CHAPTER IX.

JACOBI'S INVERSION PROBLEM.

163. IT is known what advance was made in the theory of elliptic functions by the adoption of the idea, of Abel and Jacobi, that the value of the integral of the first kind should be taken as independent variable, the variables, x and y, belonging to the upper limit of this integral being regarded as dependent. The question naturally arises whether it may not be equally advantageous, if possible, to introduce a similar change of independent variable in the higher cases. We have seen in the previous chapter that, if $u_1^{x_1 a}, \dots, u_p^{x, a}$ be any p linearly independent integrals of the first kind, the p equations

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} = - u_i^{x_{p+1}, a_{p+1}} - \dots\dots - u_i^{x_Q, a_Q}, \quad (i = 1, 2, \dots, p),$$

justify us in regarding the places $x_1, \dots, x_p$ as rationally determinable from the arbitrary places $a_1, \dots, a_Q, x_{p+1}, \dots, x_Q$; hence is suggested the problem, known as Jacobi's inversion problem*, which may be stated thus: *if $U_1, \dots, U_p$ be arbitrary quantities, regarded as variable, and $a_1, \dots, a_p$ be arbitrary fixed places, required to determine the nature and the expression of the dependence of the places $x_1, \dots, x_p$, which satisfy the p equations*

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} = U_i, \qquad (i = 1, 2, \dots, p),$$

upon the quantities $U_1, \dots, U_p$. It is understood that the path of integration from a_r to x_r is to be taken the same in each of the p equations, and is not restricted from crossing the period loops.

164. It is obvious first of all that if for any set of values $U_1, \dots, U_p$ there be one set of corresponding places $x_1, \dots, x_p$ of such general positions that no ϕ-polynomial (§ 101) vanishes in them, there cannot be another set of places, $x_1', \dots, x_p'$, belonging to the same values of $U_1, \dots, U_p$. For then we should have

$$u_i^{x_1', x_1} + \dots\dots + u_i^{x_p', x_p} = 0, \qquad (i = 1, 2, \dots, p),$$

* Jacobi, *Crelle* XIII. (1835), p. 55.

and therefore (§ 158, Chap. VIII.) there would exist a rational function having $x_1, \ldots, x_p$ as poles and $x_1', \ldots, x_p'$ as zeros, which is contrary (§ 37, Chap. III.) to the hypothesis that no ϕ-polynomial vanishes in $x_1, \ldots, x_p$.

But a further result follows from the § referred to (§ 158, Chap. VIII.). Let $2\omega_{i,1}, \ldots, 2\omega_{i,p}, 2\omega_{i,1}', \ldots, 2\omega_{i,p}'$ denote the periods of $u_i^{x,a}$, and $m_1, \ldots, m_p, m_1', \ldots, m_p'$ denote any rational integers which are the same for all values of i. On the hypothesis that the inversion problem is capable of solution for all values of the quantities $U_1, \ldots, U_p$, suppose these quantities to vary continuously from the values $U_1, \ldots, U_p$ to the values $V_1, \ldots, V_p$, where

$$V_i = U_i + 2m_1\omega_{i,1} + \ldots\ldots + 2m_p\omega_{i,p} + 2m_1'\omega_{i,1}' + \ldots\ldots + 2m_p'\omega_{i,p}', \qquad (i=1, 2, \ldots, p),$$
$$= U_i + 2\Omega_i, \text{ say,}$$

and let $z_1, \ldots, z_p$ be the places such that

$$u_i^{z_1, a_1} + \ldots + u_i^{z_p, a_p} = V_i;$$

then it follows from § 158, that the places $z_1, \ldots, z_p$ are, in some order, the same as the places $x_1, \ldots, x_p$. For this reason it is proper to write the equations of the inversion problem in the form

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} \equiv U_i,$$

where the sign $\equiv$ indicates that the two sides of the congruence differ by a quantity of the form $2\Omega_i$. And further, if the set $x_1, \ldots, x_p$ be uniquely determined by the values $U_1, \ldots, U_p$, any symmetrical function of the values of x, y at the places of this set, must be a single-valued function of $U_1, \ldots, U_p$. Denoting such a function by $\phi(U_1, \ldots, U_p)$, we have, therefore,

$$\phi(U_1 + 2\Omega_1, U_2 + 2\Omega_2, \ldots, U_p + 2\Omega_p) = \phi(U_1, \ldots, U_p).$$

The functions that arise are therefore such as are unaltered when the p variables $U_1, \ldots, U_p$ are simultaneously increased by the same integral multiples of any one of the $2p$ sets of quantities denoted by

$$2\omega_{1,r},\ 2\omega_{2,r},\ \ldots,\ 2\omega_{p,r}$$
$$2\omega_{1,r}',\ 2\omega_{2,r}',\ \ldots,\ 2\omega_{p,r}'. \qquad (r=1, 2, \ldots, p).$$

165. The sign $\equiv$ will often be employed in what follows, in the sense explained above. There is one case in which it is absolutely necessary. In what has preceded the paths of integration have not been restricted from crossing the period loops. But it is often convenient, for the sake of definiteness, to use only integrals for which this restriction is enforced. In such case the problem expressed by the equations

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} = U_i$$

may be incapable of solution for some values of $U_1, \ldots, U_p$. This can be seen as follows: if both the sets of equations

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} = U_i,$$

$$u_i^{z_1, a_1} + \ldots\ldots + u_i^{z_p, a_p} = U_i + 2\Omega_i,$$

were capable of solution, it would follow, by § 158, that the set $z_1, \ldots, z_p$ is the same as the set $x_1, \ldots, x_p$. And thence, as the paths are restricted not to cross the period loops, we should have

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} = u_i^{z_1, a_1} + \ldots\ldots + u_i^{z_p, a_p},$$

and thence

$$2\Omega_1 = 2m_1\omega_{i,1} + \ldots\ldots + 2m_p\omega_{i,p} + 2m_1'\omega_{i,1}' + \ldots\ldots + 2m_p'\omega_{i,p}' = 0;$$

but these equations are reducible to

$$m_i + m_1'\tau_{i,1} + \ldots\ldots + m_p'\tau_{i,p} = 0,$$

and, therefore, there would exist a function, expressed by

$$e^{2\pi i (m_1' v_1^{x, a} + \ldots\ldots + m_p' v_p^{x, a})},$$

(where $v_1^{x, a}, \ldots, v_p^{x, a}$ are Riemann's elementary integrals of the first kind), everywhere finite and without periods. Such a function must be a constant; thus the conclusion would involve that $v_1^{x, a}, \ldots, v_p^{x, a}$ are not linearly independent, which is untrue.

Hence when the paths of integration are restricted not to cross the period loops, the equations of the inversion problem must be written

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} \equiv U_i;$$

in this case the integral sum on the left-hand side is not capable of assuming all values; and the particular period which must be added to the right-hand side to make the two sides of the congruence equal is determined by the solution of the problem.

166. Before passing to the proof that Jacobi's inversion problem does admit of solution, another point should be referred to. It is not at first sight apparent why it is necessary to take p arguments, $U_1, \ldots, U_p$, and p dependent places $x_1, \ldots, x_p$. It may be thought, perhaps, that a single equation

$$u^{x, a} = U,$$

wherein $u^{x, a}$ is any definite integral of the first kind, suffices to determine the place x as a function of the argument U. We defer to a subsequent place the enquiry whether this is true when the path of integration on the left hand is not allowed to cross the period loops of the Riemann surface; it is obvious enough that in such a case all conceivable values of U would not arise, for instance $U = \infty$ would not arise, and the function of U obtained would only be defined for restricted values of the argument. But it is possible to see that when the path of integration is not limited, the place x cannot be definitely determinate from U. For, then, putting $x = f(U)$, we must have $f(U + 2\Omega) = f(U)$, wherein

$$\Omega = m_1\omega_1 + \ldots\ldots + m_p\omega_p + m_1'\omega_1' + \ldots\ldots + m_p'\omega_p',$$

$m_1, \ldots, m_p'$ being arbitrary rational integers, and $2\omega_1, \ldots, 2\omega_p'$ being the periods of $u^{x, a}$; and it can be shewn, when $p > 1$, that in general it is possible to choose the integers $m_1, \ldots, m_p'$ so that Ω shall be within assigned nearness of any prescribed arbitrary value whatever. Thus not only would the function $f(U)$ have infinitesimal periods, but any assigned value of this function would arise for values of the argument lying within assigned nearness of any value whatever. We shall deal later with the possibility of the existence of infinitesimal periods; for the present such functions are excluded from consideration.

The arithmetical theorem referred to * may be described thus; if a_1, a_2 be any real quantities, the values assumed by the expression $N_1a_1 + N_2a_2$, when N_1, N_2 take all possible rational integer values independently of one another, are in general infinite in number; exception arises only in the case when the ratio a_1/a_2 is rational; and it is in general possible to find rational integer values of N_1 and N_2 to make $N_1a_1 + N_2a_2$ approach within assigned nearness of any prescribed real quantity. Similarly if $a_1, a_2, a_3, b_1, b_2, b_3$ be real quantities, of the expressions $N_1a_1 + N_2a_2 + N_3a_3$, $N_1b_1 + N_2b_2 + N_3b_3$, where N_1, N_2, N_3 take all possible rational integer values independently of one another, there are, in general, values which lie within assigned nearness respectively to two arbitrarily assigned real quantities a, b. More generally, if $a_1, \ldots, a_k, b_1, \ldots, b_k, \ldots\ldots, c_1, \ldots, c_k$ be any $(k-1)$ sets each of k real quantities, and $a, b, \ldots, c$ be $(k-1)$ arbitrary real quantities, it is in general possible to find rational integers $N_1, \ldots, N_k$ such that the $(k-1)$ quantities

$$N_1a_1 + \ldots\ldots + N_ka_k - a,\ N_1b_1 + \ldots\ldots + N_kb_k - b, \ldots, N_1c_1 + \ldots\ldots + N_kc_k - c,$$

are all within assigned nearness of zero.

Hence it follows, taking $k = 2p$, that we can choose values of the integers $m_1, \ldots, m_p'$, to make $p-1$ of the quantities

$$\Omega_r = m_1\omega_{r,1} + \ldots\ldots + m_p\omega_{r,p} + m_1'\omega_{r,1}' + \ldots\ldots + m_p'\omega_{r,p}',$$

say $\Omega_1, \ldots, \Omega_{p-1}$, approach within assigned nearness of any $(p-1)$ prescribed values, and at the same time to make the real part of the remaining quantity Ω_p approach within assigned nearness of any prescribed value; but the imaginary part of Ω_p will thereby be determined. We cannot therefore

* Jacobi, *loc. cit.*; Hermite, *Crelle*, LXXXVIII. p. 10.

expect to obtain an intelligible inversion by taking less than p new variables $U_1, U_2, \ldots$; and it is manifest that we ought to use the same number of dependent places $x_1, x_2, \ldots$. On the other hand, the proof which has been given that there can in general only be one set of places $x_1, \ldots, x_p$ corresponding to given values of $U_1, \ldots, U_p$ would not remain valid in case the left-hand sides of the equations of the problem of inversion consisted of a sum of more than p integrals; for it is generally possible to construct a rational function with $p+1$ assigned poles.

167. It follows from the argument here that when $p > 1$ an integral of the first kind, $u^{x, a}$, is capable, for given positions of the extreme limits, x, a, of the integration, of assuming values within assigned nearness of any prescribed value whatever. Though not directly connected with the subject here dealt with it is worth remark that it does not thence follow that the integral is capable of assuming all possible values. For the values represented by an expression of the form

$$m_1\omega_1 + \ldots\ldots + m_p\omega_p + m_1'\omega_1' + \ldots\ldots + m_p'\omega_p',$$

for all values of the integers $m_1, \ldots, m_p, m_1', \ldots, m_p'$, form an enumerable aggregate—that is, they can be arranged in order and numbered $-\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty$. To prove this we may begin by proving that all values of the form $m_1\omega_1 + m_2\omega_2$ form an enumerable aggregate; the proof is identical with the proof that all rational fractions form an enumerable aggregate; and may then proceed to shew that all values of the form $m_1\omega_1 + m_2\omega_2 + m_3\omega_3$ form an enumerable aggregate, and so on, step by step. Since then the aggregate of all conceivable complex values is not an enumerable aggregate, the statement made is justified.

The reader may consult Harkness and Morley, *Theory of Functions*, p. 280, Dini, *Theorie der Functionen einer reellen Grösse* (German edition by Luröth and Schepp), pp. 27, 191, Cantor, *Acta Math.* II. pp. 363—371, Cantor, *Crelle*, LXXVII. p. 258, *Rendiconti del Circolo Mat. di Palermo*, 1888, pp. 197, 135, 150, where also will be found a theorem of Poincaré's to the effect that no multiform analytical function exists whose values are not enumerable.

168. Consider now* the equations

$$\text{(A)} \qquad u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} = U_i, \qquad (i = 1, 2, \ldots, p)$$

wherein, denoting the differential coefficient of $u_i^{x, a}$ in regard to the infinitesimal at x by $\mu_i(x)$, the fixed places $a_1, \ldots, a_p$ are supposed to be such that the determinant of p rows and columns whose (i, j)th element is $\mu_j(a_i)$ does not vanish; wherein also the p paths of integration a_1 to $x_1, \ldots, a_p$ to x_p, are to be the same in all the p equations, and are not restricted from crossing the period loops.

When $x_1, \ldots, x_p$ are respectively in the neighbourhoods of $a_1, \ldots, a_p$ and $U_1, \ldots, U_p$ are small, these equations can be written

$$\left[t_1\mu_i(a_1) + \frac{t_1^2}{\underline{|2}}\mu_i'(a_1) + \ldots\ldots\right] + \ldots\ldots + \left[t_p\mu_i(a_p) + \frac{t_p^2}{\underline{|2}}\mu_i'(a_p) + \ldots\ldots\right] = U_i,$$

* The argument of this section is derived from Weierstrass; see the references given in connection with § 170.

wherein t_r is the infinitesimal in the neighbourhood of the place a_r, and $\mu_r'(x)$ is derived from $\mu_r(x)$ by differentiation. From these equations we obtain

$$t_r = \nu_{r,1} U_1 + \dots\dots + \nu_{r,p} U_p + U_r^{(2)} + U_r^{(3)} + \dots\dots, \qquad (r = 1, 2, \dots, p),$$

where, if Δ denote the determinant whose (i, j)th element is $\mu_j(a_i)$, $\nu_{i,j}$ denotes the minor of this element divided by Δ, and $U_r^{(k)}$ denotes a homogeneous integral polynomial in $U_1, \dots, U_p$ of the kth degree. These series will converge provided $U_1, \dots, U_p$ be of sufficient, not unlimited, smallness. Hence also, so long as the place x_r lies within a certain finite neighbourhood of the place c_r, the values of the variables x_r, y_r associated with this place, which are expressible by convergent series of integral powers of t_r, are expressible by series of integral powers of $U_1, \dots, U_p$ which are convergent for sufficiently small values of $U_1, \dots, U_p$.

Suppose that the values of $U_1, \dots, U_p$ are such that the places $x_1, \dots, x_p$ thus obtained are not such that the determinant whose (i, j)th element is $\mu_j(x_i)$ is zero; then if $U_1', \dots, U_p'$ be small quantities, it is similarly possible to obtain p places $x_1', \dots, x_p'$, lying respectively in the neighbourhoods of $x_1, \dots, x_p$, such that

$$u_i^{x_1', x_1} + \dots\dots + u_i^{x_p', x_p} = U_i', \qquad (i = 1, 2, \dots, p);$$

by adding these equations to the former we therefore obtain

$$u_i^{x_1', a_1} + \dots\dots + u_i^{x_p', a_p} = U_i + U_i', \qquad (i = 1, 2, \dots, p).$$

Since all the series used have a finite range of convergence, we are thus able, step by step, to obtain places $x_1, \dots, x_p$ to satisfy the p equations

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} = U_i, \qquad (i = 1, 2, \dots, p),$$

for any finite values of the quantities $U_1, \dots, U_p$ which can be reached from the values $0, 0, \dots, 0$ without passing through any set of values for which the corresponding positions of $x_1, \dots, x_p$ render a certain determinant zero.

169. The method of continuation thus sketched has a certain interest; but we can arrive at the required conclusion in a different way. Let $U_1, \dots, U_p$ be any finite quantities; and let m be a positive integer. When m is large enough, the quantities $U_1/m, \dots, U_p/m$ are, in absolute value, as small as we please. Hence there exist places $z_1, \dots, z_p$, lying respectively in the neighbourhoods of the places $a_1, \dots, a_p$, such that

$$u_i^{z_1, a_1} + \dots\dots + u_i^{z_p, a_p} = -U_i/m \qquad (i = 1, 2, \dots, p).$$

In order then to obtain places $x_1, \dots, x_p$, to satisfy the equations

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} = U_i, \qquad (i = 1, 2, \dots, p),$$

it is only necessary to obtain places $x_1, \dots, x_p$, such that

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} + mu_i^{z_1, a_1} + \dots\dots + mu_i^{z_p, a_p} = 0, \quad (i = 1, 2, \dots, p);$$

and it has been shewn (Chap. VIII. § 158), that these equations express only that the set of $mp + p$ places formed of $z_1, \dots, z_p$, each m times repeated and the places $x_1, \dots, x_p$, are coresidual with the set of $(m+1)p$ places formed of $a_1, \dots, a_p$ each $(m+1)$ times repeated.

Now, when $(m+1)p$ places are not zeros of a ϕ-polynomial, we may (Chap. VI.) arbitrarily assign all but p of the places of a set of $(m+1)p$ places which are coresidual with them; and the other p places will be algebraically and rationally determinable from the mp assigned places.

Hence with the general positions assigned to the places $a_1, \dots, a_p$, it follows, if Z denote any rational function, that the values of Z at the places $x_1, \dots, x_p$ are the roots of an algebraical equation,

$$Z^p + Z^{p-1} R_1 + \dots\dots + R_p = 0,$$

whose coefficients $R_1, \dots, R_p$ are rationally determinable from the places $z_1, \dots, z_p$, and are therefore, by what has been shewn, expressible by series of integral powers of $U_1/m, \dots, U_p/m$, which converge for sufficiently large values of m. Thus the problem expressed by the equations

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_p, a_p} = U_i, \qquad (i = 1, 2, \dots, p),$$

is always capable of solution, for any finite values of $U_1, \dots, U_p$.

It has already been shewn (§ 164), that for general values of $U_1, \dots, U_p$ the set $x_1, \dots, x_p$ obtained is necessarily unique; the same result follows from the method of the present article. It is clear in § 164, in what way exception can arise; to see how a corresponding peculiarity may present itself in the present article the reader may refer to the concluding result of § 99 (Chap. VI.). (See also Chap. III. § 37, Ex. ii.)

In case the places $a_1, \dots, a_p$ in the equations (A) be such that the determinant denoted by Δ vanishes, we may take places $b_1, \dots, b_p$, for which the corresponding determinant is not zero, and follow the argument of the text for the equations

$$u_i^{x_1, b_1} + \dots\dots + u_i^{x_p, b_p} = V_i,$$

in which $V_i = U_i + u_i^{a_1, b_1} + \dots\dots + u_i^{a_p, b_p}$.

We do not enter into the difficulty arising as to the solution of the inversion problem expressed by the equations (A) in the case where $U_1, \dots, U_p$ have such values that $x_1, \dots, x_p$ are zeros of a ϕ-polynomial. This point is best cleared up by actual examination of the functions which are to be obtained to express the solution of the problem (cf.* § 171, and

* See also Clebsch and Gordan, *Abel. Functnen.*, pp. 184, 186.

Props. xiii. and xv., Cor. iii., of Chap. X.). But it should be noticed that the method of § 168 shews that a solution exists in all cases in which the fixed places $a_1, \ldots, a_p$ do not make the determinant Δ vanish; the peculiarity in the special case is that instead of an unique solution $x_1, \ldots, x_p$, all the $\infty^{\tau+1}$ sets coresidual with $x_1, \ldots, x_p$ are equally solutions, $\tau+1$ being the number of linearly independent ϕ-polynomials which vanish in $x_1, \ldots, x_p$. This follows from §§ 154, 158.

170. We consider now how to form functions with which to express the solution of the inversion problem.

Let $P^{x,a}_{\xi,\gamma}$ denote any elementary integral of the third kind, with infinities at the arbitrary fixed places ξ, γ. Then if $a_1, \ldots, a_p, x_1, \ldots, x_p$ denote the places occurring on the left hand in equation (A), it can be shewn that the function

$$T = P^{x_1, a_1}_{\xi,\gamma} + \ldots\ldots + P^{x_p, a_p}_{\xi,\gamma}$$

is the logarithm of a single valued function of $U_1, \ldots, U_p$, and that the solution of the inversion problem can be expressed by this function; and further that, if $I^{x,a}$ denote any Abelian integral, the sum

$$I^{x_1, a_1} + \ldots\ldots + I^{x_p, a_p}$$

can also* be expressed by the function T.

It is clear that in this statement it is immaterial what integral of the third kind is adopted. For the difference between two elementary integrals of the third kind with infinities at ξ, γ is of the form

$$\lambda_1 u_1^{x,a} + \ldots\ldots + \lambda_p u_p^{x,a} + \lambda,$$

where $\lambda_1, \ldots, \lambda_p, \lambda$ may depend on ξ, γ but are independent of x; hence the difference between the two corresponding values of T is of the form

$$\lambda_1 U_1 + \ldots\ldots + \lambda_p U_p + \lambda;$$

and this is a single-valued function of $U_1, \ldots, U_p$.

For definiteness we may therefore suppose that $P^{x,a}_{\xi,\gamma}$ denotes the integral of the third kind obtained in Chap. IV. (§ 45. Also Chap. VII. § 134).

Then, firstly, when $x_1, \ldots, x_p$ are very near to $a_1, \ldots, a_p$, and $U_1, \ldots, U_p$ are small, T is given by

$$\sum_{i=1}^{p} \left\{ t_i \left[(a_i, \xi) - (a_i, \gamma) \right] \frac{da_i}{dt} + \frac{t_i^2}{\underline{|2}} D^2_{a_i} P^{a_i, c}_{\xi,\gamma} + \ldots\ldots \right\},$$

* The introduction of the function T is, I believe, due to Weierstrass. See *Crelle*, LII. p. 285 (1856) and *Mathem. Werke* (Berlin, 1894), I. p. 302. The other functions there used are considered below in Chaps. XI., XIII.

where t_i denotes the infinitesimal in the neighbourhood of the place a_i, c is an arbitrary place, and the notation is as in § 130, Chap. VII. It is intended of course that neither of the places ξ or γ is in the neighbourhood of any of the places $a_1, \ldots, a_p$. Now we have shewn that the infinitesimals $t_1, \ldots, t_p$ are expressible as convergent series in $U_1, \ldots, U_p$. Thus T is also expressible as a convergent series in $U_1, \ldots, U_p$ when $U_1, \ldots, U_p$ are sufficiently small.

Nextly, suppose the places $x_1, \ldots, x_p$ are not near to the places $a_1, \ldots, a_p$; determine, as in § 168, places to satisfy the equations

$$u_i^{z_1, a_1} + \ldots\ldots + u_i^{z_p, a_p} = - U_i/m,$$
$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_p, a_p} = U_i,$$

m being a large positive integer; then we shall also have (§ 158, Chap. VIII.)

$$P_{\xi, \gamma}^{x_1, a_1} + \ldots\ldots + P_{\xi, \gamma}^{x_p, a_p} + m\,(P_{\xi, \gamma}^{z_1, a_1} + \ldots\ldots + P_{\xi, \gamma}^{z_p, a_p}) = \log \frac{Z(\xi)}{Z(\gamma)},$$

where $Z(x)$ denotes the rational function which has a pole of the $(m+1)$th order at each of the places $a_1, \ldots, a_p$, and has a zero of the mth order at each of the places $z_1, \ldots, z_p$. The function $Z(x)$ has also a simple zero at each of the places $x_1, \ldots, x_p$, but this fact is not part of the definition of the function.

This equation can be written in the form

$$e^T = e^{-mT_0} \frac{Z(\xi)}{Z(\gamma)},$$

wherein T_0 denotes the sum

$$P_{\xi, \gamma}^{z_1, a_1} + \ldots\ldots + P_{\xi, \gamma}^{z_p, a_p}.$$

It follows by the proof just given that T_0 is expressible as a series of integral powers of the variables $U_1/m, \ldots, U_p/m$, which converges for sufficiently great values of m; and it is easy to see that the expression $Z(\xi)/Z(\gamma)$ is also expressible by series of integral powers of $U_1/m, \ldots, U_p/m$. For let the most general rational function having a pole of the $(m+1)$th order in each of $a_1, \ldots, a_p$ be of the form

$$Z(x) = \lambda_1 Z_1(x) + \ldots\ldots + \lambda_{mp} Z_{mp}(x) + \lambda,$$

wherein $Z_1(x), \ldots, Z_{mp}(x)$ are definite functions, and $\lambda, \lambda_1, \ldots, \lambda_{mp}$ are arbitrary constants. Then the expression of the fact that this function vanishes to the mth order at each of the places $z_1, \ldots, z_p$ will consist of mp equations determining $\lambda_1, \ldots, \lambda_{mp}$ rationally and symmetrically in terms of the places $z_1, \ldots, z_p$. Hence (by § 168) $\lambda_1, \ldots, \lambda_{mp}$ are expressible as series of integral powers of $U_1/m, \ldots, U_p/m$. Hence $Z(\xi)/Z(\gamma)$ is expressible by series of integral powers of $U_1/m, \ldots, U_p/m$.

Hence, for any finite values of $U_1, \dots, U_p$ the function e^T is expressible by series of integral powers of $U_1, \dots, U_p$. It is also obvious, from the method of proof adopted, that the series obtained for any set of values of $U_1, \dots, U_p$ are independent of the range of values for $U_1, \dots, U_p$ by which the final values are reached from the initial set $0, 0, \dots, 0$; so that the function e^T is a single valued function of $U_1, \dots, U_p$. The function e^T reduces to unity for the initial set $0, 0, \dots, 0$.

171. An actual expression of the function e^T, in terms of $U_1, \dots, U_p$, will be obtained in the next chapter (§ 187, Prop. xiii.). We shew here that if that expression be known, the solution of the inversion problem can also be given in explicit terms. Let $\Pi_{\xi,\gamma}^{x,a}$ denote the normal elementary integral of the third kind (Chap. II., § 14). Then if K denote the sum

$$K = \Pi_{\xi,\gamma}^{x_1,a_1} + \dots\dots + \Pi_{\xi,\gamma}^{x_p,a_p},$$

it follows, as here, that e^K is a single valued function of $U_1, \dots, U_p$, whose expression is known when that of e^T is known, and conversely. Denote e^K by $V(U_1, \dots, U_p;\ \xi, \gamma)$. Let $Z(x)$ denote any rational function whatever, its poles being the places $\gamma_1, \dots, \gamma_k$; and let the places at which $Z(x)$ takes an arbitrary value X be denoted by $\xi_1, \dots, \xi_k$. Then, from the equation (Chap. VIII., § 154),

$$\Pi_{x_i,a_i}^{\xi_1,\gamma_1} + \dots\dots + \Pi_{x_i,a_i}^{\xi_k,\gamma_k} = \log\frac{Z(x_i) - X}{Z(a_i) - X}, \qquad (i = 1, 2, \dots, p),$$

we obtain *

$$V(U_1, \dots, U_p;\ \xi_1, \gamma_1) \dots V(U_1, \dots, U_p;\ \xi_k, \gamma_k) = \frac{[X - Z(x_1)] \dots [X - Z(x_p)]}{[X - Z(a_1)] \dots [X - Z(a_p)]};$$

the left-hand side of this equation has, we have said, a well ascertained expression, when the values of $U_1, \dots, U_p$, the function $Z(x)$, and the value X, are all given; hence, substituting for X in turn any p independent values, we can calculate the expression of any symmetrical function of the quantities

$$Z(x_1), \dots, Z(x_p),$$

and this will constitute the complete solution of the inversion problem.

It has been shewn in § 152, Chap. VIII. that any Abelian integral $I^{x,a}$ can be written as a sum of elementary integrals of the third kind and of differential coefficients of such integrals, together with integrals of the first kind. Hence, when the expression of $V(U_1, \dots, U_p;\ \xi, \gamma)$ is obtained, that of the sum

$$I^{x_1,a_1} + \dots\dots + I^{x_p,a_p}$$

can also be obtained.

* Clebsch u. Gordan, *Abels. Functionen*, (1866), p. 175.

172. The consideration of the function

$$\Pi_{\xi,\gamma}^{x_1, a_1} + \ldots\ldots + \Pi_{\xi,\gamma}^{x_p, a_p},$$

which is contained in this chapter is to be regarded as of a preliminary character. It will appear in the next chapter that it is convenient to consider this function as expressed in terms of another function, the theta function. It is possible to build up the theta function in an *à priori* manner, which is a generalization of that, depending on the equation

$$\wp u = -\frac{d^2}{du^2} \log \sigma(u),$$

whereby, in the elliptic case, the σ-function may be supposed derived from the function $\wp(u)$. But this process is laborious, and furnishes only results which are more easily evident *à posteriori*. For this reason we proceed now immediately to the theta functions; formulae connecting these functions with the algebraical integrals so far considered are given in chapters X. XI. and XIV.