1}{2}\,\Omega_{\beta, \alpha};$$

hence the function

$$\Theta\,(v^{x, m} - v^{z_1, m_1'} - \ldots\ldots - v^{z_p, m_p'};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha),$$

$$= e^{\pi i\beta\alpha}\,\Theta\,(v^{z_1, m_1'} + \ldots\ldots + v^{z_p, m_p'} - v^{x, m};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha),$$

$$= e^{\pi i\beta\alpha}\,\Theta\,(u - \tfrac{1}{2}\Omega_{\beta, \alpha};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha),$$

where

$$u_s = v_s^{z_1, m_1} + \ldots\ldots + v_s^{z_p, m_p} - v^{x, m}, \qquad (s = 1, 2, \ldots, p);$$

the function is therefore equal to

$$e^{\pi i\beta\alpha - \pi i\alpha(u - \frac{1}{4}\tau\alpha)}\,\Theta\,(u),$$

by equation (C), § 175; thus *the function* $\Theta(v^{x, m} - v^{z_1, m_1'} - \ldots\ldots - v^{z_p, m_p'};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$ *vanishes when x is at either of the places $z_1, \ldots, z_p$.*

We can similarly prove that

$$\Theta\,(v^{x, m} - v^{z_1, m_1'} - \ldots\ldots - v^{z_p, m_p'}) = e^{-\pi i\alpha(u + \frac{1}{2}\beta + \frac{1}{4}\tau\alpha)}\,\Theta\,(-u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha).$$

It has been remarked (§ 175) that there are effectively 2^{2p} theta functions, corresponding to the 2^{2p} sets of values of the integers α, β in which each is either 0 or 1. The present proposition enables us to associate each of the functions with one of the solutions of the equivalence (G). When the function $\Theta\,(v^{x, m};\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$ does not vanish identically in respect to x, its zeros are the places $m_1', \ldots, m_p'$. Therefore, instead of the function $\Theta\,(u)$, we may regard the function $\Theta\,(u;\ \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$ as fundamental, and shall only be led to the places $m_1', \ldots, m_p'$, instead of $m_1, \ldots, m_p$.

(XI.) The sets of places $m_1', \ldots, m_p'$ which are connected with the places $m_1, \ldots, m_p$ by means of the equations

$$v_s^{m_1', m_1} + \ldots\ldots + v_s^{m_p', m_p} \equiv \tfrac{1}{2}\Omega_{\beta, \alpha}, \qquad \text{(H)},$$

wherein $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p$ denote in turn all the 2^{2p} sets of values in which each element is either 0 or 1, may be divided into two categories, according as the integer $\beta\alpha, = \beta_1\alpha_1 + \ldots\ldots + \beta_p\alpha_p$, is even or odd. We have remarked, in Proposition (IX.), that they may be divided into two categories according as they are the zeros, of the second order, of a proper polynomial $\lambda\psi_0 + \Delta\phi$, or consist of the $p - 1$ zeros, each of the second order, of a ϕ-polynomial together with the place m. *When the fundamental Riemann surface is perfectly general these two methods of division of the 2^{2p} sets entirely agree. When $\beta\alpha$ is odd, $m_1', \ldots, m_p'$ consist of the place m and the $p - 1$ zeros, each of the second order, of a ϕ-polynomial. When $\beta\alpha$ is even, $m_1', \ldots, m_p'$*

consist of the zeros, each of the second order, of a proper polynomial ψ. In the latter case we may speak of the places $m_1', \dots, m_p'$ as a set of *tangential derivatives* of the place m.

For by the equations (D), (A), (§ 175), we have

$$e^{\pi i a u}\,\Theta\left(\tfrac{1}{2}\Omega_{\beta,\,a}+u\right)/e^{-\pi i a u}\,\Theta\left(\tfrac{1}{2}\Omega_{\beta,\,a}-u\right)=e^{-\pi i\beta a};$$

hence, when βa is odd, $e^{\pi i a u}\,\Theta\left(\tfrac{1}{2}\Omega_{\beta,\,a}+u\right)$ is an odd function of u, and must vanish when u is zero; since then $\Theta\left(\tfrac{1}{2}\Omega_{\beta,\,a}\right)$ vanishes, there exist, by Proposition (VII.), places $n_1, \dots, n_{p-1}$, such that

$$-\tfrac{1}{2}\Omega_{\beta,\,a}\equiv v^{m_p,\,m}-v^{n_1,\,m_1}-\dots\dots-v^{n_{p-1},\,m_{p-1}}, \qquad \text{(K)},$$

or

$$2\left(v^{n_1,\,m_1}+\dots\dots+v^{n_{p-1},\,m_{p-1}}+v^{m,\,m_p}\right),\ =\Omega_{\beta,\,a},\ \equiv 0.$$

Hence (Chap. VIII. § 158) we have

$$(m^2, n_1^2, \dots, n_{p-1}^2)\equiv(m_1^2, \dots, m_p^2),$$

so that, by equation (G), the places $n_1, \dots, n_{p-1}$ are the zeros of a ϕ-polynomial, each being of the second order.

When βa is even, the function $e^{\pi i a u}\,\Theta\left(\tfrac{1}{2}\Omega_{\beta,\,a}+u\right)$ is an even function, and it is to be expected that it will not vanish for $u=0$. This is generally the case, but exception may arise when the fundamental Riemann surface is of special character. We are thus led to make a distinction between the general case, which, noticing that $\Theta\left(\tfrac{1}{2}\Omega_{\beta,a}+u\right)$ is equal to $e^{-\pi i a(u+\frac{1}{2}\beta-\frac{1}{4}\tau a)}\,\Theta\left(u;\ \tfrac{1}{2}\beta,\ \tfrac{1}{2}a\right)$, may be described as that in which no even theta function vanishes for zero values of the argument, and special cases in which one or more even theta functions do vanish for zero values of the argument.

Suppose then, firstly, that no even theta function vanishes for zero values of the argument. Then if $n_1', \dots, n'_{p-1}$ be places which, repeated, are the zeros of a ϕ-polynomial, we have

$$(m^2, n_1'^2, \dots, n'^2_{p-1})\equiv(m_1^2, m_2^2, \dots, m_p^2);$$

hence the argument

$$v^{m_p,\,m}-v^{n_1',\,m_1}-\dots\dots-v^{n'_{p-1},\,m_{p-1}}$$

is a half-period, $\equiv -\tfrac{1}{2}\Omega_{\beta',\,a'}$, say. Thus, by the result (F), $\Theta\left(\tfrac{1}{2}\Omega_{\beta',a'}\right)$ is zero; therefore, by the hypothesis $\beta' a'$ is an odd integer. So that, in this case, every odd half-period corresponds to a ϕ-polynomial of which all the zeros are of the second order, and conversely.

Further, in this case it is immediately obvious that the places $m_1, \dots, m_p$ do not consist of the place m and the zeros of a ϕ-polynomial whose zeros are of the second order; for if $m_1, \dots, m_p$ were the places $n_1, \dots, n_{p-1}, m$, then, by the result (F), the function $\Theta\left(v^{z_1,\,n_1}+\dots\dots+v^{z_{p-1},\,n_{p-1}}\right)$ would vanish for all positions of $z_1, \dots, z_{p-1}$, and therefore $\Theta(0)$ would vanish.

185. If, however, nextly, there be even theta functions which vanish for zero values of the argument, it does not follow as above that every ϕ-polynomial with double zeros corresponds to an odd half-period; there will still be such ϕ-polynomials corresponding to the $2^{p-1}(2^p-1)$ odd half-periods, but there will also be such ϕ-polynomials corresponding to even half-periods.

For if $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p$ be integers such that $\beta\alpha$ is even, and $\Theta(u+\frac{1}{2}\Omega_{\beta,\alpha})$ vanishes for $u=0$, the first differential coefficients, in regard to $u_1, \ldots, u_p$, of the even function $e^{\pi i\alpha u}\Theta(u+\frac{1}{2}\Omega_{\beta,\alpha})$, being odd functions, will vanish for $u=0$. By an argument which, for convenience, is postponed to Prop. XIV., it follows that then the function $\Theta(v^{x,z}+\frac{1}{2}\Omega_{\beta,\alpha})$ vanishes identically for all positions of x and z. Therefore, by Prop. V., there is at least a single infinity of places $z_1, \ldots, z_{p-1}$ satisfying the equations

$$-\tfrac{1}{2}\Omega_{\beta,\alpha} \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}};$$

these equations are equivalent to

$$(m^2, z_1^2, \ldots, z_{p-1}^2) \equiv (m_1^2, m_2^2, \ldots, m_p^2);$$

hence there is a single infinity of ϕ-polynomials with double zeros corresponding to the even half-period $\frac{1}{2}\Omega_{\beta,\alpha}$, and their $p-1$ zeros form coresidual sets with multiplicity at least equal to 1.

By similar reasoning we can prove another result*; the argument is repeated in the example which follows; *if, for any set of values of the integers $\beta_1, \ldots, \beta_p, \alpha_1, \ldots, \alpha_p$, it is possible to obtain more than one set of places $n_1, \ldots, n_{p-1}$ to satisfy the equations*

$$-\tfrac{1}{2}\Omega_{\beta,\alpha} \equiv v^{m_p, m} - v^{n_1, m_1} - \ldots\ldots - v^{n_{p-1}, m_{p-1}},$$

then it is, of course, possible to obtain an infinite number of such sets. Let ∞^q be the number of sets obtainable. Then $\beta\alpha \equiv q+1$ (mod. 2). And this may be understood to include the general cases when (i) *for an even value of $\beta\alpha$, no solution of the congruence is possible* ($q=-1$), (ii), *for an odd value of $\beta\alpha$, only a single solution is possible* ($q=0$).

As an example of the exceptional case here referred to, consider the hyperelliptic surface; and first suppose $p=3$, the equation associated with the surface being

$$y^2=(x-a_1)\ldots\ldots(x-a_8);$$

then we clearly have $\binom{8}{2}=28=2^{p-1}(2^p-1)$ ϕ-polynomials, each of the form $(x-a_i)(x-a_j)$, of which the zeros are both of the second order. We have, however, also, a ϕ-polynomial, of the form $(x-c)^2$, in which c is arbitrary, of which the zeros are both of the second order; denote these zeros by c and $\bar{c}$; then if $\frac{1}{2}\Omega_{\beta,\alpha}$ be a proper half-period

$$-\tfrac{1}{2}\Omega_{\beta,\alpha} \equiv v^{m_3, m} - v^{c, m_1} - v^{\bar{c}, m_2};$$

* Weber, *Math. Ann.* XIII. p. 42.

but, since, if e be any other place, the function $(x-c)/(x-e)$ is a rational function, it follows that $(c, \bar{c}) \equiv (e, \bar{e})$, and therefore that in the value just written for $\frac{1}{2}\Omega_{\beta, \alpha}$, c may be replaced by e, and therefore, regarded as quite arbitrary. By the result (F), the function $\Theta(u)$ vanishes when u is replaced by $\frac{1}{2}\Omega_{\beta, \alpha}$, and therefore $\Theta(v^{x, z} - \frac{1}{2}\Omega_{\beta, \alpha})$, which is equal to $\Theta(v^{x, m} - v^{c, m_1} - v^{\bar{c}, m_2} - v^{z, m_3})$, vanishes when x is at c; since c is arbitrary the function $\Theta(v^{x, z} - \frac{1}{2}\Omega_{\beta, \alpha})$ vanishes identically in regard to x, for all positions of z. If the function $\Theta(v^{x, z} + v^{x_1, z_1} - \frac{1}{2}\Omega_{\beta, \alpha})$ vanished identically, it would, by Prop. VI., be possible, in the equation

$$-\tfrac{1}{2}\Omega_{\beta, \alpha} \equiv v^{m_3, m} - v^{z_1, m_1} - v^{z_2, m_2}$$

to choose both z_1 and z_2 arbitrarily. As this is not the case, it follows, by Prop. XIV. below, that the function $\Theta(u + \frac{1}{2}\Omega_{\beta, \alpha})$, and its first, but not its second differential coefficients, vanish for $u=0$. Hence $\frac{1}{2}\Omega_{\beta, \alpha}$ is an even half-period. (See the tables for the hyperelliptic case, given in the next chapter, §§ 204, 205.)

There is therefore, in the hyperelliptic case in which $p=3$, one even theta function which vanishes for zero values of the argument.

In any hyperelliptic case in which p is odd, the equation associated with the surface being

$$y^2 = (x-a_1) \ldots\ldots (x - a_{2p+2})$$

ϕ-polynomials with double zeros are given by

(i) the $\binom{2p+2}{p-1}$ polynomials such as $(x-a_1) \ldots\ldots (x-a_{p-1})$. As there is no arbitrary place involved, the q of the theorem enunciated (§ 185) is zero, and the half-period given by the equation

$$-\tfrac{1}{2}\Omega_{\beta, \alpha} \equiv v^{m_p, m} - v^{n_1, m_1} - \ldots\ldots - v^{n_{p-1}, m_{p-1}},$$

where $n_1^2, \ldots, n_{p-1}^2$ are the zeros of the ϕ-polynomial under consideration, is consequently odd.

(ii) the $\binom{2p+2}{p-3}$ polynomials such as $(x-a_1) \ldots\ldots (x-a_{p-3})(x-c)^2$, wherein c is arbitrary. Here $q=1$ and $\beta\alpha \equiv 0$ (mod. 2).

(iii) the $\binom{2p+2}{p-5}$ polynomials such as $(x-a_1) \ldots\ldots (x-a_{p-5})(x-c)^2(x-e)^2$, for which $q=2$, $\beta\alpha \equiv 1$ (mod. 2); and so on. And, finally,

the single polynomial of the form $(x-c_1)^2 \ldots\ldots (x-c_{\frac{p-1}{2}})^2$, in which all of $c_1, \ldots, c_{\frac{p-1}{2}}$ are arbitrary; in this case $q = \frac{p-1}{2}$, $\beta\alpha \equiv \frac{p+1}{2}$ (mod. 2).

On the whole there arise

$$\binom{2p+2}{p-1} + \binom{2p+2}{p-5} + \ldots\ldots + 1, \text{ or } \binom{2p+2}{p-1} + \binom{2p+2}{p-5} + \ldots\ldots + \binom{2p+2}{2}$$

ϕ-polynomials corresponding to odd half-periods, according as $p \equiv 1$ or 3 (mod. 4).

Now in fact, when $p \equiv 1$ (mod. 4)

$$1 + \binom{2p+2}{4} + \ldots\ldots + \binom{2p+2}{p-1}, \ = \tfrac{1}{8}\left[(1+x)^{2p+2} + (1-x)^{2p+2} + (1+ix)^{2p+2} + (1-ix)^{2p+2}\right]_{x=1},$$

is equal to

$$\tfrac{1}{8}\left(2^{2p+2}+2^{p+2}\cos\frac{p+1}{2}\pi\right) \text{ or } 2^{2p-1}-2^{p-1} \text{ or } 2^{p-1}(2^p-1),$$

while, when $p\equiv 3 \pmod{4}$

$$\binom{2p+2}{2}+\binom{2p+2}{6}+\ldots\ldots+\binom{2p+2}{p-1},$$

$$=\tfrac{1}{8}\left[(1+x)^{2p+2}+(1-x)^{2p+2}-(1+ix)^{2p+2}-(1-ix)^{2p+2}\right]_{x=1},$$

is equal to $\tfrac{1}{8}\left(2^{2p+2}-2^{p+2}\cos\dfrac{p+1}{2}\pi\right)$, and therefore, also to $2^{p-1}(2^p-1)$.

Thus all the odd half-periods are accounted for. And there are

$$\binom{2p+2}{p-3}+\binom{2p+2}{p-7}+\ldots\ldots$$

even half-periods which reduce the theta function to zero. This number is equal to

$$-\tfrac{1}{2}\binom{2p+2}{p+1}+\{2^{2p}-2^{p-1}(2^p-1)\},$$

namely to $2^{p-1}(2^p+1)-\dbinom{2p+1}{p}$. This is the number of even theta functions which vanish for zero values of the argument. It is easy to see that the same number is obtained when p is even. For instance when $p=4$, there are 10 even theta functions which vanish for zero values of the argument. They correspond to the 10 ϕ-polynomials of the form $(x-c)^2(x-a_1)$, wherein c is arbitrary, and a_1 is one of the 10 branch places. There are therefore $\dbinom{2p+1}{p}$ even theta functions which do not vanish for zero values of the argument.

In regard to the places $m_1, \ldots, m_p$ in the hyperelliptic case the following remark may conveniently be made here. Suppose the place m taken at the branch place a_{2p+2}; using the geometrical rule given in § 183, we may take for the polynomial Δ, of grade μ, the polynomial $x-a_{2p+2}$, of grade 1; its remaining $n\mu-2$, $=0$, zeros, give no conditions for the polynomial ψ of grade $(n-1)\sigma+n-3+\mu$, $=(2-1)p+2-3+1$, $=p$. Since $\sigma+1$, the dimension of y, is $p+1$, the only possible form for ψ is that of an integral polynomial in x of order p. This is to be chosen so that its $2p$ zeros consist of p repeated zeros. When $p=3$, for example, it must, therefore, be of one of the forms $(x-a_i)(x-a_j)(x-a_k)$, $(x-a_i)(x-c)^2$, where c is arbitrary. It will be seen in the next chapter that the former is the proper form.

186. Another matter* which connects the present theory with a subject afterwards (Chap. XIII.) dealt with may be referred to here. Let $\tfrac{1}{2}\Omega$ be a half-period such that the congruence

$$\tfrac{1}{2}\Omega\equiv v^{m_p,\,m}-v^{z_1,\,m_1}-\ldots\ldots-v^{z_{p-1},\,m_{p-1}}$$

can be satisfied by ∞^q coresidual sets of places $z_1, \ldots, z_{p-1}$ (as in Proposition VI.). Then we have

$$(m^2, z_1^2, \ldots, z^2_{p-1})=(m_1^2, \ldots, m_p^2),$$

so that (Prop. IX.) $z_1, \ldots, z_{p-1}$, each repeated, are the zeros of a ϕ-polynomial; denote this polynomial by ϕ. If $z_1', \ldots, z'_{p-1}$ be another set, which, repeated, are the zeros of a ϕ-polynomial ϕ', and are such that

$$\tfrac{1}{2}\Omega\equiv v^{m_p,\,m}-v^{z_1',\,m_1}-\ldots\ldots-v^{z'_{p-1},\,m_{p-1}},$$

* Cf. Weber, *Math. Annal.* XIII. p. 35; Noether, *Math. Annal.* XVII. 263.

then we have

$$0 \equiv 2v^{m_p, m} - v^{z_1, m_1} - v^{z_1', m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}} - v^{z'_{p-1}, m_{p-1}},$$

so that $z_1, \ldots, z_{p-1}, z_1', \ldots, z'_{p-1}$ are the zeros of a ϕ-polynomial; denote this polynomial by ψ.

The rational functions ψ/ϕ, ϕ'/ψ have the same poles, the places $z_1, \ldots, z_{p-1}$, and the same zeros, the places $z_1', \ldots, z'_{p-1}$. Therefore, absorbing a constant multiplier in ψ, we have

$$\psi^2 = \phi\phi', \text{ and } \phi'/\phi = (\psi/\phi)^2,$$

and thus the function $\sqrt{\phi'/\phi}$ may be regarded as a rational function if a proper sign be always attached. The function has $z_1, \ldots, z_{p-1}$ for poles and $z_1', \ldots, z'_{p-1}$ for zeros. Conversely any rational function having $z_1, \ldots, z_{p-1}$ for poles can be written in this form. For if $z_1'', \ldots, z''_{p-1}$ be the zeros of such a function, we have

$$v^{z_1'', z_1} + \ldots\ldots + v^{z''_{p-1}, z_{p-1}} \equiv 0,$$

and therefore, by the first equation of this §, also

$$\tfrac{1}{2}\Omega \equiv v^{m_p, m} - v^{z_1'', m_1} - \ldots\ldots - v^{z''_{p-1}, m_{p-1}};$$

thus q of the zeros can be taken arbitrarily; and if Φ be any ϕ-polynomial whose zeros $\zeta_1, \ldots, \zeta_{p-1}$ are all of the second order, and such that

$$\tfrac{1}{2}\Omega \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

we can put

$$\sqrt{\frac{\Phi}{\phi}} = \lambda + \lambda_1\sqrt{\frac{\phi_1}{\phi}} + \ldots\ldots + \lambda_q\sqrt{\frac{\phi_q}{\phi}},$$

where $\phi_1, \ldots, \phi_q$ are particular polynomials such as ϕ' or Φ, and $\lambda, \lambda_1, \ldots, \lambda_q$ are constants. In other words, corresponding to the ∞^q sets of solutions of the original equation of this §, we have an equation of the form

$$\sqrt{\Phi} = \lambda\sqrt{\phi} + \lambda_1\sqrt{\phi_1} + \ldots\ldots + \lambda_q\sqrt{\phi_q},$$

wherein proper signs are to be attached to the ratios of any two of the square roots, and any two of the $q+1$ polynomials $\phi, \phi_1, \ldots, \phi_q$, are such that their product is the square of a ϕ-polynomial. There are therefore $\frac{1}{2}q(q+1)$ linearly independent quadratic relations connecting the ϕ-polynomials. (Cf. Chap. VI. §§ 110—112.)

For example in the hyperelliptic case in which $p=3$, the vanishing of an even theta function corresponds to the existence of a ϕ-polynomial $\Phi = (x-c)^2$, such that

$$\sqrt{\Phi} = -c\sqrt{1} + \sqrt{x^2}, \ = -c\sqrt{\phi_1} + \sqrt{\phi_3},$$

where $\phi_1\phi_3, = (x)^2, = \phi_2^2$.

Ex. i. Prove, for $p=3$, that if an even theta function vanishes for zero values of the arguments the surface is necessarily hyperelliptic.

Ex. ii. Prove, for $p=4$, that if two even theta functions vanish for zero values of the arguments the surface is necessarily hyperelliptic; so that, then, eight other even theta functions also vanish for zero values of the arguments. The number, 2, of conditions thus necessary for the fundamental constants of the surface, in order that it be hyperelliptic, is the same as the difference, $9-7$, between the number, $3p-3$, of constants in the general surface of deficiency 4, and the number, $2p-1$, of constants in the general hyperelliptic surface of deficiency 4.

187. (XII.) If r denote any arguments such that $\Theta(r)=0$, and such that $\Theta(v^{x,z}+r)$ does not vanish identically for all positions of x and z, the Riemann normal integral of the third kind can be expressed in the form

$$\Pi_{a,\beta}^{x,z}=\log\left[\frac{\Theta(v^{x,a}+r)}{\Theta(v^{x,\beta}+r)}\Big/\frac{\Theta(v^{z,a}+r)}{\Theta(v^{z,\beta}+r)}\right].$$

For consider the function of x given by

$$e^{-\Pi_{a,\beta}^{x,z}}\frac{\Theta(v^{x,a}+r)\,\Theta(v^{z,\beta}+r)}{\Theta(v^{x,\beta}+r)\,\Theta(v^{z,a}+r)};$$

(α) it is single-valued on the Riemann surface dissected by the a and b period loops;

(β) it does not vanish or become infinite, for the zeros of $\Theta(v^{x,z}+r)$, other than z, do not depend upon z (by Proposition IV.);

(γ) it is unaffected by a circuit of any one of the period loops. At a loop a_i it has clearly (Equation B, § 175) the factor unity; at a loop b_i it has the factor

$$e^{-2\pi i v_i^{a,\beta}}\cdot e^{-2\pi i(v_i^{x,a}+r_i+\frac{1}{2}\tau_{i,i})}\cdot e^{2\pi i(v_i^{x,\beta}+r_i+\frac{1}{2}\tau_{i,i})},$$

which is also unity. Thus the function is single-valued on the undissected surface;

(δ) thus the function is independent of x; and hence equal to the value it has when the place x is at z, namely 1.

A particular case is obtained by taking

$$r=v^{m_p,m}-v^{z_1,m_1}-\ldots\ldots-v^{z_{p-1},m_{p-1}},$$

where $z_1, \ldots, z_{p-1}$ are any places such that $\Theta(v^{x,z}+r)$ does not vanish identically. Then by the result (F) the function $\Theta(r)$ vanishes.

Hence we have

$$\Pi_{a,\beta}^{x,z}=\log\left[\frac{\Theta(v^{x,m}-v^{z_1,m_1}-\ldots\ldots-v^{z_{p-1},m_{p-1}}-v^{a,m_p})}{\Theta(v^{x,m}-v^{z_1,m_1}-\ldots\ldots-v^{z_{p-1},m_{p-1}}-v^{\beta,m_p})}\right.$$
$$\left.\Big/\frac{\Theta(v^{z,m}-v^{z_1,m_1}-\ldots\ldots-v^{z_{p-1},m_{p-1}}-v^{a,m_p})}{\Theta(v^{z,m}-v^{z_1,m_1}-\ldots\ldots-v^{z_{p-1},m_{p-1}}-v^{\beta,m_p})}\right].$$

Another particular case, of great importance, is obtained by taking $r=\frac{1}{2}\Omega_{k,k'}$, k, k' denoting respectively p integers $k_1, \ldots, k_p$, $k_1', \ldots, k_p'$, such that kk' is odd, the assumption being made that the equations

$$\tfrac{1}{2}\Omega_{k,k'}\equiv v^{m_p,m}-v^{\zeta_1,m_1}-\ldots\ldots-v^{\zeta_{p-1},m_{p-1}}$$

are not satisfied by more than one set of places $\zeta_1, \ldots, \zeta_{p-1}$ (cf. Props. III., V.). Then the function $\Theta(v^{x,z} + \tfrac{1}{2}\Omega_{k,k'})$ does not vanish identically, and we have

$$\Pi_{\alpha,\beta}^{x,z} = \log \frac{\Theta(v^{x,\alpha} + \tfrac{1}{2}\Omega_{k,k'})\,\Theta(v^{z,\beta} + \tfrac{1}{2}\Omega_{k,k'})}{\Theta(v^{x,\beta} + \tfrac{1}{2}\Omega_{k,k'})\,\Theta(v^{z,\alpha} + \tfrac{1}{2}\Omega_{k,k'})}.$$

(XIII.) Suppose k equal to or less than p; consider the function given by the product of

$$e^{-\Pi_{\alpha_1,\beta_1}^{x,z} - \Pi_{\alpha_2,\beta_2}^{x,z} - \ldots\ldots - \Pi_{\alpha_k,\beta_k}^{x,z}}$$

and

$$\frac{\Theta(v^{x,m} - v^{\alpha_1,m_1} - \ldots\ldots - v^{\alpha_k,m_k} + r)}{\Theta(v^{x,m} - v^{\beta_1,m_1} - \ldots\ldots - v^{\beta_k,m_k} + r)} \Bigg/ \frac{\Theta(v^{z,m} - v^{\alpha_1,m_1} - \ldots\ldots - v^{\alpha_k,m_k} + r)}{\Theta(v^{z,m} - v^{\beta_1,m_1} - \ldots\ldots - v^{\beta_k,m_k} + r)},$$

wherein r denotes arguments given by

$$r = -(v^{\gamma_{k+1},m_{k+1}} + \ldots\ldots + v^{\gamma_p,m_p}),$$

and each of the sets $\alpha_1, \ldots, \alpha_k, \gamma_{k+1}, \ldots, \gamma_p, \beta_1, \ldots, \beta_k, \gamma_{k+1}, \ldots, \gamma_p$ is such that the functions involved do not vanish identically in regard to x.

This function is single-valued on the dissected Riemann surface, does not become infinite or zero, and, for example, at the period loop b_i it has the factor e^L, where

$$L, = -2\pi i(v^{\alpha_1,\beta_1} + \ldots\ldots + v^{\alpha_k,\beta_k}) - 2\pi i(v^{x,m} - v^{\alpha_1,m_1} - \ldots\ldots - v^{\alpha_k,m_k})$$
$$+ 2\pi i(v^{\alpha,m} - v^{\beta_1,m_1} - \ldots\ldots - v^{\beta_k,m_k}),$$

is zero. Thus the function has the constant value, unity, which it has when x is at z. Therefore

$$\Pi_{\alpha_1,\beta_1}^{x,z} + \ldots + \Pi_{\alpha_k,\beta_k}^{x,z} = \log\left[\frac{\Theta(v^{x,m} - v^{\alpha_1,m_1} - \ldots - v^{\alpha_k,m_k} - v^{\gamma_{k+1},m_{k+1}} - \ldots - v^{\gamma_p,m_p})}{\Theta(v^{x,m} - v^{\beta_1,m_1} - \ldots - v^{\beta_k,m_k} - v^{\gamma_{k+1},m_{k+1}} - \ldots - v^{\gamma_p,m_p})}\right.$$
$$\left.\Bigg/ \frac{\Theta(v^{z,m} - v^{\alpha_1,m_1} - \ldots\ldots - v^{\alpha_k,m_k} - v^{\gamma_{k+1},m_{k+1}} - \ldots\ldots - v^{\gamma_p,m_p})}{\Theta(v^{z,m} - v^{\beta_1,m_1} - \ldots\ldots - v^{\beta_k,m_k} - v^{\gamma_{k+1},m_{k+1}} - \ldots\ldots - v^{\gamma_p,m_p})}\right],$$

the places $\gamma_{k+1}, \ldots, \gamma_p$ being arbitrarily chosen so that $\alpha_1, \ldots, \alpha_k, \gamma_{k+1}, \ldots, \gamma_p$ are not zeros of a ϕ-polynomial, and $\beta_1, \ldots, \beta_k, \gamma_{k+1}, \ldots, \gamma_p$ are not zeros of a ϕ-polynomial.

Thus, when $k = p$, we have the expression of the function considered in § 171, Chap. IX. in terms of theta functions. For the case where $\alpha_1, \ldots, \alpha_k$ are the zeros of a ϕ-polynomial, cf. Prop. XV. Cor. iii.

188. (XIV.) We return now to the consideration of the identical vanishing of the Θ function. We have proved (Prop. VII.), that if $\Theta(v^{x_1,z_1} + \ldots\ldots + v^{x_q,z_q} + r)$ be identically zero for all positions of $x_1, \ldots, x_q, z_1, \ldots, z_q$, but $\Theta(v^{x,z} + v^{x_1,z_1} + \ldots\ldots + v^{x_q,z_q} + r)$ be not identically zero for all positions of

x and z, then there exist ∞^q sets of places $\zeta_1, \ldots, \zeta_{p-1}$, and ∞^q sets of places $\xi_1, \ldots, \xi_{p-1}$, such that

$$r = v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

and

$$-r = v^{m_p, m} - v^{\xi_1, m_1} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}}.$$

Now, if in the equation $\Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r) = 0$, we make x_q approach to and coincide with z_q, we obtain

$$\sum_{i=1}^{p} \Theta_i'(v^{x_1, z_1} + \ldots\ldots + v^{x_{q-1}, z_{q-1}} + r)\, \Omega_i(z_q) = 0,$$

wherein $\Theta_i'(u)$ is put for $\dfrac{\partial}{\partial u_i}\Theta(u)$, $\Omega_i(x)$ for $2\pi i\, D_x v_i^{x, a}$, a being arbitrary; and this equation holds for all positions of $x_1, z_1, \ldots, x_{q-1}, z_{q-1}$. Since, however, the quantities $\Omega_1(z_q), \ldots, \Omega_q(z_q)$ cannot be connected by any linear equation whose coefficients are independent of z_q, we can thence infer that the first differential coefficients of $\Theta(u)$ vanish identically when u is of the form $v^{x_1, z_1} + \ldots\ldots + v^{x_{q-1}, z_{q-1}} + r$. It follows then in the same way that the second differential coefficients of $\Theta(u)$ vanish identically when u has the form $v^{x_1, z_1} + \ldots\ldots + v^{x_{q-2}, z_{q-2}} + r$; in particular all the first and second differential coefficients vanish when $u = r$. Proceeding thus we finally infer that $\Theta(u)$ and all its differential coefficients up to and including those of the qth order vanish when $u = r$.

We proceed now to shew conversely that when $\Theta(u)$ and all its differential coefficients up to and including those of the qth order, vanish for $u = r$, then $\Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ vanishes identically for all positions of $x_1, z_1, x_2, z_2, \ldots, x_q, z_q$. By what has just been shewn $\Theta(v^{x, z} + v^{x_1, z_1} + \ldots\ldots + v^{x_q, z_q} + r)$ will not vanish identically unless the differential coefficients of the $(q+1)$th order also vanish.

We begin with the case $q = 1$. Suppose that $\Theta(u)$, $\Theta_1'(u), \ldots, \Theta_p'(u)$, all vanish for $u = r$; we are to prove that $\Theta(v^{x, z} + r)$ vanishes identically for all positions of x and z.

Let e, f be such arguments that $\Theta(e) = 0$, $\Theta(f) = 0$, but such that $\Theta_i'(e)$ are not all zero and $\Theta_i'(f)$ are not all zero, and therefore $\Theta(v^{x, z} + e)$, $\Theta(v^{x, z} + f)$ do not vanish identically; consider the function

$$\frac{\Theta(e + v^{x, z})\,\Theta(e - v^{x, z})}{\Theta(f + v^{x, z})\,\Theta(f - v^{x, z})};$$

firstly, it is rational in x and z; for, considered as a function of x, it has, at the period loop b_r, (Equation B, § 175) the factor

$$e^{-2\pi i\,(v_r^{x, z} + e + v_r^{x, z} - e) - \pi i \tau_{r, r}} \Big/ e^{-2\pi i\,(v_r^{x, z} + f + v_r^{x, z} - f) - \pi i \tau_{r, r}}$$

whose value is unity; and a similar statement holds when the expression is considered as a function of z, for the expression is immediately seen to be symmetrical in x and z; secondly, regarded as a function of x, the expression has $2(p-1)$ zeros, and the same number of poles, and these (Prop. IV.) are independent of z. Similarly as a function of z it has $2(p-1)$ zeros and poles, independent of x; therefore the expression can be written in the form $F(x)F(z)$, where $F(x)$ denotes the definite rational function having the proper zeros and poles, multiplied by a suitable constant factor, and $F(z)$ is the same rational function of z.

Putting, then, x to coincide with z, and extracting a square root, we infer

$$F(x) = \pm \frac{\sum_{i=1}^{p} \Theta_i'(e)\,\Omega_i(x)}{\sum_{i=1}^{p} \Theta_i'(f)\,\Omega_i(x)},$$

where $\Omega_i(x) = 2\pi i\, D_x v_i^{x,\,a}$, for a arbitrary, is the differential coefficient of an integral of the first kind; thence we have

$$\frac{\Theta(v^{x,\,z}+e)\,\Theta(v^{x,\,z}-e)}{\Theta(v^{x,\,z}+f)\,\Theta(v^{x,\,z}-f)} = \frac{[\Sigma\Theta_i'(e)\,\Omega_i(x)]\,[\Sigma\Theta_i'(e)\,\Omega_i(z)]}{[\Sigma\Theta_i'(f)\,\Omega_i(x)]\,[\Sigma\Theta_i'(f)\,\Omega_i(z)]}.$$

In this equation suppose that e approaches indefinitely near to r, for which $\Theta(r)=0$, $\Theta_i'(r)=0$. Then the right hand becomes infinitesimal, independently of x and z. Therefore also the left hand becomes infinitesimal independently of x and z; and hence $\Theta(v^{x,\,z}+r)$ vanishes identically, for all positions of x and z.

We have thus proved the case of our general theorem in which $q=1$. The theorem is to be inferred for higher values of q by proving that if the function $\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_{m-1},\,z_{m-1}}+r)$ vanish identically for all positions of $x_1, z_1, \ldots, x_{m-1}, z_{m-1}$, and also the differential coefficients of $\Theta(u)$, of order m, vanish for $u=r$, then the function $\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_m,\,z_m}+r)$ vanishes identically. For instance if this were proved, it would follow, putting $m=2$, from what we have just proved, that also $\Theta(v^{x_1,\,z_1}+v^{x_2,\,z_2}+r)$ vanished identically, and so on.

As before let f be such that $\Theta(f)=0$, but all of $\Theta_i'(f)$ are not zero; so that $\Theta(v^{x,\,z}+f)$ does not vanish identically in regard to x and z. Let e be such that $\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_{m-1},\,z_{m-1}}+e)$ vanishes identically for all positions of $x_1, z_1, \ldots, x_{m-1}, z_{m-1}$, but such that the differential coefficients of $\Theta(u)$ of the first order do not vanish identically for $u=v^{x_1,\,z_1}+\ldots+v^{x_{m-1},\,z_{m-1}}+e$; so that the function $\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_m,\,z_m}+e)$ does not vanish identically. Consider the product of the expressions

$$\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_m,\,z_m}+e)\,\Theta(v^{x_1,\,z_1}+\ldots\ldots+v^{x_m,\,z_m}-e)$$

$$\frac{\Pi'\Theta(v^{x_h,\,x_k}+f)\,\Theta(v^{x_h,\,x_k}-f)\,\Pi'\Theta(v^{z_h,\,z_k}+f)\,\Theta(v^{z_h,\,z_k}-f)}{\Pi\Pi\Theta(v^{x_\lambda,\,z_\mu}+f)\,\Theta(v^{x_\lambda,\,z_\mu}-f)},$$

wherein h, k in the numerator denote in turn every pair of the numbers $1, 2, \ldots, m$, so that the numerator contains $4 \cdot \frac{1}{2}m(m-1)+2 = 2(m^2-m+1)$ theta functions, and λ, μ in the denominator are each to take all the values $1, 2, \ldots, m$, so that there are $2m^2$ theta functions in the denominator.

Firstly, this product is a rational function of each of the $2m$ places $x_1, z_1, \ldots, x_m, z_m$. Consider for instance x_1; it is clear that if the product be rational in x_1, it will be entirely rational. As a function of x_1, the product has at the period loop b_r a factor $e^{-2\pi i K}$ where

$$K = 2\left(v_r^{x_1, z_1} + \ldots\ldots + v_r^{x_m, z_m} + \tfrac{1}{2}\tau_{r,r}\right) + 2\sum_{k=2}^{m}\left(v_r^{x_1, x_k} + \tfrac{1}{2}\tau_{r,r}\right) - 2\sum_{\mu=1}^{m}\left(v_r^{x_1, z_\mu} + \tfrac{1}{2}\tau_{r,r}\right),$$

and this expression is identically zero.

Secondly, considering the product as a rational function of x_1, the denominator is zero to the second order when x_1 coincides with any one of the m places $z_1, \ldots, z_m$, and is otherwise zero at $2m(p-1)$ places depending on f only; of these latter places $2(m-1)(p-1)$ are also zeros of the factors $\Pi'\Theta(v^{x_h, x_k}+f)\Theta(v^{x_h, x_k}-f)$; there are then $2(p-1)$ poles of the function which depend on f only. The factors $\Pi'\Theta(v^{x_h, x_k}+f)\Theta(v^{x_h, x_k}-f)$ have also the zeros $x_2, \ldots, x_m$, each of the second order. The factors $\Theta(v^{x_1, z_1}+\ldots+v^{x_m, z_m}+e)\Theta(v^{x_1, z_1}+\ldots+v^{x_m, z_m}-e)$ have, by the hypothesis as to e, the zeros $z_1, z_2, \ldots, z_m$, each of the second order, as well as $2(p-m)$ other zeros depending on e only. On the whole then, regarded as a function of x_1, the product has

for zeros, $2(p-m)$ zeros depending on e, as well as the zeros $x_2, \ldots, x_m$, each of the second order,

for poles, $2(p-1)$ poles depending on f;

the function is thus of order $2(p-1)$; and it is determined, save for a factor independent of x_1, by the assignation of its zeros and poles. It is to be noticed that these do not depend on $z_1, z_2, \ldots, z_m$.

It is easy now to see that the product, regarded as a function of z_1, depends on $z_2, \ldots, z_m, e, f$ in just the same way as, regarded as a function of x_1, it depends on $x_2, \ldots, x_m, e, f$.

The expression is therefore of the form $F(x_1, x_2, \ldots, x_m)F(z_1, z_2, \ldots, z_m)$, wherein F denotes a rational function of all the variables involved.

The form of F can be determined by supposing $x_1, \ldots, x_m$ to approach indefinitely near to $z_1, \ldots, z_m$ respectively; then we obtain

$$\Theta(v^{x_1, z_1}+\ldots+v^{x_m, z_m}+e) = \frac{1}{2\pi i}t_m \sum_{i=1}^{p}\Theta_i'(v^{x_1, z_1}+\ldots+v^{x_{m-1}, z_{m-1}}+e)\,\Omega_i(z_m),$$

where t_m is the infinitesimal for the neighbourhood of the place z_m,

$$\Theta_i'(v^{x_1, z_1}+\ldots\ldots+v^{x_{m-1}, z_{m-1}}+e)$$
$$= \frac{1}{2\pi i}t_{m-1}\sum_{j=1}^{p}\Theta'_{i,j}(v^{x_1, z_1}+\ldots\ldots+v^{x_{m-1}, z_{m-1}}+e)\,\Omega_j(z_{m-1}),$$

where t_{m-1} is the infinitesimal for the neighbourhood of the place z_{m-1}, and so on, and eventually,

$$\Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_m, z_m} + e)$$
$$= \frac{t_1 t_2 \ldots t_m}{(2\pi i)^m} \sum_{i_m=1}^{p} \ldots \sum_{i_1=1}^{p} \Theta'_{i_1, i_2, \ldots i_m}(e)\, \Omega_{i_1}(z_1)\, \Omega_{i_2}(z_2) \ldots \Omega_{i_m}(z_m).$$

Similarly

$$\Pi\Pi\Theta(v^{x_\lambda, z_\mu} + f) = \Pi'\Theta(v^{z_h, z_k} + f)\, \Theta(v^{z_h, z_k} - f) \left[\frac{t_1 t_2 \ldots t_m}{(2\pi i)^m} \prod_{\mu=1}^{m} \sum_{i=1}^{p} \Theta_i'(f)\, \Omega_i(z_\mu)\right],$$

where h, k refers to all pairs of different numbers from among 1, 2, ..., m.

Therefore, dividing by a factor

$$(-)^m\, \Pi'\Theta^2(v^{z_h, z_k} + f)\, \Theta^2(v^{z_h, z_k} - f) \left[\frac{t_1 \ldots t_m}{(2\pi i)^m}\right]^2,$$

which is common to numerator and denominator, and taking the square root, we have

$$F(z_1, \ldots, z_m) = \frac{\sum\limits_{i_m=1}^{p} \ldots \sum\limits_{i_1=1}^{p} \Theta'_{i_1, i_2, \ldots, i_m}(e)\, \Omega_1(z_1)\, \Omega_2(z_2) \ldots \Omega_m(z_m)}{\prod\limits_{\mu=1}^{m} \left[\sum\limits_{i=1}^{p} \Theta_i'(f)\, \Omega_i(z_\mu)\right]}.$$

On the whole therefore we have the equation

$$\Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_m, z_m} + e)\, \Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_m, z_m} - e)$$
$$\cdot \frac{\Pi'\Theta(v^{x_h, x_k} + f)\, \Theta(v^{x_h, x_k} - f)\, \Pi'\Theta(v^{z_h, z_k} + f)\, \Theta(v^{z_h, z_k} - f)}{\Pi\Pi\Theta(v^{x_\lambda, z_\mu} + f)\, \Theta(v^{x_\lambda, z_\mu} - f)}$$
$$= \frac{\Psi(x_1, \ldots, x_m, e)\, \Psi(z_1, \ldots, z_m, e)}{\prod\limits_{1}^{p} \Phi(x_\mu, f) \prod\limits_{1}^{p} \Phi(z_\mu, f)},$$

where

$$\Phi(x, f) = \sum_{i=1}^{p} \Theta_i'(f)\, \Omega_i(x),$$

$$\Psi(x_1, \ldots, x_m, e) = \sum_{i_m=1}^{p} \ldots \sum_{i_1=1}^{p} \Theta'_{i_1, i_2, \ldots, i_m}(e)\, \Omega_{i_1}(x_1) \ldots \Omega_{i_m}(x_m).$$

Suppose now that e_i is made to approach to r_i; then the conditions we have imposed for e are satisfied, and there is added the further condition that the differential coefficients of order m, $\Theta'_{i_1, i_2, \ldots, i_m}$, also vanish. Hence it follows that $\Theta(v^{x_1, z_1} + \ldots\ldots + v^{x_m, z_m} + r)$ vanishes identically.

The whole theorem enunciated is thus demonstrated.

(XV.) The remarkable investigation of Prop. XIV. is due to Riemann; it is worth while to give a separate statement of one of the results obtained. Using q instead of $m-1$, we have proved that if the equations

$$e \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}}$$

are satisfied by ∞^q sets of places $\zeta_1, \ldots, \zeta_{p-1}$, so that also the equations

$$-e \equiv v^{m_p, m} - v^{\xi_1, m_1} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}}$$

are satisfied by ∞^q sets of places $\xi_1, \ldots, \xi_{p-1}$, then their exists a rational function, which has (i) for *poles*, the $2(p-1)$ places $t_1, \ldots, t_{p-1}, z_1, \ldots, z_{p-1}$, which satisfy the equations

$$f \equiv v^{m_p, m} - v^{t_1, m_1} - \ldots\ldots - v^{t_{p-1}, m_{p-1}}$$

$$-f \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}},$$

f being supposed such that these equations have one and only one set of solutions, and has (ii) for *zeros*, the arbitrary places $x_1, \ldots, x_q$, each of the second order, together with $2(p-1-q)$ places $\zeta_{q+1}, \ldots, \zeta_{p-1}, \xi_{q+1}, \ldots, \xi_{p-1}$, satisfying the equations

$$e \equiv v^{m_p, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_q, m_q} - v^{\zeta_{q+1}, m_{q+1}} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

$$-e \equiv v^{m_p, m} - v^{x_1, m_1} - \ldots\ldots - v^{x_q, m_q} - v^{\xi_{q+1}, m_{q+1}} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}},$$

and the function can be given in the form

$$\Psi(x_1, x_2, \ldots, x_q, x, e) \div \Phi(x, f),$$

the notation being that employed at the conclusion of Proposition (XIV.). The expressions Ψ, Φ occurring here have the zeros of certain ϕ-polynomials, to which they are proportional.

Corollary i. If we take $p-1$ places $\zeta_1, \ldots, \zeta_{p-1}$, so situated that only one ϕ-polynomial vanishes in all of them, and define e by the equations

$$e \equiv v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}},$$

there will be no other set $\zeta_1, \ldots, \zeta_{p-1}$, satisfying these equations, or $q = 0$. If $\xi_1, \ldots, \xi_{p-1}$ be the remaining zeros of the ϕ-polynomial which vanishes in $\zeta_1, \ldots, \zeta_{p-1}$, we have (Prop. IX.)

$$(m^2, \zeta_1, \ldots, \zeta_{p-1}, \xi_1, \ldots, \xi_{p-1}) \equiv (m_1^2, \ldots, m_p^2),$$

and therefore

$$-e \equiv v^{m_p, m} - v^{\xi_1, m_1} - \ldots\ldots - v^{\xi_{p-1}, m_{p-1}}.$$

Similarly if $t_1, \ldots, t_{p-1}$ be arbitrary places which are the zeros of only one ϕ-polynomial, we can put

$$f \equiv v^{m_p, m} - v^{t_1, m_1} - \ldots\ldots - v^{t_{p-1}, m_{p-1}}$$

$$-f \equiv v^{m_p, m} - v^{z_1, m_1} - \ldots\ldots - v^{z_{p-1}, m_{p-1}}.$$

Then the rational function having $t_1, \ldots, t_{p-1}, z_1, \ldots, z_{p-1}$ for poles, and $\zeta_1, \ldots, \zeta_{p-1}, \xi_1, \ldots, \xi_{p-1}$ for zeros is given by $\Phi(x, e) \div \Phi(x, f)$. Thus the ϕ-polynomial which vanishes in $\zeta_1, \ldots, \zeta_{p-1}, \xi_1, \ldots, \xi_{p-1}$ is given by

$$\sum_{i=1}^{p} \Theta_i'\left(v^{m_p, m} - v^{\zeta_1, m_1} - \ldots\ldots - v^{\zeta_{p-1}, m_{p-1}}\right) \phi_i(x),$$

where $\phi_1(x), \ldots, \phi_p(x)$ are the ϕ-polynomials occurring in the differential coefficients of Riemann's normal integrals of the first kind.

Hence if $n_1, \ldots, n_{p-1}$ be places which, repeated, are all the zeros of a ϕ-polynomial, the form of this polynomial is known. Since, then, we have (Prop. XI. p. 269)

$$\tfrac{1}{2}\Omega \equiv v^{m_p, m} - v^{n_1, m_1} - \ldots\ldots - v^{n_{p-1}, m_{p-1}},$$

we can write this polynomial

$$\sum_{i=1}^{p} \Theta_i'(\tfrac{1}{2}\Omega)\, \phi_i(x),$$

$\tfrac{1}{2}\Omega$ being an odd half-period.

If another ϕ-polynomial than this one vanished in $n_1, \ldots, n_{p-1}$, there would be other places $n_1', \ldots, n'_{p-1}$, such that

$$\tfrac{1}{2}\Omega \equiv v^{m_p, m} - v^{n_1', m_1} - \ldots\ldots - v^{n'_{p-1}, m_{p-1}},$$

and therefore (Prop. VI.) the function $\Theta(v^{x, z} + \tfrac{1}{2}\Omega)$ would vanish identically; in that case (Prop. XIV. p. 276) the coefficients $\Theta_i'(\tfrac{1}{2}\Omega)$ would vanish.

We can express the ϕ-polynomial in terms of any integrals of the first kind; if $V_1^{x, m}, \ldots, V_p^{x, m}$ be any linearly independent integrals of the first kind, expressible in terms of the Riemann normal integrals $v_1^{x, m}, \ldots, v_p^{x, m}$ by linear equations of the form

$$v_i^{x, m} = \lambda_{i, 1} V_1^{x, m} + \ldots\ldots + \lambda_{i, p} V_p^{x, m}, \qquad (i = 1, 2, \ldots, p),$$

and the function $\Theta(u)$ be regarded as a function of $U_1, \ldots, U_p$ given by

$$u_i = \lambda_{i, 1} U_1 + \ldots\ldots + \lambda_{i, p} U_p, \qquad (i = 1, 2, \ldots, p),$$

and, so regarded, be written $\vartheta(U)$, the ϕ-polynomial which has zeros of the second order at $n_1, \ldots, n_{p-1}$ can be written

$$\sum_{i=1}^{p} \vartheta_i'(\tfrac{1}{2}\bar{\Omega})\, \psi_i(x),$$

where $\psi_1(x), \ldots, \psi_p(x)$ are the ϕ-polynomials corresponding to $V_1^{x, m}, \ldots, V_p^{x, m}$, and $\tfrac{1}{2}\bar{\Omega}$ denotes a set of simultaneous half-periods of the integrals $V_1^{x, m}, \ldots, V_p^{x, m}$. If $\tfrac{1}{2}\Omega$ stand for p quantities of which a general one is

$$\tfrac{1}{2}(k_i + k_1'\tau_{i, 1} + \ldots\ldots + k_p'\tau_{i, p}), \qquad (i = 1, 2, \ldots, p),$$

and $\omega_{r,s}$, $\omega'_{r,s}$ be $2p^2$ quantities given by

$$\left.\begin{matrix}1\\0\end{matrix}\right\} = 2\lambda_{i,1}\omega_{1,s} + 2\lambda_{i,2}\omega_{2,s} + \dots\dots + 2\lambda_{i,p}\omega_{p,s}, \quad (i, s = 1, 2, \dots, p),$$

$$\tau_{i,s} = 2\lambda_{i,1}\omega'_{1,s} + 2\lambda_{i,2}\omega'_{2,s} + \dots\dots + 2\lambda_{i,p}\omega'_{p,s},$$

where, in the first equation, we are to take 1 or 0 according as $i = s$ or $i \neq s$, then $\frac{1}{2}\overline{\Omega}$ will stand for p quantities of which one is

$$k_1\omega_{i,1} + \dots\dots + k_p\omega_{i,p} + k_1'\omega'_{i,1} + \dots\dots + k_p'\omega'_{i,p}, \quad (i = 1, 2, \dots, p).$$

For example when the fundamental Riemann surface is that whose equation may be interpreted as the equation of a plane quartic curve, every double tangent is associated with an odd half-period and its equation may be put into the form

$$x\vartheta_1'(\tfrac{1}{2}\overline{\Omega}) + y\vartheta_2'(\tfrac{1}{2}\overline{\Omega}) + \vartheta_3'(\tfrac{1}{2}\Omega) = 0.$$

Corollary ii. If the equations

$$e \equiv v^{m_p, m} - v^{x_1, m_1} - v^{\zeta_2, m_2} - \dots\dots - v^{\zeta_{p-1}, m_{p-1}}$$

can be satisfied with an arbitrary position of x_1 and suitable positions of $\zeta_2, \dots, \zeta_{p-1}$, and therefore, also, the equations

$$-e \equiv v^{m_p, m} - v^{x_1, m_1} - v^{\xi_2, m_2} - \dots\dots - v^{\xi_{p-1}, m_{p-1}}$$

can be satisfied, then a ϕ-polynomial vanishing at x_1 to the second order, and otherwise vanishing in $\zeta_2, \dots, \zeta_{p-1}, \xi_2, \dots, \xi_{p-1}$, is given by

$$\sum_{i=1}^{p} \Omega_i(x) \sum_{j=1}^{p} \Theta'_{i,j}(e)\,\Omega_j(x_1) = 0.$$

Ex. In the case of a plane quintic curve having two double points, this gives us the equation of the straight lines joining these double points to an arbitrary point x_1, of the curve.

Corollary iii. We have seen (Chap. VI. § 98) that any rational function of which the multiplicity (q) is greater than the excess of the order of the function over the deficiency of the surface, say, $q = Q - p + \tau + 1$, can be expressed as the quotient of two ϕ-polynomials. If the function have $\zeta_1, \dots, \zeta_Q$ for zeros, and $\xi_1, \dots, \xi_Q$ for poles, and the common zeros of the ϕ-polynomials expressing the function be $z_1, \dots, z_R$, where $R = 2p - 2 - Q$, the function is in fact expressed by

$$\sum_{i=1}^{p} \Theta_i'(e)\,\Omega_i(x) \div \sum_{i=1}^{p} \Theta_i'(f)\,\Omega_i(x),$$

where (cf. § 93, Chap. VI.)

$$e \equiv v^{m_p, m} - v^{z_1, m_1} - \dots\dots - v^{z_{R-\tau}, m_{R-\tau}} - v^{\zeta_1, m_{R-\tau+1}} - \dots\dots - v^{\zeta_q, m_{p-1}},$$

$$f \equiv v^{m_p, m} - v^{z_1, m_1} - \dots\dots - v^{z_{R-\tau}, m_{R-\tau}} - v^{\xi_1, m_{R-\tau+1}} - \dots\dots - v^{\xi_q, m_{p-1}}.$$

189. Before concluding this chapter it is convenient to introduce a slightly more general function * than that so far considered; we denote by $\vartheta(u\,;\,q,q')$, or by $\vartheta(u,q)$, the function

$$\vartheta(u\,;\,q,q')=\Sigma e^{au^2+2hu(n+q')+b(n+q')^2+2i\pi q(n+q')},$$

wherein the summation extends to all positive and negative integer values of the p integers $n_1, \ldots, n_p$, a is any symmetrical matrix whatever of p rows and columns, h is any matrix whatever of p rows and columns, in general not symmetrical, b is any symmetrical matrix whatever of p rows and columns, such that the real part of the quadratic form bm^2 is necessarily negative for all real values of the quantities $m_1, \ldots, m_p$, other than zero, and q, q' denote two sets, each of p constant quantities, which constitute the characteristic of the function. In the most general case the matrix b depends on $\frac{1}{2}p(p+1)$ independent constants; if however we put $i\pi\tau$ for b, τ being the symmetrical matrix hitherto used, depending only on $3p-3$ constants, and denote the p quantities hu by U, we shall obtain

$$\vartheta(u\,;\,q,q')=e^{au^2}\Theta(U\,;\,q,q').$$

We make consistent use of the notation of matrices (see Appendix ii.). If u denote a row (or column) letter of p elements, and h denote any matrix of p rows and columns, then hu is a row letter; we shall generally write huv for $hu\,.\,v$; and we have $huv=\bar{h}vu$, where $\bar{h}$ is the matrix obtained from h by transposition of rows and columns. Further if k be any matrix of p rows and columns, $hu\,.\,kv=\bar{h}kvu=khuv$. For the present every matrix denoted by a single letter is a square matrix of p rows and columns.

Now let $\omega, \omega', \eta, \eta'$ be any such matrices, and P, P' be row letters of elements $P_1, \ldots, P_p, P_1', \ldots, P_p'$. Then, by the sum of the two row letters $\omega P+\omega' P'$ we denote a row letter consisting of p elements, each being the sum of an element of ωP with the corresponding element of $\omega' P'$. This row letter, with every element multiplied by 2, will be denoted by Ω_P, so that

$$\Omega_P=2\omega P+2\omega' P';$$

in a similar way we define a row letter of p elements by the equation

$$H_P=2\eta P+2\eta' P';$$

then $u+\Omega_P$ will denote a row letter of p elements, like u.

The equation we desire to prove, subject to proper relations connecting $\omega, \omega', \eta, \eta'$, is the following,

$$\vartheta(u+\Omega_P,q)=e^{H_P(u+\frac{1}{2}\Omega_P)-\pi iPP'+2\pi i(Pq'-P'q)}\,e^{-2\pi iPq'}\,\vartheta(u,P+q), \qquad \text{(L)},$$

which is a generalization of some of the fundamental equations given for $\Theta(u)$.

* Schottky, *Abriss einer Theorie der Abelschen Functionen von drei Variabeln*, Leipzig, 1880. The introduction of the matrix notation is suggested by Cayley, *Math. Annal.* (XVII.), p. 115.

In order that this equation may hold it is sufficient that the terms on the two sides of the equation, which contain the same values of the summation letters $n_1, \ldots, n_p$, should be equal; this will be so if

$$a(u+\Omega_P)^2+2h(u+\Omega_P)(n+q')+b(n+q')^2+2\pi iq(n+q')$$
$$=H_P(u+\tfrac{1}{2}\Omega_P)-\pi iPP'-2\pi iP'q+au^2+2hu(n+q'+P')+b(n+q'+P')^2$$
$$+2\pi i(P+q)(n+q'+P');$$

picking out in this conditional equation respectively the terms involving squares, first powers, and zero powers of $n_1, \ldots, n_p$, we require

$$b=b,$$
$$h(u+\Omega_P)+\bar{b}q'+\pi iq=hu+\bar{b}(q'+P')+\pi i(P+q),$$

and

$$a(u+\Omega_P)^2+2h(u+\Omega_P)q'+bq'^2+2\pi iqq'=H_P(u+\tfrac{1}{2}\Omega_P)-\pi iPP'-2\pi iP'q$$
$$+au^2+2hu(q'+P')+b(q'+P')^2+2\pi i(P+q)(q'+P').$$

190. In working out these conditions it will be convenient at first to neglect the fact that a and b are symmetrical matrices, in order to see how far it is necessary.

The second of these conditions gives

$$h\Omega_P=\pi iP+\bar{b}P',$$

and therefore gives the two conditions $h\omega=\frac{1}{2}\pi i$, $h\omega'=\frac{1}{2}\bar{b}$, whereby ω, ω' are determined in terms of the matrices h, b. In particular when $h=\pi i$ and $b=i\pi\tau$, as in the case of the function $\Theta(u)$, we have $2\omega=1$, $2\omega'=\tau$, namely 2ω, $2\omega'$ are the matrices of the periods of the Riemann normal integrals of the first kind, respectively at the first kind, and at the second kind of period loops.

The third condition gives

$$2au\Omega_P+a\Omega^2_P+2h\Omega_Pq'=H_P(u+\tfrac{1}{2}\Omega_P)$$
$$-\pi iPP'-2\pi iP'q+2huP'+b(2q'P'+P'^2)+2\pi i(qP'+Pq'+PP'),$$

that is

$$(2\bar{a}\Omega_P-H_P-2\bar{h}P')u+(a\Omega_P-\tfrac{1}{2}H_P)\Omega_P-\pi iPP'-bP'^2$$
$$+2(h\Omega_P-\pi iP-\bar{b}P')q'=0;$$

in order that this may be satisfied for all values of $u_1, \ldots, u_p$, we must have, referring to the equation already obtained from the second condition,

$$H_P=2\bar{a}\Omega_P-2\bar{h}P',$$

and

$$(a\Omega_P-\tfrac{1}{2}H_P)\Omega_P=(\pi iP+bP')P';$$

from the first of these, by the equation already obtained, we have

$$(\bar{a}\Omega_P-\tfrac{1}{2}H_P)\Omega_P=\bar{h}P'\Omega_P=h\Omega_PP'=(\pi iP+\bar{b}P')P';$$

subtracting this from the second equation, there results

$$(\bar{a} - a)\,\Omega^2_P = (\bar{b} - b)\,P'^2,$$

and in order that this may hold independently of the values assigned to P, P' it is necessary that $\bar{a} = a$, $b = \bar{b}$; when this is so, these two equations give, in addition to the one already obtained, only the equation

$$H_P = 2a\Omega_P - 2\bar{h}P',$$

leading to

$$\eta = 2a\omega, \quad \eta' = 2a\omega' - 2\bar{h},$$

which express the matrices η and η' in terms of the matrices a and h. These equations, with

$$h\Omega_P = \pi i P + bP',$$

or

$$h\omega = \tfrac{1}{2}\pi i, \quad h\omega' = \tfrac{1}{2}b,$$

are all the conditions necessary, and they are clearly sufficient. When they are satisfied we have

$$\vartheta(u + \Omega_P, q) = e^{\lambda_P(u) - 2\pi i P'q}\,\vartheta(u\,;\,q + P), \qquad \text{(L)},$$

where

$$\lambda_P(u) = H_P(u + \tfrac{1}{2}\Omega_P) - \pi i PP'.$$

Ex. Weierstrass's function σu is given by

$$A\sigma u = \Sigma e^{\frac{\eta}{2\omega}u^2 + \frac{2\pi i u}{2\omega}(n+\frac{1}{2}) + i\pi\tau(n+\frac{1}{2})^2 + \pi i(n+\frac{1}{2})}$$

where A is a certain constant.

The equations obtained express the $4p^2$ elements of the matrices $\omega, \omega', \eta, \eta'$ in terms of the $p^2 + p(p+1)$ quantities occurring in the matrices a, h, b; there must therefore be $2p^2 - p$ relations connecting the quantities in ω, ω', η, η'. The equations are in fact of precisely the same form as those already obtained in § 140, Chap. VII., equation (A), and precisely as in § 141 it follows that the necessary relations connecting ω, ω', η, η' may be expressed by either of the equations (B), (C) of § 140. Using the notation of matrices in greater detail we may express these relations in a still further way.

For

$$\begin{aligned}\tfrac{1}{2}(H_P\Omega_Q - H_Q\Omega_P) &= (a\Omega_P - \bar{h}P')\,\Omega_Q - (a\Omega_Q - \bar{h}P')\,\Omega_P \\ &= -\bar{h}P'\Omega_Q + \bar{h}Q'\Omega_P \\ &= h\Omega_P\,.\,Q' - h\Omega_Q\,.\,P' \\ &= (\pi i P + bP')\,Q' - (\pi i Q + bQ')\,P',\end{aligned}$$

so that

$$H_P\Omega_Q - H_Q\Omega_P = 2\pi i\,(PQ' - P'Q);$$

this relation includes all the $2p^2 - p$ necessary relations; for it gives

$$(\eta P + \eta' P')(\omega Q + \omega' Q') - (\eta Q + \eta' Q')(\omega P + \omega' P') = \tfrac{1}{2}\pi i\,(PQ' - P'Q),$$

or (using the matrix relation already quoted in the form $hu.kv = \bar{h}kvu = \bar{k}huv$)

$$(\bar{\omega}\eta - \bar{\eta}\omega) PQ + (\bar{\omega}\eta' - \bar{\eta}\omega') P'Q + (\bar{\omega}'\eta - \bar{\eta}'\omega) PQ' + (\bar{\omega}'\eta' - \bar{\eta}'\omega') P'Q' = \tfrac{1}{2}\pi i (PQ' - P'Q),$$

and expressing that this equation holds for all values of P, Q, P', Q', we obtain the Weierstrassian equations ((B) § 140).

Similarly the Riemann equations ((C) § 140) are all expressed by

$$(2\bar{\omega}'P + 2\bar{\eta}'Q)(2\bar{\omega}P' + 2\bar{\eta}Q') - (2\bar{\omega}P + 2\bar{\eta}Q)(2\bar{\omega}'P' + 2\bar{\eta}'Q') = 2\pi i (PQ' - P'Q).$$

Ex. i. If we substitute for the variables u in the ϑ function linear functions of any p new variables v, with non-vanishing determinant of transformation, and L_P be formed from the new form of the ϑ function, regarded as a function of v, just as H_P was formed from the original function, prove that $L_P v = H_P u$, and that $\lambda_P(u)$ remains unaltered.

Ex. ii. Prove that

$$\lambda_P(u + \Omega_M) + \lambda_M(u) - 2\pi i M'P = \lambda_Q(u + \Omega_N) + \lambda_N(u) - 2\pi i N'Q,$$

provided

$$M + P = N + Q.$$

The equation (L) is simplified when P, P' both consist of integers. For if M, M' be rows of integers, it is easy (putting a new summation letter, m, for $n + M'$, in the exponent of the general term of $\vartheta(u;\ q + M,\ q' + M')$,) to verify that

$$\vartheta(u;\ q + M,\ q' + M') = e^{2\pi i Mq'}\vartheta(u;\ q,\ q').$$

Therefore, if m, m' consist of integers, we find

$$\vartheta(u + \Omega_m,\ q) = e^{\lambda_m(u) + 2\pi i (mq' - m'q)}\vartheta(u,\ q),$$

and in particular

$$\vartheta(u + \Omega_m) = e^{\lambda_m(u)}\vartheta(u),$$

where $\vartheta(u)$ is written for $\vartheta(u;\ 0,\ 0)$. The reader will compare the equations obtained at the beginning of this chapter, where $a = 0$, $\eta = 0$, $\eta' = -2\pi i$, $\omega = \tfrac{1}{2}$, $\omega' = \tfrac{1}{2}\tau$, $\Omega_P = P + \tau P'$, $H_P = -2\pi i P'$, $\lambda_P(u) = -2\pi i P'(u + \tfrac{1}{2}P + \tfrac{1}{2}\tau P') - \pi i PP'$.

One equation, just used, deserves a separate statement; we have

$$\vartheta(u;\ q + M) = e^{2\pi i Mq'}\vartheta(u;\ q),$$

where M stands for a row of integers $M_1, \ldots, M_p, M_1', \ldots, M_p'$.

191. Finally, to conclude these general explanations as to the function $\vartheta(u)$, we may enquire in what cases $\vartheta(u)$ can be an odd or even function.

When m, m' are rows of integers the general formula gives

$$\vartheta(-u + \Omega_m,\ q) = e^{\lambda_m(-u) + 2\pi i (mq' - m'q)}\vartheta(-u,\ q);$$

hence when $\vartheta(u, q)$ is odd, or is even, since $\lambda_m(-u) = \lambda_{-m}(u)$, we have

$$\vartheta(u - \Omega_m, q) = e^{\lambda_{-m}(u) + 2\pi i(mq' - m'q)}\, \vartheta(u, q);$$

therefore, by equation (L),

$$\begin{aligned}\vartheta(u + \Omega_m, q), &= \vartheta(u - \Omega_m, q) \,.\, e^{\lambda_{2m}(u - \frac{1}{2}\Omega_m) + 4\pi i(mq' - m'q)}, \\ &= \vartheta(u, q)\, e^{\lambda_{-m}(u) + \lambda_{2m}(u - \frac{1}{2}\Omega_m) + 6\pi i(mq' - m'q)},\end{aligned}$$

while also, by the same equation,

$$\vartheta(u + \Omega_m, q) = \vartheta(u, q)\, e^{\lambda_m(u) + 2\pi i(mq' - m'q)}.$$

Thus the expression

$$\lambda_{2m}(u - \tfrac{1}{2}\Omega_m) + \lambda_{-m}(u) - \lambda_m(u) + 4\pi i(mq' - m'q)$$

must be an integral multiple of $2\pi i$. This is immediately seen to require only that $2(mq' - m'q - mm')$ be integral for all integral values of m, m'. Hence the necessary and sufficient condition is that q and q' consist of half-integers. In that case we prove as before that $\vartheta(u, q)$ is odd or even according as $4qq'$ is an odd or even integer.

192. In what follows in the present chapter we consider only the case in which $b = i\pi\tau$, τ being the matrix of the periods of Riemann's normal integrals at the second kind of period loops. And if $u_1^{x,a}, \dots, u_p^{x,a}$ denote any p linearly independent integrals of the first kind, such as used in §§ 138, 139, Chap. VII., the matrix h is here taken to be such that

$$2\pi i v_i^{x,a} = h_{i,1} u_1^{x,a} + \dots\dots + h_{i,p} u_p^{x,a}, \qquad (i = 1, 2, \dots, p),$$

so that h is as in § 139, and

$$\vartheta(u^{x,a}, q) = e^{au^2}\, \Theta(v^{x,a}, q),$$

where $u = u^{x,a}$.

From the formula

$$\vartheta(u + \Omega_m) = e^{H_m(u + \frac{1}{2}\Omega_m) - \pi i m m'}\, \vartheta(u),$$

wherein m, m' denote rows of integers, we infer, using the abbreviation

$$\zeta_i(u) = \frac{\partial}{\partial u_i} \log \vartheta(u),$$

that

$$\zeta_i(u + \Omega_m) - \zeta_i(u) = 2(\eta_{i,1} m_1 + \dots\dots + \eta_{i,p} m_p + \eta'_{i,1} m_1' + \dots\dots + \eta'_{i,p} m_p');$$

particular cases of this formula are

$$\begin{aligned}\zeta_i(u_1 + 2\omega_{1,r}, \dots, u_p + 2\omega_{p,r}) &= \zeta_i(u) + 2\eta_{i,r}, \\ \zeta_i(u_1 + 2\omega'_{1,r}, \dots, u_p + 2\omega'_{p,r}) &= \zeta_i(u) + 2\eta'_{i,r}.\end{aligned}$$

Thus if u_s be the argument

$$u_s^{x, m} - u_s^{x_1, m_1} - \ldots\ldots - u_s^{x_p, m_p},$$

where $u_1^{x, a}, \ldots, u_p^{x, a}$ are any p linearly independent integrals of the first kind, and the matrix a here used in the definition of $\vartheta(u)$ be the same as that previously used (Chap. VII. § 138) in the definition of the integral $L_i^{x, a}$, so that the matrices η, η' will be the same in both cases, then it follows that the periods of the expression

$$\zeta_i(u) + L_i^{x, a},$$

regarded as a function of x, are zero.

193. And in fact, when the matrix a is thus chosen, there exists the equation

$$-\zeta_i(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) + \zeta_i(u^{a, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p})$$
$$= L_i^{x, a} + \sum_{r=1}^{p} \tilde{\nu}_{r, i}\left[(x_r, x) - (x_r, a)\right] \frac{dx_r}{dt},$$

wherein $\tilde{\nu}_{r, i}$ denotes the minor of the element $\mu_i(x_r)$ in the determinant whose (r, i)th element is $\mu_i(x_r)$, divided by this determinant itself; thus $\tilde{\nu}_{r, i}$ depends on the places $x_1, \ldots, x_p$ exactly as the quantity $\nu_{r, i}$ (Chap. VII. § 138) depends on the places $c_1, \ldots, c_p$.

For we have just remarked that the two sides of this equation regarded as functions of x have the same periods; the left-hand side is only infinite at the places $x_1, \ldots, x_p$; if in $L_i^{x, a}$, which does not depend on the places $c_1, \ldots, c_p$ used in forming it (Chap. VII. § 138), we replace $c_1, \ldots, c_p$ by $x_1, \ldots, x_p$, it takes the form

$$\tilde{\nu}_{1, r}\Gamma_{x_1}^{x, a} + \ldots\ldots + \tilde{\nu}_{p, i}\Gamma_{x_p}^{x, a} - 2(a_{i, 1} u_1^{x, a} + \ldots\ldots + a_{i, p} u_p^{x, a}),$$

and becomes infinite only at the places $x_1, \ldots, x_p$. Hence the difference of the two sides of the equation is a rational function with only p poles, $x_1, \ldots, x_p$, having arbitrary positions. Such a function is a constant (Chap. III. § 37, and Chap. VI.); and by putting $x = a$, we see that this constant is zero.

194. It will be seen in the next chapter that in the hyperelliptic case the equation of § 193 enables us to obtain a simple expression for $\zeta_i(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p})$ in terms of algebraical integrals and rational functions only. In the general case we can also obtain such an expression*;

* See Clebsch und Gordan, *Abels. Functnen.* p. 171, Thomae, *Crelle*, LXXI. (1870), p. 214, Thomae, *Crelle*, CI. (1887), p. 326, Stahl, *Crelle*, CXI. (1893), p. 98, and, for a solution on different lines, see the latter part of chapter XIV. of the present volume.

though not of very simple character (§ 196). In the course of deriving that expression we give another proof of the equation of § 193.

The function of x given by $\vartheta(u^{x,m}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$ will have p zeros, unless $\vartheta(u^{x,m} + \tfrac{1}{2}\Omega_{\beta,\alpha})$ vanish identically (§§ 179, 180); we suppose this is not the case. Denote these zeros by $m_1', \dots, m_p'$. Then (Prop. X. § 184) the function $\vartheta(u^{x,m} - u^{x_1,m_1'} - \dots\dots - u^{x_p,m_p'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)$ will vanish when x coincides with $x_1, x_2, \dots,$ or x_p. Determining $m_1, \dots, m_p$ so that

$$u^{m_1, m_1'} + \dots\dots + u^{m_p, m_p'} \equiv \tfrac{1}{2}\Omega_{\beta,\alpha},$$

and supposing the exact value of the left-hand side to be $\tfrac{1}{2}\Omega_{\beta,\alpha} + \Omega_{k,h}$, where k, h are integral, this function is equal to

$$\vartheta(u^{x,m} - u^{x_1,m_1} - \dots\dots - u^{x_p,m_p} - \tfrac{1}{2}\Omega_{\beta,\alpha} - \Omega_{k,h}; \tfrac{1}{2}\beta. \tfrac{1}{2}\alpha),$$

and this, by equation (L) is equal to

$$e^{-\frac{1}{2}H_{\beta,\alpha}(u - \frac{1}{4}\Omega_{\beta,\alpha}) + \frac{1}{4}\pi i\beta\alpha}\,\vartheta(u),$$

where $u = u^{x,m} - u^{x_1,m_1} - \dots\dots - u^{x_p,m_p} - \Omega_{k,h}$.

Therefore (§ 190) the expression

$$\Phi, = \frac{\vartheta(u^{x,m} - u^{x_1,m_1'} - \dots\dots - u^{x_p,m_p'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)}{\vartheta(u^{\mu,m} - u^{x_1,m_1'} - \dots\dots - u^{x_p,m_p'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)} \Bigg/ \frac{\vartheta(u^{x,m} - u^{\mu_1,m_1'} - \dots\dots - u^{\mu_p,m_p'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)}{\vartheta(u^{\mu,m} - u^{\mu_1,m_1'} - \dots\dots - u^{\mu_p,m_p'}; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha)},$$

is equal to

$$\frac{\vartheta(u^{x,m} - u^{x_1,m_1} - \dots\dots - u^{x_p,m_p})}{\vartheta(u^{\mu,m} - u^{x_1,m_1} - \dots\dots - u^{x_p,m_p})} \Bigg/ \frac{\vartheta(u^{x,m} - u^{\mu_1,m_1} - \dots\dots - u^{\mu_p,m_p})}{\vartheta(u^{\mu,m} - u^{\mu_1,m_1} - \dots\dots - u^{\mu_p,m_p})};$$

we may write this in the form

$$\frac{\vartheta(U-r)}{\vartheta(V-r)} \Bigg/ \frac{\vartheta(U-s)}{\vartheta(V-s)};$$

the expression is therefore equal to

$$e^L \frac{\Theta(v^{x,m} - v^{x_1,m_1} - \dots\dots - v^{x_p,m_p})}{\Theta(v^{\mu,m} - v^{x_1,m_1} - \dots\dots - v^{x_p,m_p})} \Bigg/ \frac{\Theta(v^{x,m} - v^{\mu_1,m_1} - \dots\dots - v^{\mu_p,m_p})}{\Theta(v^{\mu,m} - v^{\mu_1,m_1} - \dots\dots - v^{\mu_p,m_p})},$$

where

$$L, = a(U-r)^2 - a(V-r)^2 - a(U-s)^2 + a(V-s)^2,$$

is equal to

$$-2aU(r-s) + 2aV(r-s),$$

or

$$-2a(U-V)(r-s),$$

that is

$$-2au^{x,\mu}(u^{x_1,\mu_1} + \dots\dots + u^{x_p,\mu_p}),$$

which denotes

$$-\sum_{r=1}^{p}\left(\sum_{i,j}\sum 2a_{i,j}\,u_j^{x,\mu}\,u_i^{x_r,\mu_r}\right).$$

Hence, by Prop. XIII. § 187, supposing that the matrix a, here used, is the same as that used in § 138, Chap. VII., and denoting the canonical integral

$$\Pi_{z,c}^{x,a}-2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r,s}\,u_r^{x,a}\,u_s^{z,c},$$

which has already occurred (page 194), by $R_{z,c}^{x,a}$, we have

$$R_{x_1,\mu_1}^{x,\mu}+\ldots\ldots+R_{x_p,\mu_p}^{x,\mu}=\log\Phi.$$

195. From the formula

$$\sum_{r=1}^{p} R_{x,\mu}^{x_r,\mu_r}=\log\frac{\vartheta\,(u^{x,m}-u^{x_1,m_1}-\ldots-u^{x_p,m_p})}{\vartheta\,(u^{x,m}-u^{\mu_1,m_1}-\ldots-u^{\mu_p,m_p})}\Big/\frac{\vartheta\,(u^{\mu,m}-u^{x_1,m_1}-\ldots-u^{x_p,m_p})}{\vartheta\,(u^{\mu,m}-u^{\mu_1,m_1}-\ldots-u^{\mu_p,m_p})},$$

since

$$R_{x,\mu}^{x_r,\mu_r}=P_{x,\mu}^{x_r,\mu_r}+\sum_{i=1}^{p} u_i^{x_r,\mu_r}\,L_i^{x,\mu},$$

we obtain

$$\sum_{r=1}^{p} P_{x,\mu}^{x_r,\mu_r}+\sum_{i=1}^{p}\sum_{r=1}^{p} u_i^{x_r,\mu_r}L_i^{x,\mu}=\log\frac{\vartheta\,(u^{x,m}-U)}{\vartheta\,(u^{x,m}-U_0)}\Big/\frac{\vartheta\,(u^{\mu,m}-U)}{\vartheta\,(u^{\mu,m}-U_0)},$$

where

$$U=u^{x_1,m_1}+\ldots\ldots+u^{x_p,m_p},$$

$$U_0=u^{\mu_1,m_1}+\ldots\ldots+u^{\mu_p,m_p},$$

and therefore

$$U-U_0=\sum_{r=1}^{p} u^{x_r,\mu_r}.$$

Hence, differentiating,

$$\sum_{r=1}^{p}\frac{\partial x_r}{\partial U_i}\left[(x_r,x)-(x_r,\mu)\right]+L_i^{x,\mu}=-\zeta_i(u^{x,m}-U)+\zeta_i(u^{\mu,m}-U),$$

where

$$\zeta_i(u)=\frac{\partial}{\partial u_i}\log\vartheta(u);$$

but, from

$$dU_i=Du_i^{x_1,m_1}.\,dx_1+\ldots\ldots+Du_i^{x_p,m_p}.\,dx_p,$$

where $dx_1,\ldots,dx_p$ denote the infinitesimals at $x_1,\ldots,x_p$, we obtain

$$\frac{\partial x_r}{\partial U_i}=\tilde{\nu}_{r,i}\frac{dx_r}{dt};$$

thus

$$-\zeta_i(u^{x,m}-U)+\zeta_i(u^{\mu,m}-U)=L_i^{x,m}+\sum_{r=1}^{p}\tilde{\nu}_{r,i}\left[(x_r,x)-(x_r,\mu)\right]\frac{dx_r}{dt},$$

which is the equation of § 193.

196. From the equation

$$R_{z_1, \mu_1}^{x, \mu} + \ldots\ldots + R_{z_p, \mu_p}^{x, \mu} = \log \Phi,$$

differentiating in regard to x, we obtain an equation which we write in the form

$$\sum_{r=1}^{p} F_x^{z_r, \mu_r} = \sum_{r=1}^{p} \mu_r(x) \left[\zeta_r(u^{x, m} - U) - \zeta_r(u^{x, m} - U_0)\right],$$

where $U = u^{z_1, m_1} + \ldots\ldots + u^{z_p, m_p}$, $U_0 = u^{\mu_1, m_1} + \ldots\ldots + u^{\mu_p, m_p}$.

Thus, if we take for $\mu_1, \ldots, \mu_p$ places determined from x just as $m_1, \ldots, m_p$ are determined from m, so that

$$(m, \mu_1, \ldots, \mu_p) \equiv (x, m_1, \ldots, m_p),$$

the arguments $u^{x, m} - U_0$ will be $\equiv 0$; as the odd function $\zeta_r(u)$ vanishes for zero values of the argument, we therefore have (§ 192), writing Ω_P for the exact value of $u^{x, m} - U_0$,

$$\begin{aligned} F_x^{z_1, \mu_1} + \ldots\ldots + F_x^{z_p, \mu_p} &= \sum_{r=1}^{p} \mu_r(x) \left[\zeta_r(u^{x, m} - u^{z_1, m_1} - \ldots\ldots - u^{z_p, m_p}) - (H_P)_r\right] \\ &= \sum_{r=1}^{p} \mu_r(x)\, \zeta_r(u^{x, m} - u^{z_1, m_1} - \ldots - u^{z_p, m_p} - \Omega_P) \\ &= -\sum_{r=1}^{p} \mu_r(x)\, \zeta_r(u^{z_1, \mu_1} + \ldots + u^{z_p, \mu_p}). \end{aligned}$$

If in this equation we put x at m we derive

$$F_m^{z_1, m_1} + \ldots\ldots + F_m^{z_p, m_p} = -\sum_{r=1}^{p} \mu_r(m)\, \zeta_r(u^{z_1, m_1} + \ldots\ldots + u^{z_p, m_p}), \qquad \text{(M)},$$

where $z_1, \ldots, z_p$ are arbitrary.

If however we put x in turn at p independent places $c_1, \ldots, c_p$, and denote the places determined from c_i, as $m_1, \ldots, m_p$ are determined from m, by $c_{i, 1}, \ldots, c_{i, p}$, so that

$$(c_i, m_1, \ldots, m_p) \equiv (m, c_{i, 1}, \ldots, c_{i, p}),$$

we obtain p equations of the form

$$F_{c_i}^{z_1, c_{i, 1}} + \ldots\ldots + F_{c_i}^{z_p, c_{i, p}} = -\sum_{r=1}^{p} \mu_r(c_i)\, \zeta_r(u^{z_1, c_{i, 1}} + \ldots\ldots + u^{z_p, c_{i, p}}).$$

Suppose then that $x, x_1, \ldots, x_p$ are arbitrary independent places; for $z_1, \ldots, z_p$ put the places $x_{i, 1}, \ldots, x_{i, p}$ determined by the congruence

$$(x, x_{i, 1}, \ldots, x_{i, p}) \equiv (c_i, x_1, \ldots, x_p);$$

then, if Ω_Q denote a certain period, $-u^{x_{i, 1}, c_{i, 1}} - \ldots - u^{x_{i, p}, c_{i, p}}$ is equal to $\Omega_Q + u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}$, and we have

$$F_{c_i}^{x_{i, 1}, c_{i, 1}} + \ldots + F_{c_i}^{x_{i, p}, c_{i, p}} = \sum_{r=1}^{p} \mu_r(c_i)\, \zeta_r(\Omega_Q + u^{x, m} - u^{x_1, m_1} - \ldots - u^{x_p, m_p});$$

therefore

$$\zeta_i(\Omega_Q + u^{x,\,m} - u^{x_1,\,m_1} - \dots - u^{x_p,\,m_p}) = \sum_{r=1}^{p} \nu_{r,\,i}\,[F_{c_r}^{x_r,\,1,\;c_{r,\,1}} + \dots + F_{c_r}^{x_r,\,p,\;c_{r,\,p}}],$$

where $\nu_{r,\,i}$ is the minor of $\mu_i(c_r)$ in the determinant whose (r, s)th element is $\mu_s(c_r)$, divided by the determinant itself.

In particular, when the differential coefficients $\mu_1(x), \dots, \mu_p(x)$ are those already denoted (§ 121, Chap. VII.) by $\omega_1(x), \dots, \omega_p(x)$, and $V_i^{x,\,a} = \int_a^x \omega_i(x)\,dt_x$, and the paths of integration are properly taken, we have*

$$\frac{\partial}{\partial V_i} \log \vartheta\,(V^{x,\,m} - V^{x_1,\,m_1} - \dots\dots - V^{x_p,\,m_p}) = F_{c_i}^{x_i,\,1,\;c_{i,\,1}} + \dots\dots + F_{c_i}^{x_i,\,p,\;c_{i,\,p}}.$$

197. A further result should be given. Let $x, x_1, \dots, x_p$ be fixed places. Take a variable place z, and thereby determine places $z_1, \dots, z_p$, functions of z, such that

$$(x, z_1, \dots, z_p) \equiv (z, x_1, \dots, x_p).$$

Then from the formula

$$\begin{aligned} - \zeta_i(u^{z,\,m} - u^{z_1,\,m_1} - \dots\dots - u^{z_p,\,m_p}) + \zeta_i(u^{a,\,m} - u^{z_1,\,m_1} - \dots\dots - u^{z_p,\,m_p}) \\ = L_i^{z,\,a} + \sum_{s=1}^{p} \nu_{s,\,i}\,[(z_s, z) - (z_s, a)]\,\frac{dz_s}{dt}, \end{aligned}$$

wherein $\nu_{s,\,i}$ is formed with $z_1, \dots, z_p$, we have, by differentiating in regard to z and denoting $-\frac{\partial}{\partial u_j}\zeta_i(u)$ by $\wp_{i,\,j}(u)$,

$$\begin{aligned} &\sum_{j=1}^{p} \wp_{i,\,j}(U)\left[\mu_j(z) - \mu_j(z_1)\frac{dz_1}{dz} - \dots\dots - \mu_j(z_p)\frac{dz_p}{dz}\right] \\ &\qquad\qquad - \sum_{j=1}^{p} \wp_{i,\,j}(\overline{U})\left[-\mu_j(z_1)\frac{dz_1}{dz} - \dots\dots - \mu_j(z_p)\frac{dz_p}{dz}\right] \\ &= D_z L_i^{z,\,a} + \sum_{s=1}^{p}[(z_s, z) - (z_s, a)]\frac{dz_s}{dt}\sum_{r=1}^{p}\frac{dz_r}{dz}\frac{d}{dz_r}(\nu_{s,\,i}) \\ &\quad + \sum_{s=1}^{p}\nu_{s,\,i}\left[\frac{d}{dz_s}\left((z_s, z)\frac{dz_s}{dt}\right) - \frac{d}{dz_s}\left((z_s, a)\frac{dz_s}{dt}\right)\right]\frac{dz_s}{dz} + \sum_{s=1}^{p}\nu_{s,\,i}\,D_z\left((z_s, z)\frac{dz_s}{dt}\right), \end{aligned}$$

where $U = u^{z,\,m} - u^{z_1,\,m_1} - \dots\dots - u^{z_p,\,m_p}$, $\overline{U} = u^{a,\,m} - u^{z_1,\,m_1} - \dots\dots - u^{z_p,\,m_p}$.

In this equation a is arbitrary. Let it now be put to coincide with z; hence

$$\sum_{j=1}^{p}\mu_j(z)\,\wp_{i,\,j}(U) = D_z L_i^{z,\,a} + \sum_{s=1}^{p}\nu_{s,\,i}\,D_z\left[(z_s, z)\frac{dz_s}{dt}\right].$$

* This form is used by Noether, *Math. Annal.* XXXVII. (1890), p. 488.

Therefore

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\mu_i(k)\,\mu_j(z)\,\wp_{i,j}(U)$$
$$=\sum_{i=1}^{p}\mu_i(k)\,D_z L_i^{z,a}+\sum_{s=1}^{p}\sum_{i=1}^{p}\nu_{s,i}\,\mu_i(k)\,D_z\left[(z_s,z)\frac{dz_s}{dt}\right]$$
$$=\sum_{i=1}^{p}\mu_i(k)\,D_z L_i^{z,a}+\sum_{s=1}^{p}\omega_s(k)\,D_z\left[(z_s,z)\frac{dz_s}{dt}\right]$$
$$=D_z'\left\{\sum_{i=1}^{p}\mu_i(k)\,L_i^{z,a}+\sum_{s=1}^{p}\omega_s(k)\,[(z_s,z)-(z,a)]\frac{dz_s}{dt}\right\},$$

where D_z' means a differentiation taking no account of the fact that $z_1, \ldots, z_p$ are functions of z,

$$=D_z'\left\{\sum_{i=1}^{p}\mu_i(k)\,L_i^{z,a}-\psi(z,a;\,k,z_1,\ldots,z_p)+\left[(k,z)-(k,a)\frac{dk}{dt}\right]\right\},$$
$$=D_z'\left\{D_k R_{z,a}^{k,c}-\psi(z,a;\,k,z_1,\ldots,z_p)\right\},$$

in which form the expression is algebraically calculable when the integrals $L_i^{x,a}$ are known (Chap. VII. § 138),

$$=D_z'\left\{\Gamma_k^{z,a}-\psi(z,a;\,k,z_1,\ldots,z_p)-2\Sigma\Sigma a_{r,s}\,\mu_r(k)\,u_s^{z,c}\right\},$$

where c is an arbitrary place; and this (cf. Ex. iv. § 125)

$$=-\,W(z;\,k,z_1,\ldots,z_p)-2\sum_{r=1}^{p}\sum_{s=1}^{p}a_{r,s}\,\mu_r(z)\,\mu_r(k).$$

If now

$$(k,z_1,\ldots,z_p)\equiv(z,k_1,\ldots,k_p),$$

so that

$$U\equiv u^{x,m}-u^{x_1,m_1}-\ldots\ldots-u^{x_p,m_p}\equiv u^{z,m}-u^{z_1,m_1}-\ldots\ldots-u^{z_p,m_p}$$
$$\equiv u^{k,m}-u^{k_1,m_1}-\ldots\ldots-u^{k_p,m_p},$$

and

$$(x,z_1,\ldots,z_p)\equiv(z,x_1,\ldots,x_p),$$
$$(x,k_1,\ldots,k_p)\equiv(k,x_1,\ldots,x_p),$$

then the formula is

$$-\sum_i\sum_j\wp_{i,j}(U)\,.\,\mu_i(k)\,\mu_j(z)=W(z;\,k,z_1,\ldots,z_p)+2\sum_{r=1}^{p}\sum_{s=1}^{p}a_{r,s}\,\mu_r(z)\,\mu_s(k),$$
$$=W(k;\,z,k_1,\ldots,k_p)+2\sum_{r=1}^{p}\sum_{s=1}^{p}a_{r,s}\,\mu_r(z)\,\mu_s(k),$$

by Ex. iv. § 125.

By the congruences

$$u^{z_1, x_1} + \ldots\ldots + u^{z_p, x_p} \equiv u^{z, x}$$

the places $z_1, \ldots, z_p$ are algebraically determinable from the places $x, x_1, \ldots, x_p, z$, and therefore the function $W(z; k, z_1, \ldots, z_p)$ can be expressed by $x, x_1, \ldots, x_p, k, z$ only. In fact we have

$$\psi(z_1, x; z, x_1, \ldots, x_p) = 0, \ldots\ldots, \psi(z_p, x; z, x_1, \ldots, x_p) = 0.$$

The interest of the formula lies in the fact that the left-hand side is a multiply periodic function of the arguments $U_1, \ldots, U_p$.

A particular way of expressing the right-hand side in terms of $x, x_1, \ldots, x_p, z, k$ is to put down $\frac{1}{2}p(p+1)$ linearly independent particular cases of this equation, in which the right-hand side contains only $x, x_1, \ldots, x_p, z, k$, and then to solve for the $\frac{1}{2}p(p+1)$ quantities $\wp_{i,j}$. Since $\psi(z, a; k, z_1, \ldots, z_p)$ vanishes when $k = z_p$, we clearly have, as one particular case,

$$\sum_i \sum_j \wp_{i,j}(u^{z, m} - u^{z_1, m_1} - \ldots\ldots - u^{z_p, m_p}) \mu_i(z) \mu_j(z_p) = D_z D_{z_p} R^{z, a}_{z_p, c},$$

and therefore

$$\sum_i \sum_j \wp_{i,j}(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) \mu_i(x) \mu_j(x_r) = D_x D_{x_r} R^{x, a}_{x_r, c}, \qquad \text{(N)}$$

and there are p equations of this form, in which $x_1, \ldots, x_p$ occur instead of x_r.

If we determine $x_1', \ldots, x'_{p-1}$ by the congruences

$$u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p} \equiv -[u^{x_p, m} - u^{x_1', m_1} - \ldots\ldots - u^{x'_{p-1}, m_{p-1}} - u^{x, m_p}],$$

so that $x_1', \ldots, x'_{p-1}$ are the other zeros of a ϕ-polynomial vanishing in $x_1, \ldots, x_{p-1}$, we can infer $p-1$ other equations, of the form

$$\sum_i \sum_j \wp_{i,j}(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) \mu_i(x_p) \mu_j(x_r') = D_{x_p} D_{x'_r} R^{x_p, a}_{x_r', a},$$

where $r = 1, 2, \ldots, (p-1)$. Here the right-hand side does not depend upon the place x. And we can obtain p such sets of equations.

We have then sufficient * equations. For the hyperelliptic case the final formula is given below (§ 217, Chap. XI.).

198. *Ex.* i. Verify the formula (N) for the case $p = 1$.

Ex. ii. Prove that

$$\zeta_i(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) + L^{x, a}_i - L^{x_1, a}_i - \ldots\ldots - L^{x_p, a}_i$$

is a rational function of $x, x_1, \ldots, x_p$.

Ex. iii. Prove that if

$$(x, z_1, \ldots, z_p) \equiv (z, x_1, \ldots, x_p) \equiv (a, a_1, \ldots, a_p),$$

then

$$\psi(x, a; z, x_1, \ldots, x_p) = \Gamma^{x, a}_z + \Gamma^{z_1, a_1}_z + \ldots\ldots + \Gamma^{z_p, a_p}_z.$$

Deduce the first formula of § 193 from the final formula of § 196.

* The function $\wp_{i,j}(u)$, here employed, is remarked, for the hyperelliptic case, by Bolza, *Göttinger Nachrichten*, 1894, p. 268.

Ex. iv. Prove that if

$$Q_i = \Gamma_{c_i}^{x_{i,1}, a_1} + \ldots\ldots + \Gamma_{c_i}^{x_{i,p}, a_p},$$

where $a_1, \ldots, a_p$ are arbitrary places, and

$$V_r = V_r^{x, m} - V_r^{x_1, m_1} - \ldots\ldots - V_r^{x_p, m_p} = V_r^{c_i, m} - V_r^{x_{i,1}, m_1} - \ldots\ldots - V_r^{x_{i,p}, m_p},$$

then

$$\frac{\partial Q_i}{\partial V_r} = W(c_i;\ c_r, x_{i,1}, \ldots, x_{i,p}),$$

where W denotes the function used in Ex. iv. § 125 ; it follows therefore by that example, that $\frac{\partial Q_i}{\partial V_r} = \frac{\partial Q_r}{\partial V_i}$. Hence the function

$$Q_1 dV_1 + \ldots\ldots + Q_p dV_p$$

is a perfect differential ; it is in fact, by the final equation of § 196, practically equivalent to the differential of the function $\log \Theta(V^{x, m} - V^{x_1, m_1} - \ldots\ldots - V^{x_p, m_p})$. Thus the theory of the Riemann theta functions can be built up from the theory of algebraical integrals. Cf. Noether, *Math. Annal.* XXXVII. For the step to the expression of the function by the theta series, see Clebsch and Gordan, *Abelsche Functionen* (Leipzig, 1866), pp. 190—195.

Ex. v. Prove that if

$$(m^2, x_{i,1}, \ldots, x_{i,p}, z_1, \ldots, z_p) \equiv (c_i^2, m_1^2, \ldots, m_p^2)$$

then

$$\frac{\partial}{\partial V_i} \log \Theta(V^{x, m} - V^{x_1, m_1} - \ldots\ldots - V^{x_p, m_p}) = \tfrac{1}{2}(\Gamma_{c_i}^{x_{i,1}, z_1} + \ldots\ldots + \Gamma_{c_i}^{x_{i,p}, z_p}).$$

Ex. vi. Prove that

$$-\sum_{i=1}^{p} \mu_i(z)\left[\zeta_i(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) - \zeta_i(u^{a, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p})\right]$$
$$= F_z^{x, a} - \psi(x, a;\ z, x_1, \ldots, x_p).$$

Ex. vii. If

$$T(x, a;\ x_1, \ldots, x_p) = [\psi(x, a;\ z, x_1, \ldots, x_p) - F_z^{x, a}]_{z=x},$$

prove that

$$\log \vartheta(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p})$$
$$= A + A_1 u_1^{x, a} + \ldots\ldots + A_p u_p^{x, a} + \int^x dx\, T(x, a;\ x_1, \ldots, x_p),$$

where $A, A_1, \ldots, A_p$ are independent of x.

Ex. viii. Prove that

$$-\sum_{r=1}^{p} \mu_r(x)\, \wp_{i,r}(u^{x, m} - u^{x_1, m_1} - \ldots\ldots - u^{x_p, m_p}) = \sum_{r=1}^{p} \tilde{\nu}_{r,i} D_x D_{x_r} R_{x_r, c}^{x, a},$$

where a, c are arbitrary places and the notation is as in § 193.