CHAPTER VIII.

ABEL'S THEOREM; ABEL'S DIFFERENTIAL EQUATIONS.

148. THE present chapter is mainly concerned with that theorem with which the subject of the present volume may be said to have begun. It will be seen that with the ideas which have been analysed in the earlier part of the book, the statement and proof of that theorem is a matter of great simplicity.

The problem of the integration of a rational algebraical function (of a single variable) leads to the introduction of a transcendental function, the logarithm; and the integral of any such rational function can be expressed as a sum of rational functions and logarithms of rational functions. More generally, an integral of the form

$$\int dx\, R\,(x,\, y,\, y_1,\, \dots,\, y_k),$$

wherein $x, y, y_1, y_2, \dots$ are capable of rational expression in terms of a single parameter, and R denotes any rational algebraic function, can be expressed as a sum of rational functions of this parameter, and logarithms of rational functions of the same. This includes the case of an integral of the form

$$\int dx\, R\,(x,\, \sqrt{ax^2 + bx + c}).$$

But an integral of the form

$$\int dx\, R\,(x,\, \sqrt{ax^4 + bx^3 + cx^2 + dx + e})$$

cannot, in general, be expressed by means of rational or logarithmic functions; such integrals lead in fact to the introduction of other transcendental functions than the logarithm, namely to elliptic functions; and it appears that the nearest approach to the simplicity of the case, in which the subject of integration is a rational function, is to be sought in the relations which exist for the *sums* of like elliptic integrals. For instance, we have the equation

$$\int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - \int_0^{x_3} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 0,$$

provided

$$x_3(1-k^2x_1^2x_2^2) = x_1\sqrt{(1-x_2^2)(1-k^2x_2^2)} + x_2\sqrt{(1-x_1^2)(1-k^2x_1^2)}.$$

On further consideration, however, it is clear that this is not a complete statement; and it is proper, beside the quantity x, to introduce a quantity y, such that

$$y^2-(1-x^2)(1-k^2x^2)=0,$$

and to regard y, for any value of x, as equally capable either of the positive or negative sign; in fact by varying x continuously from any value, through one of the values $x=\pm 1$, $x=\pm\frac{1}{k}$, and back to its original value, we can suppose that y varies continuously from one sign to the other. Then the theorem in question can be written thus;

$$\int_{(0,1)}^{(x_1,y_1)}\frac{dx_1}{y_1}+\int_{(0,1)}^{(x_2,y_2)}\frac{dx_2}{y_2}+\int_{(0,1)}^{(x_3,y_3)}\frac{dx_3}{y_3}=0,$$

where the limits specify the value of y as well as the value of x. The theorem holds when, in the first two integrals the variables (x, y) are taken through any continuous succession of simultaneous values, from the lower to the upper limits, the variables in the last integral being, at every stage of the integration, defined by the equations

$$-x_3(1-k^2x_1^2x_2^2)=x_1y_2+x_2y_1,$$
$$y_3(1-k^2x_1^2x_2^2)^2=y_1y_2(1+k^2x_1^2x_2^2)-x_1x_2(1-k^2x_1^2x_2^2)(1-k^2).$$

The quantity y is called an algebraical function of x; and the notion thus introduced is a fundamental one in the theorems to be considered; its complete establishment has been associated, in this volume, with a Riemann surface.

In the case where $y^2=(1-x^2)(1-k^2x^2)$ we have the general theorem that, if $R(x, y)$ be any rational function of x, y, the sum of any number, m, of similar integrals

$$\int_{(a_1,b_1)}^{(x_1,y_1)}R(x,y)\,dx+\dots\dots+\int_{(a_m,b_m)}^{(x_m,y_m)}R(x,y)\,dx$$

can be expressed by rational functions of $(x_1, y_1), \dots, (x_m, y_m)$, and logarithms of such rational functions, with the addition of an integral

$$-\int_{(a_{m+1},b_{m+1})}^{(x_{m+1},y_{m+1})}R(x,y)\,dx.$$

Herein the lower limits $(a_1, b_1), \dots, (a_m, b_m)$ represent arbitrary pairs of corresponding values of x and y, and the succession of values for the pairs $(x_1, y_1), \dots, (x_m, y_m)$ is quite arbitrary; but in the last integral x_{m+1}, y_{m+1} are each rational functions of $(x_1, y_1), \dots, (x_m, y_m)$, which must be properly determined, and it is understood that the relations are preserved at all stages of the integration, so that for example a_{m+1}, b_{m+1} are respectively taken to be the same rational functions of $(a_1, b_1), \ldots, (a_m, b_m)$. The question of what alteration is necessary in the enunciation when this convention is not observed, is the question of the change in the value of an integral

$$\int_{(a_{m+1}, b_{m+1})}^{(x_{m+1}, y_{m+1})} R(x, y)\, dx$$

when the path of integration is altered. This question is fully treated in the consideration of the Riemann surface, with the help of what have been called period loops.

149. Abel's theorem may be regarded as a generalization of the theorem just stated, and may be enunciated as follows: Let y be the algebraical function of x defined by an equation of the form

$$f(y, x) = y^n + A_1 y^{n-1} + \ldots\ldots + A_n = 0,$$

wherein $A_1, \ldots, A_n$ are rational polynomials in x, and the left-hand side of the equation is supposed incapable of resolution into the product of factors of the same rational form; let $R(x, y)$ be any rational function of x and y; then the sum of any number, m, of similar integrals

$$\int^{(x_1, y_1)} R(x, y)\, dx + \ldots\ldots + \int^{(x_m, y_m)} R(x, y)\, dx,$$

with arbitrary lower limits, is expressible by rational functions of $(x_1, y_1), \ldots, (x_m, y_m)$, and logarithms of such rational functions, with the addition of the sum of a certain number, k, of integrals,

$$-\int^{(z_1, s_1)} R(x, y)\, dx - \ldots\ldots - \int^{(z_k, s_k)} R(x, y)\, dx,$$

wherein $z_1, \ldots, z_k$ are values of x, determinable from $x_1, y_1, \ldots, x_m, y_m$ as the roots of an algebraical equation whose coefficients are rational functions of $x_1, y_1, \ldots, x_m, y_m$, and $s_1, \ldots, s_k$ are the corresponding values of y, of which any one, say s_i, is determinable as a rational function of z_i, and $x_1, y_1, \ldots, x_m, y_m$. The relations thus determining $(z_1, s_1), \ldots, (z_k, s_k)$ from $(x_1, y_1), \ldots, (x_m, y_m)$ may be supposed to hold at all stages of the integration; in particular they determine the lower limits of the last k integrals from the arbitrary lower limits of the first m integrals. The number k does not depend upon m, nor upon the form of the rational function $R(x, y)$; and in general it does not depend upon the values of $(x_1, y_1), \ldots, (x_m, y_m)$, but only upon the fundamental equation which determines y in terms of x.

150. In this enunciation there is no indication of the way in which the equations determining $z_1, s_1, \ldots, z_k, s_k$ from $x_1, y_1, \ldots, x_m, y_m$ are to be found. Let $\theta(y, x)$ be an integral polynomial in x and y, wherein some or all of the coefficients are regarded as variable. By continuous variation of these

coefficients the set of corresponding values of x and y which satisfy both the equations $f(y, x)=0$, $\theta(y, x)=0$, will also vary continuously. Then, if m be the number of variable coefficients of $\theta(y, x)$, and $m+k$ the total number of variable pairs (x, y) which satisfy both the equations $f(y, x)=0$, $\theta(y, x)=0$, the necessary relations between $(x_1, y_1), \ldots, (x_m, y_m), (z_1, s_1), \ldots, (z_k, s_k)$ are expressed by the fact that these pairs are the common solutions of the equations $f(y, x)=0$, $\theta(y, x)=0$. The polynomial $\theta(y, x)$ may have any form in which there enter m variable coefficients; by substitution, in $\theta(y, x)$, of the m pairs of values $(x_1, y_1), \ldots, (x_m, y_m)$, we can determine these variable coefficients as rational functions of $x_1, y_1, \ldots, x_m, y_m$; by elimination of y between the equations $\theta(y, x)=0$, $f(y, x)=0$, we obtain an algebraic equation for x, breaking into two factors, $P_0(x)\,P(x)=0$, one factor, $P_0(x)$, not depending on $x_1, y_1, \ldots, x_m, y_m$, and vanishing for the values of x at the fixed solutions of $f(y, x)=0$, $\theta(y, x)=0$, which do not depend on $x_1, y_1, \ldots, x_m, y_m$, the other factor, $P(x)$, having the form

$$(x-x_1)\ldots(x-x_m)(x^k+R_1x^{k-1}+\ldots+R_k),$$

where $R_1, \ldots, R_k$ are rational functions of $x_1, y_1, \ldots, x_m, y_m$. Finally, from the equations $f(s_i, z_i)=0$, $\theta(s_i, z_i)=0$ we can determine s_i rationally in terms of $z_i, x_1, y_1, \ldots, x_m, y_m$. As a matter of fact the rational functions of $x_1, y_1, \ldots, x_m, y_m$, which appear on the right-hand side of the equation which expresses Abel's theorem, are rational functions of the variable coefficients in $\theta(y, x)$.

151. When $\theta(y, x)$ is quite general save for the condition of having certain fixed zeros satisfying $f(y, x)=0$, the forms of $(z_1, s_1), \ldots, (z_k, s_k)$ as functions of $(x_1, y_1), \ldots, (x_m, y_m)$ are independent of the form of $\theta(y, x)$. This appears from the following enunciation of the theorem, which introduces ideas that have been elaborated since Abel's time, and which we regard as the final form—Let $(a_1, b_1), \ldots, (a_Q, b_Q)$ be any places of the Riemann surface whatever, such that sets coresidual therewith have a multiplicity q, and a sequence $Q-q=p-\tau-1$, where $\tau+1$ is the number of ϕ polynomials vanishing in the places $(a_1, b_1), \ldots, (a_Q, b_Q)$; let $(x_1, y_1), \ldots, (x_q, y_q)$ be q arbitrary places determining a set coresidual with $(a_1, b_1), \ldots, (a_Q, b_Q)$, and $(z_1, s_1), \ldots, (z_{p-\tau-1}, s_{p-\tau-1})$ be the sequent places of this set*; then, $R(x, y)$ being any rational function of (x, y), the sum

$$\int_{(a_1, b_1)}^{(x_1, y_1)} R(x, y)\,dx+\ldots\ldots+\int_{(a_q, b_q)}^{(x_q, y_q)} R(x, y)\,dx$$

is expressible by rational functions of $(x_1, y_1), \ldots, (x_q, y_q)$, and logarithms of such rational functions, with the addition of a sum

$$-\int_{(a_{q+1}, b_{q+1})}^{(z_1, s_1)} R(x, y)\,dx-\ldots\ldots-\int_{(a_Q, b_Q)}^{(z_{p-\tau-1}, s_{p-\tau-1})} R(x, y)\,dx$$

* See Chap. VI. § 95.

herein it is understood that the paths of integration are such that at every stage the variables form a set coresidual with $(a_1, b_1), \ldots, (a_Q, b_Q)$.

The places $(a_1, b_1), \ldots, (a_Q, b_Q)$ may therefore be regarded as the poles, and $(x_1, y_1), \ldots, (x_q, y_q), (z_1, s_1), \ldots, (z_{p-\tau-1}, s_{p-\tau-1})$ as the zeros, of the same rational function $Z(x)$; if $\theta_1(y, x)$ denote the form of the polynomial $\theta(y, x)$ when it vanishes in $(a_1, b_1), \ldots, (a_Q, b_Q)$, and $\theta_2(y, x)$ denote its form when its zeros are $(x_1, y_1), \ldots, (z_1, s_1), \ldots$, the function $Z(x)$ may be expressed in the form $\theta_2(y, x)/\theta_1(y, x)$. If the polynomials $\theta_1(y, x)$, $\theta_2(y, x)$ are not adjoint, the function will be of the kind, hitherto regarded as special, which takes the same value at all the places of the Riemann surface which correspond to a multiple point of the plane curve represented by the equation $f(y, x) = 0$; this fact does not affect the application of Abel's theorem to the case.

152. To prove the theorem thus enunciated, with the greatest possible definiteness, we shew first that it may be reduced to two simple cases.

In the neighbourhood of any place of the Riemann surface, at which t is the infinitesimal, we can express $R(x, y)\dfrac{dx}{dt}$ in a series of positive and negative powers of t, in which the number of negative powers is finite. Let the expression at some place, ξ, where negative powers actually enter, be denoted by

$$\underline{|m-1}\,\frac{A_m}{t^m} + \underline{|m-2}\,\frac{A_{m-1}}{t^{m-1}} + \ldots\ldots + \frac{A_2}{t^2} + \frac{A_1}{t} + B + B_1 t + B_2 t^2 + \ldots\ldots;$$

then, if $P^{x,c}_{\xi,\gamma}$ denote any elementary integral of the third kind, with infinities at ξ, γ, and $E^{x,c}_{\xi}$ denote the differential coefficient of $P^{x,c}_{\xi,\gamma}$ in regard to the infinitesimal at ξ, the places γ, c being arbitrary, the difference

$$\int_{(a,b)}^{(x,y)} R(x, y)\,dx - A_1 P^{x,c}_{\xi,\gamma} - A_2 E^{x,c}_{\xi} - A_3 D_{\xi} E^{x,c}_{\xi} - \ldots\ldots - A_m D^{m-2}_{\xi} E^{x,c}_{\xi},$$

wherein D_ξ denotes differentiation in regard to the infinitesimal at ξ, is finite at the place ξ. The number of places, ξ, at which negative powers of t enter in the expansion of $R(x, y)\dfrac{dx}{dt}$, is finite; dealing with each in turn we obtain an expression of the form

$$\int_{(a,b)}^{(x,y)} R(x, y)\,dx - \sum_{\xi} [A_1 P^{x,c}_{\xi,\gamma} + A_2 E^{x,c}_{\xi} + A_3 D_{\xi} E^{x,c}_{\xi} + \ldots\ldots + A_m D^{m-2}_{\xi} E^{x,c}_{\xi}],$$

wherein γ, c are taken the same for every place ξ; this is finite at all places of the Riemann surface, except possibly the place γ. If t_γ be the infinitesimal at this place the function is there infinite like $(\Sigma A_1)\log t_\gamma$. But in fact ΣA_1 is zero (Chap. II. § 17, Ex. (δ); Chap. VII. § 137, Ex. vi.). Hence the function under consideration is nowhere infinite, and is therefore necessarily* a linear aggregate of integrals of the first kind, plus a constant. Hence if $u_1^{x, a}, \ldots, u_p^{x, a}$ be a set of linearly independent integrals of the first kind, a denoting the place (a, b), and $C_1, \ldots, C_p$ be proper constants, we have

$$\int_a^x R(x, y)\, dx = \sum_\xi (A_1 + A_2 D_\xi + \ldots\ldots + A_m D_\xi^{m-2}) P_{\xi, \gamma}^{x, a} + C_1 u_1^{x, a} + \ldots\ldots + C_p u_p^{x, a}.$$

The consideration of the sum

$$\int_{a_1}^{x_1} R(x, y)\, dx + \ldots\ldots + \int_{a_Q}^{x_Q} R(x, y)\, dx,$$

wherein $a_1, \ldots, a_Q$ denote the places $(a_1, b_1), \ldots, (a_Q, b_Q)$, and $x_1, \ldots, x_Q$ denote the places $(x_1, y_1), \ldots, (x_q, y_q), (z_1, s_1), \ldots, (z_{p-\tau-1}, s_{p-\tau-1})$, is thus reduced to the consideration of the two sums

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_Q, a_Q}, \qquad (i = 1, 2, \ldots, p.)$$

$$P_{\xi, \gamma}^{x_1, a_1} + \ldots\ldots + P_{\xi, \gamma}^{x_Q, a_Q}.$$

Ex. i. By the proposition here repeated from § 20, Chap. II., it follows that any rational function can be written in the form

$$R(x, y) = \sum_\xi \{A_1 [(x, \xi) - (x, \gamma)] + A_2 D_\xi (x, \xi) + \ldots + A_m D_\xi^{m-2} (x, \xi)\}$$

$$+ [(x, 1)^{\tau_1' - 1} \phi_1(x, y) + \ldots + (x, 1)^{\tau'_{n-1} - 1} \phi_{n-1}(x, y)] / f'(y)$$

where (cf. § 45, Chap. IV.)

$$(x, \xi) = [\phi_0(x, y) + \sum_1^{n-1} \phi_r(x, y)\, g_r(\xi, \eta)] / (x - \xi) f'(y),$$

η being the value of y at the place ξ.

Ex. ii. Prove also that any rational function with simple poles at $\xi_1, \xi_2, \ldots$ can be written in the form

$$\lambda_1 [(\xi_1, x) - (\xi_1, a)] + \lambda_2 [(\xi_2, x) - (\xi_2, a)] + \ldots,$$

$\lambda_1, \lambda_2, \ldots$ being constants, and a denoting an arbitrary place (cf. § 130, Chap. VII.).

153. We shall prove, now, in regard to these two sums, under the conventions that the upper limits are coresidual with the lower limits, and that the Q paths of integration are such that at every stage the variables are at places also coresidual with the lower limits, a convention under which the paths of integration may quite well cross the period loops on the Riemann surface, that the first sum is zero for all values of i, and the second equal to $\log Z(\xi)/Z(\gamma)$, $Z(x)$ being the† rational function which has $a_1, \ldots, a_Q$ as poles and $x_1, \ldots, x_Q$ as zeros. The sense in which the logarithm is to be understood will appear from the proof of the theorem. If we suppose the lower limits arbitrarily assigned, the general function $Z(x)$, of which these

* Forsyth, *Theory of Functions*, § 234.

† If two rational functions have the same poles and the same zeros their ratio is necessarily a constant.

places $a_1, \ldots, a_Q$ are the poles, will contain $q+1$ arbitrary linear coefficients, entering homogeneously, and the assignation of q of the zeros, say $x_1, \ldots, x_q$, will determine the others, as explained.—The equations giving the determination will be such functions of $a_1, \ldots, a_Q$ as are identically satisfied by these places, $a_1, \ldots, a_Q$. Hence the general form of Abel's theorem is

$$\sum_{i=1}^{Q} \int_a^{x_i} R(x, y)\, dx = \sum_{\xi} \left[A_1 \log \frac{Z(\xi)}{Z(\gamma)} + A_2 \frac{Z'(\xi)}{Z(\xi)} + \ldots\ldots \right]$$
$$= \sum_{\xi} \left[A_1 \log Z(\xi) + A_2 \frac{Z'(\xi)}{Z(\xi)} + \ldots\ldots \right]$$

where $Z'(\xi) = D_\xi Z(\xi)$; the term $\sum_{\xi} A_1 \log Z(\gamma) = \log Z(\gamma) \Sigma A_1$ can be omitted because $\Sigma A_1 = 0$ (Chap. II. p. 20 (δ)). Herein $Z(\xi)$ is a rational function of $a_1, \ldots, a_Q$ and $x_1, \ldots, x_q$.

154. In carrying out the proof we make at first a simplification—Let $Z(x)$, or Z, be the rational function having $a_1, \ldots, a_Q$ as simple poles and $x_1, \ldots, x_Q$ as simple zeros, these places being supposed to be all different; trace on the Riemann surface an arbitrary path joining a_1 to x_1, chosen so as to avoid all places where dZ is zero to higher than the first order, and let μ be the value of Z at any place of this path; then there will be $Q-1$ other places at which Z has the same value μ; the paths traced by these $Q-1$ places as μ varies from ∞ to 0 are the paths we assign for the $Q-1$ integrals following the first. The simultaneous positions thus defined for the variables in the Q integrals are, for $q > 1$, not so general* as those allowed by the convention that the simultaneous positions are coresidual with $a_1, \ldots, a_Q$; but it will be seen that the more general case is immediately deducible from the particular one.

Consider now, for any value of μ, the rational function

$$\frac{1}{Z-\mu} \frac{dI}{dx},$$

$I, = \int R(x, y)\, dx$, being any Abelian integral whatever. In accordance with a theorem previously used (Chap. II. p. 20 (δ); Chap. VII. § 137, Ex. vi.) the sum of the coefficients of t^{-1} in the expansions of $(Z-\mu)^{-1} dI/dt$, in terms of the infinitesimal t, at all places where negative powers of t occur, is equal to zero. Of such places there are first the Q places where Z is equal to μ. We shall suppose that dI/dt is finite at all these places; then the sum of the coefficients of t^{-1} at these places is

$$\Sigma \frac{1}{d\mu/dt} \left(\frac{dI}{dt}\right), \quad = \left(\frac{dI}{d\mu}\right)_1 + \ldots\ldots + \left(\frac{dI}{d\mu}\right)_Q,$$

* Sets coresidual with two given coresidual sets have a multiplicity q; but sets equivalent with two given coresidual sets have a variability expressible by one parameter only (cf. Chap. VI. §§ 94—96).

provided $Z-\mu$ be not zero to the second order at any of the places, that is, provided dZ be not zero to higher than the first order. In accordance with the convention made as to the paths of the variables in the integrals, we suppose this condition to be satisfied.

Hence this sum is equal to the sum of the coefficients of t^{-1} in the expansions of the function $-(Z-\mu)^{-1}\, dI/dt$ at all places, only, where dI/dt is infinite; this result we may write in the form

$$\left(\frac{dI}{d\mu}\right)_1 + \ldots\ldots + \left(\frac{dI}{d\mu}\right)_Q = -\left(\frac{dI}{dt}\,\frac{1}{Z-\mu}\right)_{t^{-1}};$$

we may regard this equation as a convenient way of stating Abel's theorem for many purposes; and may suppose the case, in which an infinity of dI/dt coincides with a place at which $Z=\mu$, to be included in this equation, the left hand being restricted to all places at which $Z=\mu$ and dI/dt is not infinite.

In this equation, in case $I, = u_i^{x,\,a}$, be any integral of the first kind, the right hand vanishes; then, integrating in regard to μ from ∞ to 0, we obtain

$$u_i^{x_1,\,a_1} + \ldots\ldots + u_i^{x_Q,\,a_Q} = 0. \tag{A}$$

In case I be an integral of the third kind, $= P_{\xi,\gamma}^{x,\,c}$ say, and Z be not equal to μ either at ξ or γ, the right hand is equal to

$$-\frac{1}{Z(\xi)-\mu} + \frac{1}{Z(\gamma)-\mu};$$

hence, integrating,

$$P_{\xi,\gamma}^{x_1,\,a_1} + \ldots\ldots + P_{\xi,\gamma}^{x_Q,\,a_Q}, \; = \int_\infty^0 d\mu\left(-\frac{1}{Z(\xi)-\mu} + \frac{1}{Z(\gamma)-\mu}\right), \; = \log\frac{Z(\xi)}{Z(\gamma)}, \tag{B}$$

while, if the places at which the rational function $Z(x)$ has the values μ, ν be respectively denoted by

$$x_1', \ldots\ldots, x'_Q,$$

and

$$a_1', \ldots\ldots, a'_Q$$

we have

$$P_{\xi,\gamma}^{x_1',\,a_1'} + \ldots\ldots + P_{\xi,\gamma}^{x'_Q,\,a'_Q}, \; = \int_\nu^\mu d\mu\left(-\frac{1}{Z(\xi)-\mu} + \frac{1}{Z(\gamma)-\mu}\right),$$

$$= \log\left[\frac{Z(\xi)-\mu}{Z(\xi)-\nu}\Big/\frac{Z(\gamma)-\mu}{Z(\gamma)-\nu}\right].$$

For any Abelian integral we similarly have

$$I^{x_1',\,a_1'} + \ldots\ldots + I^{x'_Q,\,a'_Q} = \left[\frac{dI}{dt}\log\frac{Z(x)-\mu}{Z(x)-\nu}\right]_{t^{-1}},$$

which is a complete statement of Abel's theorem.

155. In the equation (B), and in the equation which follows it, the significance of the logarithm is determined by the path of μ in the integral expression which defines the logarithm; we may also define the logarithm by considering the two sides of the equation as functions of ξ.

There is no need to extend the equation (B) to the case where one of the paths of integration on the left passes through either ξ or γ, since in that case a corresponding infinite term enters on both sides of the equation.

But it is clear that the condition that no two of the upper limits $x_1, \dots, x_Q$ should be coincident is immaterial, and may be removed. And if two (or more) of the places at which Z takes any value, μ, should coincide, the equations (A) and (B) can be formed each as the sum of two equations in which the course of integration is respectively from $Z=\infty$ to $Z=\mu$ and from $Z=\mu$ to $Z=0$, and the final outcome can only be that the order in which the upper limits $x_1, \dots, x_Q$ are associated with the lower limits $a_1, \dots, a_Q$ may undergo a change. But in the general case we may equally put, for example, in equations (A), (B),

$$\int_{a_1}^{x_1} dI + \int_{a_2}^{x_2} dI, = \int_{a_1}^{x_2} dI + \int_{x_2}^{x_1} dI + \int_{x_1}^{x_2} dI + \int_{a_2}^{x_1} dI, = \int_{a_1}^{x_2} dI + \int_{a_2}^{x_1} dI,$$

with proper conventions as to the paths; hence the condition that dZ shall not be zero to higher than the first order at any stage of the integration may be discarded also, with a certain loss of definiteness. The most general form of equation (A), when each of the Q paths of integration are arbitrary, is of course

$$u_i^{x_1, a_1} + \dots\dots + u_i^{x_Q, a_Q} = M_1 \omega_{i,1} + \dots\dots + M_p \omega_{i,p} + M_1' \omega'_{i,1} + \dots\dots + M_p' \omega'_{i,p}, \quad \text{(C)}$$

where $\omega_{i,1}, \dots, \omega'_{i,p}$ are the periods of $u_i^{x,a}$ and $M_1, \dots, M_p'$ are rational integers, independent of i. We shall subsequently see that this equation is sufficient to prove that the places $x_1, \dots, x_Q$ are coresidual with the set $a_1, \dots, a_Q$.

If, in equation (B), we substitute for $Z(x)$ any one of its rational expressions, say* $\theta_2(x)/\theta_1(x)$, we shall obtain

$$P_{\xi,\gamma}^{x_1, a_1} + \dots\dots + P_{\xi,\gamma}^{x_Q, a_Q} = \log \frac{\theta_2(\xi)}{\theta_1(\xi)} \Big/ \frac{\theta_2(\gamma)}{\theta_1(\gamma)},$$

where, now, $\theta_2(x)$, $\theta_1(x)$ are any two polynomials, integral in x and y, of which, beside common zeros, $\theta_2(x)$ has $x_1, \dots, x_Q$ for zeros, and $\theta_1(x)$ has $a_1, \dots, a_Q$ for zeros. If in this equation we suppose any of the coefficients in $\theta_2(x)$ to vary infinitesimally in any way, such that the common zeros of $\theta_2(x)$

* $\theta(x)$ is, for shortness, put for what would more properly be denoted by $\theta(y, x)$.

and $\theta_1(x)$ remain fixed, $\theta_2(x)$ changing thereby into $\theta_2(x)+\delta\theta_2(x)$, the places $x_1, \dots, x_Q$ changing thereby to $x_1+dx_1, \dots, x_Q+dx_Q$, we shall obtain

$$\frac{dP_{\xi,\gamma}^{x_1,c}}{dx_1}dx_1+\dots\dots+\frac{dP_{\xi,\gamma}^{x_Q,c}}{dx_Q}dx_Q=\delta\log\frac{\theta_2(\xi)}{\theta_2(\gamma)},$$

which is slightly more general than any equation before given, in that the places $x_1+dx_1, \dots, x_Q+dx_Q$, though coresidual with $x_1, \dots, x_Q$, are not necessarily such that the function $\theta_2(x)/\theta_1(x)$ has the same value at all of them. This general equation is obtained by Abel in the course of his proof of his theorem.

For any Abelian integral we have, similarly, the equation

$$\frac{dI}{dx_1}dx_1+\dots\dots+\frac{dI}{dx_Q}dx_Q=\left[\frac{\overline{dI}}{dt}\,\delta\log\theta(x)\right]_{t^{-1}},$$

which, also, may be regarded as a complete statement of Abel's theorem.

156. In equation (B) the logarithm of the right hand will disappear if $Z(\xi)=Z(\gamma)$, namely if the infinities of the integral be places at which the function $Z(x)$ has the same value.

One case of this may be noticed; if $\psi(y, x)$ be an integral polynomial of grade $(n-1)\sigma+n-3$ (cf. Chap. VI. §§ 86, 91), which is adjoint at all places except those two, say A, A', which correspond to an ordinary double point of the curve represented by the equation $f(y, x)=0$, the integral

$$V^{x,a}, =\int_a^x \frac{\psi(y, x)}{f'(y)}dx,$$

will be an integral of the third kind having A, A' as its infinities. Hence, if in forming the function $Z(x)$, $=\theta_2(x)/\theta_1(x)$, the places A, A' have been disregarded, so that the polynomials $\theta_1(x)$, $\theta_2(x)$ do not vanish in these places, the function $Z(x)$ will take the same value at A as at A', and we shall obtain

$$V^{x_1,a_1}+\dots\dots+V^{x_Q,a_Q}=0.$$

Hence we obtain the result: if, in the formation of the integrals of the first kind for a given fundamental curve, we overlook the existence of a certain number, say δ, of double points, we shall obtain $p+\delta$ integrals, where p is the true deficiency of the curve; and these integrals will be linear aggregates of the actual integrals of the first kind and of δ integrals of the third kind. If in the formation of the rational functions also we overlook the existence of these double points, Abel's theorem will have the same form of equation for the $p+\delta$ integrals as if they were integrals of the first kind (cf. §§ 83, 90, and Abel, *Œuvres Comp.*, Christiania, 1881, Vol. I. p. 167).

For example, let $a_1, \dots, a_Q$ be arbitrary places in which $\tau+1$ ϕ-polynomials vanish (Chap. VI. §§ 101, 93). Take $q\,(=Q-p+\tau+1)$ arbitrary places $c_1, \dots, c_q$, and so determine the set $c_1, \dots, c_Q$ coresidual with $a_1, \dots, a_Q$. A rational function, $\zeta(x)$, which has the places $a_1, \dots, a_Q$ for poles and the places $c_1, \dots, c_Q$ for zeros is quite determinate save for a constant multiplier. Let $x_1, \dots, x_Q$ be any set of places at which $\zeta(x)$ has the same value, A say, so that $x_1, \dots, x_Q$ are the zeros of $\zeta(x) - A$; then, as $a_1, \dots, a_Q$ are the poles of $\zeta(x) - A$, we have

$$P_{c_1, c_2}^{x_1, a_1} + \dots\dots + P_{c_1, c_2}^{x_Q, a_Q} = \log \frac{\zeta(c_1) - A}{\zeta(c_2) - A},$$

and as $\zeta(c_1) = \zeta(c_2) = 0$, the right hand is zero.

Hence, calling the places where a definite rational function has the same value a set of *level points* for the function, we can make the statement—the level points of a definite function satisfy the equations

$$\frac{dP_{c_1, c_2}^{x_1}}{dx_1} dx_1 + \dots\dots + \frac{dP_{c_1, c_2}^{x_Q}}{dx_Q} dx_Q = 0,$$

c_1, c_2 being any two of the zeros of the function.

In particular, when $q = 1$, the sets of level points are the most general sets coresidual with the poles or zeros of the function. Hence, if $x_1, \dots, x_{p+1}$ be any set of places coresidual with a fixed set $c_1, c_2, \dots, c_{p+1}$, in which no ϕ-polynomials vanish, we have the equations

$$\frac{dP_{c_1, c_2}^{x_1}}{dx_1} dx_1 + \dots\dots + \frac{dP_{c_1, c_2}^{x_{p+1}}}{dx_{p+1}} dx_{p+1} = 0.$$

157. *Ex.* i. We give an example of the application of Abel's theorem.

For the surface associated with the equation

$$y^2 = 4x^{2p+1} - g_1 x^{2p-1} - g_2 x^{2p-2} - \dots - g_{2p}$$

the integral

$$I = \int \frac{x^p + c_1 x^{p-1} + \dots + c_p}{y} dx$$

is of the second kind, becoming infinite only at the (single) place $x = \infty$. Consider the rational function

$$Z = \frac{y + Ax^p + Bx^{p-1} + \dots + Kx + L}{y + A_0 x^p + B_0 x^{p-1} + \dots + K_0 x + L_0},$$

which, for general values of $A, \dots, L_0$, is of the $(2p+1)$th order, its zeros, for instance, being given by

$$4x^{2p+1} - g_1 x^{2p-1} - \dots - g_{2p} - (Ax^p + \dots + L)^2 = 0.$$

To evaluate the expression

$$\left(\frac{dI}{dt} \frac{1}{Z - \mu}\right)_{t^{-1}},$$

the place $x = \infty$ being the only one to be considered, we put $x = t^{-2}$ and obtain

$$y = \frac{2}{t^{2p+1}} \left(1 - \tfrac{1}{8} g_1 t^4 - \tfrac{1}{8} g_2 t^6 - \dots\dots\right),$$

$$Z=\frac{1+\frac{1}{2}At+\dots\dots}{1+\frac{1}{2}A_0t+\dots\dots}=1+\tfrac{1}{2}(A-A_0)t+\dots\dots,$$

$$\frac{1}{Z-\mu}=\frac{1}{1-\mu}-\frac{1}{2}\frac{A-A_0}{(1-\mu)^2}t+\dots\dots,$$

$$\frac{dI}{dt}=\frac{\frac{1}{t^{2p}}+\frac{c_1}{t^{2p-2}}+\dots\dots}{\frac{2}{t^{2p+1}}(1-\frac{1}{8}g_1t^4-\dots\dots)}\frac{-2}{t^3},$$

$$=-\frac{1}{p^2}(1+c_1t^2+c_2t^4+\dots)(1+\tfrac{1}{8}g_1t^4+\dots),$$

$$=-\frac{1}{t^2}-c_1-(c_2+\tfrac{1}{8}g_1)t^2-\dots\dots,$$

and therefore

$$\frac{dI}{dt}\frac{1}{Z-\mu}=-\frac{1}{1-\mu}\frac{1}{t^2}+\frac{1}{2}\frac{A-A_0}{(1-\mu)^2}\frac{1}{t}-\frac{c_1}{1-\mu}-\dots\dots,$$

wherein the coefficient of t^{-1} is $\frac{1}{2}(A-A_0)(1-\mu)^{-2}$.

Hence, if $x_1,\dots,x_{2p+1}$ be the zeros, and $a_1,\dots,a_{2p+1}$ be the poles of Z, we have

$$I^{x_1,\,a_1}+\dots+I^{x_{2p+1},\,a_{2p+1}}=-\tfrac{1}{2}(A-A_0)\int_{\infty}^{0}\frac{d\mu}{(1-\mu)^2}=-\tfrac{1}{2}(A-A_0).$$

Now the zeros of Z are zeros of the polynomial

$$y+U(x)=y+Ax^p+Bx^{p-1}+\dots\dots+Kx+L=0;$$

denoting the values of y by $y_1,\dots,y_{2p+1}$, and using $F(x)$ for $(x-x_1)\dots\dots(x-x_{p+1})$, where $(x_1,y_1),\dots,(x_{p+1},y_{p+1})$ are any $p+1$ of the places $(x_1,y_1),\dots,(x_{2p+1},y_{2p+1})$, we have, from the $p+1$ equations

$$y_i+Ax_i^p+Bx_i^{p-1}+\dots\dots+Kx_i+L=0,\qquad (i=1,2,\dots\dots,(p+1)),$$

$$\sum_{i=1}^{p+1}\frac{y_i}{F'(x_i)}=\left[\sum_{i=1}^{p+1}\frac{xy_i}{(x-x_i)F'(x_i)}\right]_{x=\infty}=-\left[\sum_{i=1}^{p+1}\frac{x\,U(x_i)}{(x-x_i)F'(x_i)}\right]_{x=\infty}=-\left[x\frac{U(x)}{F(x)}\right]_{x=\infty}=-A,$$

and hence, if $b_1,b_2,\dots$ be the values of y when $x=a_1,a_2,\dots$, and $F_0(x)=(x-a_1)\dots(x-a_{p+1})$, we have

$$I^{x_1,\,a_1}+\dots\dots+I^{x_{2p+1},\,a_{2p+1}}=\tfrac{1}{2}\sum_{i=1}^{p+1}\frac{y_i}{F'(x_i)}-\tfrac{1}{2}\sum_{i=1}^{p+1}\frac{b_i}{F_0'(a_i)}.$$

If in the integral I the term x^p be absent, the value obtained for the sum

$$I^{x_1,\,a_1}+\dots\dots+I^{x_{2p+1},\,a_{2p+1}}$$

will be zero.

The reader will notice that for $p=1$, we obtain an equation from which the equation

$$-\zeta(u_1)-\zeta(u_2)-\zeta(u_3)=\tfrac{1}{2}\frac{\wp'u_1-\wp'u_2}{\wp u_1-\wp u_2}$$

can be deduced, u_1, u_2, u_3 being arguments whose sum is zero; and that the algebraic equation whose roots are $x_1,\dots,x_{2p+1}$ gives

$$x_1+x_2+\dots\dots+x_{2p+1}=\tfrac{1}{4}A^2=\tfrac{1}{4}\left(\sum_{i=1}^{p+1}\frac{y_i}{F'(x_i)}\right)^2,$$

which for $p=1$ becomes

$$\wp(u_1)+\wp(u_2)+\wp(u_3)=\tfrac{1}{4}\left(\frac{\wp'u_1-\wp'u_2}{\wp u_1-\wp u_2}\right)^2.$$

Ex. ii. If Y, Z be any two rational functions, and u any integral of the first kind, prove by the theorem

$$\left(\frac{1}{(Y-b)(Z-c)}\frac{du}{dx}\frac{dx}{dt}\right)_{t^{-1}}=0$$

that the sum of the values of $(Y-b)^{-1}\,du/dZ$, at all places where $Z=c$, added to the sum of the values of $(Z-c)^{-1}\,du/dY$ at all places where $Y=b$, is zero.

It is assumed that all the zeros of the functions $Y-b$, $Z-c$ are of the first order.

Hence prove the equation

$$\sum_{r=1}^{Q}\int_{a_r}^{x_r}\frac{du}{x-b}=\sum_{i=1}^{n}\left(\frac{du}{dx}\log\frac{Z(x)-\mu}{Z(x)-\nu}\right)_i,$$

where $a_1, \ldots, a_Q$ are the places at which $Z(x)=\nu$, $x_1, \ldots, x_Q$ the places at which $Z(x)=\mu$, and the suffix on the right hand indicates that the values of the expression in the brackets are to be taken for the n places of the surface at which $x=b$.

It is assumed that there are no branch places for $x=b$.

Ex. iii. If $\phi(x)$ be any integral polynomial in x, $y^2=(x, 1)_{2p+2}$, $=f(x)$ say, and $M(x)$, $N(x)$ be any two integral polynomials in x of which some coefficients are variable, and

$$f(x)\,.\,M^2(x)-N^2(x)=K(x-x_1)\ldots\ldots(x-x_Q),$$

where K is a constant or an integral polynomial whose coefficients do not depend upon the variable coefficients in $M(x)$, $N(x)$, and $y_1, \ldots, y_Q$ be determined by the equations $y_i\,M(x_i)+N(x_i)=0$, then, on the hypothesis that z is not one of the quantities $x_1, \ldots, x_Q$, and is not a root of $f(x)=0$, prove that

$$\int^{x_1}\frac{\phi(x)\,dx}{(x-z)\,y}+\ldots\ldots+\int^{x_Q}\frac{\phi(x)\,dx}{(x-z)\,y}=\frac{\phi(z)}{\sqrt{f(z)}}\log\frac{N(z)+M(z)\sqrt{f(z)}}{N(z)-M(z)\sqrt{f(z)}}-R+C,$$

where C is a constant, and R is the coefficient of $\frac{1}{x}$ in the development of the function

$$\frac{\phi(x)}{(x-z)\sqrt{f(x)}}\log\frac{N(x)+M(x)\sqrt{f(x)}}{N(x)-M(x)\sqrt{f(x)}}$$

in descending powers of x; herein the signs of $\sqrt{f(x)}$, $\sqrt{f(z)}$ are arbitrary, but must be used consistently.

Shew that the statement remains valid when $f(x)$ is of order $2p+1$ (in which case the development from which r is chosen is to be regarded as a development in powers of $\sqrt{x}$); prove that r is zero when $\phi(x)$ is of order p, or of less order. Obtain the corresponding theorem when z is a root of $f(x)=0$.

Ex. iv. The result of Ex. iii. is given by Abel (*Œuvres Compl.*, Vol. i. p. 445), with a direct proof. We explain now the nature of this proof, in the general case. Let $f(y, x)=0$ be the fundamental equation, and let $\theta(y, x)$ be a polynomial of which some of the coefficients are variable; if $y_1, \ldots, y_n$ be the n conjugate roots of $f(y, x)=0$ corresponding to any general value of x, the equation

$$r(x)=\theta(y_1, x)\,\theta(y_2, x)\ldots\ldots\theta(y_n, x)=0,$$

gives the values of x at the finite zeros of the polynomial $\theta(y, x)$. Suppose that the left-hand side breaks into two factors $F_0(x)$ and $F(x)$, of which the former does not contain any of the variable coefficients of $\theta(y, x)$. Let ξ be a root of $F(x)=0$, and $\eta_1, \ldots, \eta_n$ be the corresponding values of y; then one or more of the places $(\xi, \eta_1), \ldots\ldots$,

(ξ, η_n) are zeros of $\theta(y, x)$; fix attention upon one of these, and denote it by (ξ, η). Then if, by a slight change in the variable coefficients of $\theta(y, x)$, whereby it becomes changed into $\theta(y, x)+\delta\theta(y, x)$, $F(x)$ become $F(x)+\delta F(x)$, the symbol δ referring only to the coefficients of $\theta(y, x)$, and ξ become $\xi+d\xi$, we have the equations

$$\delta F(\xi)+F'(\xi)\, d\xi=0,$$

$$F_0(\xi)\,\delta F(\xi)=\delta r(\xi)=\sum_{i=1}^{n}\theta(\eta_1, \xi)\ldots\ldots\theta(\eta_{i-1}, \xi)\,\theta(\eta_{i+1}, \xi)\ldots\ldots\theta(\eta_n, \xi)\,\delta\theta(\eta_i, \xi),$$

where $F'(\xi)=dF(\xi)/d\xi$. Denote now by $U(x)$ the rational function of x, given by

$$U(x)=\sum_{i=1}^{n}\theta(y_1, x)\ldots\ldots\theta(y_{i-1}, x)\,\theta(y_{i+1}, x)\ldots\ldots\theta(y_n, x)\,\delta\theta(y_i, x);$$

then if $R(x, y)$ be any rational function of x and y, we have

$$R(\xi, \eta)\, d\xi=-R(\xi, \eta)\frac{U(\xi)}{F_0(\xi)\,F'(\xi)},$$

where, on account of $\theta(\eta, \xi)=0$ we can write

$$U(\xi)=\frac{r(\xi)}{\theta(\eta, \xi)}\delta\theta(\eta, \xi)$$

and

$$R(\xi, \eta)\,U(\xi)=\sum_{i=1}^{n}R(\xi, \eta_i)\,\theta(\eta_1, \xi)\ldots\ldots\theta(\eta_{i-1}, \xi)\,\theta(\eta_{i+1}, \xi)\ldots\ldots\theta(\eta_n, \xi)\,\delta\theta(\eta_i, \xi)$$
$$=\phi(\xi), \text{ say,}$$

$\phi(\xi)$ being a rational function of ξ only. Taking the sum of the equations of this form, for all the zeros of $\theta(y, x)$, we have

$$\Sigma R(\xi, \eta)\, d\xi=-\Sigma\frac{\phi(\xi)}{F_0(\xi)\,F'(\xi)};$$

herein the summation on the right hand can be carried out, and the result written as the perfect differential of a function of the variable coefficients of $\theta(y, x)$, in fact in the form

$$\left[R(x, y)\frac{dx}{dt}\,\delta\log\theta(y, x)\right]_{t^{-1}},$$

as we have shewn.

For example, when

$$f(y, x)=y^3+x^3-3ayx-1,\quad \theta(y, x)=y-mx-n, \text{ we have } F_0(x)=1,$$
$$F(x)=x^3+(mx+n)^3-3ax(mx+n)-1,$$

and

$$\frac{\xi\eta d\xi}{\eta^2-a\xi}=-\frac{3\xi\eta\delta F(\xi)}{f'(\eta)\,F'(\xi)}=-\frac{3\xi\eta f'(\eta)(\xi\delta m+\delta n)}{f'(\eta)\,F(\xi)}=-\frac{3\xi(m\xi+n)(\xi\delta m+\delta n)}{F'(\xi)},\ =\frac{\psi(\xi)}{F'(\xi)}, \text{ say.}$$

Now $$\frac{\psi(x)}{F(x)}=-\frac{3m\delta m}{1+m^3}+\sum_{i=1}^{3}\frac{\psi(\xi_i)}{(x-\xi_i)F'(\xi_i)},$$

and hence $$\Sigma\frac{\xi\eta d\xi}{\eta^2-a\xi}=\left[\frac{x\psi(x)}{F(x)}+\frac{3xm\delta m}{1+m^3}\right]_{x=\infty},\ =-3\delta\left(\frac{mn-a}{1+m^3}\right),$$

as is easily seen. From this we infer

$$\sum_{i=1}^{3}\int_0^{x_i}\frac{xydx}{y^2-ax}=-3\,\frac{mn-a}{1+m^3}+3\left(\frac{mn-a}{1+m^3}\right)_{\substack{m=a\\ n=1}},\ =(x_1+x_2+x_3)\frac{x_1-x_2}{y_1-y_2}.$$

In this example it is easily seen that the integral is only infinite when x is infinite; putting $x=t^{-1}$, the equation $f(y, x)=0$ gives $y=-\omega t^{-1}-a\omega^2+At+Bt^2+\ldots\ldots$, where $\omega=1$, or $(-1\pm\sqrt{-3})/2$; then $\log\theta(y, x)\, dI/dt$, $=\log(y-mx-n)\,[xy/(y^2-ax)]\, dx/dt$, has $(a\omega^2+n)\,\omega^2/(\omega+m)$ for coefficient of t^{-1}, and we easily find

$$\frac{a+n}{m+1}+\frac{a\omega^2+n}{m+\omega}\,\omega^2+\frac{a\omega+n}{m+\omega^2}\,\omega=\frac{3\,(a-mn)}{m^3+1}.$$

Ex. v. If Y, Z denote any two rational functions (in x and y), such that there is no finite value of x for which both have infinities, and $\Sigma\,(YZ)$ denote the sum of the n conjugate values of YZ for any value of x, and $[\Sigma\,(\overline{Y}Z)]_{(x-a)^{-1}}$ denote the sum of the coefficients of $(x-a)^{-1}$ in the expansions of the rational function of x, $\Sigma\,(YZ)$, for all finite values of x for which Y is infinite, and $[\Sigma\,(YZ)]_{x^{-1}}$ denote the coefficient of x^{-1} in the expansion of $\Sigma\,(YZ)$ in descending powers of x, it is easy (cf. § 162 below) to prove that

$$\left(Y\frac{dx}{dt}\overline{Z}\right)'_{t^{-1}}=[\Sigma\,(YZ)]_{x^{-1}}-[\Sigma\,(\overline{Y}Z)]_{(x-a)^{-1}},$$

wherein, on the left hand, the dash indicates that the sum is to be taken only for the *finite* places at which Z is infinite. Hence if I be any Abelian integral, $=\int R(x, y)\, dx$, we have

$$\left(\frac{dI}{dt}\,\delta\log\theta\,(y, x)\right)'_{t^{-1}}=\left[\Sigma\left(\frac{dI}{dx}\,\delta\log\theta\,(y, x)\right)\right]_{x^{-1}}-\left[\Sigma\left(\frac{\overline{dI}}{dx}\,\delta\log\theta\,(y, x)\right)\right]_{(x-a)^{-1}}.$$

Hence, if we assume that $\theta\,(y, x)$ has no variable zeros at infinity, we can obtain Abel's theorem in the form

$$\Sigma\,\frac{dI}{dx}\,dx=-\left[\Sigma\left(\frac{dI}{dx}\,\delta\log\theta\,(y, x)\right)\right]_{x^{-1}}+\left[\Sigma\left(\frac{\overline{dI}}{dx}\,\delta\log\theta\,(y, x)\right)\right]_{(x-a)^{-1}},$$

wherein the summation on the left refers to all the zeros of $\theta\,(y, x)$.

This is the form in which the result is given by Abel (*Œuvres Compl.*, Christiania, 1881, Vol. i. p. 159, and notes, Vol. ii. p. 296), the right hand being obtained by actual evaluation of the summation which we have written, in the last example, in the form

$$-\Sigma\,\frac{\phi\,(\xi)}{F_0\,(\xi)\,F'\,(\xi)}.$$

The reader is recommended to study Abel's paper*, which, beside the theorem above, contains two important enquiries; first, as to the form necessary for the rational function dI/dx, in order that the right-hand side of the equation of Abel's theorem may reduce to a constant, next, as to the least number of the integrals in the equation of Abel's theorem, of which the upper limits may not be taken arbitrarily but must be taken as functions of the other upper limits. Though the results have been incorporated in the theory here given (§§ 156, 151, 95), Abel's investigation must ever have the deepest interest.

Ex. vi. Obtain the result of Ex. i. (§ 157) by the method explained in Ex. iv.

* Which was presented to the Academy of Sciences of Paris in Oct. 1826, and published by the Academy in 1841 (*Mémoires par divers savants*, t. vii.). During this period many papers were published in *Crelle's Journal* on Abel's theorem, by Abel, Minding, Jürgensen, Broch, Richelot, Jacobi and Rosenhain. (See *Crelle*, i—xxx. I have not examined all these papers with care. Jürgensen uses a method of fractional differentiation.)

Ex. vii. Prove that the sum of the values of the expression

$$\frac{U \cdot v}{J},$$

wherein v is any linear expression in the homogeneous coordinates x, y, z, U is any integral polynomial of degree $m+n-3$, J is the Jacobian of any two curves $f=0$, $\phi=0$, of degrees n and m, and the line $v=0$, and the sum extends to all the common points of $f=0$ and $\phi=0$, vanishes, multiple points of $f=0$, $\phi=0$ being disregarded.

Hence deduce Abel's theorem for integrals of the first kind.

(See Harnack, Alg. Diff. *Math. Annal.* t. ix.; Cayley, *Amer. Journ.* Vol. v. p. 158; Jacobi, theoremata nova algebraica, *Crelle*, t. xiv. The theorem is due to Jacobi; for geometrical applications, see also Humbert, *Liouville's Journal* (1885) Ser. iv. t. i. p. 347)*.

Ex. viii. For the surface

$$y^2 = \phi(x)\,\psi(x), \quad = f(x),$$

wherein $\phi(x)$, $\psi(x)$ are cubic polynomials in x, prove the equation

$$P_{\xi,\gamma}^{x_1, m_1} + P_{\xi,\gamma}^{x_2, m_2} + P_{\xi,\gamma}^{\bar{\xi}, \bar{\gamma}} + 2\log\{[\sqrt{\phi(\xi)\,\psi(\gamma)} + \sqrt{\phi(\gamma)\,\psi(\xi)}]/2\sqrt[4]{f(\xi)\,f(\gamma)}\} = 0,$$

wherein x_1, x_2, $\bar{\xi}$ and m_1, m_2, $\bar{\gamma}$ are coresidual with the roots of $\phi(x)=0$, and $\bar{\xi}$, $\bar{\gamma}$ are the places conjugate to ξ and γ; conjugate places being those for which the values of x are the same.

158. When the places $x_1, \ldots, x_Q$ are determined as coresidual with the fixed places $a_1, \ldots, a_Q$, $p-\tau-1$ of the places $x_1, \ldots, x_Q$ are fixed by the assignation of the others. Hence the $p+1$ relations, which are given by Abel's theorem,

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_Q, a_Q} = 0,$$

$$P_{\xi,\gamma}^{x_1, a_1} + \ldots\ldots + P_{\xi,\gamma}^{x_Q, a_Q} = \log\,[Z(\xi)/Z(\gamma)],$$

cannot be independent. We prove now first of all that the last may be regarded as a consequence of the other p equations. *In fact, if $x_1, \ldots, x_Q$ and $a_1, \ldots, a_Q$ be any two sets of places, such that, for any paths of integration,*

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_Q, a_Q} = M_1\omega_{i,1} + \ldots\ldots + M_p\omega_{i,p} + M_1'\omega'_{i,1} + \ldots\ldots + M'_p\omega'_{i,p},$$

($i = 1, 2, \ldots, p$), wherein $u_1^{x,a}, \ldots, u_p^{x,a}$ are any set of linearly independent integrals of the first kind, $\omega_{i,1}, \ldots, \omega'_{i,p}$ are the periods of the integral $u_i^{x,a}$, and $M_1, \ldots, M'_p$ are rational integers independent of i, then there exists a rational function having the places $a_1, \ldots, a_Q$ for poles and the places $x_1, \ldots, x_Q$ for zeros.

For if $v_1^{x,a}, \ldots, v_p^{x,a}$ be the normal integrals of the first kind, so that we have equations of the form,

$$v_i^{x,a} = C_{i,1}\,u_1^{x,a} + \ldots\ldots + C_{i,p}\,u_p^{x,a},$$

* Further algebraical consideration of Abel's theorem may be found in Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie* (Paris 1883) Vol. iii. Geometrical applications are given by Humbert, *Liouville's Journal*, 1887, 1889, 1890 (Ser. iv. t. iii. v. vi.).

wherein $C_{i,1}, \ldots, C_{i,p}$ are constants, and therefore, also,

$$C_{i,1}\,\omega_{1,j} + \ldots\ldots + C_{i,p}\,\omega_{p,j} = 0 \text{ or } 1, \text{ according as } i \neq j, \text{ or } i = j,$$

and

$$C_{i,1}\,\omega'_{1,j} + \ldots\ldots + C_{i,p}\,\omega'_{p,j} = \tau_{i,j},$$

we can deduce

$$v_i^{x_1, a_1} + \ldots\ldots + v_i^{x_Q, a_Q} = M_i + M_1'\tau_{i,1} + \ldots\ldots + M'_p\,\tau_{i,p}.$$

Consider now the function

$$Z(x) = e^{\Pi^{x,c}_{x_1, a_1} + \ldots\ldots + \Pi^{x,c}_{x_Q, a_Q} - 2\pi i\,(M'_1 v_1^{x,c} + \ldots\ldots + M'_p v_p^{x,c})},$$

c being an arbitrary place.

Herein an integral, $\Pi^{x,c}_{x_1, a_1}$, suffers an increment $2\pi i$ when x makes a circuit about the place x_1; but this does not alter the value of $Z(x)$. And in fact $Z(x)$ is a single-valued function of x; for the functions $\Pi^{x,a}_{x_i, a_i}$ have no periods at the first p period loops, while, if x describe a circuit equivalent to crossing the i-th period loop of the second kind, the function $Z(x)$ is only multiplied by the factor

$$e^{2\pi i\,(v_i^{x_1, a_1} + \ldots\ldots + v_i^{x_Q, a_Q}) - 2\pi i\,(M'_1\tau_{i,1} + \ldots\ldots + M'_p\tau_{i,p})},$$

or $e^{2\pi i M_i}$, whose value is unity.

Further the function $Z(x)$ has no essential singularities; for it has poles at the places $a_1, \ldots, a_Q$, and is elsewhere finite.

Since the function has zeros at $x_1, \ldots, x_Q$ and not elsewhere, the statement made above is justified.

Ex. i. It is impossible to find two places γ, ξ, such that each of the p integrals $u_i^{\xi, \gamma}$ is zero. For then there would exist a rational function, given by

$$e^{\Pi^{x,a}_{\xi, \gamma}},$$

having only one pole, at the place γ. (Cf. § 6, Chap. I.) It is also impossible that the equations

$$v_i^{\xi, \gamma} = M_1 + M'_1\,\tau_{i,1} + \ldots\ldots + M'_p\,\tau_{i,p},$$

wherein $M_1, \ldots, M_p, M'_1, \ldots, M'_p$ are rational integers independent of i, should be simultaneously true.

Ex. ii. If p equations, of the form

$$v_i^{\xi_1, \gamma_1} + v_i^{\xi_2, \gamma_2} = M_i + M'_1\,\tau_{i,1} + \ldots\ldots + M'_p\,\tau_{i,p}$$

exist, γ_1 and γ_2 are the poles of a rational function of the second order, and the surface is hyperelliptic. (Chap. V. § 52.)

159. In regard now to the equations

$$u_i^{x_1, a_1} + \ldots\ldots + u_i^{x_Q, a_Q} = 0,$$

which express that the places $x_1, \ldots, x_Q$ are coresidual with the places $a_1, \ldots, a_Q$, if $\tau + 1$ be the number of ϕ-polynomials which vanish in the places $a_1, \ldots, a_Q$ (Chap. VI. § 93), or (Chap. III. §§ 27, 37) the number of linearly independent linear aggregates of the form

$$C_1\Omega_1(x) + \ldots\ldots + C_p\Omega_p(x),$$

wherein $C_1, \ldots, C_p$ are constants, which vanish in these places, then, $Q - p + \tau + 1$ of the places $x_1, \ldots, x_Q$ can be assumed arbitrarily, and the equations are therefore equivalent to only $p - \tau - 1$ equations, determining the other places of $x_1, \ldots, x_Q$ in terms of those assumed. This can be stated also in another way: the p differential equations

$$\frac{du_i}{dx_1}dx_1 + \ldots\ldots + \frac{du_i}{dx_Q}dx_Q = 0, \quad (i = 1, 2, \ldots, p),$$

express that the places $x_1, \ldots, x_Q$ are coresidual with the places $x_1 + dx_1, \ldots, x_Q + dx_Q$; if the places $x_1, \ldots, x_Q$ have quite general positions these equations are independent; if however $\tau + 1$ linearly independent linear aggregates, of the form,

$$C_1\frac{du_1}{dx} + \ldots\ldots + C_p\frac{du_p}{dx} = 0,$$

wherein $C_1, \ldots, C_p$ are constants, vanish in the places $x_1, \ldots, x_Q$, then the p differential equations are linearly determinable from $p - \tau - 1$ of them.

Ex. i. A rational function having $x_1, \ldots, x_Q$ as poles of the first order, and such that $\lambda_1, \ldots, \lambda_p$ are the coefficients of the inverses of the infinitesimals in the expansion of the function in the neighbourhood of these places, can be written in the form

$$-\lambda_1 \Gamma_{x_1}^{x, c} - \ldots\ldots - \lambda_Q \Gamma_{x_Q}^{x, c};$$

the conditions that the periods be zero are then the p equations

$$\lambda_1\Omega_i(x_1) + \ldots\ldots + \lambda_Q\Omega_i(x_Q) = 0, \quad (i = 1, 2, \ldots, p).$$

But, if we take consecutive places coresidual with $x_1, \ldots, x_Q$, and $t_1, \ldots, t_Q$ be the corresponding values of the infinitesimals at $x_1, \ldots, x_Q$, we also have

$$\Omega_i(x_1)t_1 + \ldots\ldots + \Omega_i(x_Q)t_Q = 0;$$

thus, if the first q $(= Q - p + \tau + 1)$ of $t_1, \ldots, t_Q$ be taken proportional to $\lambda_1, \ldots, \lambda_q$, we shall have the equations

$$t_1/\lambda_1 = \ldots\ldots = t_Q/\lambda_Q.$$

Ex. ii. When the set $x_1, \ldots, x_Q$, beside being coresidual with $a_1, \ldots, a_Q$, has other specialities of position, Abel's theorem may be incompetent to express them. For instance, in the case of a Riemann surface whose equation represents a plane quartic curve with two double points, there is one finite integral; if $a_1, \ldots, a_4$ represent any 4 collinear points, and $x_1, \ldots, x_4$ represent any other 4 collinear points, the equation of Abel's theorem is

$$u^{x_1, a_1} + \ldots + u^{x_4, a_4} = 0;$$

but this equation does not express the *two* relations which are necessary to ensure that $x_1, \ldots, x_4$ are collinear; it expresses only that x_1, x_2, x_3, x_4 are on a conic, S, passing through the double points, or that x_1, x_2, x_3, x_4 are the zeros, and $a_1, \ldots, a_4$ are the poles of the rational function S/LL_0, where $L=0$ is the line containing $a_1, \ldots, a_4$ and $L_0=0$ is the line joining the double points.

160. From these results there follows the interesting conclusion that the p simultaneous differential equations

$$\frac{du_i}{dx_1}dx_1 + \ldots\ldots + \frac{du_i}{dx_Q}dx_Q = 0, \qquad (i=1, 2, \ldots, p),$$

have algebraical integrals, Q being $>p$, and $u_1, \ldots, u_p$ being a set of p linearly independent integrals of the first kind. The problem of determining these integrals consists only in the expression of the fact that $x_1, \ldots, x_Q$ constitute a set belonging to a lot of coresidual sets of places.

The most general lot will consist of the sets coresidual with Q arbitrary fixed places $a_1, \ldots, a_Q$, in which no ϕ-polynomials vanish. But the lot does not therefore depend on Q arbitrary constants; for in place of the set $a_1, \ldots, a_Q$ we can equally well use a set $A_1, \ldots, A_Q$, whereof $q, = Q-p$, places have positions arbitrarily assigned beforehand; in other words, all possible lots of sets of Q places with multiplicity q can be regarded as derived from fundamental sets of Q places in which q places are the same for all. *A lot depends therefore on $Q-q, =p$, arbitrary constants*, and this number of arbitrary constants should appear in the integrals of the equations (Chap. VI. § 96).

We may denote the Q arbitrary places, with which $x_1, \ldots, x_Q$ are coresidual, by $A_1, \ldots, A_q, a_1, \ldots, a_p$, so that $A_1, \ldots, A_q$ are arbitrarily assigned beforehand, in any way that is convenient, and the positions of $a_1, \ldots, a_p$ are the arbitrary constants of the integration.

Then one way in which we can express the integrals of the equations is as follows: form the rational function with poles, of the first order, in the places $x_1, \ldots, x_Q$, and determine the ratios of the $q+1$ homogeneous arbitrary coefficients entering therein, so that the function vanishes in $A_1, \ldots, A_q$. Then the function is determined save for an arbitrary multiplier, and must vanish also in $a_1, \ldots, a_p$. The expression of the fact that it does so gives p equations, each containing one of $a_1, \ldots, a_p$ as an arbitrary constant.

From these p equations we may suppose p of the places $x_1, \ldots, x_Q$, say $x_1, \ldots, x_p$, to be expressed in terms of $a_1, \ldots, a_p$ and $x_{p+1}, \ldots, x_Q$ (and $A_1, \ldots, A_q$). The resulting equations may be derived also by forming the general rational function with its poles in $a_1, \ldots, a_p, A_1, \ldots, A_q$ and eliminating the arbitrary constants by the condition that this function vanishes in $x_i, x_{p+1}, x_{p+2}, \ldots, x_Q$, i being in turn taken equal to $1, 2, \ldots, p$.

For example, for $Q=p+1$, if $\psi(x, a; z, c_1, \ldots, c_p)$ denote the definite rational function which has poles of the first order in the places $z, c_1, \ldots, c_p$, the coefficient of the inverse of the infinitesimal at the place z being taken $=-1$, which function also vanishes at the place a (Chap. VII. § 122), then a complete set of integrals is given by

$$\psi(a_1, A; x_{p+1}, x_1, \ldots, x_p)=0=\ldots\ldots=\psi(a_p, A; x_{p+1}, x_1, \ldots, x_p),$$

and a complete set is also given by

$$\psi(x_1, x_{p+1}; A, a_1, \ldots, a_p)=0=\ldots\ldots=\psi(x_p, x_{p+1}; A, a_1, \ldots, a_p).$$

The first of these integrals is in fact the equation

$$\begin{vmatrix} \frac{du_1}{dx_1}, & \frac{du_1}{dx_2}, & \cdot & , \frac{du_1}{dx_{p+1}} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{du_p}{dx_1}, & \frac{du_p}{dx_2}, & \cdot & , \frac{du_p}{dx_{p+1}} \\ \frac{dP}{dx_1}, & \frac{dP}{dx_2}, & \cdot & , \frac{dP}{dx_{p+1}} \end{vmatrix}=0,$$

wherein $P=P_{a_1, A}^{x_1, c}$, and may be regarded as derived by elimination of $dx_1, \ldots, dx_{p+1}$ from the p given differential equations and the differential of the equation (§ 156)

$$P_{a_1, A}^{x_1, c_1}+\ldots\ldots+P_{a_1, A}^{x_{p+1}, c_{p+1}}=0,$$

which holds when $(x_1, \ldots, x_{p+1})$, $(c_1, \ldots, c_{p+1})$, and $(A, a_1, \ldots, a_p)$ are coresidual sets.

Ex. i. For $p=1$, the fundamental equation being $y^2=(x, 1)_4=\lambda^2x^4+\ldots$, shew that the differential equation

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}=0$$

has the integral

$$\frac{y_1+b}{x_1-a}+\lambda x_1=\frac{y_2+b}{x_2-a}+\lambda x_2,$$

where $b^2=(a, 1)_4$. (Here the place A has been taken at infinity.)

Shew also that this integral expresses that the places (x_1, y_1), (x_2, y_2), $(a, -b)$, are the variable zeros of the polynomial $-y+p+qx-\lambda x^2$, when p and q are varied.

Ex. ii. For $p=2$, the fundamental equation being $y^2=(x, 1)_6=\lambda^2x^6+\ldots$, using the form of the function $\psi(x, a; z, c_1, \ldots, c_p)$ given in Ex. ii. § 132, Chap. VII., and putting the place A at infinity, obtain, for the differential equations

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}+\frac{dx_3}{y_3}=0, \qquad \frac{x_1dx_1}{y_1}+\frac{x_2dx_2}{y_2}+\frac{x_3dx_3}{y_3}=0,$$

the integral

$$\frac{y_1}{(x_1-a)F'(x_1)}+\frac{y_2}{(x_2-a)F'(x_2)}+\frac{y_3}{(x_3-a)F'(x_3)}+\frac{b}{F(a)}=-\lambda,$$

wherein $F(x)=(x-x_1)(x-x_2)(x-x_3)$, $b^2=(a, 1)_6$, and the position of the place (a, b) is the arbitrary constant of integration. By taking three positions of (a, b) we obtain a system of complete integrals.

Shew that this integral is obtained by eliminating p, q, r from the equations which express that the places (x_1, y_1), (x_2, y_2), (x_3, y_3), (a, b) are zeros of the polynomial $-y-\lambda x^3+px^2+qx+r$.

Ex. iii. For the case $(p=3)$ in which the fundamental equation is of the form

$$f(y, x)=(x, y)_4+(x, y)_3+(x, y)_2+(x, y)_1=0,$$

$(x, y)_4$ being a homogeneous polynomial of the fourth degree with general coefficients, etc., prove that an integral of the equations

$$\frac{dx_1}{f'(y_1)}+\frac{dx_2}{f'(y_2)}+\frac{dx_3}{f'(y_3)}+\frac{dx_4}{f'(y_4)}=0, \quad \frac{x_1dx_1}{f'(y_1)}+\text{etc.}=0, \quad \frac{y_1dx_1}{f'(y_1)}+\text{etc.}=0,$$

is given by

$$(2, 3, 4)\, U_1+(3, 1, 4)\, U_2+(1, 2, 4)\, U_3-(1, 2, 3)\, U_4=0,$$

where

$$(2, 3, 4)=\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{vmatrix} \text{ etc.,}$$

and

$$U_i=\frac{f\left(\frac{b}{a}x_i, x_i\right)}{x_i(x_i-a)\left(y_i-\frac{b}{a}x_i\right)},$$

$f(b, a)$ being $=0$, and the position of (a, b) being the arbitrary constant of integration. A complete system of integrals is obtained by giving (a, b) any three arbitrary positions. To obtain these equations the place A has been put at $x=0$, $y=0$.

Ex. iv. When the fundamental equation is $x^4+y^4=1$, shew, putting the place A at $x=1$, $y=0$, that, as in Ex. iii., we have integrals of the form

$$(2, 3, 4)\, U_1+(3, 1, 4)\, U_2+(1, 2, 4)\, U_3-(1, 2, 3)\, U_4=0,$$

wherein

$$U_i=\frac{x_i^2(2a^2-a+1)-x_i(a+1)^2+a^2-a+2}{(a-1)y_i-(x_i-1)b},$$

and $a^4+b^4=1$.

161. The method of forming the integrals of the differential equations which is explained in the last article may also be stated thus: take any adjoint polynomial ψ which vanishes in the Q places $A_1, \ldots, A_q, a_1, \ldots, a_p$; let $C_1, \ldots, C_R$ be the other zeros* of ψ; let the general adjoint polynomial of the same grade as ψ, which vanishes in $C_1, \ldots, C_R$, be denoted by

$$\lambda\psi+\lambda_1\psi_1+\ldots\ldots+\lambda_q\psi_q,$$

$\lambda, \lambda_1, \ldots, \lambda_q$ being arbitrary constants. By expressing that the places $x_i, x_{p+1}, x_{p+2}, \ldots, x_Q$ are zeros of this polynomial we obtain a relation whereby x_i is determined from $x_{p+1}, \ldots, x_Q$ in terms of the arbitrary positions

* Beside those where $f'(y)$ or $F'(\eta)$ vanishes (cf. Chap. VI. § 86).

$a_1, \ldots, a_p$ (and $A_1, \ldots, A_q$). By taking $i = 1, 2, \ldots, p$ we obtain a complete system* of integrals.

Now instead of regarding the set $A_1, \ldots, A_q, a_1, \ldots, a_p$ as the arbitrary quantities of the integration, we may regard the set $C_1, \ldots, C_R$ as the arbitrary quantities, or, more accurately, we may regard the p quantities upon which the lot of sets coresidual with $C_1, \ldots, C_R$ depends, as the arbitrary quantities. To this end, and under the hypothesis that no ϕ-polynomials vanish in the places $C_1, \ldots, C_R$, imagine a set of places $B_1, \ldots, B_{R-p}$, $b_1, \ldots, b_p$ determined coresidual with $C_1, \ldots, C_R$, in which $B_1, \ldots, B_{R-p}$ have any convenient positions assigned beforehand, so that the lot of sets coresidual with $C_1, \ldots, C_R$ depends upon the positions of $b_1, \ldots, b_p$. Let a general adjoint polynomial with $Q + R$ variable zeros be of the form

$$\Theta = \mu\vartheta + \mu_1\vartheta_1 + \ldots\ldots + \mu_k\vartheta_k,$$

wherein $\mu, \ldots, \mu_k$ are arbitrary constants, and k is for shortness written for $Q + R - p$. Then an integral of the differential equations under consideration is obtained by expressing that the places

$$B_1, \ldots, B_{R-p}, b_1, \ldots, b_p, x_i, x_{p+1}, x_{p+2}, \ldots, x_Q$$

are zeros of the polynomial Θ; and a complete system of integrals is obtained by putting i in turn equal to $1, 2, \ldots, p$.

Similarly a complete set of integrals is obtained by expressing that the places

$$x_1, \ldots, x_p, x_{p+1}, \ldots, x_Q, b_i, B_1, \ldots, B_{R-p}$$

are zeros of the polynomial Θ, i being taken in turn equal to $1, 2, \ldots, p$.

In this enunciation there is no restriction as to the value of R, save that it must not be less than p.

Ex. i. For the general surface of the form

$$f(y, x) = (x, y)_4 + (x, y)_3 + (x, y)_2 + (x, y)_1 + \text{constant} = 0,$$

a set of integrals of the equations

$$\sum_1^4 \frac{dx_i}{f'(y_i)} = 0, \quad \sum_1^4 \frac{x_i dx_i}{f'(y_i)} = 0, \quad \sum_1^4 \frac{y_i dx_i}{f'(y_i)} = 0,$$

is given by

$$\begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ a_i^2 & a_ib_i & b_i^2 & a_i & b_i & 1 \\ A^2 & AB & B^2 & A & B & 1 \end{vmatrix} = 0,$$

* And we can of course obtain quite similarly a set of p integrals, each connecting $x_1, \ldots, x_Q, A_1, \ldots, A_q$, and one of the arbitrary positions $a_1, \ldots, a_p$.

where $f(b_i, a_i)=0$, $f(B, A)=0$, $i=1, 2, 3$, and the place (A, B) may be taken at any convenient position.

Ex. ii. Taking as before $Q=p+1$, and considering the hyperelliptic case, the fundamental equation being

$$y^2=(x, 1)_{2p+2}=\lambda^2 x^{2p+2}+\mu u^{2p+1}+\ldots\ldots,$$

we require a polynomial having $R+p+1$ variable zeros: such an one is

$$\Theta=-y+\lambda x^{p+1}+Fx^p+Gx^{p-1}+\ldots\ldots+H,$$

R being equal to p, and we have

$$(\lambda^2 x^{2p+2}+\mu x^{2p+1}+\ldots)-(\lambda x^{p+1}+Fx^p+\ldots+H)^2=(\mu-2\lambda F)\,F(x)\,\phi(x),$$

where $F(x)=(x-x_1)\ldots\ldots(x-x_{p+1})$, $\phi(x)=(x-b_1)\ldots\ldots(x-b_p)$.

An integral of the differential equations may be obtained by eliminating $F, G, \ldots, H$ from the equations expressing that the places

$$b_1, \ldots, b_p, x_i, x_{p+1}$$

are zeros of the polynomial Θ, or from the equations expressing that

$$x_1, \ldots, x_p, x_{p+1}, b_i$$

are zeros of this polynomial, and a complete system of integrals, in either case, by taking i in turn equal to $1, 2, \ldots, p$.

Or a complete system of p integrals may be obtained by eliminating $F, G, \ldots, H$ from the $2p+1$ equations obtained by equating the coefficients of the same powers of x on the two sides of the equation.

We may of course also take Θ in the form

$$-y+Ex^{p+1}+Fx^p+\ldots\ldots+H;$$

then $R=p+1$, and the places $B_1, \ldots, B_{R-p}$ are not evanescent; putting the place B_1 at infinity we obtain $E=\lambda$, as above.

Ex. iii. The integration in the previous example may be carried out in various ways. By introducing again a set of fixed places $a_1, \ldots, a_p, A$, coresidual with $x_1, \ldots, x_p, x_{p+1}$, we can draw a particular inference as to the forms of the coefficients $F, G, \ldots, H$. For if $U(x)$ denote $\lambda x^{p+1}+Fx^p+\ldots+G$, and $U_0(x)$ denote what $U(x)$ becomes when $x_1, \ldots, x_{p+1}$ take the positions $a_1, \ldots, a_p, A$, the coefficients $F, G, \ldots, H$ being then $F_0, G_0, \ldots, H_0$, and also $F_0(x)=(x-a_1)\ldots\ldots(x-a_p)(x-A)$, then, because each of the polynomials $-y+U(x)$, $-y+U_0(x)$ vanishes in the places $b_1, \ldots, b_p$, the polynomial $U(x)-U_0(x)$ must divide by $\phi(x)$, namely $U(x)=U_0(x)+t\,\phi(x)$, where t is a variable parameter; or, if we write $\phi(x)=x^p+t_1 x^{p-1}+\ldots\ldots+t_p$, $t_1, \ldots, t_p$ being then regarded, instead of $b_1, \ldots, b_p$, as the arbitrary constants of the integration, we have

$$F=F_0+t, \quad G=G_0+tt_1, \ \ldots\ldots\ldots, \quad H=H_0+tt_p,$$

and the quantities $G-t_1F, \ldots, H-t_pF$ are constants in the integration, being unaltered when the places $x_1, \ldots, x_{p+1}$ come to $a_1, \ldots, a_p, A$. Hence we can formulate the following result: let the $p+1$ quantities $F_0, G_0, \ldots, H_0$ be determined so that the polynomial $-y+U_0(x)$ vanishes in the fixed places $a_1, \ldots, a_p, A$. Then denoting $(x-a_1)\ldots(x-a_p)(x-A)$ by $F_0(x)$, the fraction

$$[y^2-U_0^2(x)]/F_0(x)$$

is an integral polynomial; denote it by $(\mu-2F_0\lambda)(x^p+t_1x^{p-1}+\ldots\ldots+t_p)$, so that

$\phi_0, t_1, \dots, t_p$ are uniquely determined in terms of the places $a_1, \dots, a_p$, A, and put $F(x)$ for $x^p + t_1 x^{p-1} + \dots\dots + t_p$. Then $x_1, \dots, x_{p+1}$ are the roots of the equation

$$\frac{y^2 - [U_0(x) + t\,\phi(x)]^2}{\phi(x)} = (\mu - 2F_0\lambda)\,F_0(x) - 2t\,U_0(x) - t^2\phi(x) = 0\,;$$

and the set $x_1, \dots, x_{p+1}$ varies with the value of t, which is the only variable quantity in this equation. By equating the coefficients of the various powers of x in the polynomial on the left-hand side of this equation to the coefficients in the polynomial $(\mu - 2F_0\lambda)\,F(x)$, we can express each of the symmetric functions

$$h_1 = x_1 + \dots\dots + x_{p+1}$$
$$h_2 = x_1x_2 + x_1x_3 + \dots\dots + x_p\,x_{p+1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

as rational quadratic functions of a variable parameter t, containing definite rational functions of the variables at the places $a_1, \dots, a_p$, A; the place A may be given any fixed position that is convenient; the positions of the places $a_1, \dots, a_p$ are the arbitrary constants of the integration.

Ex. iv. By eliminating t between the $p+1$ equations obtained at the end of Ex. iii. we obtain the complete system of p integrals. In particular any two of the quantities $h_1, h_2, \dots$ are connected by a quadratic relation, and any three of them are connected by a linear relation (Jacobi, *Crelle*, t. 32, p. 220).

Ex. v. From the equation

$$\frac{U(x)}{F(x)} = \lambda + \sum_{r=1}^{p+1} \frac{y_r}{(x - x_r)\,F'(x_r)}$$

we infer

$$F + \lambda h_1 = \sum_{r=1}^{p+1} \frac{y_r}{F'(x_r)}, \quad \mu - 2F\lambda = \mu + 2\lambda^2 h_1 - 2\lambda \sum_{r=1}^{p+1} \frac{y_r}{F'(x_r)},$$

where $h_1 = x_1 + \dots + x_{p+1}$; hence if a be the value of x at a branch place of the surface, we have from Ex. ii.

$$-F(a)\left[\lambda + \sum_{r=1}^{p+1} \frac{y_r}{(a - x_r)\,F'(x_r)}\right]^2 = \phi(a)\left[\mu + 2\lambda^2 h_1 - 2\lambda \sum_{r=1}^{p+1} \frac{y_r}{F'(x_r)}\right],$$

and if, herein, a be put in turn at any p of the branch places of the surface, the resulting values of $\phi(a)$ may be regarded as the arbitrary constants of the integration, and the resulting equations as a complete set of integrals; and if $\lambda = 0$, as we may always suppose without loss of generality (Chap. V.), we thus obtain the p integrals

$$(a_i - x_1)\dots(a_i - x_{p+1})\left[\sum_{r=1}^{p+1} \frac{y_r}{(a_i - x_r)\,F'(x_r)}\right]^2 = C_i, \qquad (i = 1, 2, \dots\dots, p)$$

$C_1, \dots, C_p$ being the constants of integration (Richelot, *Crelle*, xxiii. (1842), p. 369. In this paper is also shewn how to obtain integrals by extension of Lagrange's method for the case $p = 1$. See Lagrange, *Theory of Functions*, Chap. II., and Cayley, *Elliptic Functions*, 1876, p. 337).

Ex. vi. By comparing coefficients of x^{2p} in the equation of Ex. ii., we obtain

$$\nu - (2\lambda G + F^2) = (\mu - 2\lambda F)\,(t_1 - h_1),$$

where $h_1 = x_1 + \dots + x_{p+1}$; hence prove that

$$\left\{\sum_{r=1}^{p+1} \frac{y_r}{F'(x_r)}\right\}^2 - \mu\,(x_1 + \dots + x_{p+1}) - \lambda^2\,(x_1 + \dots + x_{p+1})^2 = \nu - t_1\,\mu - 2\lambda\,(G - Ft_1)\,;$$

by Ex. ii. the right-hand side is a constant in the integration; hence this equation is an integral of the differential equations; in particular if $\lambda=0$, $\mu=4$, which is not a loss of generality, we have the integral

$$x_1+\ldots+x_{p+1}+C=\tfrac{1}{4}\left[\sum_{r=1}^{p+1}\frac{y_r}{F'(x_r)}\right]^2,$$

where C is a constant; this is a generalization of the equation, for $p=1$,

$$\wp u+\wp v+\wp(u+v)=\tfrac{1}{4}\left(\frac{\wp' u-\wp' v}{\wp u-\wp v}\right)^2$$

(cf. Ex. i. § 157).

Ex. vii. Shew that if the fundamental equation be

$$y^2=(x,\,1)^{2p+2}=\lambda^2x^{2p+2}+\mu x^{2p+1}+\ldots\ldots+Lx+M,$$

then another integral is

$$x_1^2\ldots x_{p+1}^2\left[\sum_{r=1}^{p+1}\frac{y_r}{x_r^2F'(x_r)}\right]^2-L\left(\frac{1}{x_1}+\ldots+\frac{1}{x_{p+1}}\right)-M\left(\frac{1}{x_1}+\ldots+\frac{1}{x_{p+1}}\right)^2=\text{Const.}$$

(Richelot, *loc. cit.*)

Ex. viii. If a_0, a_i be the values of x at two branch places of the surface, obtain the equations

$$\frac{(a_i-x_1)\ldots\ldots(a_i-x_{p+1})}{(a_i-A)\ldots\ldots(a_i-a_p)}\bigg/\frac{(a_0-x_1)\ldots\ldots(a_0-x_{p+1})}{(a_0-A)\ldots\ldots(a_0-a_p)}=(1+\mu\rho_i)^2,$$

wherein the quantities $A,\ldots,a_p$ are the values of x at fixed places coresidual with $x_1,\ldots,x_{p+1}$, ρ_i is an absolute constant, and μ is a parameter varying with the places $x_1,\ldots,x_{p+1}$. Take i in turn equal to $1, 2, \ldots, (p+1)$, and, eliminating μ, we obtain a complete set of integrals. In particular if the left-hand side of this equation be denoted by G_i we have such equations as

$$(G_i-1)\rho_j\rho_k(\rho_j-\rho_k)+(G_j-1)\rho_k\rho_i(\rho_k-\rho_i)+(G_k-1)\rho_i\rho_j(\rho_i-\rho_j)=0.$$

(Weierstrass, *Collected Works*, Vol. I. p. 267.)

162. The proof of Abel's theorem which has been given in this chapter can be extended to the case of an algebraical curve in space. Taking the case of three dimensions, and denoting the coordinates by x, y, z, we shall assume that for any finite value of x, say $x=a$, the curve is completely given by a series of equations of the form

$$\begin{array}{llll} x=a+t_1^{w_1+1}, & x=a+t_2^{w_2+1}, & \ldots\ldots\ldots, & x=a+t_k^{w_k+1}, \\ y=P_1(t_1), & y=P_2(t_2), & \ldots\ldots\ldots, & y=P_k(t_k), \\ z=Q_1(t_1), & z=Q_2(t_2), & \ldots\ldots\ldots, & z=Q_k(t_k), \end{array} \tag{D}$$

wherein $w_1+1,\ldots,w_k+1$ are positive integers, $t_1,\ldots,t_k$ are infinitesimals, and P_1, $Q_1,\ldots,P_k$, Q_k, denote power series of integral powers of the variable, with only a finite number of negative powers, which have a finite radius of convergence. The values represented by any of these k columns, for all values of the infinitesimal within the radius of convergence involved, are the coordinates of all points of the curve which lie within the neighbourhood of a single *place* (cf. § 3, Chap. I.); the sum

$$(w_1+1)+(w_2+1)+\ldots\ldots+(w_k+1)$$

is the same for all values of x, and equal to n, the order of the curve. A similar result holds for infinite values of x; we have only to write $\frac{1}{x}$ for $x-a$.

We assume further that any rational symmetric function of the n sets of values for the pair (y, z), which are represented by the equations (D), is a rational function of x.

Then we can prove that if $R(x, y, z)$ be any rational function of x, y, z, the sum of the coefficients of t^{-1} in the expression $R(x, y, z)\frac{dx}{dt}$, at all the k places of the curve represented by the equations (D), is equal to the coefficient of $\frac{1}{x-a}$ in the rational function of x,

$$U(x) = R(x, y_1, z_1) + R(x, y_2, z_2) + \ldots\ldots + R(x, y_n, z_n).$$

And further that the sum of the coefficients of t^{-1} in $R(x, y, z)\frac{dx}{dt}$ at all the places arising for $x=\infty$ is equal to the coefficient of $-\frac{1}{x}$ in the expansion of the same rational function of x, namely, equal to the coefficient of t^{-1} in $U(x)\frac{dx}{dt}$, when $x=\frac{1}{t}$.

Hence, the theorem

$$\left[U(x)\frac{dx}{dt}\right]_{t^{-1}} = 0,$$

which holds for any rational function, $U(x)$, of a single variable (as may be immediately proved by expressing the function in partial fractions in the ordinary way), enables us to infer, in the case of the curve considered, that also

$$\left[R(x, y, z)\frac{dx}{dt}\right]_{t^{-1}} = 0.$$

By this theorem, applied to the case

$$\left[\frac{1}{R(x, y, z)}\frac{d}{dx}R(x, y, z)\frac{dx}{dt}\right]_{t^{-1}} = 0,$$

we can prove that the number of poles of $R(x, y, z)$ is equal to the number of its zeros, and therefore also equal to the number of places where $R(x, y, z)$ has any assigned value μ, a place being counted as r coincident zeros when the expression, in $R(x, y, z)$, of the appropriate values for x, y, z, in terms of the infinitesimal, leads to a series in which the lowest power of t is t^r; similarly for the poles.

Hence, if I be any integral of the form $\int R(x, y, z)\, dx$, we can apply this theorem in the form

$$\left(\frac{dI}{dt}\frac{1}{Z-\mu}\right)_{t^{-1}} = 0,$$

Z being any rational function of x, y, z, and so obtain, as before (§§ 154, 155), the theorem

$$I^{x_1, a_1} + \ldots\ldots + I^{x_k, a_k} = \left(\frac{dI}{dt}\int_{\mu=\infty}^{0} \delta\mu \frac{\partial}{\partial\mu}\log(Z-\mu)\right)_{t^{-1}},$$

and if Z is of the form $\theta_2(x, y, z)/\theta_1(x, y, z)$, where θ_2, θ_1 are integral polynomials, we can put the right-hand side

$$= \left[\frac{dI}{dt}\log\frac{\theta_2(x, y, z)}{\theta_1(x, y, z)}\right]_{t^{-1}},$$

wherein $x_1, \ldots, x_k$ are the places at which $Z=0$, or $\theta_2(x, y, z)=0$, and $a_1, \ldots, a_k$ are the places where $Z=\infty$ or $\theta_1(x, y, z)=0$, and the places to be considered on the right hand are the infinities of dI/dt.

The reader may also consult the investigation given by Forsyth, *Phil. Trans.*, 1883, Part i. p. 337.

Take for example the curve which is the complete intersection of the cylinders

$$y^2 = x(1-x)$$
$$z^2 = x.$$

For any finite value of x, except $x=0$ or $x=1$, we have 4 places given by

$$y = \pm\sqrt{x(1-x)}, \quad z = \pm\sqrt{x}.$$

For infinite values of x, putting $x=\frac{1}{t^2}$, we have two places given by

$$y = i\frac{1}{t^2}+\ldots \quad , \qquad y = -i\frac{1}{t^2}+\ldots \quad ,$$
$$z = \frac{1}{t} \quad , \qquad z = \frac{1}{t} \quad .$$

For $x=1$, putting $x=1+t^2$, we have two places given by

$$y = it+\ldots \quad , \qquad y = it+\ldots \quad ,$$
$$z = +(1+\tfrac{1}{2}t^2+\ldots), \qquad z = -(1+\tfrac{1}{2}t^2+\ldots) .$$

For $x=0$, putting $x=t^2$, we have two places given by

$$y = t(1-\tfrac{1}{2}t^2-\ldots), \qquad y = -t(1-\tfrac{1}{2}t^2-\ldots),$$
$$z = t \quad , \qquad z = t \quad ,$$

and, at $x=0$, $y=0$, $z=0$, $dx : dy : dz = 2t : 1 : 1$ or $= 2t : -1 : 1 = 0 : 1 : 1$ or $= 0 : -1 : 1$ so that there is a double point with $x=0$, $y=\pm z$ for tangents.

Consider now $\Sigma\int\frac{dx}{yz}$, from the intersections of $z+ax+by=0$ to those of $z+a'x+b'y=0$.

Put $I=\int\frac{dx}{yz}$; then $\frac{dI}{dt}$, $=\frac{1}{yz}\frac{dx}{dt}$, when x is near to 0, has, for one value,

$$\frac{1}{t^2(1-\frac{1}{2}t^2\ldots)}2t=\frac{2}{t}(1+\tfrac{1}{2}t^2+\ldots),$$

while
$$\log\frac{z+a'x+b'y}{z+ax+by}=\log\frac{t+a't^2+b't(1-\frac{1}{2}t^2\ldots)}{t+at^2+bt(1-\frac{1}{2}t^2\ldots)}=\log\frac{1+b'}{1+b}\frac{1+\dfrac{a'}{1+b'}t+\ldots}{1+\dfrac{a}{1+b}t+\ldots}$$

$$=\log\frac{1+b'}{1+b}+\left(\frac{a'}{1+b'}-\frac{a}{1+b}\right)t+\ldots\ldots,$$

and the contribution to the sum $\left(\frac{dI}{dt}\log\frac{z+a'x+b'y}{z+ax+by}\right)_{t^{-1}}$ is $2\log\frac{1+b'}{1+b}$.

If we take the other place at $x=0$ we shall get, as the contribution to

$$\left(\frac{dI}{dt}\log\frac{z+a'x+b'y}{z+ax+by}\right)_{t^{-1}},$$

the quantity $-2\log\frac{1-b'}{1-b}$.

Thus, on the whole we get, at $x=0$,

$$2\log\left(\frac{1+b'}{1-b'}\bigg/\frac{1+b}{1-b}\right).$$

It is similarly seen that no contribution arises at the places $x=1$, $x=\infty$.

Thus on the whole

$$\int\frac{dx_1}{x_1\sqrt{1-x_1}}+\int\frac{dx_2}{x_2\sqrt{1-x_2}}=2\log\left(\frac{1+b'}{1-b'}\bigg/\frac{1+b}{1-b}\right).$$

Now from the equations $z_1+ax_1+by_1=0$, $z_2+ax_2+by_2=0$, we find

$$b=\frac{z_1x_2-z_2x_1}{x_1y_2-x_2y_1},$$

and thus

$$\int^{x_1}\frac{dx}{x\sqrt{1-x}}+\int^{x_2}\frac{dx}{x\sqrt{1-x}}=2\log\frac{\sqrt{x_1(1-x_2)}-\sqrt{x_2(1-x_1)}+\sqrt{x_2}-\sqrt{x_1}}{\sqrt{x_1(1-x_2)}-\sqrt{x_2(1-x_1)}-\sqrt{x_2}+\sqrt{x_1}}+\text{constant}$$

which is a result that can be directly verified.