CHAPTER VII.

Coordination of simple elements. Transcendental uniform functions.

120. We have shewn in Chapter II. (§§ 18, 19, 20), that all the fundamental functions are obtainable from the normal elementary integral of the third kind. The actual expression of this integral for any given form of fundamental equation, is of course impracticable without precise conventions as to the form of the period loops, and for numerical results it may be more convenient to use an integral which is defined algebraically. Of such integrals we have given two forms, one expressed by the fundamental integral functions (Chap. IV. §§ 45, 46), the other expressed in the terms of the theory of plane curves (Chap. VI. § 92, Ex. ix.). In the present Chapter we shew how from the integral $P_{z,\,c}^{x,\,a}$, obtained in Chap. IV.*, to determine algebraically an integral $Q_{z,\,c}^{x,\,a}$ for which the equation $Q_{z,\,c}^{x,\,a} = Q_{x,\,a}^{z,\,c}$ has place; incidentally the character of $P_{z,\,c}^{x,\,a}$, as a function of z, becomes plain; and therefore also the character of the integral of the second kind, $E_z^{x,\,a}$, which was found in Chap. IV. (§§ 45, 47).

This determination arises in close connexion with the investigation of the algebraic expression of the rational function of x which was obtained in § 49 and denoted by $\psi(x, a;\ z, c_1, \dots c_p)$. It was there shewn that every rational function of x can be expressed in terms of this function. It is shewn in this Chapter that any uniform function whatever, which has a finite number of distinct infinities, which may be essential singularities, can be expressed by such a function.

Further, it is here shewn how to obtain an uniform function of x having only one zero, at which it vanishes to the first order, and one infinity; and that any uniform function can be expressed in factors by means of this function.

* For the integral of the third kind obtained in Chap. VI. the reader may compare Clebsch and Gordan, *Theorie der Abel. Functionen* (Leipzig, 1866), p. 117, and, for other important results, Noether, *Math. Annal.* xxxvii. (1890), pp. 442, 448; also Cayley, *Amer. Journal*, v. (1882), p. 173.

121. Let $u_1^{x,a}, \ldots, u_p^{x,a}$ denote any p linearly independent integrals of the first kind, vanishing at the arbitrary place a. Let t denote the infinitesimal at x, and let $Du_1^x, \ldots\ldots, Du_p^x$ denote the differential coefficients of the integrals in regard to t, all of which are everywhere finite. Let $c_1, \ldots, c_p$ denote any p fixed places of the Riemann surface, so chosen that no linear aggregate of the form

$$\lambda_1 Du_1^x + \ldots\ldots + \lambda_p Du_p^x,$$

where $\lambda_1, \ldots, \lambda_p$ are constants, vanishes in all the places $c_1, \ldots, c_p$, but such that one linear aggregate of this form vanishes in every set of $p-1$ of these places*; and let $\omega_i(x)$ denote the linear aggregate, of this form, which vanishes in all of $c_1, \ldots, c_p$ except c_i, and is equal to 1 at the place c_i.

Then $\omega_i(x)$ is expressible as the quotient of two determinants; the denominator has $Du_s^{c_r}$ for its (r, s)th element, the numerator differs from the denominator only in the i-th row, which consists of the quantities $Du_1^x, \ldots, Du_p^x$; thus $\omega_1(x), \ldots, \omega_p(x)$ are determinable algebraically when $u_1^x, \ldots, u_p^x$ are given. Conversely the differential coefficients of the normal integrals of the first kind (§§ 18, 23) are clearly expressible by $\omega_1(x), \ldots, \omega_p(x)$, in the form

$$\Omega_i(x) = \omega_1(x)\,\Omega_i(c_1) + \ldots\ldots + \omega_p(x)\,\Omega_i(c_p).$$

We have already used $v_i^{x,a}$ as a notation for the normal integral $\frac{1}{2\pi i}\int_a^x \Omega_i(x)\,dt_x$. In this chapter we shall use the notation $V_i^{x,a} = \int_a^x \omega_i(x)\,dt_x$.

If the period of the integral $u_i^{x,a}$ at the j-th period loop of the first kind† be denoted by $C_{i,j}$, we can express $v_i^{x,a}$ as the quotient of two determinants, the denominator having $C_{j,i}$ for its (i, j)th element, and the numerator being different from the denominator only in the ith row which consists of the elements $u_1^{x,a}, \ldots, u_p^{x,a}$.

122. Consider now the function of x expressed‡ by

$$\Gamma_z^{x,a} - \sum_{r=1}^{p} \omega_r(z)\,\Gamma_{c_r}^{x,a},$$

z being any place whatever. The function is clearly infinite to the first order at the place z, like $-t_z^{-1}$, t_z being the infinitesimal at z; it is also infinite at each of the places $c_1, \ldots, c_p$, and, at c_i, like $\omega_i(z)\,t_{c_i}^{-1}$, t_{c_i} being the infinitesimal at c_i. The function has no periods at the period loops of the

* Thus there exists no rational function infinite only to the first order at each of $c_1, \ldots, c_p$. Cf. §§ 23, 26.

† $C_{i,j}$ is the quantity by which the value of $u_i^{x,a}$ on the left side of this period loop exceeds the value on the right side. See the figure, § 18, Chap. II.

‡ Klein, *Math. Annal.* XXXVI. p. 9 (1890), Neumann, *loc. cit.* p. 14, p. 259.

first kind. At the ith period loop of the second kind the function has the period

$$\Omega_i(z) - \sum_{r=1}^{p} \omega_r(z)\,\Omega_i(c_r),$$

which, as remarked (§ 121), is also zero. Hence *the function is a rational function of x.* It vanishes at the place a. We shall denote the function by $\psi(x, a;\ z, c_1, \ldots, c_p)$. It is easy to see that it entirely agrees, in character, with the function given in § 49.

For the places $c_1, \ldots, c_p$ have been chosen so that no aggregate of the form

$$\lambda_1\Omega_1(x) + \ldots\ldots + \lambda_p\Omega_p(x)$$

vanishes in all of them. Hence (Chap. III. § 37) the general rational function having poles of the first order at the places $z, c_1, \ldots, c_p$ is of the form $Ag + B$, where g is such a function, and A, B are constants. These constants can be uniquely determined so that the residue at the pole, z, is -1, and so that the function vanishes at the place a.

Ex. For the case $p=1$, if we use Weierstrass's elliptic functions, the places x, a, z, c, being represented by the arguments u, a, v, γ_1, and put $x = \wp u$, $y = \wp'(u)$ etc., we may take, supposing v not to be a half period,

$$\Gamma_z^{x,\,a} = -\frac{1}{\wp' v}\left[\zeta(u-v) - \zeta(a-v) - \frac{\eta}{\omega}(u-a)\right],\quad \omega_1(z) = \frac{\wp'(\gamma_1)}{\wp'(v)},$$

$$\Gamma_{c_1}^{x,\,a} = -\frac{1}{\wp'(\gamma_1)}\left[\zeta(u-\gamma_1) - \zeta(a-\gamma_1) - \frac{\eta}{\omega}(u-a)\right],$$

and obtain

$$\psi(x, a;\ z, c_1) = -\frac{1}{\wp' v}\{\zeta(u-v) - \zeta(u-\gamma_1) - \zeta(a-v) + \zeta(a-\gamma_1)\},$$

or

$$\psi(x, a;\ z, c_1) = \frac{1}{2\wp'(v)}\left[\frac{\wp'(u-v) + \wp'(u-\gamma_1)}{\wp(u-v) - \wp(u-\gamma_1)} - \frac{\wp'(a-v) + \wp'(a-\gamma_1)}{\wp(a-v) - \wp(a-\gamma_1)}\right];$$

and any doubly periodic function can be expressed linearly by functions of this form, in which the same value occurs for γ_1 and different values for v. (Cf. § 49, Chap. IV.)

123. Since $\omega_i(z), = \dfrac{d}{dt_z} V_i^{z,\,c}$, is a linear function of $\Omega_1(z), \ldots, \Omega_p(z)$, it follows that $\omega_i(z)\Big/\dfrac{dz}{dt}$ is a rational function of z; and $\Gamma_z^{x,\,a}, = \dfrac{d}{dt_z}\Pi_{z,\,c}^{x,\,a}$, $= \left(\dfrac{d}{dz}\Pi_{z,\,c}^{x,\,a}\right)\dfrac{dz}{dt_z}$, is such that* $\Gamma_z^{x,\,a}\Big/\dfrac{dz}{dt}$ is a rational function of z; hence

* Throughout this chapter such an expression as $f(z)\,\dfrac{dz}{dt}$ is used to denote the limit, when a variable place ξ approaches the place z, of the expression $f(\xi)\,\dfrac{d\xi}{dt}$, t being the infinitesimal for

$\psi(x, a; z, c_1, \ldots, c_p) \Big/ \dfrac{dz}{dt}$ is a rational function of z. It is easy also to see, from the determinant expression of $\omega_i(z)$, that $\omega_i(z)\dfrac{dc_i}{dt}$ is a rational function of $c_1, \ldots, c_p$.

Hence $\psi(x, a; z, c_1, \ldots, c_p) \Big/ \dfrac{dz}{dt}$ *is a rational function of the variables of all the places* $x, a, z, c_1, \ldots, c_p$.

Further, as depending upon z, $\psi(x, a; z, c_1, \ldots, c_p)$ is infinite only when $\Gamma_z^{x, a}$ is infinite; and $\Gamma_z^{x, a}, = \dfrac{d}{dt_z}\Pi_{x, a}^{z, a}$, is infinite only when z is at x or at a. At the place x, $\Gamma_z^{x, a}$ is infinite like $\dfrac{d}{dt_x}\log t_x$, namely like the inverse of the infinitesimal at the place x.

Hence $\psi(x, a; z, c_1, \ldots, c_p)$, *regarded as depending upon* z, *is infinite only when* z *is in the neighbourhood of the place* x, *or in the neighbourhood of the place* a. *At the place* x, $\psi(x, a; z, c_1, \ldots, c_p)$ *is infinite like the positive inverse of the infinitesimal, at the place* a *it is infinite like the negative inverse of the infinitesimal.* The rational function of z denoted by

$$\psi(x, a; z, c_1, \ldots, c_p) \Big/ \frac{dz}{dt}$$

will therefore be infinite at the place x like $\dfrac{1}{w_1+1}\dfrac{1}{z-x}$ and at the place a like $-\dfrac{1}{w_2+1}\dfrac{1}{z-a}$, where w_1+1, w_2+1 denote the number of sheets that wind at the places x, a respectively; and will be infinite at every branch place, like $\dfrac{A}{(w+1)t^w}$, t being the infinitesimal at the place, $w+1$ the number of sheets that wind there, and A the value of $\psi(x, a; z, c_1, \ldots, c_p)$ when z is at the branch place.

The actual expression of the function $\psi(x, a; z, c_1, \ldots, c_p)$ is given below (§ 130).

124. From the function $\psi(x, a; z, c_1, \ldots, c_p)$ we obtain a function,

$$\bar{E}(x, z) = e^{\int_c^z \psi(x, a;\, z, c_1, \ldots, c_p)\, dt_z}, = e^{\Pi_{z, c}^{x, a} - \sum_{r=1}^{p} V_r^{z, c}\Gamma_{c_r}^{x, a}},$$

wherein c is an arbitrary place, which has the following properties, as a function of x.

the neighbourhood of the place z. When z is not a branch place $\dfrac{d\xi}{dt}=1$; when $w+1$ sheets wind at z, $\dfrac{d\xi}{dt}=(w+1)t^w$ (cf. §§ 2, 3; Chap. I.). Ample practice in the notation is furnished by the examples of this chapter.

(i) It is an uniform function of x. For the exponent has no periods at the period loops of the first kind, and at the ith period loop of the second kind it has the period

$$2\pi i v_i^{z, c} - \sum_{r=1}^{p} V_r^{z, c}\, \Omega_i(c_r)$$

which, as follows from the equation

$$\Omega_i(z) = \omega_1(z)\,\Omega_i(c_1) + \ldots\ldots + \omega_p(z)\,\Omega_i(c_p),$$

is equal to zero. Further the integral multiples of $2\pi i$, which may accrue to $\Pi_{z,\,c}^{x,\,a}$ when x describes a contour enclosing one of the places z, c, do not alter the value of the function.

(ii) The function vanishes only at the place z, and to the first order.

(iii) The function has a pole of the first order at the place c.

(iv) The function is infinite at the place c_i, like $e^{V_i^{z,c} t_{c_i}^{-1}}$, t_{c_i} being the infinitesimal at the place. We may therefore speak of $c_1, \ldots, c_p$ as essential singularities of the function.

125. In order to call attention to the importance of such a function as this, we give an application. Let $R(x)$ denote a rational function, having simple poles at $\alpha_1, \ldots, \alpha_m$, and simple zeros at $\beta_1, \ldots, \beta_m$. We suppose these places different from the fixed places c, a, $c_1, \ldots, c_p$. Then the product

$$F(x) = R(x)\,\frac{\bar{E}(x, \alpha_1) \ldots\ldots \bar{E}(x, \alpha_m)}{\bar{E}(x, \beta_1) \ldots\ldots \bar{E}(x, \beta_m)},$$

is an uniform function of x, which becomes infinite only at the places $c_1, \ldots c_p$; at c_i it is infinite like a constant multiple of

$$e^{-\sum\limits_{r=1}^{m} V_i^{\alpha_r, \beta_r}\, \Gamma_{c_i}^{x,\, a}}.$$

Now, in fact, $\log F(x)$ is also an uniform function of x: for it is only infinite at the places $c_1, \ldots, c_p$, and, at the place c_i, like $-\left(\sum\limits_{r=1}^{m} V_i^{\alpha_r, \beta_r}\right)\Gamma_{c_i}^{x,\, a}$.

Hence the integral $\int d\log F(x)$, $= \int \frac{F'(x)}{F(x)}\,dx$, taken round any closed area on the Riemann surface which does not enclose any of the places $c_1, \ldots, c_p$, is certainly zero, and taken round the place c_i is equal to $-\sum\limits_{r=1}^{m} V_i^{\alpha_r, \beta_r} \int \frac{dt_{c_i}}{t_{c_i}^2}$, taken round c_i, and is, therefore, also zero.

But an uniform function of x which is infinite only to the first order at each of $c_1, \ldots, c_p$ does not exist. For the places $c_1, \ldots, c_p$ were chosen so that the conditions that the periods of a function, of the form

$$\lambda_1 \Gamma_{c_1}^{x,\, a} + \ldots\ldots + \lambda_p \Gamma_{c_p}^{x,\, a},$$

wherein $\lambda_1, \ldots, \lambda_p$ are constants, should be zero, namely the conditions

$$\lambda_1\Omega_r(c_1) + \ldots\ldots + \lambda_p\Omega_r(c_p) = 0, \qquad r = 1, 2, \ldots\ldots, p,$$

are impossible unless each of $\lambda_1, \ldots, \lambda_p$ be zero.

Hence we can infer that $\sum_{r=1}^{m} V_i^{\alpha_r, \beta_r} = 0$, for $i = 1, 2, \ldots, p$, and that $F(x)$ is a constant; this constant is clearly equal to $F(a)$, for $\bar{E}(a, z) = 1$ for all values of z.

Hence, any rational function can be expressed as a product of uniform functions of x, in the form

$$R(x) = R(a)\frac{\bar{E}(x, \beta_1)\ldots\ldots\bar{E}(x, \beta_m)}{\bar{E}(x, \alpha_1)\ldots\ldots\bar{E}(x, \alpha_m)},$$

where $\alpha_1, \ldots, \alpha_m$ are the poles and $\beta_1, \ldots, \beta_m$ the zeros of the function. We have given the proof in the case in which the poles and zeros are of the first order. But this is clearly not important.

Further, the zeros and poles of a rational function are such that

$$\sum_{r=1}^{m} V_i^{\alpha_r, c} = \sum_{r=1}^{m} V_i^{\beta_r, c}, \qquad i = 1, 2, \ldots, p,$$

c being an arbitrary place. This is a case of Abel's Theorem, which is to be considered in the next Chapter. We remark that in the definition of the function $\bar{E}(x, z)$ by means of Riemann integrals, the ordinary conventions as to the paths joining the lower and upper limits of the integrals are to be regarded; these paths must not intersect the period loops.

Ex. i. For the case $p=0$, $\Pi_{z, c}^{x, a} = \log\left(\frac{x-z}{x-c}\,\frac{a-c}{a-z}\right)$ and $\bar{E}(x, z) = \frac{(x-z)(a-c)}{(x-c)(a-z)}$.

Ex. ii. For the case $p=1$, supposing the place c represented by the argument γ, we have

$$\psi(x, a\,;\, z, c_1, \ldots, c_p) = -\frac{1}{\wp'(v)}\{\zeta(u-v) - \zeta(u-\gamma_1) - \zeta(a-v) + \zeta(a-\gamma_1)\}$$

$$\log\bar{E}(x, z) = \int_c^z \psi(x, a\,;\, z, c_1, \ldots, c_p)\,dz = -\int_\gamma^v dv\,\{\zeta(u-v) - \zeta(u-\gamma_1) - \zeta(a-v) + \zeta(a-\gamma_1)\}$$

$$= \log\frac{\sigma(u-v)}{\sigma(u-\gamma)}\frac{\sigma(a-\gamma)}{\sigma(a-v)} + (v-\gamma)\left[\zeta(u-\gamma_1) - \zeta(a-\gamma_1)\right],$$

and therefore

$$\bar{E}(x, z) = \frac{\sigma(u-v)\,\sigma(a-\gamma)}{\sigma(u-\gamma)\,\sigma(a-v)}\,e^{(v-\gamma)[\zeta(u-\gamma_1)-\zeta(a-\gamma_1)]}.$$

Ex. iii. Prove, if a', c' denote any places whatever, that

$$\frac{\bar{E}(x, z)\,\bar{E}(a', c')}{\bar{E}(x, c')\,\bar{E}(a', z)} = e^{\Pi_{z, c'}^{x, a'} - \sum_{i=1}^{p} V_i^{z, c'}\Gamma_{c_i}^{x, a'}}.$$

Ex. iv. The rational function of x, $\psi(x, \zeta\,;\, z, c_1, \ldots, c_p)$, will, beside ζ, have p zeros, say $\gamma_1, \ldots, \gamma_p$, such that the set $\zeta, \gamma_1, \ldots, \gamma_p$ is equivalent with or coresidual with the set $z, c_1, \ldots, c_p$ (§§ 94, 96, Chap. VI.). Hence, in the product

$$\psi(x, \zeta\,;\, z, c_1, \ldots, c_p)\,\psi(x, z\,;\, \zeta, \gamma_1, \ldots, \gamma_p),$$

the zeros of either factor are the poles of the other, and the product is therefore a constant. To find the value of this constant, let x approach to the place z. Then the product becomes equal to

$$-t_x^{-1} \cdot t_x [D_x \psi(x, z; \zeta, \gamma_1, \ldots, \gamma_p)]_{x=z}.$$

It is clear from the expression of $\psi(x, a; z, c_1, \ldots, c_p)$ which has been given, that $D_x \psi(x, a; z, c_1, \ldots, c_p)$ does not depend upon the place a. Thus, by the symmetry, we have the result

$$\begin{aligned}\psi(x, \zeta; z, c_1, \ldots, c_p)\, \psi(x, z; \zeta, \gamma_1, \ldots, \gamma_p) &= -D_z \psi(z, a; \zeta, \gamma_1, \ldots, \gamma_p) \\ &= -D_\zeta \psi(\zeta, a; z, c_1, \ldots, c_p),\end{aligned}$$

where a is a perfectly arbitrary place, and the sets $z, c_1, \ldots, c_p, \zeta, \gamma_1, \ldots, \gamma_p$ are subject to the condition of being coresidual.

Hence also if $W(x; z, c_1, \ldots, c_p)$ denote the expression

$$D_x \left[\psi(x, a; z, c_1, \ldots, c_p) - \Gamma_z^{x, a}\right],$$

we have

$$W(z; \zeta, \gamma_1, \ldots, \gamma_p) = W(\zeta; z, c_1, \ldots, c_p),$$

provided only the set $z, c_1, \ldots, c_p$ be coresidual with the set $\zeta, \gamma_1, \ldots, \gamma_p$.

Ex. v. Prove, with the notation of Ex. iv., that

$$\psi(x, a; z, c_1, \ldots, c_p)\, \psi(z, a; \zeta, \gamma_1, \ldots, \gamma_p) = \psi(x, \zeta; z, c_1, \ldots, c_p)\, \psi(x, a; \zeta, \gamma_1, \ldots, \gamma_p).$$

126. These investigations can be usefully modified*; we can obtain a rational function $\psi(x, a; z, c)$, having the same general character as $\psi(x, a; z, c_1, \ldots, c_p)$ but simpler in that its poles occur only at two distinct places z, c, of the Riemann surface, and we can obtain an uniform function $E(x, z)$ having only one zero, of the first order, at the place z, which is infinite at only one place, c, of the surface.

The limit, when the place x approaches the place c, of the rth differential coefficient of $\Omega_i(x)$ in regard to the infinitesimal at the place c, will be denoted by $\Omega_i^{(r)}(c)$, or simply by $\Omega_i^{(r)}$. We have shewn (Chap. III. § 28) that there are certain numbers $k_1, \ldots, k_p$, such that no rational function exists, infinite only at the place c, to the orders $k_1, \ldots, k_p$. The periods of a function of the form

$$D_c^{k-1} \Gamma_c^{x, a} - \lambda_1 D_c^{k_1 - 1} \Gamma_c^{x, a} - \ldots\ldots - \lambda_p D_c^{k_p - 1} \Gamma_c^{x, a},$$

wherein $\lambda_1, \ldots, \lambda_p$ are constants, and $D_c^{k-1} \Gamma_c^{x, a}$ denotes† the limit, when z approaches c, of the kth differential coefficient of the function $\Pi_{z, \mu}^{x, a}$ in regard to the infinitesimal at c, μ being an arbitrary place, are all of the form

$$\Omega_i^{(k-1)} - \lambda_1 \Omega_i^{(k_1 - 1)} - \ldots\ldots - \lambda_p \Omega_p^{(k_p - 1)}, \qquad (i = 1, 2, \ldots, p).$$

These periods cannot all vanish when k is any one of the numbers $k_1, \ldots, k_p$; thus the determinant formed with the p^2 quantities $\Omega_i^{(k_r - 1)}$ does

* Günther, *Crelle*, cix. p. 199 (1892).

† For purposes of calculation, when c is a branch place, it is necessary to have care as to the definition.

not vanish; but $\lambda_1, \ldots, \lambda_p$ can be chosen to make all these periods vanish when k is not one of the numbers $k_1, \ldots, k_p$.

127. Consider now the function

$$\psi(x, a;\, z, c) = \begin{vmatrix} \Gamma_z^{x,\,a} & , \ \Omega_1(z) & , \ . \ . & , \ \Omega_p(z) \\ . & . & . & . \\ D_c^{k_r-1}\,\Gamma_c^{x,\,a} & , \ \Omega_1^{(k_r-1)} & , \ . \ . & , \ \Omega_p^{(k_r-1)} \\ . & . & . & . \end{vmatrix} \div \begin{vmatrix} \Omega_1^{(k_1-1)} & , \ . \ . & , \ \Omega_p^{(k_1-1)} \\ . & . & . \\ \Omega_1^{(k_p-1)} & , \ . \ . & , \ \Omega_p^{(k_p-1)} \end{vmatrix}$$

wherein $r = 1, 2, \ldots, p$.

Since the period of $\Gamma_z^{x,\,a}$, at the ith period loop of the second kind, is $\Omega_i(z)$, the periods of the elements of the first column of the first determinant are the elements of the various other columns of that determinant. Thus the function is a rational function of x.

We shall denote the minors of the elements of the first column of the first determinant, divided by the second determinant, by $1, -\omega_1(z), \ldots, -\omega_p(z)$, although that notation has already (§ 121) been used in a different sense. Before, $\omega_i(z)$ was such that $\omega_i(c_r) = 0$ unless $r = i$ in which case $\omega_i(c_i) = 1$; now, as is easy to see, $\left[D_z^{k_r-1}\,\omega_i(z)\right]_{z=c}$ is 0 or 1 according as r is not equal or is equal to i. The integrals $\int^z \omega_i(z)\, dt_z$ are linearly independent integrals of the first kind (cf. Chap. III. § 36).

Then the function can be written

$$\psi(x, a;\, z, c) = \Gamma_z^{x,\,a} - \sum_{i=1}^{p} \omega_i(z)\, D_c^{k_i-1}\,\Gamma_c^{x,\,a}\,;$$

the function is infinite at z like $-t_z^{-1}$, t_z being the infinitesimal at the place z, and is infinite at c like*

$$\underline{|k_1-1}\;\omega_1(z)\, t_c^{-k_1} + \ldots\ldots + \underline{|k_p-1}\,.\,\omega_p(z)\, t_c^{-k_p},$$

t_c being the infinitesimal at the place c. It is not elsewhere infinite. The function vanishes when x approaches the place a. As before (§ 123) $\psi(x, a;\, z, c)\Big/\dfrac{dz}{dt}$ is a rational function of all the quantities involved; and $\psi(x, a;\, z, c)$, as depending upon z, is infinite only at the places x, a, in each case to the first order.

* This is clear when c is not a branch place, since then, when x is near to c, $\Gamma_c^{x,\,a}$ is infinite like $-\dfrac{1}{x-c}$; and the $(k-1)$th differential coefficient of this in regard to c is $-\underline{|k-1}\,(x-c)^{-k}$. When c is a branch place, exactly similar reasoning applies if we first make a conformal representation of the neighbourhood of the place, as explained in Chap. II. §§ 16, 19.

128. If now $R(x)$ be a rational function with poles of the first order at the places $z_1, \ldots, z_m$, it is possible to choose the constants $\lambda_1, \ldots, \lambda_p$ so that the difference

$$R(x) - \lambda_1\psi(x, a; z_1, c) - \lambda_2\psi(x, a; z_2, c) - \ldots\ldots - \lambda_m\psi(x, a; z_m, c)$$

is not infinite at any of the places $z_1, \ldots, z_m$; this difference is therefore infinite only at the place c, and is infinite at c like

$$-(A_1 \underline{|k_1 - 1}\, t_c^{-k_1} + \ldots\ldots + A_p \underline{|k_p - 1}\, t_c^{-k_p}),$$

where

$$A_i = \lambda_1\omega_i(z_1) + \ldots\ldots + \lambda_m\omega_i(z_m), \qquad (i = 1, 2, \ldots, p).$$

But, a rational function whose only infinity is that given by this expression, can be taken to have a form

$$A + A_1 D_c^{k_1-1}\Gamma_c^{x, a} + \ldots\ldots + A_p D_c^{k_p-1}\Gamma_c^{x, a},$$

wherein A is a constant; and we have already remarked (§ 126) that the periods of this function cannot all be zero unless each of $A_1, \ldots, A_p$ be zero. Hence this is the case, and we have the equation

$$R(x) = A + \lambda_1\psi(x, a; z_1, c) + \ldots\ldots + \lambda_m\psi(x, a; z_m, c),$$

whereby any rational function with poles of the first order is expressed by means of the function $\psi(x, a; z, c)$. It is immediately seen that the equations $A_1 = 0 = \ldots = A_p$ enable us to reduce the constants $\lambda_1, \ldots, \lambda_m$ to the number given by the Riemann-Roch Theorem (Chap. III. § 37).

When some of the poles of the function $R(x)$ are multiple, the necessary modification consists in the introduction of the functions

$$D_z\psi(x, a; z, c),\ D_z^2\psi(x, a; z, c), \ldots\ldots.$$

Ex. If $\bar{\omega}_1(x), \ldots, \bar{\omega}_p(x)$ denote what are called $\omega_1(x), \ldots, \omega_p(x)$ in § 121, and the notation of § 127 be preserved, prove that

$$\bar{\omega}_r(z) = \sum_{i=1}^{p} \omega_i(z)\, D_c^{k_i-1}\bar{\omega}_r(c),$$

and that

$$\psi(x, a; z, c) = \psi(x, a; z, c_1, \ldots, c_p) - \Sigma\omega_i(z)\, D_c^{k_i-1}\psi(x, a; c, c_1, \ldots, c_p)$$
$$\psi(x, a; z, c_1, \ldots, c_p) = \psi(x, a; z, c) - \Sigma\bar{\omega}_i(z)\,\psi(x, a; c_i, c).$$

129. From the function $\psi(x, a; z, c)$ we derive a function of x, given by

$$E(x, z) = e^{\int_c^z \psi(x, a; z, c)\, dt_z}, \quad = e^{\Pi_{z, c}^{x, a} - \sum_{r=1}^{p} V_r^{z, c} D^{k_r-1}\Gamma_c^{x, a}},$$

where, in the notation of § 127, $V_r^{z, c} = \int_c^z \omega_r(z)\, dt_z$, which has the following properties:

(i) It is an uniform function of x; there exists in fact an equation

$$2\pi i v_i^{z, c} = \sum_{r=1}^{p} V_r^{z, c}\,\Omega_i^{(k_r-1)}.$$

(ii) The function vanishes to the first order when the place x approaches the place z; and is equal to unity at the place a.

(iii) The function is infinite only at the place c, and there like

$$t_c^{-1}\, e^{\sum_{r=1}^{p} V_r^{z,c} \,\underline{|k_r - 1}\, t_c^{-k_r}}.$$

As before we can shew that any rational function $R(x)$, with poles at $a_1, \ldots, a_m$, and zeros at $\beta_1, \ldots, \beta_m$, can be written in the form

$$R(a)\,\frac{E(x, \beta_1) \ldots E(x, \beta_m)}{E(x, a_1) \ldots E(x, a_m)},$$

this being still true when some of the places $a_1, \ldots, a_m$, or some of the places $\beta_1, \ldots, \beta_m$ are coincident.

130. We pass now to the algebraical expression of the functions which have been described here*. We have already (Chap. IV. § 49) given the expression of the function $\psi(x, a; z, c_1, \ldots, c_p)$ in the case when all the places $a, z, c_1, \ldots, c_p$ are ordinary finite places. In what follows we shall still suppose these places to be finite places; the necessary modifications when this is not so can be immediately obtained by a transformation of the form $x = (\xi - k)^{-1}$, or by the use of homogeneous variables (cf. § 46, Chap. IV., § 85, Chap. VI.).

If, s being the value of y when $x = z$, we denote the expression

$$\frac{\phi_0(s, z) + \sum_{r=1}^{n-1} \phi_r(s, z)\, g_r(y, x)}{(z - x) f'(s)}$$

by† (z, x), and use the integrands $\omega_1(x), \ldots, \omega_p(x)$ defined in § 121, the rational expression of $\psi(x, a; z, c_1, \ldots, c_p)$, which was given in § 49, can be put into the form

$$\psi(x, a; z, c_1, \ldots, c_p) = (z, x) - (z, a) - \sum_{r=1}^{p} \omega_r(z)\,[(c_i, x) - (c_i, a)].$$

In case z be a branch place, the expression (z, x) is identically infinite in virtue of the factor $f'(s)$ in the denominator, and this expression can no longer be valid. But, then, the limit, as ζ approaches z, of the expression

* It is known (Klein, *Math. Annal.* xxxvi. p. 9 (1890); Günther, *Crelle*, cix. p. 199 (1892)) that the actual expressions of functions having the character of the functions $\psi(x, a; z, c_1, \ldots, c_p)$, $E(x, z)$, $Q^{z,a}_{z,c}$, have been given by Weierstrass, in lectures. Unfortunately these expressions have not yet (August, 1895) been published, so far as the writer is aware. Indications of some value are given by Hettner, *Götting. Nachr.* 1880, p. 386; Bolza, *Götting. Nachr.* 1894, p. 268; Weierstrass, *Gesamm. Werke*, Bd. ii. p. 235 (1895), and in the *Jahresbericht der Deuts. Math.-Vereinigung*, Bd. iii. (Nov. 1894), pp. 403—436. But it does not appear how far the last of these is to be regarded as authoritative; and it has not been used here. The reader is recommended to consult the later volumes of Weierstrass's works.

† This notation has already been used (§ 45). It will be adhered to.

$(\zeta, x)\dfrac{d\zeta}{dt}$, wherein t is the infinitesimal at the place z, is finite*; if we denote this limit by $(z, x)\dfrac{dz}{dt}$, and introduce a similar notation for the places $c_1, \dots, c_p$, we obtain the expression

$$\psi(x, a; z, c_1, \dots, c_p) = [(z, x) - (z, a)]\frac{dz}{dt} - \sum_{r=1}^{p} \omega_r(z) . [(c_i, x) - (c_i, a)]\frac{dc_i}{dt},$$

which, as in § 49, has the necessary behaviour, for all finite positions of $z, a, c_1, \dots, c_p$.

From this expression we immediately obtain (§ 45)

$$\overline{E}(x, z), = e^{\int_c^z \psi(x, a; z, c_1, \dots, c_p)\, dt_z}, = e^{P_{x,a}^{z,c} - \sum\limits_{r=1}^{p} V_r^{z,c}[(c_i, x) - (c_i, a)]\frac{dc_i}{dt}}.$$

131. In a precisely similar way it can be seen (see § 127) that

$$\psi(x, a; z, c) = [(z, x) - (z, a)]\frac{dz}{dt} - \sum_{r=1}^{p} \omega_r(z)\, D_c^{k_r-1}\left\{[(c, x) - (c, a)]\frac{dc}{dt}\right\},$$

wherein $D_c^{k_r-1}\left\{[(c, x) - (c, a)]\dfrac{dc}{dt}\right\} = \text{limit}_{\zeta=c}\left[\left(\dfrac{d}{dt_c}\right)^{k_r-1}\left\{[(\zeta, x) - (\zeta, a)]\dfrac{d\zeta}{dt_c}\right\}\right]$;

for this expression can be written as the quotient of two determinants, in the manner of § 49, and the integrands $\Omega_1(z), \dots, \Omega_p(z)$ are linear functions of the p integrands

$$\frac{\phi_1(z)}{f'(s)}\frac{dz}{dt}, \frac{z\phi_1(z)}{f'(s)}\frac{dz}{dt}, \dots, \frac{z^{r_1-1}\phi_1(z)}{f'(s)}\frac{dz}{dt}, \frac{\phi_2(z)}{f'(s)}\frac{dz}{dt}, \dots\dots;$$

these latter quantities can therefore be introduced in the determinants in place of $\Omega_1(z), \dots, \Omega_p(z)$, the same change being made, at the same time, for the quantities $\Omega_1(c), \dots, \Omega_p(c)$, throughout. Then it can be shewn precisely as in § 49 that the expression is not infinite when x is at infinity. In regard to finite places, it is clear that the expression

$$D_c^{k_r-1}\left\{[(c, x) - (c, a)]\frac{dc}{dt}\right\}, = D_c^{k_r} P_{x,a}^{c,y},$$

regarded as a function of x, has the same character, when x is near to c, as the function $D_c^{k_r-1}\Gamma_c^{x,a}$.

Hence, also, it follows that $E(x, z)$ has the form

$$E(x, z) = e^{P_{x,a}^{z,c} - \sum\limits_{r=1}^{p} V_r^{z,c} D_c^{k_r-1}\left\{[(c, x) - (c, a)]\frac{dc}{dt}\right\}}.$$

* $f'(\eta)$, when η is very nearly s, vanishes to order $i+w$, and $d\zeta/dt$ to order w (see Chap. VI. § 87). Or the result may be seen from the formula

$$(z, x) - (z, a) = \frac{d}{dz} P_{x,a}^{z,c}$$

(Chap. IV. § 45).

132. *Ex.* i. For the case $(p=1)$ where the surface is associated with the equation

$$y^2=(x, 1)_4,$$

if the values of the variables x, y at the place a be respectively a, b, and the values at the place c_1 be c_1, d_1 respectively, then

(α) when (c_1, d_1) is not a branch place $\omega_1(z)=\frac{d_1}{s}\frac{dz}{dt}$, $(z_1, x)=\frac{s+y}{2s(z-x)}$

and

$$\psi(x, a;\ z, c_1)=\left[\frac{s+y}{2s(z-x)}-\frac{s+b}{2s(z-a)}\right]\frac{dz}{dt}-\frac{d_1}{s}\frac{dz}{dt}\left[\frac{d_1+y}{2d_1(c_1-x)}-\frac{d_1+b}{2d_1(c_1-a)}\right]$$
$$=\frac{1}{2s}\frac{dz}{dt}\left[\frac{s+y}{z-x}-\frac{s+b}{z-a}-\frac{d_1+y}{c_1-x}+\frac{d_1+b}{c_1-a}\right];$$

(β) when (c_1, d_1) is a branch place, in the neighbourhood of which

$$x=c_1+t^2,\ y=At+\dots,\ \omega_1(z)=\frac{A}{2s}\frac{dz}{dt},\ (c_1, x)\frac{dc_1}{dt}=\text{limit of }\frac{At+y}{2At(c_1-x)}\cdot 2t=\frac{y}{A(c_1-x)},$$

and

$$\psi(x, a;\ z, c_1)=\left[\frac{s+y}{2s(z-x)}-\frac{s+b}{2s(z-a)}\right]\frac{dz}{dt}-\frac{A}{2s}\frac{dz}{dt}\left\{\frac{y}{A(c_1-x)}-\frac{b}{A(c_1-a)}\right\}$$
$$=\frac{1}{2s}\frac{dz}{dt}\left\{\frac{s+y}{z-x}-\frac{s+b}{z-a}-\frac{y}{c_1-x}+\frac{b}{c_1-a}\right\}.$$

If (s, z) be not a branch place, $\frac{1}{2s}\frac{dz}{dt}=\frac{1}{2s}$; if (s, z) be a branch place, in the neighbourhood of which $x=z+t^2$, $y=Bt+\dots$, $\frac{1}{2s}\frac{dz}{dt}$, $=$limit of $\frac{1}{2Bt}2t$, $=\frac{1}{B}$.

Ex. ii. For the case $(p=2)$ where the surface is associated with the equation $y^2=f(x)$, where $f(x)$ is an integral function of x of the sixth order, we shall form the function $\psi(x, a;\ z, c_1, c_2)$ for the case where c_1, c_2 are branch places, so that $f(c_1)=f(c_2)=0$, and shall form the function $\psi(x, a;\ z, c)$ for the case when c is a branch place, so that $f(c)=0$.

When c_1, c_2 are branch places, in the neighbourhood of which, respectively, $x=c_1+t_1^2$, $y=A_1t_1+\dots$, and $x=c_2+t_2^2$, $y=A_2t_2+\dots$, so that $A_1^2=f'(c_1)$, $A_2^2=f'(c_2)$, we have

$$\omega_1(z)=\frac{z-c_2}{c_1-c_2}\frac{A_1}{2s}\frac{dz}{dt},\quad \omega_2(z)=\frac{z-c_1}{c_2-c_1}\frac{A_2}{2s}\frac{dz}{dt},\quad [(c_1, x)-(c_1, a)]\frac{dc_1}{dt}=\frac{1}{A_1}\left(\frac{y}{c_1-x}-\frac{b}{c_1-a}\right),$$

and

$$\psi(x, a;\ z, c_1, c_2)=\left[\frac{s+y}{2s(z-x)}-\frac{s+b}{2s(z-a)}\right]\frac{dz}{dt}-\frac{1}{2s}\frac{dz}{dt}\left\{\frac{z-c_2}{c_1-c_2}\left(\frac{y}{c_1-x}-\frac{b}{c_1-a}\right)\right.$$
$$\left.+\frac{z-c_1}{c_2-c_1}\left(\frac{y}{c_2-x}-\frac{b}{c_2-a}\right)\right\}.$$

When c is a branch place, in the neighbourhood of which $x=c+t^2$, $y=At+Bt^3+\dots$, so that $A^2=f'(c)$, the numbers k_1, k_2 are 1, 3 respectively (Chap. V. § 58, Ex. ii.). In the definition of the forms $\omega_1(z)$, $\omega_2(z)$ (§ 127) we may, by linear transformation of the 2nd, 3rd, ..., $(p+1)$th columns of the numerator determinant, and the same linear transformation of the columns of the denominator determinant, replace $\Omega_1(z), \dots, \Omega_p(z)$ by the differential coefficients of any linearly independent integrals of the first kind. In the case now under consideration we may replace them by the differential coefficients of the integrals $\int\frac{dz}{2s}$, $\int\frac{zdz}{2s}$. Hence the denominator determinant becomes

$$\text{limit}_{x=c}\begin{vmatrix} \frac{1}{2y}\frac{dx}{dt}, & \frac{x}{2y}\frac{dx}{dt} \\ D_x^2\left(\frac{1}{2y}\frac{dx}{dt}\right), & D_x^2\left(\frac{x}{2y}\frac{dx}{dt}\right)\end{vmatrix} = \text{limit}_{x=c}\begin{vmatrix} \frac{2t}{2At}, & \frac{c+t^2}{2At}2t \\ \left(\frac{d}{dt}\right)^2\left(\frac{t}{At+Bt^3+\dots}\right), & \left(\frac{d}{dt}\right)^2\left(\frac{t(c+t^2)}{At+Bt^3+\dots}\right)\end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{A}, & \frac{c}{A} \\ -\frac{2B}{A^2}, & \frac{2}{A}\left(1-\frac{cB}{A}\right)\end{vmatrix}, = 2/A^2.$$

Hence $$\omega_1(z)\frac{2}{A^2} = \text{limit}_{x=c}\begin{vmatrix} \frac{1}{2s}\frac{dz}{dt}, & \frac{z}{2s}\frac{dz}{dt} \\ D_x^2\left(\frac{1}{2y}\frac{dx}{dt}\right), & D_x^2\left(\frac{x}{2y}\frac{dx}{dt}\right)\end{vmatrix} = \left\{\frac{2B}{A^2}z+\frac{2}{A}\left(1-\frac{cB}{A}\right)\right\}\frac{1}{2s}\frac{dz}{dt}$$

$$= \frac{1}{A^2}[A+B(z-c)]\frac{1}{s}\frac{dz}{dt},$$

and

$$\omega_2(z)\frac{2}{A^2} = -\text{limit}_{x=c}\begin{vmatrix} \frac{1}{2s}\frac{dz}{dt}, & \frac{z}{2s}\frac{dz}{dt} \\ \frac{1}{2y}\frac{dx}{dt}, & \frac{x}{2y}\frac{dx}{dt}\end{vmatrix} = -\frac{1}{2s}\frac{dz}{dt}\frac{c-z}{A}.$$

Hence

$$\omega_1(z) = [A+B(z-c)]\frac{1}{2s}\frac{dz}{dt}, \qquad \omega_2(z) = \tfrac{1}{2}A(z-c)\frac{1}{2s}\frac{dz}{dt}.$$

Further $$[(c, x)-(c, a)]\frac{dc}{dt} = \frac{1}{A}\left(\frac{y}{c-x}-\frac{b}{c-a}\right),$$ as in Example i.,

but

$$D_c^2\left\{[(c, x)-(c, a)]\frac{dc}{dt}\right\} = \text{limit}_{t=0}\left[\left(\frac{d}{dt}\right)^2\left\{\left[\frac{At+Bt^3+y}{2(At+Bt^3)(c-x+t^2)}\right.\right.\right.$$

$$\left.\left.\left.-\frac{At+Bt^3+b}{2(At+Bt^3)(c-a+t^2)}\right]2t\right\}\right] = -\frac{2y}{A^2(x-c)^2}[A-B(x-c)]+\frac{2b}{A^2(a-c)^2}[A-B(a-c)].$$

Hence the function $\psi(x, a; z, c)$ is given by the expression

$$\left[\frac{s+y}{2s(z-x)}-\frac{s+b}{2s(z-a)}\right]\frac{dz}{dt}-\frac{A+B(z-c)}{A}\frac{1}{2s}\frac{dz}{dt}\left(\frac{y}{c-x}-\frac{b}{c-a}\right)$$
$$+\frac{z-c}{A}\frac{1}{2s}\frac{dz}{dt}\left(\frac{A-B(x-c)}{(x-c)^2}y-\frac{A-B(a-c)}{(a-c)^2}b\right).$$

Ex. iii. Apart from the algebraical determination of the function $\psi(x, a; z, c_1, \dots, c_p)$ which is here explained, it will in many cases be very easy* to determine the function by the methods of Chapter VI. It is therefore of interest to remark that, when the function $\psi(x, a; z, c_1, \dots, c_p)$ is once obtained the forms of independent integrals of the first and second kinds can be immediately obtained as the coefficients in the first few terms of the expansion of the function in the neighbourhood of its poles, in terms of the infinitesimals at these poles.

* An adjoint polynomial Ψ of grade $(n-1)\sigma+n-2$ which vanishes in the $p+1$ places $z, c_1, \dots, c_p$ will vanish in $n+p-3$ other places. The general adjoint polynomial of grade $(n-1)\sigma+n-2$ which vanishes in these $n+p-3$ places will be of the form $\lambda\Psi+\mu\Theta$, where λ and μ are constants. The function $\psi(x, a; z, c_1, \dots, c_p)$ is obtained from $\lambda+\mu\Theta/\Psi$, by determining λ and μ properly. Cf. Noether (*loc. cit.*) Math. Annal. xxxvii.

In fact, if t_i be the infinitesimal in the neighbourhood of the place c_i, and $M_{r,i}$ denote

$$D_{c_i}\left[(c_r, c_i)\frac{dc_r}{dt}\right], \; M_{i,i} \text{ denoting } \left\{D_x\left[(c_i, x)\frac{dc_i}{dt}+\frac{1}{t_i}\right]\right\}_{x=c_i},$$

the expansion of $\psi(x, a; z, c_1, \ldots, c_p)$, as a function of x, in the neighbourhood of the place c_i, has, as the coefficient of t_i^{-1}, the expression $\omega_i(z)$, which is one of a set of linearly independent integrands of the first kind, while the coefficient of t_i is

$$D_{c_i}\left[(z, c_i)\frac{dz}{dt}\right]-\sum_{r=1}^{p}\omega_r(z)M_{r,i}.$$

Now the elementary integral of the second kind obtained in Chap. IV. (§§ 45, 47) with its pole at a place c, when z is the current place, is $E_c^{z,a}=\int_a^z dz\,D_c(z, c)$, whether c be a branch place or not, and when z is near a branch place this must be taken in the form

$$E_c^{z,a}=\int_a^z dt_z D_c\left[(z, c)\frac{dz}{dt}\right].$$

Hence the coefficient of t_i in the expansion of $\psi(x, a; z, c_1, \ldots, c_p)$, when x is near to c_i, is equal to

$$D_z E_{c_i}^{z,a}-\sum_{r=1}^{p}\omega_r(z)M_{r,i}.$$

This is the differential coefficient of an integral of the second kind, with its pole at c_i, the current place being z. We shall see that the integral of the second kind with its pole at any place z can be expressed by means of the functions $E_{c_1}, \ldots, E_{c_p}$ (§ 135, Equation x.).

Ex. iv. Similar results hold for the expansion of the function $\psi(x, a; z, c)$, as a function of x, when x is in the neighbourhood of the place c. If t_c be the infinitesimal at this place, the terms involving negative powers are

$$\frac{\lfloor k_1-1}{t_c^{k_1}}\omega_1(z)+\ldots+\frac{\lfloor k_p-1}{t_c^{k_p}}\omega_p(z),$$

of which the coefficients of the various powers of t_c are differential coefficients of linearly independent integrals of the first kind; the terms involving positive powers are

$$\sum_{k=1}\frac{t_c^k}{\lfloor k}\left\{D_c^k\left((z, c)\frac{dz}{dt}\right)-\sum_{i=1}^{p}\omega_i(z)P_{i,k}\right\},$$

where $P_{i,k}$ is the limit, when the place x approaches the place c of the expression

$$D_x^k\left\{D_c^{k_i-1}\left[(c, x)\frac{dc}{dt}\right]+\frac{\lfloor k_i-1}{t_c^{k_i}}\right\}.$$

Among the coefficients of these positive powers of t_c, only those are important for which k is one of the numbers $k_1, \ldots, k_p$. This follows from the fact that $D_c^{k-1}\Gamma_c^{x,a}$, when k is not one of the numbers $k_1, \ldots, k_p$, is expressible by those of

$$D_c^{k_1-1}\Gamma_c^{x,a}, \ldots, D_c^{k_p-1}\Gamma_c^{x,a},$$

of which the indices $k_1-1, k_2-1, \ldots$, are less than $k-1$, together with a rational function of x (Chap. III. § 28).

Ex. v. In the expansion of the function $\psi(x, a\,;\, z, c)$ whose expression is given in Example ii., the terms involving negative powers are

$$\frac{A+B(z-c)}{2s}\frac{dz}{dt}\cdot\frac{1}{t_c}+\frac{z-c}{2s}A\frac{dz}{dt}\cdot\frac{1}{t_c^{\,3}};$$

and the terms involving positive powers are

$$\frac{1}{2s}\frac{dz}{dt}\cdot t\left[\frac{A}{z-c}+B+C(z-c)\right]+\tfrac{1}{2}\frac{dz}{dt}t^2\frac{1}{(z-c)^2}+\frac{1}{2s}\frac{dz}{dt}t^3\left[\frac{A}{(z-c)^2}+\frac{B}{z-c}+C+D(z-c)\right]$$
$$+\tfrac{1}{2}\frac{dz}{dt}t^4\frac{1}{(z-c)^3}+\frac{1}{2s}\frac{dz}{dt}t^5\left[\frac{A}{(z-c)^3}+\frac{B}{(z-c)^2}+\frac{C}{z-c}+D+E(z-c)\right]+\dots,$$

where the quantities $A, B, \dots, E$ are those occurring in the expansion of y in the neighbourhood of the place c; this expansion is of the form $y=At+Bt^3+Ct^5+Dt^7+Et^9+\dots$.

Ex. vi. If in Ex. v. the integrals of the coefficients of t, t^3 and t^5 be denoted by $F_1^{\,z}$, $F_3^{\,z}$, $F_5^{\,z}$, find the equation of the form

$$F_5^{\,z}=\lambda F_1^{\,z}+\mu F_3^{\,z}+\text{integrals of the first kind}+\text{rational function of }(s, z)$$

which is known to exist (Chap. III. §§ 28, 26; Chap. V. § 57, Ex. ii.), λ and μ being constants.

Prove, in fact, if the surface be associated with the equation

$$y^2=(x-c)^6+p_1(x-c)^5+p_2(x-c)^4+p_3(x-c)^3+p_4(x-c)^2+p_5(x-c)$$

that

$$\int^z\frac{dz}{2s}\left[\frac{3p_3(z-c)^2+4p_4(z-c)+5p_5}{(z-c)^3}+2p_2+p_1(z-c)\right]=-\frac{s}{(z-c)^3}+\text{constant}.$$

133. We pass now to a comparison of the two forms we have obtained for each of the rational functions $\psi(x, a\,;\, z, c_1, \dots, c_p)$, $\psi(x, a\,;\, z, c)$, one of which was expressed by the Riemann integrals, the other in explicit algebraical form.

The cases of the two functions are so far similar that it will be sufficient to give the work only for one case $\psi(x, a\,;\, z, c_1, \dots, c_p)$, and the results for the other case.

From the two equations (§§ 122, 130)

$$\psi(x, a\,;\, z, c_1, \dots, c_p)=\Gamma_z^{x, a}-\sum_{i=1}^{p}\omega_i(z)\,\Gamma_{c_i}^{x, a},$$

$$\psi(x, a\,;\, z, c_1, \dots, c_p)=[(z, x)-(z, a)]\frac{dz}{dt}-\sum_{i=1}^{p}\omega_i(z)\,[(c_i, x)-(c_i, a)]\frac{dc_i}{dt},$$

we infer, denoting the function

$$\Gamma_z^{x, a}-[(z, x)-(z, a)]\frac{dz}{dt} \qquad \text{(i)}$$

by $H_z^{x, a}$, that

$$H_z^{x, a}=\sum_{i=1}^{p}\omega_i(z)\,H_{c_i}^{x, a} \qquad \text{(ii).}$$

The function $H_z^{x, a}$ is not infinite at the place z, but is algebraically infinite at infinity; it has the same periods as $\Gamma_z^{x, a}$. The equation (ii) shews that $H_z^{x, a} \Big/ \frac{dz}{dt}$ is a rational function of z, while the equation

$$\Gamma_z^{x, a} = [(z, x) - (z, a)] \frac{dz}{dt} + \sum_{i=1}^{p} \omega_i(z) H_{c_i}^{x, a} \qquad \text{(iii)}$$

gives the form of $\Gamma_z^{x, a} \Big/ \frac{dz}{dt}$ as a rational function of z.

Integrating the equation (iii) in regard to z, we obtain

$$\Pi_{z, c}^{x, a} = P_{x, a}^{z, c} + \sum_{i=1}^{p} V_i^{z, c} H_{c_i}^{x, a} \qquad \text{(iv)},$$

where c is an arbitrary place, and $P_{x, a}^{z, c}$ is the integral of the third kind, as a function of z, which was determined in Chap. IV. (§§ 45, 46).

Since the integral of the second kind $E_z^{x, a}$, obtained in Chap. IV. (§§ 45, 46), is equal to $D_z P_{z, c}^{x, a}$, we deduce from the last equation, interchanging x and z, and also a and c, and then differentiating in regard to z,

$$E_z^{x, a} + \sum_{i=1}^{p} V_i^{x, a} D_z H_{c_i}^{z, c} = D_z \Pi_{x, a}^{z, c}, = D_z \Pi_{z, c}^{x, a}, = \Gamma_z^{x, a} \qquad \text{(v)},$$

and thence, using equation (iii) to express $\Gamma_z^{x, a}$,

$$E_z^{x, a} = [(z, x) - (z, a)] \frac{dz}{dt} + \sum_{i=1}^{p} [\omega_i(z) H_{c_i}^{x, a} - V_i^{x, a} D_z H_{c_i}^{z, c}] \qquad \text{(vi)},$$

which* gives the form of $E_z^{x, a} \Big/ \frac{dz}{dt}$ as a rational function of z.

The difference of two elementary integrals of the second kind must needs be a function which is everywhere finite, and therefore an aggregate of integrals of the first kind. The equation (v) expresses the difference of $E_z^{x, a}$ and $\Gamma_z^{x, a}$ in this way. But it should be noticed that the coefficients of the integrals of the first kind in this equation, which depend upon z, become infinite for infinite values of z. They are the quantities

$$D_z H_{c_i}^{z, c}.$$

From the equation (iv) we have

$$P_{z, c}^{x, a} = \Pi_{z, c}^{x, a} - \sum_{i=1}^{p} V_i^{x, a} H_{c_i}^{z, c},$$

wherein the coefficients of $V_i^{x, a}$ on the right may be characterised as integrals of the second kind. From this equation also, if the periods of $V_i^{x, a}$ at the jth period loops of the

* An equation of this form is given by Clebsch and Gordan, *Abel. Functnen.* (Leipzig, 1866), p. 120.

first and second kind be denoted by $C_{i,j}$ and $C'_{i,j}$ respectively, we obtain, as the corresponding periods of $P_{z,c}^{x,a}$

$$\left(P_{z,c}^{x,a}\right)_j = \qquad - \sum_{i=1}^{p} C_{i,j} H_{c_i}^{z,c}$$

$$\left(P_{z,c}^{x,a}\right)'_j = 2\pi i v_j^{z,c} - \sum_{i=1}^{p} C'_{i,j} H_{c_i}^{z,c};$$

from these equations the periods of $E_z^{x,a}$ are immediately obtainable. These equations may be used to express the integrals $H_{c_i}^{z,c}$ in terms of the periods of $P_{z,c}^{x,a}$ at the period loops of the first kind.

134. But all these equations are in the nature of transition equations; they connect functions which are algebraically derivable with functions whose definition depends upon the form of the period loops. We proceed further to eliminate these latter functions as far as is possible, replacing them by certain constants, which, in the nature of the case, are not determinable algebraically.

The function of x expressed by $H_z^{x,a}$ is not infinite at the place z. Hence we may define p^2 finite constants $A_{i,r}$ by the equation

$$A_{i,r} = D_{c_r} H_{c_i}^{c_r, c},$$

where c is an arbitrary place. And if, as in § 132, Ex. iii., we use the algebraically determinable quantities given by

$$M_{i,r} = D_{c_r}\left[(c_i, c_r)\frac{dc_i}{dt}\right], \quad M_{i,i} = \left\{D_x\left[(c_i, x)\frac{dc_i}{dt} + \frac{1}{t_i}\right]\right\}_{x=c_i},$$

we have

$$M_{i,r} + A_{i,r} = D_{c_r}\Gamma_{c_i}^{c_r, c} = D_{c_i}\Gamma_{c_r}^{c_i, c} = M_{r,i} + A_{r,i},$$

and

$$M_{i,i} + A_{i,i} = \left[D_x\left(\Gamma_{c_i}^{x,c} + \frac{1}{t_i}\right)\right]_{x=c_i}.$$

Then, from equation (v), putting therein c_r for z,

$$H_{c_r}^{x,a} = \Gamma_{c_r}^{x,a} - [(c_r,x)-(c_r,a)]\frac{dc_r}{dt} = E_{c_r}^{x,a} - [(c_r,x)-(c_r,a)]\frac{dc_r}{dt} + \sum_{i=1}^{p} A_{i,r} V_i^{x,a} \qquad \text{(vii)}$$

and thence, since $E_{c_r}^{x,a} = \int_a^x dx\, D_{c_r}(x, c_r)$

$$D_x H_{c_r}^{x,a} = D_{c_r}\left[(x, c_r)\frac{dx}{dt}\right] - D_x\left[(c_r, x)\frac{dc_r}{dt}\right] + \sum_{i=1}^{p} A_{i,r}\,\omega_i(x).$$

If in this equation we replace x by z and i by r and then substitute in equation (v), we obtain

$$\Gamma_z^{x,a} = E_z^{x,a} + \sum_{i=1}^{p} V_i^{x,a}\left\{D_{c_i}\left[(z, c_i)\frac{dz}{dt}\right] - D_z\left[(c_i, z)\frac{dc_i}{dt}\right] + \sum_{r=1}^{p} A_{r,i}\,\omega_r(z)\right\};$$

and thus, if we define an, algebraically determinable, integral by the equation

$$G_z^{x,a} = E_z^{x,a} + \sum_{i=1}^{p} V_i^{x,a} \left\{ D_{c_i}\left[(z, c_i)\frac{dz}{dt}\right] - D_z\left[(c_i, z)\frac{dc_i}{dt}\right] - \tfrac{1}{2}\sum_{r=1}^{p}(M_{r,i} - M_{i,r})\,\omega_r(z)\right\} \quad \text{(viii)},$$

we have

$$\Gamma_z^{x,a} = G_z^{x,a} + \sum_{i=1}^{p} V_i^{x,a} \sum_{r=1}^{p}(A_{r,i} + \tfrac{1}{2}M_{r,i} - \tfrac{1}{2}M_{i,r})\,\omega_r(z),$$

or

$$\Gamma_z^{x,a} = G_z^{x,a} + \tfrac{1}{2}\sum_{i=1}^{p} V_i^{x,a} \sum_{r=1}^{p}(A_{r,i} + A_{i,r})\,\omega_r(z), \quad \text{(viii)}',$$

from which, by integration in regard to z, we obtain an equation

$$Q_{z,c}^{x,a} = \int_c^z G_z^{x,a}\,dt_z = \Pi_{z,c}^{x,a} - \tfrac{1}{2}\sum_{i=1\ldots p}^{r=1\ldots p}\sum (A_{r,i} + A_{i,r})\,V_i^{x,a}\,V_r^{z,c} \quad \text{(ix)},$$

either of these expressions being, by equation (viii), also equal to

$$P_{z,c}^{x,a} + \sum_{i=1}^{p} V_i^{x,a}\left[E_{c_i}^{z,c} - [(c_i, z) - (c_i, c)]\frac{dc_i}{dt}\right] + \tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}(V_i^{x,a}\,V_r^{z,c} - V_r^{x,a}\,V_i^{z,c})(M_{i,r} - M_{r,i}) \quad \text{(ix)}'.$$

The equation (ix) shews that the integral $Q_{z,c}^{x,a}$ is such that

$$Q_{z,c}^{x,a} = Q_{x,a}^{z,c}$$

while every term of (ix)$'$ is capable of algebraic determination.

135. From the equation (ix), when none of the places $x, z, c_1, \ldots, c_p$ are branch places, we obtain

$$\frac{\partial^2 Q_{z,c}^{x,a}}{\partial x\,\partial z} = \frac{\partial}{\partial z}(x, z) + \sum_{i=1}^{p}\omega_i(x)\left[\frac{\partial}{\partial c_i}(z, c_i) - \frac{\partial}{\partial z}(c_i, z)\right] + \tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}[\omega_i(x)\,\omega_r(z) - \omega_r(x)\,\omega_i(z)]\,[M_{i,r} - M_{r,i}] \quad \text{(x)},$$

and hence, from the characteristic property $\dfrac{\partial^2}{\partial x\,\partial z}Q_{z,c}^{x,a} = \dfrac{\partial^2}{\partial x\,\partial z}Q_{x,a}^{z,c}$, we infer

$$\frac{\partial}{\partial z}(x,z) - \frac{\partial}{\partial x}(z,x) + \sum_{i=1}^{p}\left\{\omega_i(x)\left[\frac{\partial}{\partial c_i}(z, c_i) - \frac{\partial}{\partial z}(c_i, z)\right] - \omega_i(z)\left[\frac{\partial}{\partial c_i}(x, c_i) - \frac{\partial}{\partial x}(c_i, x)\right]\right\} + \tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}[\omega_i(x)\,\omega_r(z) - \omega_r(x)\,\omega_i(z)]\,[M_{i,r} - M_{r,i}] = 0 \quad \text{(xi)},$$

wherein every quantity which occurs is defined algebraically. The form when some of the places are branch places is obtainable by slight modifications. *This is then the general algebraic relation underlying the fundamental property of the interchange of argument and parameter, which was originally denoted, in this volume, by the equation* $\Pi_{z,c}^{x,a} = \Pi_{x,a}^{z,c}$.

The relation is of course independent of the places $c_1, \ldots, c_p$. For an expression in which these places do not enter, see § 138, Equation 17.

The equation (xi) can be obtained in an algebraic manner (§ 137, Ex. vi.). The method followed here gives the relations connecting the Riemann normal integrals and the particular integrals obtained in Chap. IV., with the canonical integrals $G_z^{x,a}$, $Q_{z,c}^{x,a}$.

It should be noticed, in equation (xi), that in the last summation each term occurs twice. By a slight change of notation the factor $\frac{1}{2}$ can be omitted.

The interchange of argument and parameter was considered by Abel; some of his formulae, with references, are given in the examples in § 147.

136. From the equation (viii)′ we have

$$\Gamma_{c_s}^{x,a} = G_{c_s}^{x,a} + \tfrac{1}{2}\sum_{i=1}^{p} (A_{s,i} + A_{i,s})\, V_i^{x,a}.$$

From this equation, and the equation (viii)′, we infer that

$$G_z^{x,a} - \sum_{s=1}^{p} \omega_s(z)\, G_{c_s}^{x,a} = \Gamma_z^{x,a} - \sum_{s=1}^{p} \omega_s(z)\, \Gamma_{c_s}^{x,a}$$
$$= \psi(x, a;\ z, c_1, \ldots, c_p) \qquad \text{(xii)},$$

which result may be regarded as giving an expression of the function $\psi(x, a;\ z, c_1, \ldots, c_p)$ in terms of the integrals G; but, written in the form

$$G_z^{x,a} = \sum_{s=1}^{p} \omega_s(z)\, \Gamma_{c_s}^{x,a} + [(z, x) - (z, a)]\frac{dz}{dt} - \sum_{i=1}^{p} \omega_i(z)\, [(c_i, x) - (c_i, a)]\frac{dc_i}{dt},$$

the equation (xii) has another importance; if we call $Q_{z,c}^{x,a}$ an elementary *canonical* integral of the third kind, and $G_z^{x,a}$, $= D_z Q_{z,c}^{x,a}$, an elementary canonical integral of the second kind, we may express the result in words thus—*The elementary canonical integral of the second kind with its pole at any place z is expressible in the form*

$$\sum_{s=1}^{p} \omega_s(z)\, G_{c_s}^{x,a} + (\text{rational function of } x, z, c_1, \ldots, c_p)\Big/\frac{dz}{dt},$$

wherein the elementary canonical integrals occurring, have their poles at p arbitrary independent places $c_1, \ldots, c_p$.

Further, by equation (xii) the function $\bar{E}(x, z)$, of § 124, can be written in the form

$$\bar{E}(x, z) = e^{Q_{z,c}^{x,a} - \sum_{s=1}^{p} V_s^{z,c} G_{c_s}^{x,a}} \qquad \text{(xiii)}.$$

If we put

$$K_z^{x,a} = G_z^{x,a} - [(z, x) - (z, a)] \frac{dz}{dt} \qquad \text{(xiv)},$$

the equation following equation (xii) gives

$$K_z^{x,a} = \sum_{i=1}^{p} \omega_i(z) K_{c_i}^{x,a} \qquad \text{(xv)},$$

and therefore, also

$$Q_{z,c}^{x,a} = P_{z,c}^{x,a} + \sum_{i=1}^{p} V_i^{x,a} K_{c_i}^{z,c} \qquad \text{(xvi)},$$

and

$$D_x\left((z, x)\frac{dz}{dt}\right) - D_z\left((x, z)\frac{dx}{dt}\right) = \sum_{i=1}^{p} [\omega_i(x) D_z K_{c_i}^{z,c} - \omega_i(z) D_x K_{c_i}^{x,a}] \quad \text{(xvii)}$$

which is another form of equation (xi).

It is easy to see that

$$G_{c_s}^{x,a} = E_{c_s}^{x,a} - \tfrac{1}{2} \sum_{i=1}^{p} (M_{i,s} - M_{s,i}) V_i^{x,a}.$$

137. *Ex.* i. Prove that the most general elementary integral of the third kind, with its infinities at the places z and c, and vanishing at the place a, which is unaltered when x, z are interchanged and also a and c, is of the form

$$\Pi_{z,c}^{x,a} - \sum_{i=1}^{p} \sum_{r=1}^{p} a_{i,r} V_i^{x,a} V_r^{z,c},$$

wherein $a_{i,r}$ are constants satisfying the equations $a_{i,r} = a_{r,i}$.

Ex. ii. If the integral of Ex. i. be denoted by $\overline{Q}_{z,c}^{x,a}$, and $D_z \overline{Q}_{z,c}^{x,a}$ be denoted by $\overline{G}_z^{x,a}$, prove that

$$\psi(x, a; z, c_1, \ldots, c_p) = \overline{G}_z^{x,a} - \sum_{s=1}^{p} \omega_s(z) \overline{G}_{c_s}^{x,a}.$$

Ex. iii. If, in particular, $\overline{Q}_{z,c}^{x,a}$ be given by

$$\overline{Q}_{z,c}^{x,a} = Q_{z,c}^{x,a} - \tfrac{1}{2} \sum_{i=1}^{p} \sum_{r=1}^{p} (M_{r,i} + M_{i,r}) V_i^{x,a} V_r^{z,c},$$

prove that

$$\overline{G}_{c_i}^{z,a} = E_{c_i}^{z,a} - \sum_{r=1}^{p} M_{r,i} V_r^{z,a}.$$

This is the integral, in regard to z, of the coefficient of t_i in the expansion of $\psi(x, a; z, c_1, \ldots, c_p)$, as a function of x, in the neighbourhood of the place c_i (§ 132, Ex. iii.).

The integral $Q_{z,c}^{x,a}$ is algebraically simpler than the integral $\overline{Q}_{z,c}^{x,a}$, of this example, in that its calculation does not require the determination of the limits denoted by $M_{i,i}$.

Ex. iv. For the case $p = 1$, when the fundamental equation is of the form

$$y^2 = (x, 1)_4,$$

if the variables at the place c_1 be denoted by $x=c_1$, $y=d_1$, the place not being a branch place, prove that

$$\frac{\partial}{\partial z}(x, z)-\frac{\partial}{\partial x}(z, x)=\frac{1}{24ys}[f^{(\mathrm{III})}(z)\,.\,(x-z)+\tfrac{1}{2}f^{(\mathrm{IV})}(z)\,.\,(x-z)^2],$$

and calculate $Q_{z,\,c}^{x,\,a}$, from the equation xi, in the form

$$Q_{z,\,c}^{x,\,a}=\int_a^x\int_c^z \frac{ys+f(x, z)}{2(x-z)^2}\frac{dx}{y}\frac{dz}{s}-\frac{1}{24}f''(c_1)\int_a^x\frac{dx}{y}\int_c^z\frac{dz}{s},$$

where, if $y^2=f(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$, the symbol $f(x, z)$ denotes the symmetrical expression

$$x^2(a_0z^2+2a_1z+a_2)+2x(a_1z^2+2a_2z+a_3)+(a_2z^2+2a_3z+a_4).$$

Prove also that in this case $M_{1,\,1}=-f'(c_1)/4f(c_1)$.

Calculate the integral $Q_{z,\,c}^{x,\,a}$ when the place c_1 is a branch place, and prove that in that case $M_{1,\,1}$, $=\text{limit}_{t=0}\left(\frac{1}{A}\frac{y}{c_1-x}+\frac{1}{t}\right)$, wherein $x=c_1+t^2$, $y=At+Bt^3+\ldots$, vanishes.

Ex. v. For the case $(p=2)$ in which the fundamental equation is

$$y^2=f(x),$$

where $f(x)$ is a sextic polynomial, taking c_1, c_2 to be the branch places $(c_1, 0)$, $(c_2, 0)$, in the neighbourhood of which, respectively, $x=c_1+t_1^2$, $y=A_1t_1+B_1t_1^3+\ldots$, and $x=c_2+t_2^2$, $y=A_2t_2+B_2t_2^3+\ldots$, prove that

$$E_{c_1}^{z,\,c}=\int_c^z\frac{dz}{2s}\frac{A_1}{z-c_1}\,;\ \omega_1(z)=A_1\frac{z-c_2}{c_1-c_2}\frac{1}{2s}\frac{dz}{dt}\,;\ (c_1, z)\frac{dc_1}{dt}=\frac{1}{A_1}\frac{s}{c_1-z}\,;\ M_{1,\,2}=\frac{A_2}{A_1(c_1-c_2)}$$

and infer that

$$[\omega_1(x)\,\omega_2(z)-\omega_2(x)\,\omega_1(z)]\,[M_{2,\,1}-M_{1,\,2}]=-\frac{A_1^2+A_2^2}{(c_1-c_2)^2}(x-z)\frac{1}{2s}\frac{dz}{dt}\frac{1}{2y}\frac{dx}{dt}.$$

Supposing x and z have general positions, deduce from equation (ix) that

$$\begin{aligned}4ys(x-z)^2\frac{\partial^2 Q}{\partial x\partial z}-2ys=&+\tfrac{1}{2}\frac{f'(c_1)+f'(c_2)}{(c_1-c_2)^2}(x-z)^3+f'(z)(x-z)+2f(z)\\&+(x-z)^2\left\{\frac{[f'(c_1)+f'(z)](z-c_1)-2f(z)}{(z-c_1)^2}\frac{x-c_2}{c_1-c_2}-\frac{[f'(c_2)+f'(z)][z-c_2]-2f(z)}{(z-c_2)^2}\frac{x-c_1}{c_1-c_2}\right\},\end{aligned}$$

where A_1^2, A_2^2 have been replaced by $f'(c_1)$, $f'(c_2)$ respectively.

Prove that this form leads to

$$Q_{z,\,c}^{x,\,a}=\int_c^z\int_a^x\frac{sy+f(x, z)}{2(x-z)^2}\frac{dx}{y}\frac{dz}{s}+\int_c^z\int_a^x\frac{dx}{2y}\frac{dz}{2s}[Lxz+M(x+z)+N],$$

where, if $f(x)$ be $a_0x^6+6a_1x^5+15a_2x^4+20a_3x^3+15a_4x^2+6a_5x+a_6$, $f(x, z)$ denotes the expression

$$\begin{aligned}&x^3(a_0z^3+3a_1z^2+3a_2z+a_3)+3x^2(a_1z^3+3a_2z^2+3a_3z+a_4)\\&\qquad+3x(a_2z^3+3a_3z^2+3a_4z+a_5)+(a_3z^3+3a_4z^2+3a_5z+a_6),\end{aligned}$$

and L, M, N are certain constants depending upon c_1 and c_2.

Ex. vi. Let $R(x)$ be any rational function. By expressing the fact that the value of the integral $\int R(x)\,dx$ taken round the complete boundary of the Riemann surface, is equal to the sum of its value taken round all the places of the surface at which the integral is infinite, we shall (cf. also p. 232) obtain the theorem

$$\Sigma \left[R(x)\frac{dx}{dt}\right]_{t^{-1}}=0,$$

where the summation extends to all places at which the expansion of $R(x)\frac{dx}{dt}$, in terms of the infinitesimal, contains negative powers of t, and $\left[R(x)\frac{dx}{dt}\right]_{t^{-1}}$ means the coefficient of t^{-1} in the expansion. If all the poles of $R(x)$ occur for finite values of x, this summation will contain terms arising from the fact that $\frac{dx}{dt}$ contains negative powers of t when x is infinite, as well as terms arising at the finite poles of $R(x)$. If however $R(x)$ be of the form $U(x)\frac{d}{dx}V(x)$, wherein $U(x)$, $V(x)$ are rational functions of x, whose poles are at finite places of the surface, there will be no terms arising from the infinite places of the surface.

Now let ξ denote the current variable, and x, z denote fixed finite places: prove, by applying the theorem to the case* when

$$R(\xi)=\psi(\xi, a;\ z, c_1, \dots, c_p)\frac{d}{d\xi}\psi(\xi, a;\ x, c_1, \dots, c_p),$$

that

$$D_x\psi(x, z)-D_z\psi(z, x)=\sum_{i=1}^{p}\{\omega_i(x)[\psi(x, z)]^x_{t_{c_i}}-\omega_i(z)[\psi(z, x)]^z_{t_{c_i}}\},$$

where $\psi(x, z)$ is written for shortness for $\psi(x, a;\ z, c_1, \dots, c_p)$, and $[\psi(x, z)]^x_{t_{c_i}}$ denotes the coefficient of t_{c_i} in the expansion of $\psi(x, z)$, regarded as a function of x, in the neighbourhood of the place c_i.

Shew, when all the involved places are ordinary places, that this equation is the same as equation (xii) obtained in the text.

Prove also that

$$D_xD_zQ^{x, a}_{z, c}-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}\omega_i(x)\,\omega_r(z)\,(M_{r, i}+M_{i, r})=D_z\psi(z, x)+\sum_{i=1}^{p}\omega_i(x)[\psi(x, z)]^x_{t_{c_i}}.$$

Hence, as the forms $\omega_i(x)$ are also obtainable by expansion of the function $\psi(z, x)$, *every term on the right hand is immediately calculable when the form of the function $\psi(x, z)$ is known;* then by integrating the right hand in regard to x and z we obtain an integral of the third kind for which the property of the interchange of argument and parameter holds. (Cf. Ex. iii. p. 180.)

Ex. vii. By comparison of the two forms given for the function $\psi(x, a;\ z, c)$ (§§ 126, 131), we can obtain results analogous to those obtained in §§ 133—136 for the function $\psi(x, a;\ z, c_1, \dots, c_p)$.

Putting, as before, $H^{x, a}_z=\Gamma^{x, a}_z-[(z, x)-(z, a)]\frac{dz}{dt}$, and, when z is a branch place, understanding by $D^{k-1}_z H^{x, a}_z$ the expression $D^k_z(\Pi^{x, a}_{z, c}-P^{z, c}_{x, a})$, and, further, putting

$$B_{i, r}=(D^{k_r}_z D^{k_i-1}_c H^{z, m}_c)_{z=c},\quad N_{i, r}=\left[D^{k_r}_z D^{k-1}_c\left((c, z)\frac{dc}{dt}+\frac{1}{t_c}\right)\right]_{z=c},$$

* Günther, *Crelle*, cix. p. 206.

wherein m is an arbitrary place and t_c the infinitesimal at the place c, so that

$$K_{i,r}=N_{i,r}-N_{r,i}=B_{r,i}-B_{i,r}=\left\{D_x^{k_i-1}D_z^{k_r-1}\left[D_z\left((x,z)\frac{dx}{dt}\right)-D_x\left((z,x)\frac{dz}{dt}\right)\right]\right\}_{\substack{x=c\\z=c}},$$

prove, in order, the following equations, which are numbered as the corresponding equations in §§ 133—136 ;

$$H_z^{x,a}=\sum_{i=1}^{p}\omega_i(z)\,D_c^{k_i-1}\,H_c^{x,a} \qquad \text{(ii)},$$

$$P_{z,m}^{x,a}=\Pi_{z,m}^{x,a}-\sum_{i=1}^{p}V_i^{x,a}\,D_c^{k_i-1}\,H_c^{z,m} \qquad \text{(iv)},$$

$$E_z^{x,a}=[(z,x)-(z,a)]\frac{dz}{dt}+\sum_{i=1}^{p}\left[\omega_i(z)\,D_c^{k_i-1}\,H_c^{x,a}-V_i^{x,a}\,D_z\,D_c^{k_i-1}\,H_c^{z,m}\right] \qquad \text{(vi)},$$

$$D_c^{k_r-1}\,H_c^{x,a}=D_c^{k_r-1}\left\{E_c^{x,a}-[(c,x)-(c,a)]\frac{dc}{dt}\right\}+\sum_{i=1}^{p}B_{i,r}\,V_i^{x,a} \qquad \text{(vii)},$$

wherein, when c is a branch place, the first term of the right hand is to be interpreted as

$$D_c^{k_r}\left(P_{c,m}^{x,a}-P_{x,a}^{c,m}\right);$$

also the equations

$$D_z D_c^{k_i-1}\,H_c^{z,m}=D_c^{k_i-1}\left[D_c\left((z,c)\frac{dz}{dt}\right)-D_z\left((c,z)\frac{dc}{dt}\right)\right]+\sum_{r=1}^{p}B_{r,i}\,\omega_r(z),$$

$$\Gamma_z^{x,a}=E_z^{x,a}+\sum_{i=1}^{p}V_i^{x,a}\,D_c^{k_i-1}\left[D_c\left((z,c)\frac{dz}{dt}\right)-D_z\left((c,z)\frac{dc}{dt}\right)\right]$$
$$+\sum_{i=1}^{p}\sum_{r=1}^{p}A_{r,i}\,V_i^{x,a}\,\omega_r(z),$$

and thence, that the algebraically determinable integral

$$G_z^{x,a}=E_z^{x,a}+\sum_{i=1}^{p}V_i^{x,a}\,D_c^{k_i-1}\left[D_c\left((z,c)\frac{dz}{dt}\right)-D_z\left((c,z)\frac{dc}{dt}\right)\right]$$
$$-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}(N_{r,i}-N_{i,r})\,V_i^{x,a}\,\omega_r(z),$$

is equal to

$$\Gamma_z^{x,a}-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}V_i^{x,a}\,\omega_r(z)\,(B_{r,i}+B_{i,r}) \qquad \text{(viii)};$$

and, finally, that the integral

$$Q_{z,m}^{x,a}=\Pi_{z,m}^{x,a}-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}V_i^{x,a}\,V_r^{z,m}\,(B_{r,i}+B_{i,r}) \qquad \text{(ix)},$$

which, clearly, is such that $Q_{z,m}^{x,a}=Q_{x,a}^{z,m}$, can be algebraically defined by the equation

$$Q_{z,m}^{x,a}=P_{z,m}^{x,a}+\sum_{i=1}^{p}V_i^{x,a}\,D_c^{k_i-1}\left\{E_c^{z,m}-[(c,z)-(c,m)]\frac{dc}{dt}\right\}$$
$$-\tfrac{1}{2}\sum_i\sum_r\left(V_i^{x,a}\,V_r^{z,m}-V_r^{x,a}\,V_i^{z,m}\right)K_{r,i} \qquad \text{(ix)}'.$$

Further shew that the function $\psi(x, a\,;\, z, c)$ can be written in the form

$$\psi(x,a\,;\,z,c)=G_z^{x,a}-\sum_{s=1}^{p}\omega_s(z)\,D_c^{k_s-1}\,G_c^{x,a} \qquad \text{(xii)}.$$

The algebraical formula expressing the property of interchange of argument and parameter is to be obtained from the equation

$$D_x D_z Q_{z,m}^{x,a} = D_z\left((x, z)\frac{dx}{dt}\right) + \sum_{i=1}^{p} \omega_i(x) D_c^{k_i-1}\left\{D_c\left((z, c)\frac{dz}{dt}\right) - D_z\left((c, z)\frac{dc}{dt}\right)\right\}$$
$$+\tfrac{1}{2}\Sigma\Sigma\left[\omega_i(x)\,\omega_r(z) - \omega_r(x)\,\omega_i(z)\right] K_{i,r} \qquad \text{(x).}$$

Lastly, if $L_k(z)$ denote the coefficient of $t^k/\underline{|k}$ (k positive) in the expansion of the function $\psi(x, a\,;\, z, c)$ as a function of x in the neighbourhood of the place c, so that (Ex. iv. § 132)

$$L_k(z) = D_c^k\left((z, c)\frac{dz}{dt}\right) - \sum_{i=1}^{p}\omega_i(z) P_{i,k},$$

where $P_{i,k}$ denotes a certain constant such that P_{i,k_r} is $N_{i,r}$, prove, by equating to zero the sum of the coefficients of the first negative powers of the infinitesimals in the expansions of the function of ξ, $\psi(\xi, a\,;\, z, c)\, D_\xi\, \psi(\xi, a\,;\, x, c)$, at all places where negative powers occur, that

$$D_x\psi(x, a\,;\, z, c) - D_z\psi(z, a\,;\, x, c) = \sum_{i=1}^{p}\left[\omega_i(x) L_{k_i}(z) - \omega_i(z) L_{k_i}(x)\right] \qquad \text{(A),}$$

wherein, on the right, only functions $L_k(z)$ occur for which k is one of the p numbers $k_1, k_2, \ldots, k_p$, and that

$$D_x D_z Q_{z,m}^{x,a} - \tfrac{1}{2}\sum_{i=1}^{p}\sum_{r=1}^{p}\omega_i(x)\,\omega_r(z)\,(N_{r,i} + N_{i,r}) = D_z\psi(z, a\,;\, x, c) + \sum_{i=1}^{p}\omega_i(x) L_{k_i}(z) \qquad \text{(B)};$$

thus an elementary integral of the third kind, permitting interchange of argument and parameter, is obtained immediately from the function $\psi(x, a\,;\, z, c)$ by integrating the right hand of equation (B) in regard to x and z.

Prove also, that if

$$K_z^{x,a} = G_z^{x,a} - [(z, x) - (z, a)]\frac{dz}{dt},$$

we have the formulae

$$K_z^{x,a} = \sum_{i=1}^{p}\omega_i(z) D^{k_i-1} K_c^{x,a} \qquad \text{(xv)}$$

$$Q_{z,m}^{x,a} = P_{z,m}^{x,a} + \sum_{i=1}^{p} V_i^{x,a} D_c^{k_i-1} K_c^{z,m} \qquad \text{(xvi)}$$

$$D_x\left((z, x)\frac{dz}{dt}\right) - D_z\left((x, z)\frac{dx}{dt}\right) = \sum_{i=1}^{p}\left[\omega_i(x) D_z D_c^{k_i-1} K_c^{z,m} - \omega_i(z) D_x D_c^{k_i-1} K_c^{x,a}\right] \qquad \text{(xvii).}$$

Ex. viii. To calculate the integral $Q_{z,m}^{x,a}$ for the case ($p=2$) where the fundamental equation is

$$y^2 = f(x),$$

wherein $f(x)$ is a sextic polynomial divisible by $x - c$, which is expansible in the form

$$f(x) = A^2(x-c) + Q(x-c)^2 + R(x-c)^3 + \ldots,$$

we may use the equation (xi) of Ex. vii. When x, z are near the place c, putting

$$x = c + t_1^2,\quad z = c + t_2^2,\quad y = At_1 + \frac{Q}{2A}t_1^3 + \ldots,\quad s = At_2 + \frac{Q}{2A}t_2^3 + \ldots,$$

prove that

$$D_z\left((x, z)\frac{dx}{dt}\right) - D_x\left((z, x)\frac{dz}{dt}\right) = \frac{R}{A^2}(t_1^2 - t_2^2) + \text{cubes and higher powers of } t_1 \text{ and } t_2,$$

and thence (see Ex. ii. § 132) that

$$K_{12}[\omega_1(x)\,\omega_2(z)-\omega_2(x)\,\omega_1(z)]=\frac{R(x-z)}{4yz}\,\frac{dx}{dt}\,\frac{dz}{dt}.$$

Also, when z is not a branch place, if c_1 be a place near to c, and the expansion of the function $\left[\frac{\partial}{\partial c_1}(z, c_1)-\frac{\partial}{\partial z}(c_1, z)\right]\frac{dc_1}{dt}$ in powers of the infinitesimal at c, contain the terms $M+\,.\,+Nt^2+\dots$, so that

$$M=\left[\frac{\partial}{\partial c}(z, c)-\frac{\partial}{\partial z}(c, z)\right]\frac{dc}{dt}, \qquad 2N=D_c^{\,2}\left\{\left[\frac{\partial}{\partial c}(z, c)-\frac{\partial}{\partial z}(c, z)\right]\frac{dc}{dt}\right\},$$

prove that

$$M=\frac{[A^2+f'(z)]\,[z-c]-2f(z)}{2As\,(z-c)^2},$$

$$N=\frac{3A^2(z-c)\,[A^2+\tfrac{1}{2}\,Q\,(z-c)]+f'(z)\,.\,(z-c)\,[A^2-\tfrac{1}{2}\,Q\,(z-c)]-2f(z)\,[2A^2-\tfrac{1}{2}\,Q\,(z-c)]}{2A^3s\,(z-c)^3};$$

substituting these results in the formula (xi) of Ex. vii., prove that

$$\frac{\partial^2 Q}{\partial x\partial z}=\frac{ys+f(x, z)}{2ys\,(x-z)^2}-\frac{1}{240}\left\{(x-c)\,(z-c)\frac{\partial^4 f}{\partial c^4}+6\,(x+z-2c)\frac{\partial^3 f}{\partial c^3}+12\frac{\partial^2 f}{\partial c^2}\right\}\Big/ ys,$$

where $f(x, z)$ has the same signification as in Example v. The part within the brackets $\{\ \ \}$ is of the form $ys\,\Sigma\Sigma a_{i,r}\,\omega_i(x)\,\omega_r(z)$, where $a_{i,r}=a_{r,i}$.

Obtain the same result by the formula (B) of Ex. vii., using the form of $\psi(x, a;\, z, c)$ found in Ex. ii. § 132.

138. The formulæ in §§ 133—136 enable us to express the form of a canonical integral of the third kind, in the most general case; and to calculate the integral for any fundamental algebraic equation, when the integral functions are known. But they have the disadvantage of presenting the result in a form in which there enter p arbitrary places $c_1, \dots, c_p$. We proceed now to shew how to formulate the theory in a more general way; though the results obtained are not so explicit as those previously given, they are in some cases more suitable for purposes of calculation.

Let $u_1^{x,a}, \dots, u_p^{x,a}$ denote any p linearly independent integrals of the first kind; denote $D_x u_i^{x,a}$ by $\mu_i(x)$. Let the matrix whose (i, j)th element is $\mu_j(c_i)$ be denoted by μ, $c_1, \dots, c_p$ being the places used (§ 121) to define the quantities $\omega_1(x), \dots, \omega_p(x)$. Let $\nu_{i,j}$ denote the minor of the (i, j)th element in the determinant of the matrix μ, divided by the determinant of μ; so that the matrix inverse * to μ is that whose (i, j)th element is $\nu_{j,i}$. Then we clearly have

$$\omega_i(x)=\nu_{i,1}\,\mu_1(x)+\dots\dots+\nu_{i,p}\,\mu_p(x) \qquad (i=1, 2, \dots, p).$$

* Since $u_1^{x,a}, \dots, u_p^{x,a}$ are linearly independent, and the places $c_1, \dots, c_p$ are independent (see §§ 23, 121), the matrix μ^{-1} can always be formed.

Let a denote any symmetrical matrix of p^2 quantities, $a_{i,j}$, in which $a_{i,j}=a_{j,i}$. Then we define p quantities by the p equations

$$L_i^{x,a} = \nu_{1,i} H_{c_1}^{x,a} + \nu_{2,i} H_{c_2}^{x,a} + \dots + \nu_{p,i} H_{c_p}^{x,a} - 2\,(a_{i,1} u_1^{x,a} + \dots + a_{i,p} u_p^{x,a}),$$

and call them fundamental integrals of the second kind *associated* with the integrals $u_1^{x,a}, \dots, u_p^{x,a}$. For instance when $\mu_i(x)=\omega_i(x)$, $\nu_{i,j}=0$ unless $i=j$, in which case $\nu_{i,i}=1$. Thus by taking $a_{i,j}=\frac{1}{4}(A_{i,j}+A_{j,i})$, the integrals $K_{c_1}^{x,a}, \dots, K_{c_p}^{x,a}$ (p. 187. xiv.) are a fundamental system associated with the set $V_1^{x,a}, \dots, V_p^{x,a}$.

It will be convenient in what follows to employ the notation of matrices to express the determinant relations of which we avail ourselves *. We shall therefore write the definition given above in the form

$$L^{x,a} = \bar{\nu} H^{x,a} - 2a u^{x,a},$$

wherein $L^{x,a}$ stands for the row of p quantities $L_1^{x,a}, \dots, L_p^{x,a}$, $H^{x,a}$ stands for the row of p quantities $H_{c_1}^{x,a}, \dots, H_{c_p}^{x,a}$, and $\bar{\nu}$ denotes the matrix obtained by changing the rows of ν into its columns, and is in fact equal to the matrix denoted by μ^{-1}, so that we may also write

$$L^{x,a} = \mu^{-1} H^{x,a} - 2a u^{x,a}, \quad = \mu^{-1} K^{x,a} - 2a' u^{x,a},$$

where (§ 137)

$$H_{c_i}^{x,a} = K_{c_i}^{x,a} + \tfrac{1}{2}\sum_{r=1}^{p} (A_{r,i} + A_{i,r})\, V_r^{x,a}.$$

Explicit forms of the integrals $K_{c_i}^{x,a}$ have been given (§§ 134, 136).

Then, from the equations defining the integrals $L_i^{x,a}$, we have

$$\begin{aligned}\sum_{i=1}^{p} \mu_i(z)\, L_i^{x,a} &= \sum_{j=1}^{p} H_{c_j}^{x,a} \sum_{i=1}^{p} \nu_{j,i}\,\mu_i(z) - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r,s}\, u_r^{x,a}\, \mu_s(z),\\ &= \sum_{j=1}^{p} \omega_j(z)\, H_{c_j}^{x,a} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r,s}\, u_r^{x,a}\, \mu_s(z),\\ &= H_z^{x,a} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r,s}\, u_r^{x,a}\, \mu_s(z);\end{aligned}$$

and this is an important result. For, putting for z in turn any p independent places, the p functions $L_i^{x,a}$ are determined by this equation. Thus the functions $L_1^{x,a}, \dots, L_p^{x,a}$ *do not depend upon the places* $c_1, c_2, \dots, c_p$.

* See for instance Cayley, *Collected Works*, vol. ii. p. 475, and the Appendix II. to the present volume, where other references are given.

Also, from this equation we infer

$$D_x\left[(z, x)\frac{dz}{dt}\right] - D_z\left[(x, z)\frac{dx}{dt}\right] = D_z H_x^{z, c} - D_x H_z^{x, a}$$

$$= \sum_{i=1}^{p}\left[\mu_i(x)\, D_z L_i^{z, c} - \mu_i(z)\, D_x L_i^{x, a}\right] \quad (17),$$

c being any arbitrary place. Now it is immediately seen that if $R_1(x), \ldots, R_p(x)$ be any rational functions of x such that

$$\sum_{i=1}^{p}\left[\mu_i(x)\, D_z L_i^{z, c} - \mu_i(z)\, D_x L_i^{x, a}\right] = \sum_{i=1}^{p}\left[\mu_i(x) R_i(z) - \mu_i(z)\, R_i(x)\right],$$

then $R_i(x)$ can only be a form of $D_x L_i^{x, a}$, obtained from $D_x L_i^{x, a}$ by altering the values of the constant elements of the symmetrical matrix a. Hence the equation (17) furnishes a method of calculating the integrals $L_i^{x, a}$, whenever it is possible to put the left-hand side into the form of the right-hand side.

The equation (17) shews that the expression

$$D_z\left((x, z)\frac{dx}{dt}\right) + \sum_{i=1}^{p}\mu_i(x)\, D_z L_i^{z, c},$$

is unaltered by the interchange of x and z. This expression is also equal to

$$D_z\left((x, z)\frac{dx}{dt}\right) + D_z H_x^{z, c} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r, s}\,\mu_r(x)\,\mu_s(z)$$

and, therefore, to

$$D_z \Gamma_x^{z, c} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r, s}\,\mu_r(x)\,\mu_s(z).$$

Hence, the formula (§ 134, ix.)

$$R_{z, c}^{x, a} = P_{z, c}^{x, a} + \sum_{i=1}^{p} u_i^{x, a} L_i^{z, c} = \Pi_{z, c}^{x, a} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r, s}\, u_r^{x, a}\, u_s^{z, c}$$

$$= Q_{z, c}^{x, a} + \tfrac{1}{2}\sum_{r=1}^{p}\sum_{s=1}^{p}(A_{r, s} + A_{s, r})\, V_r^{x, a}\, V_s^{z, c} - 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r, s}\, u_r^{x, a}\, u_s^{z, c}$$

gives us a form of canonical integral of the third kind not depending upon the places $c_1, \ldots, c_p$, and immediately calculable when the forms of the functions $L_i^{x, a}$ are found.

The formula

$$\Gamma_z^{x, a} = [(z, x) - (z, a)]\frac{dz}{dt} + \sum_{\mu=1}^{p}\mu_i(z)\, L_i^{x, a} + 2\sum_{r=1}^{p}\sum_{s=1}^{p} a_{r, s}\, u_r^{x, a}\,\mu_s(z)$$

serves to express any integral of the second kind in terms of the integrals $L_1, \ldots, L_p$

Ex. i. For the surface $y^2=f(x)$, where $f(x)$ is a rational polynomial of order $2p+2$, the function

$$R(\xi)=\frac{\eta}{s(\xi-z)}\cdot\frac{d}{d\xi}\left(\frac{\eta}{y(\xi-x)}\right), =\frac{1}{2ys}\left\{\frac{f'(\xi)}{(\xi-x)(\xi-z)}-\frac{2f(\xi)}{(\xi-x)^2(\xi-z)}\right\},$$

wherein $s^2=f(z)$, $\eta^2=f(\xi)$, is a rational function of ξ (without η). Prove by applying the theorem, $\Sigma\left[R(\xi)\frac{d\xi}{dt}\right]_{t-1}=0$, (Ex. vi, § 137) that

$$\frac{\partial}{\partial x}(z, x)-\frac{\partial}{\partial z}(x, z)=\sum_k\sum_{k'}(k'-k)\lambda_{k+k'+2}\left(\frac{x^k}{2y}\frac{z^{k'}}{2s}-\frac{x^{k'}}{2y}\frac{z^k}{2s}\right),$$

where k, k' represent in turn every pair of unequal numbers from $0, 1, 2, \ldots, 2p$, whose sum is not greater than $2p$, k' being greater than k, and the coefficients λ are given by the fact that

$$y^2=f(x)=\lambda+\lambda_1x+\lambda_2x^2+\ldots+\lambda_{2p+1}x^{2p+1}+\lambda_{2p+2}x^{2p+2}.$$

Hence, a set of integrals of the second kind associated with the integrals of the first kind

$$\int_a^x\frac{dx}{y},\ \int_a^x\frac{xdx}{y},\ \ldots\ldots,\ \int_a^x\frac{x^{p-1}dx}{y},$$

is given by

$$L_i^{x,a}=\int_a^x\frac{dx}{4y}\sum_{k=i}^{k=2p+1-i}\lambda_{k+1+i}(k+1-i)x^k,\qquad (i=1, 2, \ldots, p);$$

and a canonical integral of the third kind is given by

$$\int_a^x\int_c^z\frac{dz}{2s}\frac{dx}{2y}\left[\frac{2ys+2f(z)+f'(z)(x-z)}{(x-z)^2}+\sum_{i=1}^{p}x^{i-1}\sum_{k=1}^{2p+1-i}(k+1-i)\lambda_{k+1+i}z^k\right].$$

This is equal to

$$\int_a^x\int_c^z\frac{dz}{2s}\frac{dx}{2y}\frac{2ys+\sum\limits_{i=0}^{p+1}x^iz^i\left[2\lambda_{2i}+\lambda_{2i+1}(x+z)\right]}{(x-z)^2},$$

which is clearly symmetric in x and z.

The value of $\frac{\partial}{\partial x}(z, x)-\frac{\partial}{\partial z}(x, z)$ used in this example is given by Abel, *Œuvres Complètes* (Christiania, 1881), Vol. i. p. 49.

Ex. ii. Shew in Ex. i., for $p=1$, that the integral associated with $\int_a^x\frac{dx}{y}$ is $\int_a^x\frac{\lambda_3x+2\lambda_4x^2}{4y}dx$; and express these in the notation of Weierstrass's elliptic functions when the fundamental equation is $y^2=4x^3-g_2x-g_3$.

139. Suppose now that the integrals $u_1^{x,a}, \ldots, u_p^{x,a}$ are connected with the normal integrals $v_1^{x,a}, \ldots, v_p^{x,a}$ by means of the equations

$$\pi i\mu_r(x)=\lambda_{r,1}\Omega_1(x)+\ldots\ldots+\lambda_{r,p}\Omega_p(x),$$

which, since $\Omega_i(x)=2\pi iDv_i^{x,a}$, are equivalent to

$$u_r^{x,a}=2\left(\lambda_{r,1}v_1^{x,a}+\ldots\ldots+\lambda_{r,p}v_p^{x,a}\right).$$

Then the periods of the integral $u_r^{x,a}$, at the first p period loops, form the rth row of a matrix, 2λ, and the periods of the integral $u_r^{x,a}$ at the second

p period loops form the rth row of a matrix $2\lambda\tau$; we shall write $\omega = \lambda$ and $\omega' = \lambda\tau$, so that $\omega_{i,j} = \lambda_{i,j}$. The two suffixes of the quantities $\omega_{i,j}$ will prevent confusion between them and the differential coefficients $\omega_i(x)$.

Let the periods of $L_i^{x,a}$ at the jth period loops of the first and second kind be denoted by $-2\eta_{i,j}$ and $-2\eta'_{i,j}$ respectively. The matrix whose ith row consists of the quantities $\eta_{i,1}, \dots, \eta_{i,p}$ will be denoted by η; similarly the matrix of the quantities $\eta'_{i,j}$ will be denoted by η'. The matrix of the periods of the integrals $H_{c_1}^{x,a}, \dots, H_{c_p}^{x,a}$ at the first period loops is zero; the (i, j)th element of the matrix at the second period loops is the jth period of $H_{c_i}^{x,a}$, namely $\Omega_j(c_i)$. We shall denote this matrix by Δ.

By the definitions of the integrals $L_i^{x,a}$ we therefore have

$$2\eta_{i,j} = 4(a_{i,1}\omega_{1,j} + \dots + a_{i,p}\omega_{p,j}), \qquad (i, j = 1, 2, \dots, p)$$
$$2\eta'_{i,j} = 4(a_{i,1}\omega_1{}'_{,j} + \dots + a_{i,p}\omega'_{p,j}) - \big(\nu_{1,i}\Omega_j(c_1) + \dots + \nu_{p,i}\Omega_j(c_p)\big),$$

and all these equations are contained in the equations

$$\eta = 2a\omega,$$
$$\eta' = 2a\omega' - \tfrac{1}{2}\bar{\nu}\Delta = 2a\omega' - \tfrac{1}{2}\mu^{-1}\Delta.$$

Now from the equations connecting $\mu_r(x)$ and $\Omega_s(x)$, we obtain

$$\pi i\mu_r(c_i) = \lambda_{r,1}\Omega_1(c_i) + \dots\dots + \lambda_{r,p}\Omega_p(c_i),$$

wherein $\mu_r(c_i)$ is the (i, r)th element of the matrix μ, and the right hand is the (i, r)th element of the matrix $\Delta\bar{\lambda}$; hence we may put

$$\pi i\mu = \Delta\bar{\lambda}.$$

If then we denote the matrix $\tfrac{1}{2}\mu^{-1}\Delta$ by $\bar{h}$, we have

$$2\Delta\bar{\lambda}h = 2\pi i\mu\bar{h} = \pi i\Delta = \Delta\pi i,$$

and infer that $2\lambda\bar{h} = \pi i$, and thence that $2h\lambda = \pi i$. Thus $2h\omega = \pi i$, $2h\omega' = \pi i\tau$. Also the integrals $u_1^{x,a}, \dots, u_p^{x,a}, \dots, v_1^{x,a}, \dots, v_p^{x,a}$ are connected by the equation $hu^{x,a} = 2h\lambda v^{x,a} = \pi i v^{x,a}$.

140. The four equations

$$2h\omega = \pi i, \quad 2h\omega' = \pi i\tau, \quad \eta = 2a\omega, \quad \eta' = 2a\omega' - \bar{h} \qquad \text{(A)}$$

will prove to be of fundamental importance in the theory of the theta functions. They express the periods η, η' independently of the places $c_1, \dots, c_p$, used in defining $L_i^{x,a}$.

If beside the symmetrical matrix τ, and the arbitrary symmetrical matrix a, we suppose the matrix h, which is in general unsymmetrical, to be arbitrarily given, the integrals $u_1^{x,a}, \ldots, u_p^{x,a}$ being then determined by the equation $hu^{x,a} = \pi v i^{x,a}$, the first equation, $2h\omega = \pi i$, gives rise to p^2 equations whereby the p^2 quantities $\omega_{i,j}$ are to be found, and similarly the other equations give rise each to p^2 equations determining respectively the quantities $\omega'_{i,j}$, $\eta_{i,j}$, $\eta'_{i,j}$. But, thereby, the $4p^2$ quantities thus involved are determined in terms of less than $4p^2$ given quantities. For the symmetrical matrices a, τ involve each only $\frac{1}{2}p(p+1)$ quantities, and the number of given quantities is thus only $p(p+1)+p^2$. There are therefore, presumably,

$$4p^2 - [p^2 + p(p+1)], \;= 2p^2 - p,$$

relations connecting the $4p^2$ quantities $\omega_{i,j}$, $\omega'_{i,j}$, $\eta_{i,j}$, $\eta'_{i,j}$; we can in fact express these relations in various forms.

One of these forms is

$$\bar{\omega}\eta = \bar{\eta}\omega, \quad \bar{\omega}'\eta' = \bar{\eta}'\omega', \quad \bar{\eta}\omega' - \bar{\omega}\eta' = \tfrac{1}{2}\pi i = \bar{\omega}'\eta - \bar{\eta}'\omega, \qquad \text{(B)}$$

of which, for instance, the first equation is equivalent to the $\frac{1}{2}p(p-1)$ equations

$$\sum_{r=1}^{p} (\omega_{r,i}\,\eta_{r,j} - \eta_{r,i}\,\omega_{r,j}) = 0,$$

in which $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, p$, and i is not equal to j. The second equation is similarly equivalent to $\frac{1}{2}p(p-1)$ equations, and the third to p^2 equations. The total number of relations thus obtained is therefore the right number $p^2 + p(p-1)$, In this form the equations are known as Weierstrass's equations.

Another form in which the $2p^2 - p$ relations can be expressed is

$$\omega\bar{\omega}' = \omega'\bar{\omega}, \quad \eta\bar{\eta}' = \eta'\bar{\eta}, \quad \omega'\bar{\eta} - \omega\bar{\eta}' = \tfrac{1}{2}\pi i = \eta\bar{\omega}' - \eta'\bar{\omega} \qquad \text{(C)}$$

These equations are distinguished from the equations (A) as Riemann's equations.

141. The equations (B) and (C) are entirely equivalent; either set can be deduced from the equations (A) or from the other set. A natural way of obtaining the set (B) is to use the equation (17). A natural way of obtaining the set (C) is to make use of the Riemann method of contour integration.

The equations (A) give, recalling that $\bar{a}=a$, $\omega'=\omega\tau$, $\bar{\tau}=\tau$,

$$\bar{\omega}\eta = 2\bar{\omega}a\omega \;, = \beta, \text{ say, a symmetrical matrix,}$$

$$\bar{\omega}\eta' = 2\bar{\omega}a\omega' - \bar{\omega}h = 2\bar{\omega}a\omega\tau - \overline{h\omega} = \beta\tau - \tfrac{1}{2}\pi i.$$

Hence $\qquad \bar{\eta}\omega' = \bar{\eta}\omega\tau = \bar{\beta}\tau = \beta\tau,$

and because $\bar{\omega}' = \tau\bar{\omega}$,

$$\bar{\omega}'\eta' = \tau\bar{\omega}\eta' = \tau\beta\tau - \tfrac{1}{2}\pi i\tau,$$

and thus, as $\overline{\tau\beta\tau} = \tau\beta\tau$, we have

$$\bar{\omega}\eta = \bar{\eta}\omega, \; \bar{\omega}'\eta' = \bar{\eta}'\omega', \; \bar{\eta}\omega' - \bar{\omega}\eta' = \tfrac{1}{2}\pi i = \bar{\omega}'\eta - \bar{\eta}'\omega,$$

which are the equations (B). And it should be noticed that these results are all derived from the three $\omega'=\omega\tau$, $\bar{\omega}\eta=\beta$, $\bar{\omega}\eta'=\beta\tau-\frac{1}{2}\pi i$, assuming only that β and τ are symmetrical.

From the equations (B), putting $\bar{\omega}\eta=\beta$, $\bar{\omega}'\eta'=\gamma$, so that β and γ are symmetrical matrices, we obtain*

$$\eta=(\bar{\omega})^{-1}\beta,\ \bar{\eta}'=\gamma(\omega')^{-1},\ \text{and thence } \bar{\omega}'(\bar{\omega})^{-1}\beta-\gamma(\bar{\omega}')^{-1}\omega=\tfrac{1}{2}\pi i.$$

Hence, if $\omega^{-1}\omega'=\kappa$, so that $\omega\kappa=\omega'$, $\bar{\omega}'=\bar{\kappa}\bar{\omega}$, $\bar{\omega}'(\bar{\omega})^{-1}=\bar{\kappa}$, and $\kappa^{-1}=(\omega')^{-1}\omega$, we have

$$\bar{\kappa}\beta-\gamma\kappa^{-1}=\tfrac{1}{2}\pi i,\ \text{or}\ \bar{\kappa}\beta\kappa-\gamma=\tfrac{1}{2}\pi i\kappa,$$

and therefore, as the matrices $\bar{\kappa}\beta\kappa$ and γ are symmetrical, so also is the matrix κ; and thus

$$\omega^{-1}\omega'=\bar{\omega}'(\bar{\omega})^{-1},\ \text{and therefore}\ \omega\bar{\omega}'=\omega'\bar{\omega},$$

which is one of the equations (C).

Further

$$\bar{\omega}\eta'=\bar{\eta}\omega'-\tfrac{1}{2}\pi i=\bar{\eta}\omega\kappa-\tfrac{1}{2}\pi i=\beta\kappa-\tfrac{1}{2}\pi i,$$

and therefore

$$\bar{\eta}'\omega=\bar{\kappa}\bar{\beta}-\tfrac{1}{2}\pi i=\kappa\beta-\tfrac{1}{2}\pi i,$$

leading to

$$\bar{\omega}\eta\bar{\eta}'\omega=\beta\kappa\beta-\tfrac{1}{2}\pi i\beta,$$

and the right hand is a symmetrical matrix, and therefore equal to $\bar{\omega}\eta'\bar{\eta}\omega$; thus also

$$\eta\bar{\eta}'=\eta'\bar{\eta},$$

which is the second of the equations (C).

Finally $(\omega'\bar{\eta}-\omega\bar{\eta}')\omega=\omega'\bar{\eta}\omega-\omega(\bar{\omega}'\eta-\tfrac{1}{2}\pi i)-\omega'\bar{\omega}\eta-\omega\bar{\omega}'\eta+\tfrac{1}{2}\pi i\omega=(\omega'\bar{\omega}-\omega\bar{\omega}')\eta+\tfrac{1}{2}\pi i\omega$

$$=\tfrac{1}{2}\pi i\omega,$$

and thus

$$\omega'\bar{\eta}-\omega\bar{\eta}'=\tfrac{1}{2}\pi i,\ =\ ,\ \text{therefore},\ \eta\bar{\omega}'-\eta'\bar{\omega},$$

which is the third of equations (C).

We have deduced both the equations (B) and (C) from the equations (A). A similar method can be used to deduce the equations (B) from the equations (C).

Other methods of obtaining the equations (B) and (C) are explained in the Examples which follow (§ 142, Exx. ii—v).

142. *Ex.* i. Shew that the p integrals given by the equation

$$\Lambda_i^{x,a}=t_{1,i}H_{c_1}^{x,a}+\dots+t_{p,i}H_{c_p}^{x,a},$$

where $t_{i,j}$ is the minor of $\Omega_j(c_i)$ in the determinant of the matrix Δ (§ 139), divided by the determinant of Δ, namely by the equation

$$\Lambda^{x,a}=\Delta^{-1}H^{x,a},$$

are a set of fundamental integrals of the second kind associated with the set of integrals of the first kind $2\pi i v_1^{x,a}$, ..., $2\pi i v_p^{x,a}$, and are such that

$$D_x\left((z,x)\frac{dz}{dt}\right)-D_z\left((x,z)\frac{dx}{dt}\right)$$

$$=\sum_{i=1}^{p}\left(\omega_i(x)D_zH_{c_i}^{z,c}-\omega_i(z)D_xH_{c_i}^{x,a}\right)=\sum_{i=1}^{p}\left(\omega_i(x)D_zK_{c_i}^{z,c}-\omega_i(z)D_xK_{c_i}^{x,a}\right)$$

$$=\sum_{i=1}^{p}\left(\Omega_i(x)D_z\Lambda_i^{z,c}-\Omega_i(z)D_x\Lambda_i^{x,a}\right)=\sum_{i=1}^{p}\left(\mu_i(x)D_zL_i^{z,c}-\mu_i(z)D_xL_i^{x,a}\right).$$

* The determinant of the matrix ω, $=\lambda$, cannot vanish, because $u_1^{x,a}$, .., $u_p^{x,a}$ are linearly independent. The determinant of the matrix τ does not vanish, since otherwise we could determine an integral of the first kind with no periods at the period loops of the second kind (cf. Forsyth, *Theory of Functions*, § 231, p. 440).

Prove that the function $\Lambda_i^{x,a}$ has only one period, namely at the ith period loop of the second kind, and that this period is equal to 1. For the sets

$$2\pi i v_1^{x,a}, \ldots, 2\pi i v_p^{x,a}, \Lambda_1^{x,a}, \ldots, \Lambda_p^{x,a},$$

we have in fact $\qquad \omega=\pi i, \; \omega'=\pi i \tau, \; \eta=0, \; \eta'=-\tfrac{1}{2}.$

Shew that these values satisfy the equations (B) and (C).

Ex. ii. From Ex. i. we deduce

$$2\pi i \sum_{i=1}^{p} (v_i^{x,a} \Lambda_i^{z,c} - v_i^{z,c} \Lambda_i^{x,a}) = \sum_{i=1}^{p} (u_i^{x,a} L_i^{z,c} - u_i^{z,c} L_i^{x,a}).$$

Hence, supposing x and z separately to pass, on the dissected Riemann surface, respectively from one side to the other* of the rth period loop of the first kind, and from one side to the other of the sth period loop of the first kind, we obtain, for the increment of the right-hand side

$$-4 \sum_{i=1}^{p} (\omega_{i,r} \eta_{i,s} - \eta_{i,r} \omega_{i,s}),$$

which is the (r, s)th element of the matrix $-4(\bar{\omega}\eta - \bar{\eta}\omega)$. For the functions on the left-hand side the matrix $\bar{\omega}\eta - \bar{\eta}\omega$ vanishes (Ex. i.). Hence the same is true for those on the right hand.

Supposing x to pass from one side to the other of the rth period loops of the first kind, and z from one side to the other of the sth period loop of the second kind, we similarly prove that $\bar{\omega}\eta' - \bar{\eta}\omega'$ has the same value for the functions on the two sides of the equation, and therefore, as we see by considering the functions on the left hand, has the value $-\tfrac{1}{2}\pi i$.

While, if both x and z pass from one side to the other of period loops of the second kind we are able to infer $\qquad \bar{\omega}'\eta' = \bar{\eta}'\omega'.$

We thus obtain Weierstrass's equations (B).

Ex. iii. If $U_1^{x,a}, \ldots, U_p^{x,a}$ be any integrals, the periods of $U_i^{x,a}$ at the jth period loops of the first and second kind be respectively $\zeta_{i,j}$, $\zeta'_{i,j}$, and the matrices of these elements be respectively denoted by ζ, ζ'; and $W_1^{x,a}, \ldots, W_p^{x,a}$ be other integrals for which the corresponding matrices are ξ and ξ', prove that the integral $\int U_i^{x,a}\, dW_j^{x,a}$, taken positively round all the period-loop-pairs has the value

$$\sum_{r=1}^{p} (\zeta_{i,r} \xi'_{j,r} - \zeta'_{i,r} \xi_{j,r}),$$

which is the (i, j)th element of the matrix $\zeta\bar{\xi}' - \zeta'\bar{\xi}$.

Ex. iv. If $R_i(x)$ denote the rational function of x given by

$$R_i(x) = \sum_{r=1}^{p} \nu_{r,i} [(c_r, x) - (c_r, a)] \frac{dc_r}{dt},$$

the function $L_i^{x,a} + R_i(x)$ is infinite only at $c_1, \ldots, c_p$, and has the same periods $L_i^{x,a}$, Denote this function by $Y_i^{a,a}$.

* To that side for which the periods count positively (see the diagram, § 18).

Prove that if the expansion of the integral $Y_i^{x, a}$ in the neighbourhood of the place c_s be written in the form

$$Y_i^{x, a} = -\frac{\nu_{s, i}}{t} + f_{i, s} + g_{i, s}\, t + \dots,$$

then

$$g_{i, s} = \nu_{1, i}(A_{1, s} + M_{1, s}) + \dots + \nu_{p, i}(A_{p, s} + M_{p, s}),$$

where $A_{i, s}$, $M_{i, s}$ are as defined in § 134, and are such that $A_{i, s} + M_{i, s} = A_{s, i} + M_{s, i}$.

Hence shew that the sum of the values of the integral $\int Y_i^{x, a}\, dY_j^{x, a}$ taken round all the places $c_1, \dots, c_p$ is zero.

Ex. v. Infer from Exs. iii. and iv., by taking

(α) $U_i^{x, a} = u_i^{x, a} = W_i^{x, a}$, that $\omega\bar{\omega}' = \omega'\bar{\omega}$,

(β) $U_i^{x, a} = Y_i^{x, a}$, $W_i^{x, a} = u_i^{x, a}$, that $\eta\bar{\omega}' - \eta'\bar{\omega} = \frac{1}{2}\pi i$,

(γ) $U_i^{x, a} = Y_i^{x, a} = W_i^{x, a}$ that $\eta\bar{\eta}' = \eta'\bar{\eta}$.

These are Riemann's equations.

Ex. vi. If instead of the places $c_1, \dots, c_p$ and the matrix μ, we use a matrix depending only on one place c, the ith row being formed with the elements $D_c^{k_i - 1}\mu_1(c), \dots, D_c^{k_i - 1}\mu_p(c)$, we can similarly obtain a set $L_1^{x, a}, \dots, L_p^{x, a}$ associated with the set $u_1^{x, a}, \dots, u_p^{x, a}$.

Shew that the periods of $L_1^{x, a}, \dots, L_p^{x, a}$ thus determined are independent of the position of the place c.

Ex. vii. If the differential coefficients $\mu_1(x), \dots, \mu_p(x)$, be those derived from a set of p independent places $b_1, b_2, \dots, b_p$, just as $\omega_1(x), \dots, \omega_p(x)$ are derived from $c_1, \dots, c_p$, so that $\mu_i(b_i) = 1$, $\mu_i(b_r) = 0$, prove that $\nu_{r, i} = \omega_r(b_i)$ and that

$$L_i^{x, a} = H_{b_i}^{x, a} - 2(a_{i1} u_1^{x, a} + \dots + a_{ip} u_p^{x, a}).$$

143. We conclude this chapter with some applications* of the functions $\psi(x, a; z, c)$, $E(x, z)$ to the expression of functions which are single-valued on the (undissected) Riemann surface. Such functions include, but are more general than, rational functions, in that they may possess essential singularities.

Consider first a single-valued function which is infinite only at one place; denote the place by m, and the function by $F(x)$.

Since $\psi(x, a; z, c) \Big/ \frac{dz}{dt}$ is a rational function of z, the integral

$$\int F(z) \left[\psi(x, a; z, c) \Big/ \frac{dz}{dt}\right] dz, \text{ or } \int F(z)\, \psi(x, a; z, c)\, dt_z,$$

taken round the edges of the period-pair-loops, has zero for its value. But this integral is also equal to the sum of its values taken round the place m,

* Appell, *Acta Math.* i. pp. 109, 132 (1882), Günther, *Crelle* cix. p. 199 (1892).

where $F(z)$ is infinite, and the places x and a at which $\psi(x, a; z, c)$ is infinite.

Now, when z is in the neighbourhood of the place m, since $\psi(x, a; z, c) \Big/ \dfrac{dz}{dt}$ is a rational function of z, we can put

$$\psi(x, a; z, c) = \sum_{r=0}^{\infty} \frac{t_m^r}{\lfloor r} D_m^r \psi(x, a; m, c),$$

where t_m is the infinitesimal at the place m.

Thus the integral $\int F(z) \psi(x, a; z, c)\, dt_z$, taken round the place m, gives

$$2\pi i \sum_{r=0}^{\infty} \frac{A_r}{\lfloor r} D_m^r \psi(x, a; m, c),$$

where A_r is the value of the integral $\dfrac{1}{2\pi i}\displaystyle\int t_m^r F(z)\, dt_z$ taken round the place m.

When z is in the neighbourhood of the place x, $\psi(x, a; z, c)$ is infinite like t_x^{-1}, t_x being the infinitesimal at the place x, and therefore, taken round the place x, the integral

$$\int F(x) \psi(x, a; z, c)\, dt_z$$

gives

$$2\pi i F(x).$$

Similarly round the place a, the integral gives $-2\pi i F(a)$.

Hence the function $F(x)$ can be expressed in the form

$$F(x) = F(a) - \sum_{r=0}^{\infty} \frac{A_r}{\lfloor r} D_m^r \psi(x, a; m, c),$$

the places a and c being arbitrary (but not in the neighbourhood of the place m).

For example, when $p=0$, $\psi(x, a; z, c) = -\left(\dfrac{1}{x-z} - \dfrac{1}{a-z}\right)$, and

$$F(x) - F(a) = \sum_{r=0}^{\infty} A_r \left[\frac{1}{(x-m)^{r+1}} - \frac{1}{(a-m)^{r+1}}\right],$$

wherein

$$A_r = \frac{1}{2\pi i}\int (z-m)^r F(z)\, dz, \text{ the integral being taken round the place } m.$$

A similar result can be obtained for the case of a single valued function with only a finite number of essential singularities. When one of these singularities is only a pole, say of order μ, the integral $\int t_m^r F(z)\, dz$, taken round this pole, will vanish when $r \geqq \mu$, and the corresponding series of functions $D_m^r \psi(x, a; m, c)$ will terminate.

144. We can also obtain a generalization of Mittag Leffler's Theorem. If $c_1, c_2, \dots$ be a series of distinct places, of infinite number, which converge* to one place c, and $f_1(x), f_2(x), \dots$ be a corresponding series of rational functions, of which $f_i(x)$ is infinite only at the place c_i, then we can find a single valued function $F(x)$, with one essential singularity (at the place c), which is otherwise infinite only at the places $c_1, c_2, \dots$, and in such a way that the difference $F(x) - f_i(x)$ is finite in the neighbourhood of the place c_i.

Since $f_i(x)$ is a rational function, infinite only at the place c_i, and $\psi(x, a; z, c)$ does not become infinite when z comes to c, we can put

$$f_i(x) = f_i(a) - \sum_{r=0}^{\lambda_i} \frac{A_r}{\underline{|r}} D_{c_i}^r \psi(x, a; c_i, c), \qquad \text{(A)}$$

wherein a is an arbitrary place not in the neighbourhood of any of the places $c_1, c_2, \dots, c$, and λ_i is a finite positive integer, and A_r a constant.

Also, when z is sufficiently near to c, and x is not near to c, we can put

$$\psi(x, a; z, c) = \sum_{k=0}^{\infty} \frac{t_c^k}{\underline{|k}} [D_z^k \psi(x, a; z, c)]_{z=c},$$

wherein t_c is the infinitesimal at the place c. Thus also, when z is near to c,

$$D_z^r \psi(x, a; z, c) = \sum_{k=0}^{\infty} t_c^k R_k(x), \qquad \text{(B)},$$

wherein $R_k(x)$ is a rational function, which is only infinite at the place c. There are p values of k which do not enter on the right hand; for it can easily be seen that if $k_1, \dots, k_p$ denote the orders of non-existent rational functions infinite only at the place c, each of the functions

$$[D_z^{k_1-1} \psi(x, a; z, c)]_{z=c}, \dots\dots, [D_z^{k_p-1} \psi(x, a; z, c]_{z=c}$$

vanishes identically. Let the neighbourhood of the place c, within which z must lie in order that the expansions (B) may be valid, be denoted by M.

Of the places $c_1, c_2, \dots$, an infinite number will be within the region M; let these be the places $c_{s+1}, c_{s+2}, \dots$; then s will be finite and, when $i > s$, we have

$$D_{c_i}^r \psi(x, a; c_i, c) = \sum_{k=0}^{\infty} t_i^k R_{i,k}(x),$$

wherein t_i is the value of t_c, in the equation (B), when z is at c_i. Hence also, from the equation (A), wherein there are only a finite number of terms on the right hand, we can put

$$f_i(x) - f_i(a) = \sum_{k=0}^{\infty} t_i^k S_{i,k}(x), \qquad \text{(C)},$$

wherein $S_{i,k}$ is a rational function, $i > s$, and x is not near to the place c.

* so that c is what we may call the *focus* of the series $c_1, c_2, \dots$ (Häufungsstelle).

It is the equation (C) which is the purpose of the utilisation of the function $\psi(x, a; z, c)$ in the investigation. The functions $S_{i,k}(x)$ will be infinite only at the place c. The series (C) are valid so long as x is outside a certain neighbourhood of c. We may call this the region M'.

Let now $\epsilon_{s+1}, \epsilon_{s+2}, \ldots$ be any infinite series of real positive quantities, such that the series

$$\epsilon_{s+1} + \epsilon_{s+2} + \epsilon_{s+3} + \ldots$$

is convergent; let μ_i be the smallest positive integer such that, for $i > s$, the terms

$$\sum_{k=\mu_i+1}^{\infty} t_i^k S_{i,k}(x),$$

taken from the end of the convergent series (C), are, in modulus, less than ϵ_i, for all the positions of x outside M'; then, defining a function $g_i(x)$, when $i > s$, by the equation

$$g_i(x) = f_i(x) - f_i(a) - \sum_{k=0}^{\mu_i} t_i^k S_{i,k}(x),$$

we have, for $i > s$,

$$| g_i(x) | < \epsilon_i.$$

Thus the series

$$\sum_{i=1}^{s} [f_i(x) - f_i(a)] + \sum_{i=s+1}^{\infty} g_i(x)$$

is absolutely and uniformly convergent for all positions of x not in the neighbourhood of the places $c, c_1, c_2, \ldots$, and represents a continuous single valued function of x. When x is near to c_i, the function represented by the series is infinite like $f_i(x)$.

The function is not unique; if $\psi(x)$ denote any single-valued function which is infinite only at the place c, the addition of $\psi(x)$ to the function obtained will result in a function also having the general character required in the enunciation of the theorem. As here determined the function vanishes at the arbitrary place a; but that is an immaterial condition.

For instance when $p=0$, and the place m is at infinity, the places $m_1, m_2, m_3, \ldots$, being $0, 1, \omega, 1+\omega, \ldots, p+q\omega, \ldots$, wherein ω is a complex quantity and p, q are any rational integers, let the functions $f_1(x), f_2(x), \ldots$ be $x^{-1}, (x-1)^{-1}, (x-\omega)^{-1}, \ldots, (x-p-q\omega)^{-1}, \ldots$.

Here $$\psi(x, a; z, c) = -\left(\frac{1}{x-z} - \frac{1}{a-z}\right) = \frac{x-a}{z^2} + \frac{x^2-a^2}{z^3} + \frac{x^3-a^3}{z^4} + \ldots$$

when z is great enough and $|x| < |z|$, $|a| < |z|$.

Also

$$\frac{1}{x-m_i} = \frac{1}{a-m_i} - \psi(x, a; m_i, c)$$
$$= \frac{1}{a-m_i} - \left(\frac{x-a}{m_i^2} + \frac{x^2-a^2}{m_i^3} + \ldots\right),$$

when m_i is great enough, and $|x| < |m_i|$, $|a| < |m_i|$.

Now the series

$$\Sigma \left| \frac{1}{m_i^3} \right| = \Sigma \Sigma \left| \frac{1}{(p+q\omega)^3} \right|$$

is convergent. Hence when x and a are not too great

$$\left| \frac{x^2-a^2}{m_i^3} + \frac{x^3-a^3}{m_i^4} + \dots \right| < \epsilon_i,$$

where ϵ_i is a term of a convergent series of positive quantities. This equation holds for all values of i except $i=1$, in which case $m_i=0$.

Hence we may write

$$g_i(x) = \frac{1}{x-m_i} + \frac{1}{a-m_i} + \frac{x-a}{m_1^2}$$

and obtain the function

$$\frac{1}{x} - \frac{1}{a} + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left[\frac{1}{x-p-q\omega} - \frac{1}{a-p-q\omega} + \frac{x-a}{(p+q\omega)^2} \right],$$

which has the property required. This function is in fact equal, in the notation of Weierstrass's elliptic functions, to $\zeta(x \mid 1, \omega) - \zeta(a \mid 1, \omega)$.

145. We can always specify a rational function of x which, beside being infinite at the place c, is infinite at a place c_i like an expression of the form

$$\frac{A_0}{t_{c_i}} + \frac{A_1}{t_{c_i}^2} + \dots\dots + \frac{A_{\lambda_i}}{t_{c_i}^{\lambda_i+1}},$$

namely, such a function is

$$- \sum_{r=0}^{\lambda_i} \frac{A_r}{\underline{|r}} D_{c_i}^r \psi(x, a;\ c_i, c),$$

and this may be used in the investigation instead of the function $f_i(x) - f_i(a)$.

Hence, in the enunciation of the theorem of § 144, it is not necessary that the expressions of the rational functions $f_i(x)$ be known, or even that there should exist rational functions infinite only at the places c_i in the assigned way. All that is necessary is that the character of the infinity of the function F, at the pole c_i, should be assigned.

Conversely, any single-valued function F whose singularities consist of one essential singularity and an infinite number of distinct poles which converge to the place of the essential singularity, can be represented by a series of rational functions of x, which beside the essential singularity have each only one pole.

146. Let the places $c_1, c_2, \dots, c$ be as in § 144. We can construct a single-valued function, having the places $c_1, c_2, \dots,$ as zeros, of assigned positive integral orders $\lambda_1, \lambda_2, \dots,$ which is infinite only at the place c, where it has an essential singularity.

For the function $$E(x, z) = e^{\int_c^z \psi(x, a;\, z, c)\, dt_z}$$

is zero at the place z and infinite only at the place c. When z is near to c we can put

$$D_z \log E(x, z) = \sum_{r=0}^{\infty} \frac{t_c^r}{\underline{|r}} \left[D_z^r \psi(x, a;\, z, c)\right]_{z=c},$$

and therefore, when c_i is near to c, and x is not near to the place c, we can put

$$\lambda_i \log E(x, c_i) = \sum_{k=0}^{\infty} t_i^k R_{i,k}(x),$$

wherein $R_{i,k}(x)$ is a rational function of x which is infinite only at the place c, and t_i has the same significance as in § 144.

Let the least value of i for which this equation is valid be denoted by $s + 1$, and, taking $\epsilon_{s+1}, \epsilon_{s+2}, \ldots$ any positive quantities such that the series

$$\epsilon_{s+1} + \epsilon_{s+2} + \ldots,$$

is convergent, let μ_i be the least number such that, for $i > s$,

$$\left| \sum_{k=\mu_i+1}^{\infty} t_i^k R_{i,k}(x) \right| < \epsilon_i.$$

Then the series

$$\sum_{i=1}^{s} \lambda_i \log E(x, c_i) + \sum_{i=s+1}^{\infty} \left(\lambda_i \log E(x, m_i) - \sum_{k=0}^{\mu_i} t_i^k R_{i,k}(x) \right)$$

consists of single-valued finite functions provided x is not near to any of $c_1, c_2, \ldots, c$, and, by the condition as to the numbers μ_i, is absolutely and uniformly convergent.

Hence the product

$$\prod_{i=1}^{s} [E(x, c_i)]^{\lambda_i} \prod_{i=s+1}^{\infty} \left\{ [E(x, c_i)]^{\lambda_i} e^{-\sum_{k=0}^{\mu_i} t_i^k R_{i,k}(x)} \right\}$$

represents a single-valued function, which is infinite only at c where it has an essential singularity, which is moreover zero only at the places $c_1, c_2, \ldots$ respectively to the orders $\lambda_1, \lambda_2, \ldots$.

With the results obtained in §§ 144—146, the reader will compare the well-known results for single-valued functions of one variable (Weierstrass, *Abhandlungen aus der Functionenlehre*, Berlin, 1886, pp. 1—66, or *Mathem. Werke*, Bd. ii. pp. 77, 189).

147. The following results possess the interest that they are given by Abel; they are related to the problems of this chapter. (Abel, *Œuvres Complètes*, Christiania, 1881, vol. i. p. 46 and vol. ii. p. 46.)

Ex. i. If $\phi(x)$ be a rational polynomial in x, $=\Pi(x+a_k)^{\beta_k}$,

and $f(x)$ be a rational function of x, $=\Sigma\gamma_k x^k+\Sigma\dfrac{\delta_k}{(x+\epsilon_k)^{\mu_k}}$,

then

$$\int\frac{e^{f(x)-f(z)}\phi(x)}{(x-z)\phi(z)}dx-\int\frac{e^{f(x)-f(z)}\phi(x)}{(z-x)\phi(z)}dz=\sum_k\sum_{k'}k\gamma_k\int\frac{e^{-f(z)}z^{k'}}{\phi(z)}\int e^{f(x)}\phi(x)\,.\,x^{k-k'-2}\,dx$$

$$-\sum_k\beta_k\int\frac{e^{-f(z)}}{(z+a_k)\phi(z)}\int\frac{e^{f(x)}\phi(x)}{x+a_k}dx+\sum_k\sum_{k'}\mu_k\delta_k\int\frac{e^{-f(z)}\,dz}{(z+\epsilon_k)^{\mu_k-k'+2}\phi(z)}\int\frac{e^{f(x)}\phi(x)}{(x+\epsilon_k)^{k'}}dx.$$

The theorem can be obtained most directly by noticing that if $\phi(x,z)=\dfrac{e^{f(x)-f(z)}\phi(x)}{\phi(z)(x-z)}$ then

$$\phi(X,z)\frac{d}{dX}\phi(x,X)=\frac{e^{f(x)-f(z)}\phi(x)}{\phi(z)}\left\{\frac{f'(X)+\dfrac{\phi(X)}{\phi(X)}}{(X-x)(X-z)}+\frac{1}{(X-z)(X-x)^2}\right\}$$

is a rational function of X. Denoting it by $R(X)$ and applying the theorem

$$\Sigma\left[R(X)\frac{dX}{dt}\right]_{t^{-1}}=0,$$

we obtain Abel's result.

Ex. ii. With the same notation, but supposing $f(x)$ to be an integral polynomial, prove that

$$\int\phi(x,z)\,dx+\int\frac{\psi(x)}{\psi(z)}\phi(x,z)\,dz=\Sigma\Sigma A_{k,k'}\int\frac{e^{-f(z)}z^{k'}\,dz}{\phi(z)\psi(z)}\int e^{f(x)}\phi(x)\,x^k\,dx,$$

wherein $A_{k,k'}$, is a certain constant, and $\psi(x)$ is the product of all the simple factors of $\phi(x)$.

This result may be obtained from the rational function

$$R(X)=\frac{\psi(X)}{\psi(z)}\phi(X,z)\frac{d}{dX}\phi(x,X)$$

as in the last example.

Ex. iii. Obtain the theorem of Ex. ii. when $f(x)=0$, and $\phi(x)=[\psi(x)]^m$. In the result put $m=-\frac{1}{2}$, and obtain the result of the example in § 138.

These results are extended by Abel to the case of linear differential equations. Further development is given by Jacobi, *Crelle* xxxii. p. 194, and by Fuchs, *Crelle* lxxvi. p. 177.