CHAPTER VI.

Geometrical Investigations.

80. It has already been pointed out (§ 9) that the algebraical equation, associated with a Riemann surface, may be regarded as the equation of a plane curve; for the sake of distinctness we may call this curve the fundamental curve. The most general form of a rational function on the Riemann surface is a quotient of two expressions which are integral polynomials in the variables (x, y) in terms of which the equation associated with the surface is expressed. Either of these polynomials, equated to zero, may be regarded as representing a curve intersecting the fundamental curve. Thus we may expect that a comparison of the theory of rational functions on the Riemann surface with the theory of the intersection of a fundamental curve with other variable curves, will give greater clearness to both theories.

In the present chapter we shall make full use of the results obtainable from Riemann's theory and seek to deduce the geometrical results as consequences of that theory.

81. The converse order of development, though of more elementary character, requires much detailed preliminary investigation, if it is to be quite complete, especially in regard to the theory of the multiple points of curves. But the following account of this order of development may be given here with advantage (§§ 81—83). Let the term of highest aggregate degree in the equation of the fundamental curve $f(y, x) = 0$ be of degree n; and, in the usual way, regard the equation as having its most general form when it consists of all terms whose aggregate degree, in x and y, is not greater than n; this general form contains therefore $\frac{1}{2}(n+1)(n+2)$ terms. Suppose, further, that the curve has no multiple points other than ordinary double points and cusps, δ being the number of double points and κ of cusps. Consider now another curve, $\psi(x, y) = 0$, of order m, whose coefficients are at our disposal. By proper choice of these coefficients in ψ we can determine ψ to pass through any given points of f, whose number is not greater than the number of disposeable coefficients in ψ. Let k be the number of the prescribed points, and interpret the infinite intersections of f and ψ, in the usual way, so that their total number of intersections is mn. Then there

remain $mn-k$ intersections of f and ψ which are determined by the others already prescribed. We proceed to prove that if $m > n-3$, and if we utilise all the coefficients of ψ to prescribe as many of the intersections of ψ and f as possible, and introduce further the condition that ψ shall pass once through each cusp and double point of f, then the number of remaining intersections which are determined by the others will be $p = \frac{1}{2}(n-1)(n-2) - \delta - \kappa$*, for all values of m. For, if $m \geqq n$, the intersections of ψ with f are the same as those of a curve

$$\psi + U_{m-n}f = 0,$$

wherein U_{m-n} is any integral polynomial in the coordinates x and y, in which no term of higher aggregate dimension than $m-n$ occurs. By suitable choice of the $\frac{1}{2}(m-n+1)(m-n+2)$ coefficients which occur in the general form of U_{m-n} we can reduce $\frac{1}{2}(m-n+1)(m-n+2)$ coefficients in $\psi + U_{m-n}f$ to zero†. It will therefore contain, in its new form,

$$M + 1 = 1 + \tfrac{1}{2}m(m+3) - \tfrac{1}{2}(m-n+1)(m-n+2)$$

arbitrary coefficients. M is therefore the number of the intersections of ψ with f which we can dispose of at will, by choosing the coefficients in ψ suitably. Of these intersections, by hypothesis, $2(\delta+\kappa)$ are to be taken at the double points and cusps of the curve f. This can be effected by the disposal of $\delta+\kappa$ of the arbitrary coefficients. There remain then

$$1 + \tfrac{1}{2}m(m+3) - \tfrac{1}{2}(m-n+1)(m-n+2) - \delta - \kappa$$

disposeable coefficients and $mn - 2(\delta+\kappa)$ intersections. Of these, therefore,

$$mn - 2(\delta+\kappa) - [\tfrac{1}{2}m(m+3) - \tfrac{1}{2}(m-n+1)(m-n+2) - \delta - \kappa]$$

is the number of intersections determined by the others which are at our disposal; and this number is

$$\tfrac{1}{2}(n-1)(n-2) - (\delta+\kappa).$$

In case $m < n$, of the $mn - 2(\delta+\kappa)$ intersections of ψ with f, which are not at the double points or cusps of f, we can, by means of the $\frac{1}{2}m(m+3) - \delta - \kappa$ coefficients of ψ which remain arbitrary when ψ is prescribed to vanish at each double point and cusp, dispose of all except

$$mn - 2(\delta+\kappa) - [\tfrac{1}{2}m(m+3) - (\delta+\kappa)];$$

when $m = n-1$ or $n-2$ it is easily seen that this is the same as before.

82. Let us assume now that the polynomials which occur, as the numerator and denominator, in the expression of a rational function, have the

* Reasons are given, Forsyth, *Theory of Functions*, p. 356, § 182, for the conclusion that this number is the deficiency of the Riemann surface having $f(y, x) = 0$ as an associated equation. We shall assume this result.

† As, for instance, the coefficients of y^m, y^{m-1}, $y^{m-1}x$, ..., y^n, $y^n x$, ..., $y^n x^{m-n}$, in which case the highest power of y, in $\psi + U_{m-n}f$, that remains, is y^{n-1}.

property here assigned to ψ, of vanishing once at each double point and cusp of f. Without attempting to justify this assumption completely, we remark that if it is not verified at any particular double point, the rational function will clearly take the same value at the double point by whichever of the two branches of the curve f the double point be approached. As a matter of fact this is not generally the case. Suppose then we wish to obtain a general form of rational function which has Q given finite points of f, $A_1, \ldots, A_Q$, as poles of the first order. Draw through these poles, $A_1, \ldots, A_Q$, any curve ψ whatever, of degree greater than $n-3$, which passes once through each double point and cusp of f. Then ψ will intersect f in

$$mn - 2(\delta + \kappa) - Q$$

other points $B_1, B_2, \ldots$. Through these other points $B_1, B_2, \ldots$ of f, and through the double points, draw another curve, ϑ, of the same degree as ψ. The curve ϑ will in general not be entirely determined by the prescription of the $mn - 2(\delta+\kappa) - Q$ points $B_1, B_2, \ldots$. Let the number of its coefficients which still remain arbitrary be denoted by $q+1$. Then it would be possible by the prescription of, in all,

$$mn - 2(\delta + \kappa) - Q + q$$

points of ϑ, to determine ϑ completely. But by what has just been proved, ϑ is determined completely when all but p of its intersections are prescribed. Wherefore

$$mn - 2(\delta+\kappa) - Q + q = mn - 2(\delta+\kappa) - p.$$

Hence $Q - q = p$, and ϑ has the form

$$\lambda\psi + \lambda_1\vartheta_1 + \ldots + \lambda_q\vartheta_q,$$

where $\lambda, \lambda_1, \ldots, \lambda_q$ are arbitrary constants and $\psi, \vartheta_1, \ldots, \vartheta_q$ are $q+1$ linearly independent curves, all passing through the $mn - 2(\delta+\kappa) - Q$ points $B_1, B_2, \ldots$, as well as through the double points and cusps; and the general rational function with the Q prescribed poles will have the form

$$\lambda + \lambda_1 R_1 + \ldots + \lambda_q R_q,$$

where $R_i = \vartheta_i/\psi$; and this function contains $q+1$ arbitrary coefficients.

83. In this investigation, which is given only for purposes of illustration, we have assumed that the prescription of a point of a curve determines one of its coefficients in terms of the remaining coefficients, and that the prescription of this one point does not of itself necessitate that the curve pass through other points; and we have obtained not the exact form of the Riemann-Roch Theorem (Chap. III. § 37), but the first approximation to that theorem which is expressed by $Q-q=p$; this result is true for all cases only when $Q > n(n-3) - 2(\delta+\kappa)$.

We may illustrate the need of the hypothesis that the curves ψ and ϑ pass through the double points and cusps, by considering the more particular case when the fundamental curve

$$f = (x, y)_2 + (x, y)_3 + (x, y)_4 = 0,$$

wherein $(x, y)_2$ is an integral homogeneous polynomial in x and y of the second degree, etc., is a quartic with a double point at the origin $x=0$, $y=0$. Since here $n=4$ and $\delta+\kappa=1$, we have

$$p=\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa=\tfrac{1}{2}\,.\,3\,.\,2-1=2,$$

and therefore (in accordance with Chap. III. §§ 23, 24, etc.) there exists a rational function having any three prescribed points as poles of the first order. Let us attempt to express this function in the form ϑ/ψ, wherein ϑ, ψ are curves, of degree m, $(m>1)$, which do not vanish at the double point. Beside the three prescribed poles A_1, A_2, A_3 of the function, ψ will intersect f in $4m-3$ points B_1, B_2, ... The intersections with f of the general curve ϑ of degree m, are the same as those of a curve

$$\vartheta - U_{m-4}f=0,$$

provided $m \nless 4$, and are therefore determined by $\tfrac{1}{2}m(m+3)-\tfrac{1}{2}(m-4+1)(m-4+2)$, or $4m-3$ of them. And it is easily seen that the same result follows when $m=3$ or 2. Hence no curve ϑ can be drawn through the points B_1, B_2, ... other than the curve ψ, which already passes through them; and the rational function cannot be determined in the way desired. It will be found moreover that this is still true when the hypothesis, here made, that ψ and ϑ shall be of the same degree, is allowed to lapse. As in the general case, this hypothesis is made in order that the function obtained may be finite for infinite values of x and y.

A curve which passes through each double point and cusp of the fundamental curve f is said to be *adjoint*. When f has singularities of more complicated kind there is a corresponding condition, of greater complexity. For example in the case of the curve

$$f=y^2-(1-x^2)(1-k^2x^2)=0,$$

which, in the present point of view, we regard as a quartic, there is a singularity at the infinite end of the axis of y. If, in the usual way, we introduce the variable z to make the equation homogeneous, and then* put $y=1$, whereby the equation becomes

$$z^2=(z^2-x^2)(z^2-k^2x^2),$$

we see that the branches are, approximately, given by $z=\pm kx^2$, namely there is a point of self contact, the common tangent being $z=0$. If we assume that it is legitimate to regard this self contact as the limit of two coincident double points, we shall infer that the condition of adjointness for a curve ψ is that it shall touch the two branches of f at the point. For example this condition is satisfied by the parabola

$$y=ax^2+bx+c,$$

which, by the same transformation as that above, reduces to

$$z=ax^2+bxz+cz^2,$$

and it is obvious that the four intersections with f of this parabola, other than those at the singular point, are determined by all but p of them, p being in this case equal to 1.

We shall see in this chapter that we can obtain these results in a somewhat different way: the equation $y^2=(1-x^2)(1-k^2x^2)$ is a good example of those in which it is *not* convenient to regard the equation as a particular case of a curve of degree equal to the highest degree which occurs. Though this method, of regarding any given curve as a particular case of one whose degree is the degree of the highest term which occurs in the given equation of the curve, is always allowable, it is often cumbersome.

Ex. 1. Prove that the theorem, that the intersections with f of a variable curve ψ are determined by all but p of them, may be extended to the case where f has multiple points

* This process is equivalent to projecting the axis $y=0$ to infinity.

of order k, with separated tangents, by assuming that the condition of adjointness is that ψ should have a multiple point of order $k-1$ at every such multiple point of f, whose tangents are distinct from each other and from those of f. (In this case any such multiple point of f furnishes a contribution $\frac{1}{2}k(k-1)$ to the number $\delta+\kappa$ of f.)

Ex. 2. The curve $y^2=(x, 1)_6$ may be regarded as a sextic. Shew that the singular point at infinity may be regarded as the limit of eight double points, and that a general adjoint curve is

$$(x, 1)^{4+\mu}+y(x, 1)^{\mu+1}=0.$$

Ex. 3. Shew that for the curve $y^2=(x, 1)_{2p+2}$ a general adjoint curve is

$$(x, 1)^{\mu+2p}+y(x, 1)^{\mu+p-1}=0.$$

For further information on this subject consult Salmon, *Higher Plane Curves* (Dublin, 1879), pp. 42—48, and the references given in this volume, § 9 note, § 93, § 97, § 112 note, § 119.

84. In the remaining analytical developments of this chapter we suppose* the equation associated with the Riemann surface to be given in the form

$$f(y, x)=y^n+y^{n-1}(x, 1)_{\lambda_1}+\dots+y(x, 1)_{\lambda_{n-1}}+(x, 1)_{\lambda_n}=0,$$

so that y is an integral function of x. Let $\sigma+1$ be the dimension of y; then $\sigma+1$ is the least positive integer such that $y/x^{\sigma+1}$ is finite when x is infinite; thus if we put $x=1/\xi$ and $y=\eta/\xi^{\sigma+1}$, $\sigma+1$ is the least positive integer, such that η is an integral function of ξ. This substitution gives $f(y, x)=\xi^{-n(\sigma+1)}F(\eta, \xi)$, where

$$F(\eta, \xi)=\eta^n+\eta^{n-1}\xi^{\sigma+1-\lambda_1}(1, \xi)_{\lambda_1}+\dots+\eta\xi^{(n-1)(\sigma+1)-\lambda_{n-1}}(1, \xi)_{\lambda_{n-1}}$$
$$+\xi^{n(\sigma+1)-\lambda_n}(1, \xi)_{\lambda_n},$$

so that $\sigma+1$ is the least positive integer which is not less than any of the quantities

$$\lambda_1, \lambda_2/2, \dots, \lambda_{n-1}/(n-1), \lambda_n/n.$$

Ex. 1. For the case

$$y^4+y^2x^2(x, 1)_3+yx^3(x, 1)_4+x^4(x, 1)_5=0$$

the dimension of y as an integral function of x is 3. Writing $y=\eta/\xi^3$, where $x=1/\xi$, the equation becomes

$$\eta^4+\eta^2\xi(1, \xi)_3+\eta\xi^2(1, \xi)_4+\xi^3(1, \xi)_5=0$$

and η is an integral function of ξ of dimension 2. In fact $y_1=\eta/\xi^2=y/x$ satisfies the equation

$$y_1^4+y_1^2(x, 1)_3+y_1(x, 1)_4+(x, 1)_5=0$$

and is finite when $\xi=\infty$, or $x=0$.

Ex. 2. Shew that in the case in which the equation associated with the Riemann surface contains y to a degree equal to the highest aggregate degree which occurs, $\sigma=0$.

* Chap. IV. § 38.

Whenever we are considering the places of the surface for which $x=\infty$, we shall consider the surface in association with the equation $F(\eta, \xi)=0$; and shall speak of the infinite places as given by $\xi=0$. The original equation is practically unaffected by writing $x-c$ for x, c being a constant. We may therefore suppose the equation so written that at $x=0$, the n sheets of the surface are distinct; and may speak of the places $x=0$ as the places $\xi=\infty$.

85. By the simultaneous use of the equations $f(y, x)=0$, $F(\eta, \xi)=0$, we shall be better able to formulate our results in accordance with the view, hitherto always adopted, whereby the places $x=\infty$ are regarded as exactly like any finite places. But it should be noticed that both these equations may be regarded as particular cases of another in which homogeneous variables, of a particular kind*, are used. For put $x=\omega/z$, $y=u/z^{\sigma+1}$; we obtain $f(y, z)=z^{-n(\sigma+1)}\,U(u\,;\,\omega,\,z)$, where

$$U(u;\ \omega,\ z)=u^n+u^{n-1}z^{\sigma+1-\lambda_1}(\omega,\ z)_{\lambda_1}+\dots+uz^{(n-1)(\sigma+1)-\lambda_{n-1}}(\omega,\ z)_{\lambda_{n-1}}$$
$$+\,z^{n(\sigma+1)-\lambda_n}(\omega,\ z)_{\lambda_n},$$

and it is clear that $U(u;\ \omega,\ z)$ is changed into $f(y, x)$ by writing $u=y$, $\omega=x$, $z=1$, and is changed into $F(\eta, \xi)$ by writing $u=\eta$, $\omega=1$, $z=\xi$. We may speak of ω, z as *forms*, of degree 1, and suppose that they do not become infinite, the values $x=\infty$ being replaced by the values $z=0$. When ω, z are replaced by $t\omega$, tz, t being any quantity whatever, u is replaced by $t^{\sigma+1}u$, y and x remaining unaltered. We may therefore speak of u as a *form* of degree $\sigma+1$.

Similarly $U(u\,;\,\omega,\,z)$ is a form of degree $n(\sigma+1)$, being multiplied by $t^{n(\sigma+1)}$ when u, ω, z are replaced by $t^{\sigma+1}u$, $t\omega$, tz respectively. That there is some advantage in using such homogeneous forms to express the results of our theory will sufficiently appear; but it seems proper that the results should first be obtained independently, in order that the implications of the notation may be made clear. We shall adopt this course.

Some examples of the change which our expressions will undergo when the results are expressed by homogeneous forms, may be fitly given here:—Instead of $f(y, x)$ we shall have $U(u\,;\,\omega,\,z)$ which is equal to $z^{n(\sigma+1)}f(y, x)$; instead of $f'(y)$ we shall have $U'(u)=z^{(n-1)(\sigma+1)}f'(y)$; instead of the integral function† g_i, of dimension τ_i+1, an integral form $\bar{g}_i$ of degree τ_i+1, equal to $z^{\tau_i+1}g_i$, will arise; since $\Sigma(\tau_i+1)=n+p-1$, it is easy to see that the determinant‡ $\Delta(1, \bar{g}_1, \dots, \bar{g}_{n-1})$ is equal to $z^{2n+2p-2}\Delta(1, g_1, \dots, g_{n-1})$. In accordance with § 48, Chap. IV. the former determinant will have a factor

* This homogeneous equation is used by Hensel. See the references given in Chap. IV. (§ 42). It may be regarded as a generalization of the familiar case when $\sigma=0$.

† Chap. IV. § 42.

‡ Chap. IV. § 43.

$(\omega - cz)^r$ corresponding to a finite branch place of order r where $x = c$, and a factor z^s corresponding to a branch place of order s at $x = \infty$. Further, if, by the formula (H) of page 63, we calculate the form $\bar{\phi}_i(u, \omega, z)$ from $\bar{g}_1, \ldots, \bar{g}_{n-1}$, as $\phi_i(x, y)$ is there calculated from $g_1, \ldots, g_{n-1}$, it is easy to see that we obtain a form, $\bar{\phi}_i(u, \omega, z)$, which is equal to $z^{(n-1)(\sigma+1)-(\tau_i+1)}\phi_i(x, y)$. Hence also, if u_1, ω_1, z_1 denote special values of u, ω, z, the integral

$$\int \frac{z\,d\omega - \omega\,dz}{U'(u)} \frac{\mu^{-1}\bar{\phi}_0(u, \omega, z) + \Sigma \mu^{\tau_r}\bar{\phi}_r(u, \omega, z)\,\bar{g}_r(u_1, \omega_1, z_1)}{\omega z_1 - \omega_1 z},$$

wherein $\mu = (b\omega - az)/(b\omega_1 - az_1)$, a and b being arbitrary constants, is equal to

$$\int \frac{z^2 dx \,.\, z^{(n-1)(\sigma+1)}}{z^{(n-1)(\sigma+1)} f'(y)} \cdot \frac{\mu^{-1}\phi_0(x, y) + \Sigma \mu^{\tau_r}(z_1/z)^{\tau_r+1}\phi_r(x, y)\,g_r(x_1, y_1)}{zz_1(x - x_1)},$$

and is thus equal to

$$\int \frac{dx}{f'(y)} \cdot \frac{\lambda^{-1}\phi_0(x, y) + \Sigma \lambda^{\tau_r}\phi_r(x, y)\,g_r(x_1, y_1)}{x - x_1},$$

where $\lambda = \mu z_1/z = (bx - a)/(bx_1 - a)$.

If in this we put $b = 0$, we obtain the form which we have already shewn to be part of the expression of an integral of the third kind (Chap. IV. p. 67). But if we put $b = 1$, the integral is exactly what we have already deduced (Chap. IV. p. 70, Ex. 1) by the ordinary process of putting $x = 1/(\xi - a)$ and regarding ξ as the independent variable.

We may, if we please, further specialise the quantities ω, z, of which hitherto only the ratio has been used, supposing* them defined by $\omega = x/(x - c)$, $z = 1/(x - c)$, where c is a constant. Then $\omega - cz = 1$.

Ex. 1. The integral of the first kind obtained in Chap. IV. § 45, p. 67, can similarly be written

$$\int \frac{z\,d\omega - \omega\,dz}{U'(u)} \left[(\omega, z)^{\tau_1 - 1}\bar{\phi}_1(u, \omega, z) + \ldots\ldots + (\omega, z)^{\tau_{n-1} - 1}\bar{\phi}_{n-1}(u, \omega, z)\right].$$

Ex. 2. In the case $y^2 = (x, 1)_{2p+2}$, wherein y is of dimension $p+1$, the equation $U(u;\, \omega, z) = 0$ is

$$u^2 = (\omega, z)_{2p+2}$$

obtained by putting $y = u/z^{p+1}$, $x = \omega/z$.

86. We shall be largely concerned here with rational polynomials which are integral in x and y. The values of such a polynomial here considered are only those which it has for values of y and x satisfying the fundamental equation. We shall therefore suppose every integral polynomial in x and y reduced, by means of the fundamental equation, to a form in which the highest power of y which enters is y^{n-1}, say to a form

$$\psi(y, x) = y^{n-1}(x, 1)_{\mu_0} + \ldots + y^{n-1-i}(x, 1)_{\mu_i} + \ldots + (x, 1)_{\mu_{n-1}}.$$

* In this view ω and z are functions. If we regard c as throughout undetermined, we may regard these functions as having no *definite* infinities.

If herein we write $y=\eta/\xi^{\sigma+1}$, $x=1/\xi$, $\sigma+1$ being, as before, the dimension of y as an integral function of x, we shall obtain $\psi(y, x)=\xi^{-G}\Psi(\eta, \xi)$, where $\Psi(\eta, \xi)$ is an integral polynomial in η and ξ of which a representative term is

$$\eta^{n-1-i}\,\xi^{G-(n-1-i)(\sigma+1)-\mu_i}(1, \xi)_{\mu_i}, \qquad i=0, 1, \ldots\ldots, (n-1)$$

and G is the positive integer equal to the greatest of the quantities

$$(n-1-i)(\sigma+1)+\mu_i.$$

Thus G is the highest dimension occurring for the terms of $\psi(y, x)$, and $\Psi(\eta, \xi)$ is not identically divisible by ξ. The dimension of the integral function $\psi(y, x)$ may be G; but if $\Psi(\eta, \xi)$ vanish in every sheet at $\xi=0$, the dimension of $\psi(y, x)$ will be less than G. For this reason we shall speak of G as the *grade* of $\psi(y, x)$. It is clear that if all the values of η for $\xi=0$ be distinct, that is, if $F'(\eta)$ do not vanish for any place $\xi=0$, the polynomial $\Psi(\eta, \xi)$, of order $n-1$ in η, cannot vanish for all the n places $\xi=0$. In that case the grade and the dimension of $\psi(y, x)$ are necessarily the same. Further, by the vanishing of one of the coefficients, a polynomial of grade G may reduce to one of lower grade. In this sense a polynomial of low grade may be regarded as a particular case of one of higher grade.

In what follows we shall consider all polynomials whose grade is lower than $(n-1)\sigma+n-3$ or $(n-1)(\sigma+1)-2$, as particular cases of polynomials of grade $(n-1)\sigma+n-3$: the general expression of the grade will therefore* be $(n-1)\sigma+n-3+r$, or $(n-1)(\sigma+1)+r-2$, where r is zero or a positive integer. The most general form of a polynomial of grade $(n-1)(\sigma+1)+r-2$ is easily seen to be

$$\begin{aligned}\psi(y, x)=&\,y^{n-1}(x, 1)_{r-2}+y^{n-2}(x, 1)_{r-1}+\ldots+y^{n-1-i}(x, 1)_{r-1}+\ldots+(x, 1)_{r-1}\\&+x^r\{y^{n-2}(x, 1)_{\sigma+1-2}+\ldots\ldots+y^{n-1-i}(x, 1)_{i(\sigma+1)-2}+\ldots\ldots+(x, 1)_{(n-1)(\sigma+1)-2}\},\end{aligned}$$

wherein the first line is to be entirely absent if $r=0$, the first term of the first line is to be absent if $r=1$, and the first term of the second line is to be absent if $\sigma=0$.

Hence when $r>0$, the general polynomial of grade $(n-1)\sigma+n-3+r$ contains

$$nr-1+\tfrac{1}{2}(n-1)(n-2+n\sigma)$$

terms, this being still true if $\sigma=0$; but when $r=0$, the general polynomial of grade $(n-1)\sigma+n-3$ contains

$$\tfrac{1}{2}(n-1)(n-2+n\sigma)$$

terms. This is not the number obtained by putting $r=0$ in the number obtained for $r>0$.

* The number is written in the former way to point out the numbers for the common case when $\sigma=0$.

Further, putting

$$\psi(y, x) = \xi^{-(n-1)\sigma-(n-3)-r}\,\Psi(\eta, \xi),$$

and denoting the aggregate number of zeros of $\Psi(\eta, \xi)$ at $\xi = 0$ by μ, it is clear that the aggregate number of infinities of $\psi(y, x)$ at $x = \infty$ is $[(n-1)\sigma + n - 3 + r]\,n - \mu$. Since $\psi(y, x)$ is only infinite for $x = \infty$, this is also the total number of zeros of $\psi(y, x)$. We shall find it extremely convenient to introduce a certain artificiality of expression, and to speak of the sum of the number of zeros of $\psi(y, x)$ and the number of zeros of $\Psi(\eta, \xi)$ at $\xi = 0$ as the number of *generalized* zeros of $\psi(y, x)$. This number is then $n(n-1)(\sigma+1) + n(r-2)$.

If by a change in the values of the coefficients in $\psi(y, x)$, $\Psi(\eta, \xi)$ should take the form $\xi\Psi_1(\eta, \xi)$ where $\Psi_1(\eta, \xi)$ is an integral polynomial in η and ξ, so that $\psi(y, x)$ is equal to $\xi^{-(n-1)\sigma-(n-3)-(r-1)}\,\Psi_1(\eta, \xi)$, the sum of the number of finite zeros of $\psi(y, x)$ and the number of zeros of $\Psi_1(\eta, \xi)$ is $n(n-1)(\sigma+1) + n(r-3)$. But, since $\Psi(\eta, \xi)$ is equal to $\xi\Psi_1(\eta, \xi)$, the number of zeros of $\Psi(\eta, \xi)$ at $\xi = 0$ is n more than the number of zeros of $\Psi_1(\eta, \xi)$ at $\xi = 0$. Hence the sum of the number of finite zeros of $\psi(y, x)$ and the number of zeros of $\Psi(\eta, \xi)$ at $\xi = 0$, is still equal to

$$n(n-1)(\sigma+1) + n(r-2).$$

Ex. i. The number $n(n-1)(\sigma+1)+n(r-2)$ is clearly the number of zeros of the *integral form*

$$z^{(n-1)\sigma+n-3+r}\,\psi(uz^{-\sigma-1}, \omega z^{-1}).$$

Ex. ii. The generalized number of zeros of $f'(y)$, for which $r=2$, is $n(n-1)(\sigma+1)$.

Ex. iii. The general polynomial of grade d, $<(n-1)\sigma+n-3$, contains

$$\left[1+E\left(\frac{d}{\sigma+1}\right)\right]\left[1+d-\tfrac{1}{2}(\sigma+1)E\left(\frac{d}{\sigma+1}\right)\right] \text{ terms,}$$

$E(x)$ being the greatest integer in x. Its generalized number of zeros is nd.

87. We introduce now a certain speciality in the integral polynomials under consideration, that known as *adjointness*.

An integral polynomial $\psi(y, x)$ is said to be adjoint at a finite place $(x = a, y = b)$ when the integral

$$\int^x \frac{\psi(y, x)}{f'(y)}\,dx$$

is finite at this place. If t be the infinitesimal at the place (Chap. I. §§ 2, 3) the condition is equivalent to postulating that the expression

$$\frac{\psi(y, x)}{f'(y)}\,\frac{dx}{dt}$$

shall be finite at the place; or again equivalent to postulating that the expression

$$\frac{(x-a)\,\psi(y,\,x)}{f'(y)}$$

shall be zero at the place, to the first order at least.

As a limitation for the polynomial $\psi(y,\,x)$, the condition is therefore ineffective at all places where $f'(y)$ is not zero. And if at a finite place where $f'(y)$ vanishes, $i+w$ denote the order of zero of $f'(y)$, $w+1$ being the number of sheets that wind at this place*, the condition is that $\psi(y,\,x)$ vanish to at least order i at the place. We shall call $\frac{1}{2}i$ the index of the place; the condition of adjointness is therefore ineffective at all places of zero index.

If $\psi(y,\,x)$ be of grade $(n-1)\,\sigma+n-3+r$, and

$$\psi(y,\,x)=\xi^{-(n-1)\sigma-(n-3)-r}\,\Psi(\eta,\,\xi),$$

the condition of adjointness of $\psi(y,\,x)$ for infinite places, is that, at all places $\xi=0$ where $F'(\eta)=0$, the function

$$\frac{\xi\Psi(\eta,\,\xi)}{F'(\eta)}$$

should be zero, to the first order at least. It is easily seen that this is the same as the condition that the integral

$$\int^x \frac{1}{x^r}\,\frac{\psi}{f'(y)}\,dx$$

should be finite at the place considered.

When the condition of adjointness is satisfied at all finite and infinite places where $f'(y)=0$ or $F'(\eta)=0$, the polynomial $\psi(y,\,x)$ is said to be *adjoint*. If $\Pi(x-a)$ denote the integral polynomial which contains a simple factor corresponding to every finite value of x for which $f'(y)$ vanishes, and if N denote the number of these factors, it is immediately seen that the polynomial $\psi(y,\,x)$ is adjoint provided the function

$$\frac{\Pi(x-a)}{x^{N+r-1}f'(y)}\,\psi(y,\,x)$$

is zero, to the first order at least, at all the places where $f'(y)=0$ or $F'(\eta)=0$.

Ex. i. For the surface associated with the equation

$$f(y,\,x)=(x,\,y)_2+(x,\,y)_3+(x,\,y)_4=0$$

there are two places at $x=0$, at each of which $y=0$. At each of these places $f'(y)$ vanishes to the first order, and $w=0$. Hence the condition of adjointness is that $\psi(y,\,x)$ vanishes

* It is easy to see that i is not a *negative* integer. Cf. Forsyth, *Theory of Functions*, p. 169.

to the first order at each of these places. The general adjoint polynomial will therefore not contain any term independent of x and y.

Ex. ii. For the surface

$$y^4 - y^2[(1+k^2)x^2+1]+k^2x^4=0$$

there are two places at $x=0$, at each of which y is zero of the second order: they are not branch places. At each of these $f'(y)$ vanishes to the second order.

The dimension of y is 1, and the general polynomial of grade $(n-1)\sigma+n-3+1$ or 2, is*

$$Ay^2+By+C+x[Dy+Ex+F].$$

In order that this may vanish to the second order at the places in question, it is sufficient that $C=0$ and $F=0$. Then the polynomial takes the form

$$By+Ay^2+Dxy+Ex^2,$$

and if we put x/η for x and $1/\eta$ for y this becomes, save for a factor η^{-2},

$$B\eta+A+Dx+Ex^2,$$

which is therefore an adjoint polynomial for the surface

$$1-(1+k^2)x^2-\eta^2+k^2x^4=0.$$

Compare § 83.

Ex. iii. Prove that the general adjoint polynomial for the surface

$$y^2=(x-a)^3,$$

is

$$y(x, 1)_{r-2}+(x-a)(x, 1)_{r-1}=0.$$

(The index of the place at $x=a$ is 1.)

88. Since the number of generalized zeros of $f'(y)$ is $n(n-1)(\sigma+1)$, (§ 86, Ex. ii), we have, in the notation here adopted,

$$\Sigma(i+w)=n(n-1)(\sigma+1),$$

or if I denote Σi and W denote Σw, the summation extending to all finite and infinite places of the surface

$$I+W=n(n-1)(\sigma+1).$$

Hence, as†

$$W=2n+2p-2,$$

we can infer

$$p=\tfrac{1}{2}(n-1)(n-2+n\sigma)-\tfrac{1}{2}I,$$

shewing that I is an even integer.

Further if X denote the number of zeros of an adjoint polynomial $\psi(y, x)$, of grade $(n-1)\sigma+n-3+r$, exclusive of those occurring at places where $f'(y)=0$ or $F'(\eta)=0$, and calculated on the hypothesis that the adjoint polynomial vanishes, at a place where $f'(y)$ or $F'(\eta)$ vanishes, to an order equal to twice the index of the place‡, we have the equation

$$X+I=n(n-1)(\sigma+1)+n(r-2).$$

* § 86 preceding.

† Forsyth, *Theory of Functions*, p. 349.

‡ So that a place of index $\tfrac{1}{2}i$ where $\psi(y, x)$, or $\Psi(\eta, \xi)$, vanishes to order $i+\lambda$, will furnish a contribution λ to the number X.

Thus, as

$$I = n(n-1)(\sigma+1) - 2(n-1) - 2p,$$

we have

$$X = nr + 2p - 2;$$

and this is true when $r = 0$.

These important results may be regarded as a generalization of some of Plücker's equations* for the case $\sigma = 0$.

Ex. i. The number of terms in the general polynomial of grade $(n-1)\sigma + n - 3 + r$ was proved to be $\frac{1}{2}(n-1)(n-2+n\sigma) + nr - 1$ or $\frac{1}{2}(n-1)(n-2+n\sigma)$, according as $r > 0$ or $r = 0$. This number may therefore be expressed as $p + \frac{1}{2}I + nr - 1$ or $p + \frac{1}{2}I$ in these two cases.

Ex. ii. It is easy to see, in the notation explained in § 85, that the homogeneous form $\Delta(1, u, u^2, \dots, u^{n-1})$ is of degree $n(n-1)(\sigma+1)$ in ω and z, and the form $\Delta(1, \bar{g}_1, \dots, \bar{g}_{n-1})$ of degree W. The quotient $\Delta(1, u, \dots, u^{n-1})/\Delta(1, \bar{g}_1, \dots, \bar{g}_{n-1})$ is (§ 43) an integral form in ω, z, which, by an equation proved here, is of degree I. It is the square of an integral homogeneous form ∇ whose degree in ω, z together is $\frac{1}{2}I$.

Ex. iii. It can be proved (compare § 43 *b*, Exx. 1, 2, and § 48; also Harkness and Morley, *Theory of Functions*, pp. 269, 270, 272, or Kronecker's original paper, *Crelle*, t. 91) that if for y we take the function

$$\lambda + \lambda_1 g_1 + \dots + \lambda_{n-1} g_{n-1},$$

wherein $\lambda, \lambda_1, \dots, \lambda_{n-1}$ are integral polynomials in x, of sufficient (but finite) order, the polynomial ∇ occurring in the equation,

$$\Delta(1, y, \dots, y^{n-1}) = \nabla^2 \Delta(1, g_1, \dots, g_{n-1}),$$

cannot, for general values of the coefficients in $\lambda, \lambda_1, \dots, \lambda_{n-1}$, have any repeated factor, or have any factor which is also a factor of $\Delta(1, g_1, \dots, g_{n-1})$. And the inference can be made† that for this dependent variable y, there is no place at which the index is greater than $\frac{1}{2}$, and no value of x for which two places occur at which $f'(y)$, or $F'(\eta)$, is zero.

89. We proceed, now, to shew the utility of the notion of adjoint polynomials for the solution of the problem of finding the expression of a rational function of given poles.

Let R be any rational function, and suppose, first, that none of the finite poles of R are at places where $f'(y) = 0$. Let ψ be any integral polynomial, chosen so as to be zero at every finite pole of R, to an order at least as high as the order of the pole of R, and to be adjoint at every finite place where $f'(y)$ vanishes. Denote the integral polynomial $\Pi(x-a)$, which contains a linear factor corresponding to every finite value of x for which $f'(y)$ vanishes, by μ. Then the rational function

$$\mu\Lambda(y, x) = \mu R\psi / f'(y)$$

* Salmon, *Higher Plane Curves* (Dublin, 1879), p. 65.

† See also Noether, *Math. Annal.* t. xxiii. p. 311 (Rationale Ausführung, u. s. w.), and Halphen, *Comptes Rendus*, t. 80 (1875), where a proof is given that every algebraic plane curve may be regarded as the projection of a space curve having only one multiple point at which all the tangents are distinct. But see Valentiner, *Acta Math.*, ii. p. 137.

is finite at all finite places where R is infinite, and is finite, being zero, at every finite place at which $f'(y)=0$. If $y_1, \dots, y_n$ denote the n values of y which belong to any value of x, and c be an arbitrary constant, the function

$$\sum_{i=1}^{n} \frac{(c-y_1)(c-y_2)\dots(c-y_n)}{c-y_i} \mu\Lambda(y_i, x),$$

is a symmetrical function of $y_1, \dots, y_n$ and, therefore, expressible as a rational function in x only; moreover the function is finite for all finite values of x and, therefore, expressible as an integral polynomial in x. Since this polynomial vanishes for every finite value of x which reduces the product μ to zero, it must divide by μ. Finally, the function is an integral polynomial in c, of degree $n-1$. Hence we have an equation of the form

$$\sum_{i=1}^{n} \frac{(c-y_1)\dots(c-y_n)}{c-y_i} \Lambda(y_i, x) = c^{n-1}A_0 + c^{n-2}A_1 + \dots + cA_{n-2} + A_{n-1},$$

wherein $A_0, A_1, \dots, A_{n-1}$ are integral polynomials in x.

Therefore, putting $c=y_i$, recalling the form of the function $\Lambda(y, x)$, and replacing y_i by y, we have the result

$$R\psi = y^{n-1}A_0 + y^{n-2}A_1 + \dots + yA_{n-2} + A_{n-1},$$

which we may write in the form

$$R = \vartheta/\psi,$$

ϑ being an integral polynomial in x and y.

Since

$$\frac{(x-a)\vartheta}{f'(y)} = R\frac{(x-a)\psi}{f'(y)},$$

ϑ, like ψ, is adjoint at every finite place where $f'(y)$ vanishes.

Suppose, next, that the function R has finite poles at places where $f'(y)$ vanishes. Then the polynomial ψ is to be chosen so that $R(x-a)\psi/f'(y)$ is zero at such a place, a being the value of x at the place. This may be stated by saying that ψ is adjoint at such a place and, *besides*, satisfies the condition of being zero at the place to as high order as R is infinite.

Corollary. Suppose R to be an integral function; and for a finite place, $x=a$, $y=b$, where $f'(y)$ vanishes, suppose $t+1$ to be the least positive integer such that $(x-a)^{t+1}/f'(y)$ has limit zero at the place. Then the polynomial ψ of the preceding investigation may be replaced by the product $\Pi(x-a)^t$, extended to all the finite values of x for which $f'(y)$ is zero. Hence, any integral function is expressible in the form

$$\vartheta/\Pi(x-a)^t,$$

where ϑ is an integral polynomial in x and y, which is adjoint at every finite place where $f'(y)$ vanishes.

If the order of a zero of $f'(y)$ be represented as before by $i+w$, it is clear that the corresponding value of $t+1$ is the least positive integer for which $(t+1)(w+1)>i+w$, or, for which $t>(i-1)/(w+1)$. Hence the denominator $\Pi\,(x-a)^t$ only contains factors corresponding to places at which the index $\frac{1}{2}i$ is greater than zero; if the index be zero at all the finite places at which $f'(y)$ vanishes, every integral function is expressible integrally.

It does not follow that when the index is zero at all finite places, the functions $1, y, \ldots, y^{n-1}$, form a fundamental system of integral functions for which the condition of dimensions is satisfied. For the sum of the dimensions of $1, y, \ldots, y^{n-1}$ is greater than $p+n-1$ by the sum of the indices at all the places $x=\infty$.

It is clear that if R be any rational function whatever, it is possible to find an integral polynomial in x only, say λ, such that λR is an integral function. To this integral function we may apply the present Corollary. The reader who recalls Chapter IV. will compare the results there obtained.

90. Let the polynomial ψ be of grade $(n-1)\sigma+n-3+r$, and the polynomial ϑ of grade $(n-1)\sigma+n-3+s$, so that

$$\psi=\xi^{-(n-1)\sigma-(n-3)-r}\Psi,\quad \vartheta=\xi^{-(n-1)\sigma-(n-3)-s}\Theta,$$

and
$$R=\xi^{r-s}\Theta/\Psi,$$

Θ, Ψ being integral polynomials in η and ξ.

If R have poles for $\xi=0$, it will generally be convenient to choose the polynomial ψ so that $R\Psi$ is finite at all places $\xi=0$; if $F'(\eta)$ vanish for any places $\xi=0$, it is also convenient, as a rule, to choose ψ so that $\xi\Psi/F'(\eta)$ vanishes at every place $\xi=0$ where $F'(\eta)$ vanishes, namely, so that ψ is adjoint at infinity. When both R is infinite and $F'(\eta)$ vanishes at a place where $\xi=0$, we may suppose ψ so chosen that $\xi R\Psi/F'(\eta)$ is zero at the place. Let ψ be chosen to satisfy these conditions. Then, since $R\Psi$, $=R\psi\,.\,\xi^{(n-1)\sigma+n-3+r}$, is finite at every place, except $\xi=\infty$, and $(1-a\xi)\,\Psi/F'(\eta)$, $=\xi^{r-1}(x-a)\,\psi/f'(y)$, vanishes at every place $x=a$, $y=b$, where x is finite, at which $f'(y)$ vanishes, except $\xi=\infty$, it follows, as here, that R can be written in a form

$$R=\Theta_1/\Psi,$$

wherein Θ_1 is an integral polynomial in η and ξ.

Hence $\Theta_1=\xi^{r-s}\Theta$, and therefore $r-s$ is not negative: namely, the polynomial ϑ which occurs in the expression of a rational function in the form $R=\vartheta/\psi$, is not of higher grade than the denominator ψ, provided ψ be chosen to be adjoint at infinity, and, at the same time, to compensate the poles of R which occur for $x=\infty$. Since a polynomial of low grade is a particular case of one of higher grade we may regard ϑ and ψ as of the same grade.

Hence we can formulate a rule for the expression of a rational function of assigned poles as follows—*Choose any integral polynomial ψ which is adjoint at all finite places and is adjoint at infinity, which, moreover, vanishes at every finite place and at every infinite place* where R is infinite, to as high order as that of the infinity of R. If a pole of R fall at a place where $f'(y)$, or $F'(\eta)$, vanishes, these two conditions may be replaced by a single one in accordance with the indications of the text. Then, choose an integral polynomial ϑ, of the same grade as ψ, also adjoint at all finite and infinite places, which, moreover, vanishes at every zero of the polynomial ψ other than the poles of R, to as high order as the zero of ψ at that place. Then the function can be expressed in the form ϑ/ψ.*

91. We may apply the rule just given to determine the form of the integrals of the first kind.

If v be any integral of the first kind, dv/dx is a rational function having no poles, for finite values of x, except at the branch places of the surface. If a be the value of x at one of these branch places, the product $(x-a)\,dv/dx$ vanishes at the place. Hence we may apply to dv/dx the same reasoning as was applied to the function $\Lambda(y, x)$ in § 89, and obtain the result, that dv/dx can be expressed in the form

$$\frac{dv}{dx}=\frac{y^{n-1}A_0+y^{n-2}A_1+\ldots+yA_{n-2}+A_{n-1}}{f'(y)},$$

wherein $A_0, \ldots, A_{n-1}$ are integral polynomials in x. Denote the numerator by ϕ, and let its grade be denoted by $(n-1)\sigma+n-3+r$; then

$$-\frac{dv}{d\xi}=\xi^{-2}\frac{dv}{dx}=\xi^{-2}\frac{\xi^{-(n-1)\sigma-(n-3)-r}\Phi}{\xi^{-(n-1)\sigma-(n-1)}F'(\eta)}=\frac{\xi^{-r}\Phi}{F'(\eta)}.$$

But, as a function of ξ, $dv/d\xi$ has exactly the same character as has dv/dx as a function of x. Thus by a repetition of the argument $F'(\eta)\,dv/d\xi$ is expressible as an integral function of η and ξ. Thus r is either zero or negative.

Wherefore, $f'(y)\dfrac{dv}{dx}$ is an integral polynomial in x and y, of grade $(n-1)\sigma+n-3$ or less. It is clearly adjoint at all finite places, and, reckoned as a particular case of a polynomial of grade $(n-1)\sigma+n-3$, it is clearly also adjoint at infinity.

Conversely, it is immediately seen, that if ϕ be any integral polynomial of

* That is, if the polynomial be ψ, of grade $(n-1)\sigma+n-3+r$ and $\psi=\Psi\xi^{-(n-1)r-(n-3)-r}$, Ψ vanishes at $\xi=0$ to the order stated. A similar abbreviated phraseology is constantly employed.

grade $(n-1)\sigma+n-3$, which is adjoint at all finite and infinite places, the integral

$$\int \frac{\phi}{f'(y)}\,dx,$$

is an integral of the first kind.

Corollary. We have seen that the general adjoint polynomial of grade $(n-1)\sigma+n-3$ contains $p+\frac{1}{2}I$ terms, and we know that there are just p linearly independent integrals of the first kind. We can therefore make the inference

The condition of adjointness, for a polynomial of grade $(n-1)\sigma+n-3$, is equivalent to $\frac{1}{2}I$ linearly independent conditions for the coefficients of the polynomial, and reduces the number of terms in the polynomial to p.

92. We have shewn that a general polynomial of grade $(n-1)\sigma+n-3+r$ is of the form

$$\psi_{n-3+r}=y^{n-1}(x,\,1)_{r-2}+y^{n-2}(x,\,1)_{r-1}+\ldots+y\,(x,\,1)_{r-1}+(x,\,1)_{r-1}+x^r\psi_{n-3}.$$

We shall assume in the rest of this chapter that the condition of adjointness for a general polynomial of grade $(n-1)\sigma+n-3+r$ is equivalent to as many independent linear conditions as for a general polynomial of grade $(n-1)\sigma+n-3$. Thence, *the general adjoint polynomial of grade $(n-1)\sigma+n-3+r$ contains $nr-1+p$ terms.*

Further we shewed that the adjoint polynomial of grade $(n-1)\sigma+n-3$ has $2p-2$ zeros exclusive of those falling at places where $f'(y)=0$, or $F'(\eta)=0$.

Hence, the $2p-2$ zeros of the differential dv (Chap. II. § 21) are the zeros of the polynomial $f'(y)\,dv/dx$, exclusive of those where $f'(y)=0$, or $F'(\eta)=0$.

It is in fact an obvious corollary from the condition of adjointness that

$$dv/dt=[\phi/f'(y)]\frac{dx}{dt}$$

only vanishes when ϕ vanishes. For, at a place where $f'(y)=0$, ϕ vanishes i times, $\frac{dx}{dt}$ vanishes w times, and $f'(y)$ vanishes $i+w$ times.

Ex. i. For the surface associated with the equation

$$f(y,\,x)=y^4+y^3(x,\,1)_1+y^2(x,\,1)_2+y\,(x,\,1)_3+(x,\,1)_4=0,$$

where $(x,\,1)_1$, ... are integral polynomials in x of the degrees indicated by their suffixes, $\sigma=0$; and the general polynomial of grade $(n-1)\sigma+n-3$ or 1, is of the form (§ 86)

$$Ay+Bx+C.$$

The indices of the places where $f'(y)=0$ are easily seen to be everywhere zero—there are no places, beside branch places, at which $f'(y)$ vanishes. Hence p is equal to the number of terms in this polynomial, or $p=3$. And this polynomial vanishes in $2p-2=4$ places. These results may be modified when the coefficients in the equation have special values.

Ex. ii. For the more particular case when the equation is

$$f(y, x)=y^4+y^3(x, 1)_1+y^2(x, 1)_2+yx(x, 1)_2+x^2(x, 1)_2=0$$

there are two places at $x=0$ at which $y=0$. For general values of the coefficients in the equation these are not branch places and $f'(y)$ vanishes to the first order at each; the index at each place is therefore $\frac{1}{2}i$ where $i=1$, and the condition for adjointness of the general polynomial of grade 1, is that it shall vanish once at each of these places. These conditions are equivalent to one condition only, that $C=0$. Hence, as there are no other places where the index is greater than zero, the general integral of the first kind is

$$\int (Ay+Bx)\,dx/f'(y)$$

and $p=2$; the polynomial $Ay+Bx$ vanishes in $2p-2$ or 2 places other than the places $x=0$, $y=0$ at which $f'(y)=0$.

Ex. iii. In general when the equation of the surface represents a plane curve with a double point, the condition of adjointness at the places which correspond to this double point, is the one condition that the adjoint polynomial vanish at the double point*.

Ex. iv. Prove that for each of the surfaces

$$y^3+y^2(x, 1)_1+y(x, 1)_2+(x, 1)_4=0,$$
$$y^3+y^2(x, 1)_2+y(x, 1)_4+(x, 1)_7=0,$$

there is only one place at infinity and the index there, in both cases, is 1.

Shew that the index at the infinite place of Weierstrass's canonical surface† is in all cases

$$\tfrac{1}{2}(a-1)\left(a\left|\frac{r}{a}\right|-r-1\right),$$

where $\left|\dfrac{r}{a}\right|$ means the least integer greater than r/a, and that the deficiency is given by

$$p=\tfrac{1}{2}(r-1)(a-1)-I',$$

where I' denotes the sum of the indices at all finite places of the surface.

Cf. *Camb. Phil. Trans.* xv. iv. p. 430. The practical method of obtaining adjoint polynomials of grade $(n-1)\sigma+n-3$ which is explained in that paper (pp. 414—416) is often of great use.

Ex. v. In the notation of Chap. IV. the polynomial

$$(x, 1)^{\tau_1-1}\phi_1+\dots+(x, 1)^{\tau_{n-1}-1}\phi_{n-1}$$

is an adjoint polynomial of grade $(n-1)\sigma+n-3$.

Ex. vi. We can prove in exactly the same way as in the text that an integral of the third kind infinite only at the ordinary finite places (x_1, y_1), (x_1', y_1'), at the former like $C\log(x-x_1)$ and at the latter like $-C\log(x-x_1')$, C being a constant, can be written in the form

$$P=\int\frac{\psi}{(x-x_1)(x-x_1')}\frac{dx}{f'(y)},$$

where ψ is an adjoint integral polynomial in x and y, of grade $(n-1)\sigma+n-1$, which

* The sum of the indices at the k places of the surface corresponding to an ordinary k-ple point of the curve is $\frac{1}{2}k(k-1)$; the index at each of the places is in fact $\frac{1}{2}(k-1)$. Cf. § 83, Ex. i.

† Chap. V. § 64.

vanishes at the $(n-1)$ places $x=x_1$ where y is not equal to y_1 and at the $(n-1)$ places $x=x_1'$ where y is not equal to y_1'. Putting ψ in the form

$$\psi=(x-x_1')(C_0y^{n-1}+C_1y^{n-2}+\dots+C_{n-1})-(x-x_1)(C_0'y^{n-1}+C_1'y^{n-2}+\dots+C'_{n-1})$$
$$+(x-x_1)(x-x_1')(R_0y^{n-1}+R_1y^{n-2}+\dots+R_{n-1}),$$

where $C_0, \dots, C_{n-1}, C_0', \dots, C'_{n-1}$ are constants, it follows, since $(x-x_1')y^{n-1}$ is of grade $(n-1)\sigma+n$, and $(R_0y^{n-1}+R_1y^{n-2}+\dots+R_{n-1})(x-x_1)(x-x_1')$ is of grade $(n-1)\sigma+n+1$ at least, that R_0 is zero and $C_0=C_0'$. Further, if the equation associated with the surface be written

$$f(y, x)=y^n+Q_1y^{n-1}+Q_2y^{n-2}+\dots+Q_{n-1}=0,$$

and $\chi_i(x)$ denote

$$y^i+Q_1y^{i-1}+\dots+Q_i,$$

it follows, from the condition for ψ which ensures that the integral P is not infinite at all the n places $x=x_1$, that the factors of the polynomial

$$C_0y^{n-1}+C_1y^{n-2}+\dots+C_{n-1}$$

are the same as those of $f(y, x)/(y-y_1)$, or of

$$y^{n-1}+\chi_1(x_1)\,.\,y^{n-2}+\chi_2(x_1)\,.\,y^{n-3}+\dots+\chi_{n-1}(x_1).$$

Hence, save for a constant multiplier, P has the form

$$P=\int\frac{dx}{f'(y)}\left[(x, x_1)-(x, x_1')+y^{n-2}(x, 1)_{\sigma-1}+y^{n-3}(x, 1)_{2\sigma}+\dots+(x, 1)_{(n-1)\sigma+n-3}\right],$$

where (x, x_1) denotes

$$[y^{n-1}+y^{n-2}\chi_1(x_1)+\dots+\chi_{n-1}(x_1)]/(x-x_1),$$

so that $(x, x_1)=(x_1, x)$, and (x, x_1') denotes a similar expression.

A general polynomial ψ of grade $(n-1)\sigma+n-1$ contains $2n-1$ more terms than a general polynomial of grade $(n-1)\sigma+n-3$. In accordance with the assumption made in § 92 the general adjoint polynomial ψ of grade $(n-1)\sigma+n-1$ will contain $2n-1+p$ terms. The condition that ψ vanishes in the $2n-2$ places $x=x_1$, $x=x_1'$ other than those where $y=y_1$, $y=y_1'$ respectively, will reduce the number of terms to $p+1$. This is exactly the proper number of terms for a general integral of the third kind (cf. § 45, p. 67). The assumption of § 92 is therefore verified in this instance.

The practical determination of an integral of the third kind here sketched is often very useful. In the hyperelliptic case it gives the integral immediately.

Ex. vii. Prove that if the matrix of substitution Ω occurring on p. 62, in the equation

$$(1, y, y^2, \dots, y^{n-1})=\Omega\,(1, g, \dots, g_{n-1}),$$

be denoted by Ω_x, and the general element of the product-matrix $\Omega_x\Omega_{x_1}^{-1}$ be denoted by $c_{r,s}$, and if, for distinctness of expression, we denote the elements

$$\chi_{n-1}(x), \chi_{n-2}(x), \dots, \chi_1(x), 1, 1, y_1, y_1^2, \dots, y_1^{n-1},$$

respectively by

$$u_1, u_2, \dots, u_{n-1}, u_n, k_1, k_2, k_3, \dots, k_n,$$

then the function

$$\phi_0(x)+\phi_1(x)\,g_1(x_1)+\dots+\phi_{n-1}(x)\,g_{n-1}(x_1),$$

which occurs in the expression of an integral of the third kind given in § 45, is equal to

$$c_{11}u_1k_1+\dots+c_{ii}u_ik_i+\dots+c_{rs}u_rk_s+c_{sr}u_sk_r+\dots.$$

This takes the form $u_1k_1+\dots+u_nk_n$ obtained in Ex. vi. when $c_{rs}=0$ and $c_{ii}=1$, namely when Ω is a constant. This condition will be satisfied when the index is zero at all finite and infinite places.

Ex. viii. Prove for the surface associated with the equation

$$y^3+y^2(x, 1)_1+y(x, 1)_2+(x, 1)_4=0,$$

that the condition of adjointness for any polynomial is that it vanish to the second order at the place $\xi=0$.

Thence shew that the polynomial

$$(x-x_1')[y^2+y\chi_1(x_1)+\chi_2(x_1)]-(x-x_1)[y^2+y\chi_1(x_1')+\chi_2(x_1')]$$
$$+(Ay+Bx^2+Cx+D)(x-x_1)(x-x_1')$$

is adjoint provided $B=0$; and thence that the integral of the third kind is

$$\int\frac{dx}{f'(y)}\left[\frac{y^2+y\chi_1(x_1)+\chi_2(x_1)}{x-x_1}-\frac{y^2+y\chi_1(x_1')+\chi_2(x_1')}{x-x_1'}+Ay+Cx+D\right].$$

Ex. ix. There is a very important generalization* of the method of Ex. vi. for forming an integral of the third kind. Let μ be any positive integer. Let a general non-adjoint polynomial of grade μ be chosen so as to vanish in the two infinities of the integral, which we suppose, first of all, to be ordinary finite places. Denote this polynomial by L. It will vanish† in $n\mu-2$ other places $B_1, B_2, \ldots$. Take an adjoint polynomial ψ, of grade $(n-1)\sigma+n-3+\mu$, chosen so as to vanish in the places $B_1, B_2, \ldots$. The polynomial will presumably contain (§ 92) $n\mu-1+p-(n\mu-2)$ or $p+1$ homogeneously entering arbitrary coefficients, and will vanish (§ 88) in $n\mu+2p-2-(n\mu-2)$ or $2p$ places other than the places $B_1, B_2, \ldots$ and places where $f'(y)$, or $F'(\eta)$, vanishes. Then the integral

$$P=\int\frac{\psi}{L}\frac{dx}{f'(y)}$$

is a constant multiple of an elementary integral of the third kind.

The proof is to be carried out exactly on the lines of the proof of the form of an integral of the first kind in § 91, with reference to the investigation in § 89.

Further as we know (§ 16) that dP/dx is of the form

$$C(dP/dx)_0+\lambda_1(dv_1/dx)+\ldots+\lambda_p(dv_p/dx),$$

where $C, \lambda_1, \ldots, \lambda_p$ are arbitrary constants, $(dP/dx)_0$ is a special form of dP/dx with the proper behaviour at the infinities, and $v_1, \ldots, v_p$ are integrals of the first kind, it follows that the polynomial ψ, which is an adjoint polynomial of grade $(n-1)\sigma+n-3+\mu$, prescribed to vanish at all but two of the zeros of a non-adjoint polynomial L of grade μ, is of the form

$$\psi=\psi_0+L\phi,$$

where ψ_0 is a particular form of ψ satisfying the conditions, and ϕ is any adjoint polynomial of grade $(n-1)\sigma+n-3$; for this is the only form of ψ which will reduce dP/dx to the form specified.

Ex. x. Shew that if in Ex. ix. one or both of the infinities of the integral be places where $f'(y)=0$, the condition for L is that it vanish to the first order in each place.

Ex. xi. For the case of the surface associated with the equation

$$(y, x)_4+(y, x)_3+(y, x)_2=0,$$

* Given, for $\sigma=0$, $\mu=1$, in Clebsch and Gordan, *Abel. Functionen* (Leipzig, 1866), p. 22, and Noether, "Abel. Differentialausdrücke," *Math. Annal.* t. 37, p. 432.

† Counting zeros which occur for $x=\infty$, or supposing all the zeros to be at finite places. Zeros which occur at $x=\infty$ are to be obtained by considering $\xi^\mu L$, which is an integral polynomial in ξ and η (§ 86).

for which the dimension of y is 1, let us form the integral of the third kind with its infinities at the two places $x=0$, $y=0$ by the rules of Exs. ix. and x.; taking $\mu=1$, the general polynomial of grade 1 which vanishes at the two places in question is $\lambda x+\mu y$. The general polynomial of grade $n-3+\mu$, or 2, is of the form $ax^2+by^2+2hxy+2gx+2fy+c$. In order that this may be adjoint, c must vanish; in order that it may vanish at the two points, other than (0, 0) at which $\lambda x+\mu y$ vanishes, it must reduce to the form

$$(\lambda x+\mu y)(Ax+By+C).$$

Hence the integral of the third kind is $\int (Ax+By+C)\,dx/f'(y)$. (Cf. § 6 β, p. 19.)

Ex. xii. Obtain the other result of § 6 β, p. 19 in a similar way.

Ex. xiii. It will be instructive to compare the method of expressing rational functions which is explained here, with a method founded on the use of the integral functions obtained in Chap. IV. We consider, as example, the case of a rational function which has simple poles at k_1 places where $x=a_1$, k_2 places where $x=a_2, \dots, k_r$ places at $x=a_r$, and for simplicity we suppose all these values of x to be finite, and assume that the sheets of the surface are all distinct for each of these values of x. If R be the rational function, the function $(x-a_1)\dots(x-a_r)\,R$ is an integral function of dimension r, and is expressible in the form

$$(x,\,1)_r+(x,\,1)_{r-\tau_1-1}\,g_1+\dots+(x,\,1)_{r-\tau_{n-1}-1}\,g_{n-1}\,;$$

this form contains $(r+1)+(r-\tau_1)+\dots+(r-\tau_{n-1})$ or $nr-p+1$ coefficients; these coefficients are not arbitrary, for the function $(x-a_1)\dots(x-a_r)\,R$ must vanish at each of the $n-k_1$ places $x=a_1$ where R is not infinite, and must vanish at each of the places $x=a_2$ where R is not infinite, and so on. The number of linear conditions thus imposed is $rn-(k_1+k_2+\dots+k_r)$ or $rn-Q$, if Q be the total number of poles of the function R. Hence the number of coefficients left arbitrary is $nr-p+1-(nr-Q)$ or $Q-p+1$; this is in accordance with results already obtained.

Ex. xiv. If the differential coefficients of $\tau+1$ linearly independent integrals of the first kind vanish in the Q poles, in Ex. xiii., the conditions for the coefficients are equivalent to only $nr-Q-(\tau+1)$ independent conditions.

93. Let $A_1, \dots, A_Q$ be Q arbitrary places of the Riemann surface. We shall suppose these places so situated that a rational function exists of which they are the poles, each being of the first order*. This is a condition which is always satisfied† when $Q>p$. The general rational function in question is of the form

$$\lambda+\lambda_1 Z_1+\dots+\lambda_q Z_q,$$

wherein $\lambda, \lambda_1, \dots, \lambda_q$ are arbitrary constants and $Z_1, \dots, Z_q$ are definite rational functions whose poles, together, are the places $A_1, \dots, A_Q$.

The number q is connected with Q by an equation

$$Q-q=p-\tau-1,$$

where $\tau+1$ is‡ the number of linearly independent linear aggregates of the form

$$\mu_1\Omega(x)+\dots\dots+\mu_p\Omega_p(x),$$

* We speak as if the poles were distinct. This is unimportant.

† Cf. Chap. III.

‡ Chap. III. §§ 27, 37.

which vanish in $A_1, \ldots, A_Q$. This aggregate is the differential coefficient, in regard to the infinitesimal at the place x, of the general integral of the first kind. We have seen* that this differential coefficient only vanishes at a zero of the integral polynomial of grade $(n-1)\sigma+n-3$, which occurs in the expression of the integral of the first kind. Hence $\tau+1$ *is the number of linearly independent adjoint polynomials of grade* $(n-1)\sigma+n-3$ *which vanish in the places* $A_1, \ldots, A_Q$; in other words, $\tau+1$ is the number of coefficients in the general adjoint polynomial of grade $(n-1)\sigma+n-3$ which are left arbitrary after the prescription that the polynomial shall vanish in $A_1, \ldots, A_Q$.

Now we have proved that if any adjoint polynomial ψ, of grade $(n-1)\sigma+n-3+r$ be taken to vanish at the places $A_1, \ldots, A_Q$†, its other zeros being $B_1, \ldots, B_R$, where‡ $R = nr + 2p - 2 - Q$, and ϑ be a proper general adjoint polynomial of grade $(n-1)\sigma+n-3+r$ vanishing at $B_1, \ldots, B_R$, any rational function having $A_1, \ldots, A_Q$ as poles, is of the form ϑ/ψ. Hence the rational functions $Z_1, \ldots, Z_q$ are of the forms $\vartheta_1/\psi, \ldots, \vartheta_q/\psi$, and the general form of an adjoint polynomial of grade $(n-1)\sigma+n-3+r$ vanishing at $B_1, \ldots, B_R$ must be

$$\vartheta = \lambda\psi + \lambda_1\vartheta_1 + \ldots\ldots + \lambda_q\vartheta_q,$$

wherein $\lambda, \lambda_1, \ldots, \lambda_q$ are arbitrary constants, and $\psi, \vartheta_1, \ldots, \vartheta_q$ are special adjoint polynomials of grade $(n-1)\sigma+n-3+r$ which vanish in $B_1, \ldots, B_R$, some of them possibly vanishing also in some of $A_1, \ldots, A_Q$.

Since the general adjoint polynomial ϑ of grade $(n-1)\sigma+n-3+r$ contains $nr-1+p$ arbitrary coefficients, and these, in this case, by the prescription of the zeros $B_1, \ldots, B_R$ for ϑ, reduce to $q+1$, we may say that the places $B_1, \ldots, B_R$, as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3+r$, have the strength $nr-1+p-q-1$, or $R-(p-1)+Q-q-1$, or $R-(\tau+1)$. And, calling these places $B_1, \ldots, B_R$ the residual of the places $A_1, \ldots, A_Q$, because they are the remaining zeros of the adjoint polynomial ψ of grade $(n-1)\sigma+n-3+r$ which vanishes in $A_1, \ldots, A_Q$, we have the result:—

When Q places $A_1, \ldots, A_Q$ have the strength $p-(\tau+1)$ or $Q-q$ as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$, their residual of $R = nr+2p-2-Q$ places, which are the other zeros of any adjoint polynomial of grade $(n-1)\sigma+n-3+r$ prescribed to vanish in the places $A_1, \ldots, A_Q$, have the strength $R-(\tau+1)$ as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3+r$.

Particular cases are, (i), when no adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes in $A_1, \ldots, A_Q$; then the places $B_1, \ldots, B_R$ have a strength equal to their number; (ii), when one adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes in $A_1, \ldots, A_Q$; then

* § 92. † A condition requiring in general $Q < nr - 1 + p$. ‡ § 88.

there are $R-1$ of the places $B_1, \ldots, B_R$ such that every adjoint polynomial of grade $(n-1)\sigma+n-3+r$, vanishing at these places, vanishes at the remaining place. For an example of this case we may cite the theorem: If a cubic curve be drawn through three collinear points A_1, A_2, A_3 of a plane quartic curve, the remaining nine intersections $B_1, \ldots, B_9$ are such that every cubic through a proper set of eight of them necessarily passes through the ninth. In general any set of eight of them may be chosen.

When $\tau+1$ is greater than zero we may take the polynomial ψ itself to be of grade $(n-1)\sigma+n-3$. Since then a general polynomial ϑ of grade $(n-1)\sigma+n-3$ contains p arbitrary coefficients, we can similarly prove that

When $\tau+1$ adjoint polynomials of grade $(n-1)\sigma+n-3$ vanish in Q places $A_1, \ldots, A_Q$, so that the Q places have the strength $Q-q$ as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$, their residual $B_1, \ldots, B_R$, of $R=2p-2-Q$ places, have the strength $p-q-1$, or $R-\tau$, as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$. In this case the numbers are connected by the equations

$$Q+R=2p-2, \quad Q-R=2(q-\tau),$$

*and the characters of the sets $A_1, \ldots, A_Q$, $B_1, \ldots, B_R$ are perfectly reciprocal**.

Ex. When the strength of a set $A_1, \ldots, A_Q$, wherein $Q<p$, as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$, is equal to their number, so that the number of linearly independent adjoint polynomials of grade $(n-1)\sigma+n-3$ which vanish in the places of the set is given by $\tau+1=p-Q$, it follows that $q=0$. Thus if $B_1, \ldots, B_R$ be the residual zeros of an adjoint polynomial, ϕ, of grade $(n-1)\sigma+n-3$, which vanishes in $A_1, \ldots, A_Q$, so that $R+Q=2p-2$, only one adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes in $B_1, \ldots, B_R$, namely ϕ.

94. It is known that the number of places† of the Riemann surface at which a rational function takes an arbitrary value c, is the same as the number of places at which the function is infinite. The sets of places at which c has its different values, may be called *equivalent* sets of places for the function under consideration. For such sets we can prove the result:— *if a set of places $A_1', \ldots, A'_Q$ be equivalent to a set $A_1, \ldots, A_Q$, in the sense that a rational function g takes the value c' at each place of the former set and at no other places, and takes the value c at each of $A_1, \ldots, A_Q$ and at no other places of the Riemann surface, then the general rational function with simple poles at $A_1', \ldots, A'_Q$ contains as many linearly entering arbitrary constants as the general rational function whose poles are at $A_1, \ldots, A_Q$.*

* For the theory of such reciprocal sets from the point of view of the algebraical theory of curves, see the classical paper, Brill u. Noether, "Ueber die algebraischen Functionen u.s.w.", *Math. Annal.* vii. p. 283 (1873).

† In this Article, when a rational function g is said to have the value c at a place, it is intended that $g-c$ is zero of the first order at the place. A place where $g-c$ is zero of the k-th order is regarded as arising by the coalescence of k places where g is equal to c.

For let the general rational function with poles at $A_1, \dots, A_Q$ be denoted by G, and be given by

$$G = \nu_0 + \nu_1 G_1 + \dots\dots + \nu_q G_q,$$

where $\nu_0, \dots, \nu_q$ are arbitrary constants, and $G_1, \dots, G_q$ are particular functions whose poles are among $A_1, \dots, A_Q$—of which one, say G_1, may be taken to be the function $(g-c')/(g-c)$. Then if G' denote any function whatever having poles $A_1', \dots, A'_Q$, and not elsewhere infinite, the function $G'(g-c')/(g-c)$ is one whose poles are at $A_1, \dots, A_Q$; thus $G'(g-c')/(g-c)$ can be expressed in the form

$$G'(g-c')/(g-c) = \nu_0 + \nu_1 G_1 + \dots\dots + \nu_q G_q,$$

for proper values of $\nu_0, \dots, \nu_q$. Therefore G' can be expressed in the form

$$G' = \nu_0 \frac{g-c}{g-c'} + \nu_1 + \nu_2 G_2 \frac{g-c}{g-c'} + \dots\dots + \nu_q G_q \frac{g-c}{g-c'}.$$

Since this is true of every function whose poles are at $A_1', \dots, A'_Q$, and that the functions $G_i(g-c)/(g-c')$ are functions whose poles are at $A_1', \dots, A'_Q$, the result is obvious.

95. If the symbol ∞ be used to denote the number of values of an arbitrary (real or complex) constant, the general adjoint polynomial ϑ, of grade $(n-1)\sigma + n - 3 + r$, of the form

$$\vartheta = \lambda\psi + \lambda_1\vartheta_1 + \dots\dots + \lambda_q\vartheta_q,$$

which vanishes in the places $B_1, \dots, B_R$, gives rise to ∞^q sets of places, constituted by the zeros of ϑ other than $B_1, \dots, B_R$, each set consisting of, say, Q places. Let $A_1, \dots, A_Q$ be one of these sets.

We shall say that these sets are a *lot* of sets; that each *set* is a residual of $B_1, \dots, B_R$, and that they are co-residual with one another; in particular they are all co-residual with the set $A_1, \dots, A_Q$. Further we shall say that the multiplicity of the sets, or of the lot, is q, and that each set has the *sequence* $Q-q$; in fact an individual set is determined by q independent linear conditions, namely, of the Q places of a set, q can be prescribed and the remaining $Q-q$ are *sequent*.

It is clear then that any set, $A_1', \dots, A'_Q$, which is co-residual with $A_1, \dots, A_Q$, is equivalent with $A_1, \dots, A_Q$, in the sense of the last article; for these two sets are respectively the zeros and poles of the same rational function; in fact if ψ be the polynomial vanishing in $B_1, \dots, B_R, A_1, \dots, A_Q$, and ϑ the polynomial vanishing in $B_1, \dots, B_R, A_1', \dots, A'_Q$, the rational function ϑ/ψ has $A_1', \dots, A'_Q$ for zeros and $A_1, \dots, A_Q$ for poles. Hence by the preceding article it follows that the number $q+1$ of linear, arbitrary, coefficients in a general rational function prescribed to have its poles at $A_1, \dots, A_Q$, is the same as the number in the general function prescribed to have its poles at the co-residual set $A_1', \ldots, A'_Q$. In other words, co-residual sets of places have the same multiplicity, this being determined by the number of constants in the general rational function having one of these sets as poles; they have therefore also the same strength $Q-q$, or $p-(\tau+1)$, as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$.

96. In the determination of the sets co-residual to a given one, $A_1, \ldots, A_Q$, we have made use of a particular residual, $B_1, \ldots, B_R$. It can however be shewn that this is unnecessary—*and that, if two sets be co-residual for any one common residual, they are co-residual for any residual of one of them.* In other words, let an adjoint polynomial ψ, of grade $(n-1)\sigma+n-3+r$, be taken to vanish in a set $A_1, \ldots, A_Q$, its other zeros (besides those where $f'(y)=0$, or $F'(\eta)=0$), being $B_1, \ldots, B_R$, and an adjoint polynomial ϑ, of grade $(n-1)\sigma+n-3+r$, be taken to vanish in $B_1, \ldots, B_R$, its other zeros being the set $A_1', \ldots, A'_Q$, co-residual with $A_1, \ldots, A_Q$; then if an adjoint polynomial, ψ', of grade $(n-1)\sigma+n-3+r'$, which vanishes in $A_1, \ldots, A_Q$, have $B_1', \ldots, B'_{R'}$ for its residual zeros, R' being equal to $nr'+2p-2-Q$, it is possible to find an adjoint polynomial ϑ', of grade $(n-1)\sigma+n-3+r'$, whose zeros are the places $B_1', \ldots, B'_{R'}, A_1', \ldots, A'_Q$.

For we have shewn that any rational function having $A_1, \ldots, A_Q$ as its poles can be written as the quotient of two adjoint polynomials, of which the denominator is arbitrary save that it must vanish in the poles of the function, and be of sufficiently high grade to allow this. In particular therefore the function ϑ/ψ, whose zeros are $A_1', \ldots, A'_Q$, can be written as the quotient of two polynomials of which ψ' is the denominator, namely in the form ϑ'/ψ'. The polynomial ϑ' will therefore vanish in the places $B_1', \ldots, B'_{R'}, A_1', \ldots, A'_Q$, as stated.

Hence, not only are equivalent sets necessarily co-residual, but co-residual sets are necessarily equivalent, independently of their residual*.

97. The equivalence of the representations ϑ/ψ, ϑ'/ψ', here obtained, of the same function, has place algebraically in virtue of an identity of the form

$$\vartheta\psi'=\vartheta'\psi+Kf,$$

where $f=0$ is the equation associated with the Riemann surface and K is an integral polynomial in x and y. Reverting to the phraseology of the theory of plane curves, it can in fact be shewn that if three curves $f=0$, $\psi=0$, $H=0$ be so related that, at every common point of f and ψ, which is a multiple point of order k for f and of order l for ψ, whereat f and ψ intersect in $kl+\beta$ points, the curve H have a multiple point of order $k+l-1+\beta$, so that in particular H passes through every simple intersection of f and ψ, then there exist curves $\vartheta'=0$, $K=0$, such that, identically,

$$H=\vartheta'\psi+Kf.$$

Now in the case under consideration in the text, if the only multiple points of f be multiple points at which all the tangents are distinct, the adjointness of ψ ensures that ψ

* For the theory of co-residual sets for a plane cubic curve see Salmon, *Higher Plane Curves* (Dublin, 1879), p. 137. That theory is ascribed to Sylvester; cf. *Math. Annal.*, t. vii., p. 272 note.

has a multiple point of order $k-1$ at every multiple point of f of order k. The adjointness of the polynomials ϑ, ψ' ensures that the compound curve $\vartheta\psi'$ has a multiple point of order $2(k-1)$ or $k+k-1-1$ at every multiple point of f of order k. Further, the curve $\vartheta\psi'$ passes through the simple intersections of f and ψ, which consist of the sets $A_1, \dots, A_Q, B_1, \dots, B_R$; for ϑ passes through $B_1, \dots, B_R$, and ψ' is drawn through $A_1, \dots, A_Q$. Hence the conditions are fully satisfied in this case by taking $H=\vartheta\psi'$; thus there is an equation of the form

$$\vartheta\psi'=\vartheta'\psi+Kf,$$

from which it follows that the curve ϑ' is adjoint at the multiple points of f and passes through the remaining intersections of f and $\vartheta\psi'$, namely through $A'_1, \dots, A'_Q$ and $B'_1, \dots, B'_{R'}$. This is the result of the text.

In case of greater complication in the multiple points of f, there is need for more care in the application of the theorem here quoted from the algebraic theory of plane curves. But this theorem is of great importance. For further information in regard to it the reader may consult Cayley, *Collected Works*, Vol. I. p. 26; Noether, *Math. Annal.* vi. p. 351; Noether, *Math. Annal.* xxiii. p. 311; Noether, *Math. Annal.* xl. p. 140; Brill and Noether, *Math. Annal.* vii. p. 269. Also papers by Noether, Voss, Bertini, Brill, Baker in the *Math. Annal.* xvii, xxvii, xxxiv, xxxix, xlii respectively. See also Grassmann, *Die Ausdehnungslehre von* 1844 (Leipzig, 1878), p. 225. Chasles, *Compt. Rendus*, xli. (1853). de Jonquières, *Mém. par divers savants*, xvi. (1858).

98. From the theorem, that a lot of co-residual sets, of Q places, may be regarded as the residual of any residual of one set, S_Q, of the lot, it follows, that every lot wherein the sequence of a set is less than p, may be determined as the residual zeros of a lot of adjoint polynomials of grade $(n-1)\sigma+n-3$, which have $R=2p-2-Q$ common zeros. For the sequence $Q-q$ is equal to $p-(\tau+1)$, and when $\tau+1>0$ an adjoint polynomial (involving $\tau+1$ arbitrary coefficients) can be determined which is zero in any one set, S_Q, of the lot, and in R other places.

Hence also, when Q places are such that the most general rational function, of which they are the poles, contains more than $Q-p+1$ arbitrary constants, this general rational function can be expressed as the quotient of two adjoint polynomials of grade $(n-1)\sigma+n-3$; the same is true when the Q places are known to be zeros of an adjoint polynomial of grade $(n-1)\sigma+n-3$.

It follows from what was shewn in Chap. III. §§ 23, 27, that if p places be the poles of a rational function, an adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes in these places; and an adjoint polynomial of that grade can always be chosen to vanish in $p-1$, or a less number, of arbitrary places. Hence, *every rational function of order less than $p+1$, is expressible as the quotient of two adjoint polynomials of grade $(n-1)\sigma+n-3$.*

Ex. i. *A rational function of order $2p-2$ which contains p, or more, arbitrary constants (one being additive) is expressible as the quotient of two adjoint polynomials of grade $(n-1)\sigma+n-3$.*

Ex. ii. For a general quartic curve, co-residual sets of 4 places with multiplicity 1 are determined by variable conics having 4 given zeros; but co-residual sets of 4 places with multiplicity 2 are determined as the zeros of variable polynomials of degree 1, i.e. by straight lines.

Ex. iii. The equation of a plane quintic curve with two double points, can be written in the form $\mathfrak{S}S' - \mathfrak{S}'S = 0$, where $\mathfrak{S}$, $\mathfrak{S}'$ are cubics passing through the double points and seven other common points, and S, S' are conics passing through the double points and two other common points.

Ex. iv. When $\tau+1$ adjoint polynomials of grade $(n-1)\sigma+n-3$ vanish in a set, S_Q, of Q places, there must be $p-\tau-1$ independent places $A_1, \ldots, A_{p-\tau-1}$, in S_Q, such that every adjoint polynomial of grade $(n-1)\sigma+n-3$ which vanishes in them vanishes of itself in the remaining q places $A_{p-\tau}, \ldots, A_Q$. Let S_R be a residual of S_Q, R being equal to $2p-2-Q$. Then, regarding S_R and $A_{p-\tau}, \ldots, A_Q$, together, as forming a residual of $A_1, \ldots, A_{p-\tau-1}$, it follows (§ 93) that there is only one adjoint polynomial of grade $(n-1)\sigma+n-3$ which vanishes in S_R and in $A_{p-\tau}, \ldots, A_Q$. Hence there exists no rational function having poles only at the places $A_1, \ldots, A_{p-\tau-1}$. For such a function could be expressed as the quotient of two adjoint polynomials of grade $(n-1)\sigma+n-3$ having S_R and $A_{p-\tau}, \ldots, A_Q$ as common zeros. Compare § 26, Chap. III.

It can also be shewn, in agreement with the theory given in Chapter III., that if $B_1, \ldots, B_{\tau'+1}$ be any $\tau'+1$ independent places, τ' being less than τ, there exists no rational function having poles in S_Q and $B_1, \ldots, B_{\tau'+1}$. In fact $\tau+1-(\tau'+1)$ linearly independent adjoint polynomials of grade $(n-1)\sigma+n-3$ vanish in S_Q and $B_1, \ldots, B_{\tau'+1}$. Let $S_{R'}$, where $R'=2p-2-(Q+\tau'+1)$, be the residual zeros of one of these polynomials. Then the strength of $S_{R'}$, as determinators of adjoint polynomials of grade $(n-1)\sigma+n-3$ is (§ 93) $R'-(\tau-\tau')+1=R-\tau$, where $R=2p-2-Q$, namely the strength of $S_{R'}$ is the same as the strength of $S_{R'}$ and $B_1, \ldots, B_{\tau'+1}$ together; hence every adjoint polynomial of grade $(n-1)\sigma+n-3$ which vanishes in $S_{R'}$, vanishes also in $B_1, \ldots, B_{\tau'+1}$. Now every rational function having S_Q and $B_1, \ldots, B_{\tau'+1}$ as poles, could be expressed as the quotient of two adjoint polynomials of grade $(n-1)\sigma+n-3$ having $S_{R'}$ as common zeros; since each of these polynomials will also have $B_1, \ldots, B_{\tau'+1}$ as zeros, the result is clear.

99. The remaining Articles of this Chapter are devoted to developments more intimately connected with the algebraical theory of curves.

We have seen that an individual set of a lot of co-residual sets of Q places is determined by the prescription of a certain number, q, of the places; this number q being less than* Q and not greater than $Q-p$.

But it does not follow that *any* q places of a set are effective for this purpose; it may happen that q places, chosen at random, are ineffective to give q independent conditions.

We give an example of this which leads (§ 100) to a result of some interest.

Suppose that a set of Q places, S_Q, is given, in which no adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes; then $\tau+1$ is zero, and co-residual sets are determined by $Q-p$ places. Suppose that among the Q places there are $p+s-1$ places, forming a set which we shall denote by σ_{p+s-1}, which are common zeros of $\tau'+1$ adjoint polynomials of grade $(n-1)\sigma+n-3$; denote the other $Q-p-s+1$ or $q-s+1$ places by σ_{q-s+1}.

* The formula is $Q-q=p-(\tau+1)$; if q were Q and therefore $\tau+1=p$, all adjoint polynomials of grade $(n-1)\sigma+n-3$ would vanish in the same Q places, contrary to what is proved in § 21, Chap. II.

Take an adjoint polynomial of grade $(n-1)\sigma+n-3+r$ which vanishes in the places of the set S_Q, and let S_R denote its remaining zeros, so that $R+Q=nr+2p-2$. If we now regard the sets S_R, σ_{q-s+1} together as the residual of the set σ_{p+s-1}, it follows (§ 93) that S_R, σ_{q-s+1} together have only the strength $R+q-s+1-(\tau'+1)$, or $nr+p-2-(\tau'+s)$, as determinators of polynomials of grade $(n-1)\sigma+n-3+r$; and if we choose $s-1$ places $A_1, \ldots, A_{s-1}$ from σ_{p+s-1}, the polynomial of grade $(n-1)\sigma+n-3+r$ with zeros in S_R, which vanishes in the q places constituted by σ_{q-s+1} and $A_1, \ldots, A_{s-1}$ together, will not be entirely determined, but will contain* $\tau'+2$ arbitrary coefficients, at least†: thus $\tau'+1$ further zeros must be prescribed to make the polynomial determinate.

A particular case of this result is as follows:—Consider a lot of coresidual sets of Q, $=q+p$, places, in which no adjoint polynomial of grade $(n-1)\sigma+n-3$ vanishes. If p of the places of a set be zeros of $\tau'+1$ adjoint polynomials of grade $(n-1)\sigma+n-3$, then the other q places are not sufficient to individualise the set; $\tau'+1$ additional places are necessary.

For instance a particular set from the double infinity of sets of 5 places, on a plane quartic curve, determined by variable cubic curves having seven fixed zeros, is generally determined by prescribing 2 places of the set. But if there be one of the sets for which 3 of the five places are collinear, then the other two places do not determine this set; we require also to specify one of the three collinear places. It is easy to verify this result in an elementary way.

100. Consider now two sets S_R, S_{Q_1}, which are residual zeros of an adjoint polynomial, ψ_1, of grade $(n-1)\sigma+n-3+r_1$, so that

$$Q_1+R=nr_1+2p-2.$$

Let $X_{r-r_1}+1$ be the number of terms in the general non-adjoint polynomial of grade $r-r_1$ and N_{r-r_1} be the total number of zeros of such a non-adjoint polynomial of grade $r-r_1$. Take X_{r-r_1} independent places on the Riemann surface, forming a set which we shall denote by T_{r-r_1}, and determine a non-adjoint polynomial, χ, of grade $r-r_1$, to vanish in T_{r-r_1}. It will vanish in $N_{r-r_1}-X_{r-r_1}$ other places, U_{r-r_1}. Suppose that no adjoint polynomials of grade $(n-1)\sigma+n-3$ vanish in all the places of S_{Q_1} and T_{r-r_1}. The product of the polynomials ψ_1 and χ is an adjoint polynomial of grade $(n-1)\sigma+n-3+r$. A general adjoint polynomial of grade $(n-1)\sigma+n-3+r$ which vanishes in S_R will vanish in all the places forming S_{Q_1}, T_{r-r_1}, U_{r-r_1} together, provided we choose the polynomial to have a sufficient number of these places as zeros. Divide the set S_{Q_1} into two parts, one, $\overline{T}$, consisting of $Q_1-p+(N_{r-r_1}-X_{r-r_1})$ places, the other $\overline{U}$ consisting of $p-(N_{r-r_1}-X_{r-r_1})$

* For $nr+p-2$ is the number of independent zeros necessary to determine an adjoint polynomial of grade $(n-1)\sigma+n-3+r$.

† More if the $s-1$ places $A_1, \ldots, A_{s-1}$ be not independent of the others already chosen.

places. The sets $\overline{T}$ and T_{r-r_1} together consist of $Q_1 - p + N_{r-r_1}$, or $Q-p$, places, where

$$Q = Q_1 + N_{r-r_1}, \ = nr + 2p - 2 - R,$$

for $N_{r-r_1} = n(r-r_1)$, (§ 86, Ex. iii.); if then the sets $\overline{U}$ and U_{r-r_1} together are not zeros of any adjoint polynomial of grade $(n-1)\sigma + n - 3$, the general adjoint polynomial, of grade $(n-1)\sigma + n - 3 + r$, which vanishes in S_R, will be entirely determined by the condition of vanishing also in the places of $\overline{T}$ and T_{r-r_1}, and will of itself vanish in the remaining places $\overline{U}$ and U_{r-r_1}. If, however, $\tau' + 1$ adjoint polynomials of grade $(n-1)\sigma + n - 3 - (r - r_1)$ vanish in the places U, the products of these with the non-adjoint polynomial χ give $\tau' + 1$ adjoint polynomials of grade $(n-1)\sigma + n - 3$ vanishing in $\overline{U}$ and U_{r-r_1}. In that case, assuming that no adjoint polynomials of grade $(n-1)\sigma + n - 3$ vanish in the p places $\overline{U}$, U_{r-r_1}, other than those containing χ as a factor, the adjoint polynomial of grade $(n-1)\sigma + n - 3 + r$ which vanishes in S_R, $\overline{T}$ and T_{r-r_1}, will require $\tau' + 1$ further zeros for its complete determination (§ 99).

Since now the set T_{r-r_1} entirely determines the set U_{r-r_1}, we may drop the consideration of it, and obtain the result—

The adjoint polynomial, of grade $(n-1)\sigma + n - 3 + r$, which vanishes in all but $p - (N_{r-r_1} - X_{r-r_1})$ of the zeros of an adjoint polynomial of grade $(n-1)\sigma + n - 3 + r_1$, will have a multiplicity $\tau' + 1 + X_{r-r_1}$, where $\tau' + 1$ is the number of adjoint polynomials of grade $(n-1)\sigma + n - 3 - (r - r_1)$ which vanish in these other $p - N_{r-r_1} + X_{r-r_1}$ zeros. When $\tau' + 1$ is zero the adjoint polynomial of grade $(n-1)\sigma + n - 3 + r$ vanishes of itself in the remaining $p - N_{r-r_1} + X_{r-r_1}$ zeros of the adjoint polynomial of grade $(n-1)\sigma + n - 3 + r_1$. When $\tau' + 1$ is not zero it is necessary, for this, to prescribe $\tau' + 1$ further places of these $p - N_{r-r_1} + X_{r-r_1}$ zeros (provided $\tau' + 1 \overline{\gtrless} p - N_{r-r_1} + X_{r-r_1}$).

We have noticed (§ 86, Ex. iii.) that

$$N_{r-r_1} = n(r - r_1),$$

$$X_{r-r_1} = \left[E\left(\frac{r - r_1}{\sigma + 1}\right) + 1\right]\left[r - r_1 + 1 - \tfrac{1}{2}(\sigma + 1) E\left(\frac{r - r_1}{\sigma + 1}\right)\right] - 1,$$

where $E(x)$ denotes the greatest integer in x.

For $\sigma = 0$, therefore, the number $p - N_{r-r_1} + X_{r-r_1}$ is immediately seen to be equal to

$$\tfrac{1}{2}(\gamma - 1)(\gamma - 2) - \tfrac{1}{2}I,$$

where $\gamma = n - (r - r_1)$, and $\frac{1}{2}I$ is the sum of the indices, of the surface, for finite and infinite places (§ 88).

Thus the result, for $\sigma = 0$,—*an adjoint polynomial of degree* $n - 3 + r$ *which vanishes in all but* $\frac{1}{2}(\gamma-1)(\gamma-2) - \frac{1}{2}I$ *of the zeros of an adjoint polynomial of degree* $n - 3 + r_1$ $(r > r_1,\ \gamma = n - (r - r_1) \nless 3)$ *will have a* *multiplicity* $\tau'+1+\frac{1}{2}(n-\gamma)(n-\gamma+3)$, *where* $\tau'+1$ *is the number of adjoint polynomials of degree* $\gamma-3$ *which vanish in the* $\frac{1}{2}(\gamma-1)(\gamma-2)-\frac{1}{2}I$ *unassigned zeros; if* $\tau'+1$ *is zero this polynomial of degree* $n-3+r$ *will of itself vanish in these unassigned zeros: if* $\tau'+1>0$ *it is necessary, for this, to prescribe* $\tau'+1$ *or, if* $\tau'+1>\frac{1}{2}(\gamma-1)(\gamma-2)-\frac{1}{2}I$, *to prescribe all the unassigned zeros.*

For example let $n=5$; take as the fundamental curve a plane quintic with 2 double points ($p=4$); let the remaining point of intersection with the quintic, of the straight line drawn through these double points, be denoted by A.

(i) Take $r=2$, $r_1=1$. Then $\gamma=5-1=4$, $\gamma-3=1$; thus, an adjoint quartic curve vanishing in all but $\frac{1}{2}(\gamma-1)(\gamma-2)-2$, or 1, of the zeros of an adjoint cubic, that is, vanishing in 10 of these zeros, beside vanishing at the double points, will have a multiplicity $\tau'+1+\frac{1}{2}4$, or $\tau'+1+2$, where $\tau'+1$ is zero if the non-assigned zero be not the point A: and this quartic will then, of itself, pass through the unassigned zero. In this case, in fact, the prescription of the $10+2$ zeros of the quartic on the cubic, is a prescription of more than $4.3-p_1$, where p_1 is the deficiency of the cubic. Hence the quartic will contain the cubic wholly, as part of itself. (In general, the condition to provide against this can be seen to be $r \overset{=}{>} 3$.)

(ii) Take the same fundamental quintic, with $r=4$, $r_1=3$. Then an adjoint sextic curve, ψ, passing through all but $\frac{1}{2}3.2-2$, or 1, of the zeros of an adjoint quintic, ϑ, that is through 20 of them, will have multiplicity $\tau'+1+2$, where $\tau'+1$ is zero unless the other zero of the quintic, ϑ, be the point A.

If however the unassigned zero of the quintic, ϑ, be the point A, the 20 points are not sufficient; the sextic, ψ, has multiplicity 3 and the 20 points *plus* A are necessary to make ψ go through the remaining 7 points.

It should be noticed that an adjoint curve of degree $\gamma-3$ can always be made to pass through $\frac{1}{2}(\gamma-1)(\gamma-2)-\frac{1}{2}I-1$ places. The peculiarity in the case considered is that such curves pass through one more place.

The theorem here proved was first given by Cayley in 1843 (*Collected Works*, Vol. I. p. 25) without special reference to adjoint curves. A further restriction was added by Bacharach (*Math. Annal.* t. 26, p. 275 (1886)).

101. In the following articles of this chapter we shall speak of an adjoint polynomial of grade $(n-1)\sigma+n-3$ as a ϕ-polynomial. In chapter III. (§ 23) we have seen that the set of places constituted by the poles of a rational function, is such that one of them 'depends' upon the others; thus (§ 27) there is one place of the set such that every ϕ-polynomial vanishing in the other places, vanishes also in this. Conversely when a set of places is such that every ϕ-polynomial vanishing in all but one of the places, vanishes of necessity also in the remaining place, this remaining place depends upon the others*. When a set S is such that every ϕ-polynomial

* Or on *some* of them. For instance, if in a two-sheeted hyperelliptic surface, associated with the equation $y^2=(x, 1)_{2p+2}$, we take three places (x_1, y_1), (x_2, y_2), $(x_2, -y_2)$, every ϕ-polynomial, $(x-x_1)(x-x_2)(x, 1)_{p-3}$, of order $p-1$ in x, which vanishes in (x_1, y_1), (x_2, y_2), vanishes also in $(x_2, -y_2)$. But this last place does not, strictly, 'depend' on (x_1, y_1) and (x_2, y_2); it depends on (x_2, y_2) only.

vanishing in S, vanishes also in places $A, B, \ldots$, it will be convenient, here, to say that these places are determined by S.

Take now any $p-3$ places of the surface, which we suppose chosen in order in such a way that no one of them is determined by those preceding. Then the general ϕ-polynomial vanishing in them will be of the form $\lambda\phi + \mu\vartheta + \nu\psi$, wherein λ, μ, ν are arbitrary constants and ϕ, ϑ, ψ are ϕ-polynomials vanishing in the $p-3$ places. We desire now to find a place (x_1) such that all ϕ-polynomials vanishing in the $p-3$ given places and in x_1, shall vanish in another place x_2. For this it is sufficient that the ratios $\phi(x_1) : \vartheta(x_1) : \psi(x_1)$ be equal to the ratios $\phi(x_2) : \vartheta(x_2) : \psi(x_2)$. From the two equations thus expressed, with help of the fundamental equation of the surface, we can eliminate x_2, and obtain an equation for x_1, so that the problem is in general a determinate one and has a finite number of solutions: as a matter of fact (§ 102, p. 144, § 107) the number of positions for x_1 is $\frac{1}{2}p(p-3)$*, and each determines the corresponding position of x_2. Hence there exist on the Riemann surface ∞^{p-3} sets of $p-1$ places such that a single infinity of ϕ-polynomials vanish in them; such a set can be determined from $p-3$ quite arbitrarily chosen places, and, from them, in $\frac{1}{2}p(p-3)$ ways. Putting $Q = p-1$, $\tau + 1 = 2$, we obtain, by the Riemann-Roch Theorem $q = 1$. Hence to each set once obtained there corresponds a single infinity of co-residual sets.

102. The reasoning employed in the last article, to prove that there are a finite number of positions possible for x_1, and the reasoning subsequently to be given to determine the number of these positions, is of a kind that may be fallacious for special forms of the fundamental equation associated with the Riemann surface. An extreme case is when the surface is hyperelliptic, in which case all the ϕ-polynomials vanishing in any given place have another common zero (Chap. V. § 52). In what follows we consider only surfaces which are of perfectly general character for the deficiency assigned.

In particular we assume, what is in accordance with the reasoning of the last article, that not every set of $p-2$ places is such that the two (or more) linearly independent ϕ-polynomials vanishing in them, have another common zero†.

* This result is given in Clebsch and Gordan, *Theorie der Abel. Funct.* (Leipzig, 1866) p. 213.

† Noether (*Math. Annal.* xvii.) gives a proof that this is true for every surface which is not hyperelliptic. Take a set of $p-2$ *independent* places, denoted, say, by S, and, if every $p-2$ places determine another place, let A be the place determined by the set S. Take a further quite arbitrary place, B. When the surface is not hyperelliptic, B will not determine another place. Each of the $\frac{1}{2}(p-1)(p-2)$ sets, of $p-3$ places, which can be selected from the $p-1$ places formed by S and A, constitutes, with B, a set of $p-2$ places, and, in accordance with the hypothesis allowed, each of these sets determines another place. *It is assumed that the $p-2$ places S, and the place B, can be so chosen that the $\frac{1}{2}(p-1)(p-2)$ other places, thus determined, are different from each other and from the p places constituted by S, A and B together.* Since the places S are independent, the ϕ-polynomial vanishing in S and B is unique; and, by what we have proved,

Then it will be possible to choose $p-3$ independent places, S, as in the last article, such that there is a finite number of solutions of the problem of finding a place (x_1) such that the ϕ-polynomials vanishing in S and (x_1), have another common zero; let $p-3$ places, forming a set denoted by S, be so chosen. Let A be a place not coinciding with any of the positions possible for x_1, and not determined by S. Let ϕ, ϑ be two linearly independent ϕ-polynomials vanishing in S and A. Then the general ϕ-polynomial vanishing in S and A is of the form $\lambda\phi+\mu\vartheta$, λ and μ being arbitrary constants, and the general ϕ-polynomial vanishing in the places S only can be written in a form $\lambda\phi+\mu\vartheta+\nu\psi$, wherein ν is an arbitrary constant and ψ is a ϕ-polynomial so chosen as not to vanish at the place A.

Consider now the rational functions* $z=\phi/\psi$, $s=\vartheta/\psi$, each of the $(p+1)$th order. They both vanish at the place A.

These functions will be connected by a rational algebraic equation, $(s, z)=0$, obtained by eliminating (x, y) between the fundamental equation and the equations $z\psi=\phi$, $s\psi=\vartheta$; associated with the equation $(s, z)=0$ will be a new Riemann surface; to every place (x, y) of the old surface will belong a definite place $z=\phi/\psi$, $s=\vartheta/\psi$, of the new surface; to every place of the new surface will belong one or more places of the original surface, the number being the same for every place of the new surface†; since there is only one place of the old surface at which both z and s are zero, namely the place which was denoted by A, it follows that there is only one place of the old surface corresponding to any place of the new surface. Hence each of x, y can be expressed as rational functions of s, z, the expression being obtained from the equations $z\psi=\phi$, $s\psi=\vartheta$, $(s, z)=0$‡.

Since a linear function, $\lambda z+\mu s+\nu$, equal to $(\lambda\phi+\mu\vartheta+\nu\psi)/\psi$, vanishes* at the variable zeros of the polynomial $\lambda\phi+\mu\vartheta+\nu\psi$, namely in $p+1$ places, it follows that the equation $(s, z)=0$ may be interpreted as the equation of a plane curve of order $p+1$; the number

it vanishes in $p+\frac{1}{2}(p-1)(p-2)$ places. This number, however, is greater than $2p-2$ when $p>3$. Hence the hypothesis, that every $p-2$ places determine another is invalid. In case $p=3$ the surface is clearly hyperelliptic when every $p-2$ places determine another. In case $p=2$ or 1 the surface is always hyperelliptic. It may be remarked that when we are once assured of the existence of a rational function of p poles, we can infer the existence of a set of $p-2$ places which do not determine another (cf. § 103). We have already shewn (Chap. III. § 31) that in general a rational function of order p does exist. The reader may prove that for a hyperelliptic surface whose deficiency is an odd number there does not exist any rational function of order p.

* It must be borne in mind that, in dealing with a rational function expressed as a ratio of two adjoint polynomials, we speak of its poles as all given by the zeros of the denominator; some of these may be at $x=\infty$ (cf. § 86), and in that case their existence is to be shewn by considering (§ 84), instead of the polynomial, ψ, of grade μ, the polynomial in η and ξ, given by $\xi^{\mu}\psi$. Or we may use homogeneous variables (§ 85). For instance, for $p=3$, we may, in the text, have (§ 92, Ex. i.) $\phi=x$, $\vartheta=y$, $\psi=1$. Then $\phi:\vartheta:\psi=1:\eta:\xi=\omega:u:z$; and ψ has a zero at $z=\infty$.

† Chap. I. § 4.

‡ Or by the direct process of § 5, Chap. I.

of its double points will, therefore*, be $\frac{1}{2}p(p-1)-p$, or $\frac{1}{2}p(p-3)$, though it is not shewn here that they occur as simple double points. These double points are the transformations of the pairs of places, (x_1), (x_2), on the old surface, which were such that every ϕ-polynomial, vanishing in the $p-3$ fixed places S, and in x_1, also vanished in x_2.

Since a double point of a curve requires one condition among its coefficients, and the number of coefficients that can be introduced or destroyed, in the equation of a curve, by general linear transformation of the coordinates is 8, it follows that a curve of order m has

$$\tfrac{1}{2}m(m+3)-(\delta+\kappa)-8, \text{ or } \tfrac{1}{2}m(m+3)-\tfrac{1}{2}(m-1)(m-2)+p-8, \text{ or } 3m+p-9$$

constants which are not removeable by linear transformation. In the case under consideration here, there are $p-3$ places, S, of each of which an infinite number of positions is possible, independently of the others, and the most general linear transformation of s and z is equivalent only to adopting three new linear functions of ϕ, ϑ, ψ, instead of ϕ, ϑ, ψ, in order to express the general ϕ-polynomial through the places S. Hence there are, in the new surface (s, z) effectively

$$3(p+1)-9+p-(p-3),$$

that is, $3p-3$ intrinsic constants: this is in agreement with a result previously obtained (Chap. I. § 7).

103. The $p-3$ places S may be defined in a particular way, thus:— In general there are (Chap. III. § 31) $(p-1)p(p+1)$ places of the original surface, for each of which a rational function can be found, infinite only at such place and infinite to the pth order. Every rational function, whose order is less than $p+1$, can be expressed as the quotient of two ϕ-polynomials (§ 98). The ϕ-polynomial, ϕ, occurring in the denominator of the function, will† vanish p times at the place where the function has a pole of order p‡, and will vanish in $p-2$ other places forming a set T. The general ϕ-polynomial§ through these $p-2$ places T will not have another fixed zero, or it would be impossible to form a rational function of order p with ϕ as denominator. Let now A denote any place of the set T, the remaining $p-3$ places being denoted by S. Then we may continue the process exactly as in the last Article.

The p variable zeros of the ϕ-polynomials, of the form $\lambda\phi+\mu\vartheta$, which vanish in the $p-2$ places T will, for the transformed curve, become the variable intersections of it with the straight lines, $\lambda z+\mu s=0$, which pass through the place $s=0$, $z=0$. We enquire now how many of these straight lines will touch the new curve. This number may be found either by the ordinary methods of analytical geometry || or as the number of places where

* By the formula $p=\frac{1}{2}(n-1)(n\sigma+n-2)-\frac{1}{2}\Sigma i$, for it is clear that s is an integral function of z of dimension 1, so that $\sigma=0$. And we have remarked that i is 1 at each of the places corresponding to a double point of the curve, so that $\delta+\kappa=\frac{1}{2}\Sigma i$; cf. Forsyth, *Theory of Functions*, § 182.

† See the note (*) of § 102.

‡ This is the fact expressed by the vanishing of the determinant Δ in § 31, Chap. III.

§ Which we assume to be of the form $\lambda\phi+\mu\vartheta$, involving $q+1=2$ arbitrary coefficients. If q were greater than unity, it would be possible to construct a function of lower than the pth order. This possibility is considered below (§ 105 ff.).

|| See for example Salmon's *Higher Plane Curves*.

the differential of the function ϑ/ϕ, of order p, vanishes to the second order, namely* $2p+2p-2$. Among these tangents, however, there is one which touches the transformed curve in p points, counting as $p-1$ tangents. There are, therefore, $3p-1$ other tangents. Of the $3p$ distinct tangent lines thus obtained, there are $3p-3$ distinct cross ratios, formed from the $3p-3$ distinct sets of four of them, and these cross ratios are independent of any linear transformation of the coordinates s and z.

There are thus $3p-3$ quantities obtainable for the transformed curve. We prove, now†, that they entirely determine this curve, and may, therefore, since the transformation is reversible, be regarded as the absolute constants of the original curve. For take any arbitrary point O; draw through it 3 arbitrary straight lines and draw $3p-3$ other straight lines which form with the 3 straight lines first drawn pencils of given cross ratios. Then the coefficients of a curve of order $p+1$, which passes through O, has $\frac{1}{2}p(p-3)$ double points, and touches $3p$ straight lines through O, one of them in p consecutive points, are subject to $1+\frac{1}{2}p(p-3)+3p-1+p-1$ or $\frac{1}{2}p^2+\frac{5}{2}p-1$ linear conditions. The number of these coefficients is $\frac{1}{2}(p+1)(p+4)$ or $\frac{1}{2}p^2+\frac{5}{2}p+2$. Hence there are three coefficients left arbitrary; besides these there are five other constants in the equation of the curve, namely, those which settle the position of O and the three arbitrary straight lines through O. The eight constants thus involved in the curve can be disposed of by a linear transformation.

The reader will recognise here a verification of the argument sketched in § 7, Chap. I.; the present argument is in fact only a particular case of that, obtained by specialising the dependent variable of the new surface, and the order of the independent variable g. The restriction that the p poles of g shall be in one place can be removed, with a certain loss of definiteness and conviction.

The argument employed clearly fails for the hyperelliptic case, since then the $p-2$ fixed zeros of the polynomials ϕ and ϑ determine other places, and the function ϑ/ϕ is not of the pth order.

For $p=3$ we have the result:—If an inflexional tangent of a plane quartic curve meet the curve again in O, eight other tangents to the curve can be drawn from O. The cross ratios of the six independent sets of four tangents, which can be formed from these nine tangents, determine the curve completely—save for constants which can be altered by projection.

More generally, from any point O of the quartic, ten tangents to the curve can be drawn. The seven cross ratios of these tangents leave, by elimination of the coordinates of O, six quantities from which the curve is determinate, save for quantities altered by projection.

* Chap. I. § 6.

† Cayley, *Collected Works*, vol. VI. p. 6. Brill u. Noether, *Math. Annal.* t. VII. p. 303.

104. It is a very slight step from the process of the last Article to take the independent variable to be $g = \vartheta/\phi$, where ϑ, ϕ are ϕ-polynomials, having $p-2$ common zeros forming a set such that a single infinity of ϕ-polynomials vanish in the places of the set. And it may be convenient to take another dependent variable.

In the process of Article 102, the fixed zeros of the polynomials used are $p-3$ in number, and a double infinity of ϕ-polynomials vanish in the places of the set.

These two processes are capable of extension. If we can find a set S_Q, of Q places, in which just $(\tau+1=)\,3$ ϕ-polynomials vanish, and if the places S_Q be such that these three ϕ-polynomials have no other common zero, while the problem of finding a further place x_1, such that the two ϕ-polynomials vanishing in S_Q and x_1 have another common zero x_2, is capable of only a finite number of solutions, then we can extend the process of Article 102; we can then, in fact, transform the surface into one of $2p-2-Q$ sheets. The dependent variable in the new equation will be of dimension unity, and the equation such as represents a curve of order $2p-2-Q$. If, therefore, we can find sets S_Q in which $Q > p-3$, the new surface will have a less number of sheets, and therefore, in general, a simpler form of equation, than the surface obtained in § 102.

Similarly, if we can find a set, S_Q, which are the common zeros of $(\tau+1=)\,2$ ϕ-polynomials, say ϑ and ϕ, we can use the function $g = \vartheta/\phi$, with a suitable other function, as independent and dependent variables respectively, to obtain a new form of equation for which there are $2p-2-Q$ sheets: and if we can get $Q > p-2$ the new surface will be simpler than that obtained in § 103.

105. We are thus led to enquire what are the conditions that $\tau+1$ linearly independent ϕ-polynomials should vanish in any Q places $a_1, \dots, a_Q$.

If the general ϕ-polynomial be written in the form $\lambda_1\phi_1(x) + \dots + \lambda_p\phi_p(x)$, where $\lambda_1, \dots, \lambda_p$ are arbitrary constants, the conditions are that the Q equations

$$\lambda_1\phi_1(a_i) + \dots + \lambda_p\phi_p(a_i) = 0, \qquad (i = 1, 2, \dots, Q)$$

should be equivalent to only $p-\tau-1$ equations, for the determination of the ratios $\lambda_1 : \dots : \lambda_p$; we suppose $Q > p-\tau-1$, and further that the notation is so chosen that the independent equations are the first $p-\tau-1$ of them. Then there exist $Q-(p-\tau-1)$ sets, each of p equations, of the form

$$\phi_j(a_{p-\tau-1+\sigma}) = m_1\phi_j(a_1) + \dots + m_{p-\tau-1}\,\phi_j(a_{p-\tau-1}), \qquad (j = 1, 2, \dots, p)$$

for each value of σ from 1 to $Q-(p-\tau-1)$, the values of $m_1, \dots, m_{p-\tau-1}$ being, for any value of σ, the same for every value of j. The set, of p, of these equations, for which σ has any definite value, lead to $\tau+1$ equations, of the form

$$\begin{vmatrix} \phi_1(a_1), & \dots, & \phi_1(a_{p-\tau-1}) & , & \phi_1(a_{p-\tau-1+\sigma}) \\ \dots & \dots & \dots & \dots & \dots \\ \phi_{p-\tau-1}(a_1), & \dots, & \phi_{p-\tau-1}(a_{p-\tau-1}) & , & \phi_{p-\tau-1}(a_{p-\tau-1+\sigma}) \\ \phi_{p-\tau-1+k}(a_1), & \dots, & \phi_{p-\tau-1+k}(a_{p-\tau-1}), & & \phi_{p-\tau-1+k}(a_{p-\tau-1+\sigma}) \end{vmatrix} = 0,$$

arising for $k = 1, 2, \dots, \tau+1$.

Putting $q = Q-(p-\tau-1)$, we have therefore $q(\tau+1)$ such equations* connecting the Q places $a_1, \dots, a_Q$.

It is obvious from the method of formation that these $q(\tau+1)$ equations are in general independent; in what follows we consider only the cases in which they are independent and determinate. Then, taking $Q-q(\tau+1)$ quite arbitrary places, it is possible to determine $q(\tau+1)$ other places, such that there are $\tau+1$ linearly independent ϕ-polynomials vanishing in the total Q places.

The determination of the $q(\tau+1)$ places, from the arbitrary $Q-q(\tau+1)$ places, may be conceived of as the problem of finding $p-\tau-1-[Q-q(\tau+1)]$, or $q\tau$, places, T, to add to the $Q-q(\tau+1)$ arbitrary places, S, such that all ϕ-polynomials vanishing in the resulting $p-\tau-1$ places S, T, may have $Q-(p-\tau-1)$, or q, other common zeros. The $p-\tau-1$ places S, T are independent determinators of ϕ-polynomials.

For instance, when $Q=p-1$, $\tau+1=2$, it follows that $q=1$ and $Q-q(\tau+1)=p-3$, and hence, from the theory here given, it follows that we can determine $p-1$ places in which two ϕ-polynomials vanish, and, of these, $p-3$ places are arbitrary. The problem of determining the other two places may be conceived of as the problem of determining $p-\tau-1-[Q-q(\tau+1)]$, or *one*, other place, to add to the $p-3$ places, such that all ϕ-polynomials vanishing in the resulting $p-2$ places, which are independent determinators of ϕ-polynomials, may have $q=1$ other common zero. We have already seen reason for believing that, when the $p-3$ places are given, the other two places can be determined in $\frac{1}{2}p(p-3)$ ways.

To every set of Q places thus determined, there corresponds a co-residual lot of sets of Q places, the multiplicity of the lot being q; and every co-residual set will have the same character as the original set. The number, q, of places of a co-residual set which are arbitrary, cannot, obviously, be greater than the number, $Q-q(\tau+1)$, of the original set, which are arbitrary. Hence, the self-consistence of the theory clearly requires that $Q-q(\tau+1) \overline{\overline{>}}\, q$. From this, by means of the relation $Q-q=p-\tau-1$, we can deduce the two important results

$$p \overline{\overline{>}} (q+1)(\tau+1), \quad Q \overline{\overline{>}} q + p\frac{q}{q+1}.$$

* These equations are necessary in order that $a_1, \dots, a_Q$ should be the poles of a rational function.

Putting $Q - q(\tau+1) = q + \alpha$, we obtain

$$p = (\tau+1)(q+1) + \alpha, \quad Q = q + p\frac{q}{q+1} + \frac{\alpha}{q+1}.$$

From each such set S_Q we can deduce, as its residuals, sets, S_R, of R, $= 2p - 2 - Q$, places, in which $q+1$ ϕ-polynomials vanish, and it is immediately seen that

$$Q - q(\tau+1) - q = \alpha = R - \tau(q+1) - \tau.$$

106. If now we determine, in accordance with this theory, a set S_Q in which $\tau + 1 = 3$ ϕ-polynomials vanish, it being assumed that these three ϕ-polynomials have no other common zero, and determine ϕ, ϑ to be two ϕ-polynomials vanishing in S_Q and in one other place O, ψ being another ϕ-polynomial vanishing in S_Q but not in O, then the equations $z = \phi/\psi$, $s = \vartheta/\psi$, determine, as before, a reversible transformation of the surface, to a new surface of which the number of sheets is $R = 2p - 2 - Q$, and in which s is of dimension 1 in regard to z.

Since $R \overset{=}{>} \tau + p\tau/(\tau+1)$, the value of R is $\overset{=}{>} 2 + \frac{2}{3}p$. Thus writing $p = 3\pi$, or $3\pi + 1$, or $3\pi + 2$, according as it is a multiple of 3 or not, R is $p - \pi + 2$ in all cases.

From $R = p - \pi + 2$ follows $Q = p - 4 + \pi$; thus $q = Q - p + 3 = \pi - 1$, and $Q - q(\tau+1) = p + \pi - 4 - 3\pi + 3 = p - 2\pi - 1$. This is the number of places of the set S_Q which may be taken arbitrarily. If this number be equal to $q = \pi - 1$, it follows that, by taking two different sets of $Q - q(\tau+1)$, $= p - 2\pi - 1$, places, we get only two co-residual sets, and for the purposes of forming the functions ϕ/ψ, ϑ/ψ, one is as good as the other. If however $Q - q(\tau+1) > q$, we do not get co-residual sets by taking different arbitrary sets of $Q - q(\tau+1)$ places:—and there is a disposeableness which is expressed by the number of the arbitrary places, $Q - q(\tau+1)$, which is in excess of the number, q, which determines the sets co-residual to any given one.

Now $Q - q(\tau+1) - q = p - 2\pi - 1 - \pi + 1 = p - 3\pi$. And, in a surface of m sheets and deficiency p, the number of constants independent of linear transformations is $3m + p - 9$ (§ 102). Hence the number of unassignable quantities in the equation of the surface is

$$3(p - \pi + 2) + p - 9 - (p - 3\pi) \text{ or } 3p - 3;$$

and this is in accordance with a result previously obtained (§ 7, Chap. I.).

Ex. i. The values of π for $p=4$, 5 are 1, 1 respectively, and $p-\pi+2$, in these cases, $=5$, 6 respectively.

Hence a quintic curve with two double points ($p=4$), can be transformed into a quintic; this will also have two double points, in general, since the deficiency must be unaltered. We determine a set consisting of Q, $=1$, quite arbitrary place. Let the general conic through this place, and the two double points, be $\lambda\phi+\mu\vartheta+\nu\psi=0$. Then the formulae of transformation are $z=\phi/\psi$, $s=\vartheta/\psi$. As in the text, we may suppose ϕ, ϑ to have another common point, in which ψ does not vanish.

Ex. ii. A quintic with one double point ($p=5$) can be transformed into a sextic with, in general, $\frac{1}{2}(6-1)(6-2)-5=5$ double points. For this we take $p-2\pi-1=2$ arbitrary points; if $\lambda\phi+\mu\vartheta+\nu\psi$ be the general conic through the two points and the double point, the equations of transformation are $z=\phi/\psi$, $s=\vartheta/\psi$.

Ex. iii. Shew that the orders $p-\pi+2$ of the curves obtainable by this method to represent curves of deficiencies

$$p=6,\ 7,\ 8,\ 9$$

are respectively

$$R=6,\ 7,\ 8,\ 8.$$

107. But, as remarked (§ 104), we can also make use of sets of R places for which $\tau+1=2$, to obtain transformations of our original surface.

We can obtain such a set by taking $R-\tau(q+1)$, or $R-q-1$, arbitrary places, and determining the remaining $q+1$ such that $q+1$ ϕ-polynomials vanish in the whole set of R places.

It is proved by Brill* that the number of sets of $q+1$ thus obtainable from $R-q-1$ arbitrary places, is

$$\sum_0^{\mu}(-1)^{\lambda}\binom{p}{\lambda}\binom{2p-1-R-2\lambda}{2p-1-R-q-1},$$

where $\mu=\frac{1}{2}q$ or $\frac{1}{2}(q+1)$, according as q is even or odd, and $\binom{\lambda}{\nu}$ denotes $\lambda(\lambda-1)\ldots(\lambda-\nu+1)/\nu\,!$.

For instance with $R=p$, $q=0$, the series reduces to one term, whose value is $p-1$, which is clearly right; while, when $R=p-1$, $q=1$, the series reduces to

$$\binom{p}{p-2}-p\binom{p-2}{p-2},$$

or $\frac{1}{2}p(p-3)$, as in § 101, § 102, p. 144.

When p is even and $R=\frac{1}{2}p+1$, $q=\frac{1}{2}p-1$, this series can be summed, and is equal to

$$2\,\underline{|p-1}/\underline{|\tfrac{1}{2}p-1}\ \underline{|\tfrac{1}{2}p+1}.$$

When p is odd and $R=\frac{1}{2}(p+1)+1$, $q=\frac{1}{2}(p-1)-1$, the series can be summed, and is equal to

$$4p\,\underline{|p-2}/\underline{|\tfrac{1}{2}(p-3)}\ \underline{|\tfrac{1}{2}(p+3)}.$$

Now let $\lambda\phi+\mu\vartheta$ be the general ϕ-polynomial vanishing in a set which is residual to one of these sets of R places, λ and μ being arbitrary constants; we may transform the surface with $z=\vartheta/\phi$ as the new independent variable. The new surface obtained will have R sheets. The new dependent variable may be chosen at will, provided only the transformation be reversible.

* *Math. Annal.* xxxvi, pp. 354, 358, 369. See also Brill and Noether, *Math. Annal.* vii. p. 296.

The function $\mu z+\lambda$, $=\mu\vartheta/\phi+\lambda$, depends on $2+R-q-1$ arbitrary quantities, namely the constants λ, μ and the position of the $R-q-1$ arbitrarily taken places. There are $2R+2p-2$ places where dz is zero to the second order, namely, $2R+2p-2$ places where the curve $a\vartheta+b\phi=0$ touches the fundamental curve ; there remain then

$$2R+2p-2-(R-q+1), \ =R-1-p+q+1+3p-3, \ =3p-3$$

of the $2R+2p-2$ values which z has when dz vanishes to the second order, which are quite arbitrary. Compare § 7, Chap. I.

The least possible value of R is given by the formula $R \overline{\gtrless} \tau+p\tau/(\tau+1)$. If then p be written equal to 2π, or $2\pi+1$, according as p is even or odd, we may take* $R=p-\pi+1$, that is $\frac{1}{2}p+1$ or $\frac{1}{2}(p+1)+1$, according as p is even or odd.

Hence, when p is even, we can determine a single infinity of co-residual sets of $\frac{1}{2}p+1$ places, these sets being the zeros of ϕ-polynomials, $\lambda\phi+\mu\vartheta$, which have $\frac{3}{2}p-3$ common zeros. To determine one of these sets of $\frac{1}{2}p+1$ places, we may take one place, A, arbitrarily. The other $\frac{1}{2}p$ places can then be determined in $2\,\underline{|p-1}/\underline{|\tfrac{1}{2}p-1}\,\underline{|\tfrac{1}{2}p+1}$ ways. Let two of these ways be adopted, corresponding to one arbitrary place A ; the resulting sets of $\frac{1}{2}p+1$ places will not be co-residual; for the sets co-residual with a given set have a multiplicity 1, and therefore no two of these sets can have a place common without coinciding altogether. Let the sets co-residual to these two sets be given by $\lambda\phi+\mu\vartheta=0$, $\lambda'\phi'+\mu'\vartheta'=0$, ϕ and ϕ' being chosen so as to vanish in A : we assume that ϕ, ϕ' have no other common zero.

Then the equations $z=\phi/\vartheta$, $s=\phi'/\vartheta'$ will determine a reversible transformation, as is immediately seen in a way analogous to those already adopted. In the new equation z and s enter to a degree $\frac{1}{2}p+1$, and, since there exists* no rational function of lower order than $\frac{1}{2}p+1$, no further reduction of the degree to which z and s enter, is possible.

The new equation may be interpreted as the equation of a curve of order $p+2$: it will have the form

$$(z,\ 1)^m s^m+(z,\ 1)^m s^{m-1}+\ldots+(z,\ 1)^m=0,$$

wherein $m=\frac{1}{2}p+1$.

By putting $z=1/z_1$, $s=1/s_1$, it is reduced to the equation of a curve of order p. The form possesses the interest that it was employed by Riemann.

Ex. Obtain the 2 sets of $\frac{1}{2}p+1$ places corresponding to a given arbitrary point for a quintic curve with two double points, and transform the equation.

108. If we have a set of R places†, for which $\tau+1=4$, the co-residual places being given by the variable zeros of ϕ-polynomials of the form $\lambda\phi_1+\mu\phi_2+\nu\phi_3+\psi$, we can, by writing

$$X=\phi_1/\psi, \quad Y=\phi_2/\psi, \quad Z=\phi_3/\psi,$$

* Thus, for perfectly general surfaces of deficiency p, no rational function exists of order less than $1+\frac{1}{2}p$. Cf. Forsyth, *Theory of Functions*, p. 460. Riemann, *Gesam. Werke* (1876), p. 101.

† Wherein $R-\tau<p$, or $R<p+3$.

and eliminating x, y from these three equations and the fundamental equation associated with the Riemann surface, obtain two rational algebraic equations connecting X, Y, Z; these equations determine a curve in space, of order R; for this is the number of variable zeros of the function $\lambda X + \mu Y + \nu Z + 1$. To a point $X = X_1$, $Y = Y_1$, $Z = Z_1$ of the curve in space, will correspond the places of the surface, other than the fixed zeros of ϕ_1, ϕ_2, ϕ_3, ψ, at which

$$X_1\psi - \phi_1 = 0, \quad Y_1\psi - \phi_2 = 0, \quad Z_1\psi - \phi_3 = 0,$$

and it is generally possible to choose ϕ_1, ϕ_2, ϕ_3, ψ so that these equations have only one solution.

The lowest order possible for the space curve is given by

$$R \geqq \tau + \tau p/(\tau + 1) \geqq 3 + 3p/4.$$

If then $p = 4\pi$, or $4\pi + 1$, or $4\pi + 2$, or $4\pi + 3$, R may be taken equal to $p - \pi + 3$.

For instance with* $p=4$, $R=6$, taking a plane curve with double points at the places $x=\infty$, $y=0$ and $x=0$, $y=\infty$, given by

$$x^2y^2(x, y)_1 + xy(x, y)_2 + (x, y)_3 + (x, y)_2 + (x, y)_1 + A = 0,$$

we may† take $\lambda\phi_1 + \mu\phi_2 + \nu\phi_3 + \psi = \lambda xy + \mu x + \nu y + 1$; the places residual to the variable set of R places are, in number, $2p - 2 - 6$, $=0$. Then the equations of transformation are

$$X = xy, \quad Y = x, \quad Z = y,$$

and these give points (X, Y, Z) lying on the surfaces,

$$X = YZ,$$
$$X^2(Y, Z)_1 + X(Y, Z)_2 + (Y, Z)_3 + (Y, Z)_2 + (Y, Z)_1 + A = 0,$$

of which the first is a quadric and the second a cubic.

A set of R places with multiplicity $\tau = 3$ may of course also be used to obtain a transformation to another Riemann surface. With the same notation we may put $z = \phi_1/\psi$, $s = \phi_2/\psi$. It is clear that the resulting equation, regarded as that of a plane curve, is the orthogonal projection, on to the plane $Z = 0$, of the space curve just obtained.

A set of R places with multiplicity $\tau > 3$ may be used similarly to obtain a curve of order R in space of τ dimensions. Some considerations in this connexion will be found in the concluding articles of this chapter.

109. It has already been explained that the methods of transformation given in §§ 101—108 of this chapter are not intended to apply to surfaces which are not of general character for their deficiency, and that, in particular, hyperelliptic surfaces are excluded from consideration. We may give here a practical method of obtaining the canonical form of a hyperelliptic surface,

* Since p must be $\geqq (\tau+1)(q+1)$, this is the first case to which the theory applies.

† It is easy to shew that this is the general adjoint polynomial of degree $n-3$. We may also shew that the integrals, $\int xydx/f'(y)$, etc., are finite, or use the method given *Camb. Phil. Trans.* xv. iv. p. 413, there being no finite multiple points.

whose existence has already been demonstrated (Chap. V. § 54). Suppose first that $p > 1$. In the hyperelliptic case every ϕ-polynomial vanishing in any place A will vanish, of itself, in another place A'. Any one of these ϕ-polynomials will have $2p-4$ other zeros, forming a set which we shall denote by S. Putting $Q=2$ and $\tau+1=p-1$ in the formula $Q-q=p-\tau-1$, we find $q=1$, so that the general ϕ-polynomial vanishing in the places S will be of the form $\lambda_1\phi_1-\lambda_2\phi_2$, wherein λ_1, λ_2 are arbitrary constants; in fact these $2p-4$ places S consist of $p-2$ independent places and the other $p-2$ places determined by them, one by each. Thus a function of the second order is given by $z=\phi_1/\phi_2$. A general adjoint polynomial of grade $(n-1)\sigma+n-2$ will contain $n+p-1$ terms and vanish, in all, in $n+2p-2$ places; thus the general adjoint polynomial, of this grade, which is prescribed to vanish in a set T of $n+p-3$ arbitrary places, will be of the form $\mu_1\psi_1+\mu_2\psi_2$, μ_1, μ_2 being arbitrary constants, and will vanish in $p+1$ other places. We may suppose ψ_1 so chosen that it vanishes in one of the two zeros of ϕ_1 which are not among the set S, and we shall assume that ψ_2 does not vanish in this place, and that ψ_1 does not vanish in the other of these two zeros of ϕ_1. Then the functions $z=\phi_1/\phi_2$, $s=\psi_1/\psi_2$, are connected by a rational equation, $(s, z)=0$, with which a new Riemann surface may be associated; to any place of the old surface there corresponds only one place $z=\phi_1/\phi_2$, $s=\psi_1/\psi_2$, of the new surface; to the place $z=0$, $s=0$ of the new surface corresponds only one place of the original surface, and the same is therefore true of every place of the new surface. Thus the equation $(s, z)=0$ is of degree 2 in s and degree $p+1$ in z. The highest aggregate degree in s and z together, in the equation $(s, z)=0$, is the same as the number of zeros of functions of the form $\lambda z+\mu s+\nu$, for arbitrary values of λ, μ, ν, and therefore if the poles s be different from the poles of z, namely, if the zeros of ψ_2 other than T, be different from the zeros of ϕ_2 other than S, the aggregate degree of (s, z) in s and z together will be $p+3$; thus the equation will be included in the form

$$s^2\alpha+s\beta+\gamma=0,$$

where α, β, γ are integral polynomials in z of degree $p+1$.

If we put $\sigma=s\alpha+\frac{1}{2}\beta$, this takes the form

$$\sigma^2=\tfrac{1}{4}\alpha^2-\alpha\gamma,$$

which is of the canonical form in question.

Ex. A plane quartic curve with a double point ($p=2$) may be regarded as generated by the common variable zero A of (i) straight lines through the double point, vanishing also in variable points A and A', (ii) conics through the double point and three fixed points, vanishing also in variable points A, B, C.

When p is 1 or 0, the method given here does not apply, since then adjoint ϕ-polynomials (which in general vanish in $2p-2$ variable places) have no variable zeros. In case $p = 1$ or $p = 0$, if $\mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3$, with μ_1, μ_2, μ_3 arbitrary, be the general adjoint polynomial of grade $(n-1)\sigma + n - 2$ which vanishes in $n + p - 4$ fixed places, ψ_1, ψ_3 being chosen to have one other common zero beside these $n + p - 4$ fixed places, we may use the transformation $z = \psi_1/\psi_3$, $s = \psi_2/\psi_3$, z being a function of order $p + 1$, and s being a function of order $p + 2$. Then, since the function $\lambda z + \mu s + \nu$ vanishes in $p + 2$ places, we obtain an equation of the form*

$$s^2 (z, 1)_p + s\,(z, 1)_{p+1} + (z, 1)_{p+2} = 0,$$

of which the further reduction is immediate.

Ex. For a plane quartic curve with two double points ($p=1$) let $\mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3$ be the general conic through the double points and a further point A, ψ_1 and ψ_3 being chosen also to vanish at any point B. Then we may use the transformation $z = \psi_1/\psi_3$, $s = \psi_2/\psi_3$.

110. In the transformations which have been given we have made frequent use of the polynomials which we have called ϕ-polynomials, namely adjoint polynomials of grade $(n-1)\sigma + n - 3$. For this there is the special reason, already referred to†, that, in any reversible transformation of the surface, their ratios are changed into ratios of ϕ-polynomials belonging to the transformed surface; thus any property, or function, which can be expressed by these ϕ-polynomials only, is invariant for all birational transformations. We give now some important examples of such properties.

Let the general ϕ-polynomial be always supposed expressed in the form $\lambda_1\phi_1 + \ldots + \lambda_p\phi_p$, $\lambda_1, \ldots, \lambda_p$ being arbitrary constants. Instead of $\phi_1, \ldots, \phi_p$ we may use any p linearly independent linear functions of $\phi_1, \ldots, \phi_p$, agreed upon beforehand. A convenient method is to take p independent places $c_1, \ldots, c_p$ and define ϕ_i as the ϕ-polynomial vanishing in all of $c_1, \ldots, c_p$ except c_i; but we shall not adhere to that convention in this place. Let any general integral homogeneous polynomial in $\phi_1, \ldots, \phi_p$, of degree μ, be denoted by $\Phi^{(\mu)}$ or $\Phi'^{(\mu)}$. This polynomial contains $p(p+1)\ldots(p+\mu-1)/\mu!$ terms.

In a polynomial $\Phi^{(2)}$ there are $\frac{1}{2}p(p+1)$ products of two of $\phi_1, \ldots, \phi_p$. But these $\frac{1}{2}p(p+1)$ products of pairs are not linearly independent. For example in a hyperelliptic case, we can choose a function of the second order, z, such that the ratios of p independent ϕ-polynomials are given by

$$\phi_1 : \phi_2 : \ldots : \phi_p = 1 : z : z^2 : \ldots : z^{p-1};$$

then there will be $p - 2$ identities of the form

$$\phi_2/\phi_1 = \phi_3/\phi_2 = \ldots = \phi_p/\phi_{p-1},$$

* Further developments are given by Clebsch, *Crelle*, t. 64, pp. 43, 210. For this subject and for many other matters dealt with in this Chapter, the reader may also consult Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie* (Paris 1883), t. III.

† Chap. II. § 21.

whereby the number of linearly independent products of pairs of $\phi_1, \dots, \phi_p$ is reduced to $\frac{1}{2}p(p+1)-(p-2)$, at most. But we can in fact shew, whether the surface be hyperelliptic or not, that there are not more than $3(p-1)$ linearly independent products of pairs of $\phi_1, \dots, \phi_p$. For consider the $4(p-2)$ places in which any general quadratic polynomial, $\Phi^{(2)}$, vanishes. If $\phi_i\phi_j$ be any product of two of the polynomials $\phi_1, \dots, \phi_p$, the quotient $\phi_i\phi_j/\Phi^{(2)}$ represents a rational function having no poles except such as occur among the zeros* of $\Phi^{(2)}$; there are therefore at least as many linearly independent rational functions, with poles among the zeros of $\Phi^{(2)}$, as there are linearly independent products of pairs of $\phi_1, \dots, \phi_p$. But the general rational function having its poles among the $4(p-1)$ zeros of $\Phi^{(2)}$, contains only $4(p-1)-p+1$, $=3(p-1)$, arbitrary constants. Hence there are not more than this number of linearly independent pairs of $\phi_1, \dots, \phi_p$. In precisely the same way it follows that there are not more than $(2\mu-1)(p-1)$ linearly independent products of μ of the polynomials $\phi_1, \dots, \phi_p$.

111. But it can be further shewn that in general† there are just $(2\mu-1)(p-1)$ linearly independent products of μ of the polynomials $\phi_1, \dots, \phi_p$; so that there are

$$\frac{p(p+1)\dots(p+\mu-1)}{\lfloor\mu} - (2\mu-1)(p-1)$$

identical relations connecting the products of μ of the polynomials $\phi_1, \dots, \phi_p$.

Consider the case $\mu \doteq 2$. Take $p-2$ places such that the general ϕ-polynomial vanishing in them is of the form $\lambda\phi_1+\mu\phi_2$, λ and μ being arbitrary, and ϕ_1, ϕ_2 having no zero common beside these $p-2$ places. Let $\Phi^{(1)}$, $\Phi'^{(1)}$ denote two general linear functions of $\phi_1, \dots, \phi_p$. The polynomial

$$\phi_1\Phi^{(1)} + \phi_2\Phi'^{(1)}$$

is quadratic in $\phi_1, \dots, \phi_p$. It contains $2p$ terms. But clearly these terms are not linearly independent, for the term $\phi_2\phi_1$ occurs both in $\phi_1\Phi^{(1)}$ and in $\phi_2\Phi'^{(1)}$. Suppose, then, that there are terms, $\phi_2\Psi'^{(1)}$, occurring in $\phi_2\Phi'^{(1)}$, which are equal to terms, $\phi_1\Psi^{(1)}$, occurring in $\phi_1\Phi^{(1)}$. The necessary equation for this,

$$-\frac{\Psi'^{(1)}}{\Psi^{(1)}} = \frac{\phi_1}{\phi_2},$$

shews that $\Psi^{(1)}$ vanishes in the p zeros of ϕ_2 which are not zeros of ϕ_1. But since these p zeros form a set which is a residual of a set (of $p-2$ places)

* Here, as in all similar cases, the zeros of the polynomial are its generalised zeros when it is regarded as of its specified grade.

† Precisely, the theorem is true when the surface is sufficiently general to allow the existence of $p-2$ places such that the general ϕ-polynomial, vanishing in them, is of the form $\lambda\phi_1+\mu\phi_2$, λ and μ being arbitrary constants, and ϕ_1, ϕ_2 having no common zero other than the $p-2$ places. We have already given a proof that this is always the case when the surface is not hyperelliptic (§ 102).

in which two ϕ-polynomials vanish, it follows* that only one ϕ-polynomial vanishes in these p places; and such an one is ϕ_2. Hence $\Psi^{(1)}$ must be a multiple of ϕ_2, and therefore $\Psi'^{(1)}$ a multiple of ϕ_1. Thus the polynomial

$$\phi_1\Phi^{(1)} + \phi_2\Phi'^{(1)}$$

contains $2p-1$ linearly independent products of pairs of $\phi_1, \ldots, \phi_p$.

Let now ϕ_3 be a ϕ-polynomial not vanishing in the common zeros of ϕ_1, ϕ_2, and let $\phi_4, \ldots, \phi_p$ be chosen so that $\phi_1, \phi_2, \phi_3, \ldots, \phi_p$ are linearly independent. Consider the polynomial

$$\Phi = \phi_1\Phi^{(1)} + \phi_2\Phi'^{(1)} + \phi_3[\lambda_3\phi_3 + \ldots + \lambda_p\phi_p],$$

wherein $\lambda_3, \ldots, \lambda_p$ are arbitrary constants. Herein $\phi_3(\lambda_3\phi_3 + \ldots + \lambda_p\phi_p)$ cannot contain any terms $\phi_3(\lambda_3'\phi_3 + \ldots + \lambda_p'\phi_p)$ which are equal to terms already occurring in the part $\phi_1\Phi^{(1)} + \phi_2\Phi'^{(1)}$, or else $\lambda_3'\phi_3 + \ldots + \lambda_p'\phi_p$ would vanish in the $p-2$ common zeros of ϕ_1 and ϕ_2; and this is contrary to the hypothesis that $\lambda\phi_1 + \mu\phi_2$ is the most general ϕ-polynomial vanishing in these $p-2$ places. Hence the polynomial Φ contains $2p-1+p-2$, or $3p-3$, independent products of twos of the polynomials $\phi_1, \ldots, \phi_p$. As we have proved that a greater number does not exist, $3p-3$ is the number of such products of pairs.

Consider next the case $\mu = 3$. Since co-residual sets of $2p-1$ places have † a multiplicity $p-1$, it follows that the general polynomial, $\Psi^{(2)}$, of the second degree in $\phi_1, \ldots, \phi_p$, which vanishes in $2p-3$ fixed places, and therefore in $2p-1$ variable places, contains p arbitrary coefficients. If then the $2p-3$ fixed zeros of $\Psi^{(2)}$ be zeros of a definite polynomial, ϕ_2, it follows that $\Psi^{(2)}$ is of the form $\phi_2\Psi^{(1)}$, $\Psi^{(1)}$ being of the first degree in $\phi_1, \phi_2, \ldots, \phi_p$. Hence, as in the case $\mu = 2$, it can be proved that if ϕ_1, ϕ_2 be ϕ-polynomials with one common zero, the reduction in the number, $2(3p-3)$, of terms in a polynomial $\phi_1\Phi^{(2)} + \phi_2\Phi'^{(2)}$, which arises in consequence of the occurrence of terms, $\phi_2\Psi'^{(2)}$, in $\phi_2\Phi'^{(2)}$, which are equal to terms, $-\phi_1\Psi^{(2)}$, occurring in $\phi_1\Phi^{(2)}$, is at most equal to p. Hence the polynomial $\phi_1\Phi^{(2)} + \phi_2\Phi'^{(2)}$ contains at least $5p-6$ linearly independent products of threes of $\phi_1, \ldots, \phi_p$. Hence taking ϕ_3, and a quadratic polynomial $\Phi''^{(2)}$, such as do not vanish in the common zero of ϕ_1, ϕ_2, it follows that a cubic polynomial with at least $5p-5$ linearly independent products, is given by

$$\phi_1\Phi^{(2)} + \phi_2\Phi'^{(2)} + \phi_3\Phi''^{(2)}.$$

We have thus proved that in the cases $\mu = 2$, $\mu = 3$, the polynomial $\Phi^{(\mu)}$ contains $(2\mu-1)(p-1)$ linearly independent products. Assume now that $\Phi^{(\mu-1)}$ contains $(2\mu-3)(p-1)$ independent terms, and that $\Phi^{(\mu-2)}$

* From the formula (Chap. VI. § 93)

$$Q - R = 2(q - \tau),$$

putting $Q = p-2$, $R = p$, $\tau = 1$, we obtain $q = 0$.

† From $Q - q = p - (\tau+1)$, putting $\tau+1 = 0$ (because $2p-1 > 2p-2$) $Q = 2p-1$, $q = p-1$.

contains $(2\mu-5)(p-1)$ independent terms. A general polynomial $\Psi^{(\mu-1)}$ vanishing in the zeros of a definite ϕ-polynomial, ϕ_2, will have $2(\mu-2)(p-1)$ variable zeros; and the multiplicity of co-residual sets of $2(\mu-2)(p-1)$ places, when $\mu > 3$, is $(2\mu-5)(p-1)-1$, which by hypothesis is the same as the multiplicity of the sets of zeros of a polynomial $\phi_2\Psi^{(\mu-2)}$, in which $\Psi^{(\mu-2)}$ has its most general form possible. Hence the general polynomial $\Psi^{(\mu-1)}$ vanishing in the zeros of ϕ_2, is of the form $\phi_2\Psi^{(\mu-2)}$. If then, in a polynomial, $\phi_1\Phi^{(\mu-1)}+\phi_2\Phi'^{(\mu-1)}$, of the μth degree in $\phi_1, \ldots, \phi_p$, wherein ϕ_1, ϕ_2 have no common zeros, there be terms, $\phi_2\Psi'^{(\mu-1)}$, occurring in $\phi_2\Phi'^{(\mu-1)}$, which are equal to terms, $-\phi_1\Psi^{(\mu-1)}$, occurring in $\phi_1\Phi^{(\mu-1)}$, then $\Psi^{(\mu-1)}$ must be of the form $\phi_2\Psi^{(\mu-2)}$, and $\Psi'^{(\mu-1)}$ of the form $\phi_1\Psi'^{(\mu-2)}$, and the resulting reduction in the number, $2(2\mu-3)(p-1)$, of terms in $\phi_1\Phi^{(\mu-1)}+\phi_2\Phi'^{(\mu-1)}$, is at most equal to the number, $(2\mu-5)(p-1)$, of terms in a polynomial $\Psi^{(\mu-2)}$. Thus, there are at least

$$2(2\mu-3)(p-1)-(2\mu-5)(p-1), = (2\mu-1)(p-1),$$

linearly independent terms in the polynomial $\phi_1\Phi^{(\mu-1)}+\phi_2\Phi'^{(\mu-1)}$; as we have proved that no greater number exists, it follows that $(2\mu-1)(p-1)$ is the number of linearly independent products of μ of the polynomials $\phi_1, \ldots, \phi_p$.

112. Another most important theorem follows from the results just obtained: *Every rational function whose poles are among the zeros of a polynomial $\Psi^{(\mu)}$ can be expressed in a form $\Phi^{(\mu)}/\Psi^{(\mu)}$.* For the most general function having poles in these $2\mu(p-1)$ places contains $2\mu(p-1)-p+1$ arbitrary constants*, and we have shewn that a polynomial $\Phi^{(\mu)}$ contains just this number of terms; thus the quotient $\Phi^{(\mu)}/\Psi^{(\mu)}$, which clearly has its poles in the assigned places, is of sufficiently general character to represent any such function.

For further information on the matter here discussed the reader may consult Noether, *Math. Annal.* t. XVII. p. 263, "Ueber die invariante Darstellung algebraischer Functionen." And † *ibid.* t. XXVI. p. 143, "Ueber die Normalcurven für $p=5, 6, 7$."

In order to explain the need for the theorem just obtained, we may consider the simple case where the fundamental equation is that of a general plane quartic curve, $f(x, y, z)=0$, homogeneous coordinates being used. If we take the four polynomials,

$$\psi_1=x^2,\ \psi_2=y^2,\ \psi_3=xy,\ \psi_4=xz,$$

which are not ϕ-polynomials, from which we obtain

$$x : y : z=\psi_1 : \psi_3 : \psi_4,$$

* When $\mu>1$. The theorem has already been proved for $\mu=1$ (§ 98, Chap. VI.).

† In the present chapter all the polynomials considered in connexion with the fundamental equation have been adjoint; there is also a geometrical theory for polynomials of any grade in extension of the theory here given, in which the associated polynomials are not adjoint. For its connexion with the theory here, the reader may compare Klein, "Abel. Functionen," *Math. Annal.* t. 36, p. 60, Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie*, Paris 1883, t. III., also Lindemann, *Untersuchungen über den Riemann-Roch'schen Satz* (Teubner 1879), pp. 10, 30 etc., Noether, *Math. Annal.* t. 15, p. 507, "Ueber die Schnittpunktssysteme einer algebraischen Curve mit nicht adjungirten Curven."

then the general rational function with poles at the sixteen zeros of a polynomial, $\Psi^{(2)}$, of the second order in ψ_1, ψ_2, ψ_3, ψ_4, contains 14 homogeneously entering arbitrary constants. Now there are only ten terms in the general polynomial $\Phi^{(2)}$, of the second order in $\psi_1, \dots, \psi_4$; and these are equivalent to only nine linearly independent terms, because of the relation $\psi_1\psi_2=\psi_3^2$. Hence the rational function in question cannot be expressed in the form $\Phi^{(2)}/\Psi^{(2)}$.

113. The investigations in regard to the ϕ-polynomials $\phi_1, \dots, \phi_p$, which have been referred to in §§ 110—112, find their proper place in the consideration of the theory of algebraic curves in space of higher than two dimensions.

Let $\phi_1, \dots, \phi_p$ be linearly independent adjoint polynomials of grade $(n-1)\sigma+n-3$, defined, suppose, by the invariant condition that if $c_1, \dots, c_p$ be p independent places on the Riemann surface, ϕ_i vanishes in all of $c_1, \dots, c_p$ except c_i. Let $x_1, \dots, x_p$ be quantities whose ratios are defined by the equations

$$x_1 : x_2 : \dots : x_p = \phi_1 : \phi_2 : \dots : \phi_p.$$

We may suppose * that there is no place of the original surface at which all of $x_1, \dots, x_p$ are zero, and, since only the ratios of these quantities are defined, we may suppose that none of them become infinite.

Hence we may interpret $x_1, \dots, x_p$ as the homogeneous coordinates of a point in space of $p-1$ dimensions; we may call this the point x. Corresponding then to the one-dimensionality constituted by the original Riemann surface, we shall have a curve, in space of $p-1$ dimensions. Its order, measured by the number of zeros of a general linear function $\lambda_1 x_1 + \dots + \lambda_p x_p$, will be $2p-2$. To any place x of this curve there cannot correspond two places c, c' of the original surface, unless

$$\phi_1(c) : \phi_2(c) : \dots : \phi_p(c) = \phi_1(c') : \phi_2(c') : \dots : \phi_p(c').$$

Now, from these equations we can infer that the ϕ-polynomials corresponding to the normal integrals of the first kind, have the same mutual ratios at c as at c'; such a possibility, however, necessitates the existence of a rational function of *the second order*, expressible in the form

$$\lambda\Gamma_c^{x,} - \mu\Gamma_{c'}^{x,},$$

where λ, μ are constants whose ratio is definite, and $\Gamma_c^{x,}$, $\Gamma_{c'}^{x,}$ are normal elementary integrals of the second kind with unassigned zeros. Hence the correspondence between the original Riemann surface and the space curve, C_{2p-2}, is reversible except in the hyperelliptic case.

In the hyperelliptic case the equations of transformation are reducible to a form

$$x_1 : x_2 : \dots : x_p = 1 : z : z^2 : \dots : z^{p-1}.$$

* Chap. II. § 21.

To any point x of the space curve corresponds, therefore, not only the place (s, z) of the Riemann surface, but equally the place $(-s, z)$. The space curve may be regarded as a doubled curve of order $p-1$. (Cf. Klein, *Vorles. üb. d. Theorie der ellip. Modulfunctionen*, Leipzig, 1890, t. I. p. 569.)

For the general case in which $p=3$, the curve, C_{2p-2}, is the ordinary plane quartic curve. For the general case, $p=4$, the curve C_{2p-2} is a sextic curve in space of three dimensions, lying* on $\frac{1}{2}p(p+1)-(3p-3)$, $=1$, surface of the second order and $\frac{1}{6}p(p+1)(p+2)-(5p-5)$, $=5$, linearly independent surfaces of the third order.

Ex. If, for the case $p=4$, we suppose the original surface to be associated with the equation†

$$f(x, y) = x^2y^2(Lx+My)+xy(ax^2+2hxy+by^2)+Px^3+Qx^2y+Rxy^2$$
$$+Sy^3+Ax^2+2Hxy+By^2+Cx+Dy+1=0,$$

and put $Z=xy$, $X=x$, $Y=y$, as the non-homogeneous coordinates of the points of the curve C_{2p-2}, the single quadric surface containing the curve is clearly given by

$$U_2=Z-XY=0,$$

and one cubic surface, containing the curve, is given by

$$U_3=Z^2(LX+MY)+Z(aX^2+2hXY+bY^2)+PX^3+QX^2Y+RXY^2$$
$$+SY^3+AX^2+2HXY+BY^2+CX+DY+1=0.$$

Four other cubic surfaces, $V_1=0$, $V_2=0$, $V_3=0$, $V_4=0$, can be obtained from $U_3=0$ by replacing XY by Z, respectively in, (i) the coefficient of h, (ii) the coefficient of Q, (iii) the coefficient of R, (iv) the coefficient of H; these are linearly independent of $U_3=0$, and of one another. Other cubic surfaces can be obtained from $U_3=0$ by replacing XY by Z in two of its terms simultaneously; for instance, if we replace XY by Z in the coefficients of h and H, we obtain a surface of which the equation is $V_1-U_3+V_4=0$. Similarly all others than $U_3=0$, $V_1=0$, ..., $V_4=0$, are linearly deducible from these.

114. As an example of more general investigations, consider now the correspondence between the space curve C_{2p-2}, for $p=4$, and the original Riemann surface. Let us seek to form a rational function having $p+1=5$ given poles on the sextic curve. A surface of order μ can be drawn through 5 arbitrary points of the curve when μ is great enough; we may denote its equation by $\Psi^{(\mu)}=0$, in accordance with § 110. It was proved that the rational function can be written in the form $\Phi^{(\mu)}/\Psi^{(\mu)}$, $\Phi^{(\mu)}$ being another polynomial, of order μ in the space coordinates, which vanishes in the $6\mu-5$ zeros of $\Psi^{(\mu)}$ other than the 5 given points. Since a general surface of order μ contains $(\mu+3, 3)$‡ terms, the most general form possible for $\Phi^{(\mu)}$, when subject to the conditions enunciated, will contain

$$(\mu+3, 3)-(6\mu-5)$$

arbitrary, homogeneously entering, coefficients; the polynomials which multiply these coefficients, represent, equated to zero, all the linearly inde-

* § 111 preceding.

† Cf. § 108.

‡ Where (μ, ν) is used for the number $\mu(\mu-1)\dots(\mu-\nu+1)/\nu!$.

pendent surfaces of order μ which vanish in the $6\mu-5$ points spoken of; they will therefore include the

$$\frac{4 . 5 \ldots\ldots (p+\mu-1)}{\underline{|\mu}} - (2\mu-1)(p-1), \text{ or } (\mu-3, 3)-(6\mu-3),$$

surfaces of the μth order which* contain the sextic curve. Denote the number of these surfaces by r and their equations by $U_1=0, \ldots, U_r=0$. Then the general form of the equation of a surface, $\Phi^{(\mu)}=0$, vanishing in the $6\mu-5$ given points will be

$$\Phi^{(\mu)} = \lambda_1 U_1 + \ldots\ldots + \lambda_r U_r + \lambda \Psi^{(\mu)} + \mu U = 0,$$

wherein $\lambda_1, \ldots, \lambda_r, \lambda, \mu$ are arbitrary constants, and U is a surface of order μ, other than $\Psi^{(\mu)}$, which vanishes in the $6\mu-5$ points, and does not wholly contain the curve. The intersections of the surface $\Phi^{(\mu)}=0$ with the sextic are the same as those of the surface $\lambda\Psi^{(\mu)}+\mu U=0$; and the general form of the rational function having the $p+1=5$ given points as poles is

$$\lambda + \mu U/\Psi^{(\mu)},$$

involving the right number $(q+1=Q-p+1=5-4+1)$ of arbitrary constants.

Ex. i. There are sixteen of the surfaces $\lambda\Psi^{(\mu)}+\mu U=0$ which *touch* the sextic (in points other than the $6\mu-5$ fixed points).

For there are $2 . 5+2 . 4-2$, $=16$, places at which the differential, dz, of the rational function $z=U/\Psi^{(\mu)}$, is zero to the second order.

Ex. ii. In the example of the previous Article, prove that

$$f'(y) = \frac{\partial U_2}{\partial Y} \cdot \frac{\partial U_3}{\partial Z} - \frac{\partial U_3}{\partial Y} \cdot \frac{\partial U_2}{\partial Z}, = \Delta \text{ say},$$

and that the integrals of the first kind, expressed in terms of X, Y, Z, are given by

$$\int (\lambda_1 X + \lambda_2 Y + \lambda_3 Z + \lambda_4)\, dX/\Delta,$$

for arbitrary values of the constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$†.

115. We abstain from entering on the theory of curves in space in this place. But some general considerations on the same elementary lines as those referred to in §§ 81—83, as applicable to plane curves, may fitly conclude the present chapter‡. The general theorem considered is, that of the intersections of a curve, in space of k dimensions, which is defined as the complete locus satisfying $k-1$ algebraic equations, with a surface

* § 111.

† The canonical curve discussed by Klein, *Math. Annal.* t. 36, p. 24, is an immediate generalisation of the curve C_{2p-2} here explained. But it includes other cases also.

‡ See the note in Salmon, *Higher Plane Curves* (Dublin 1879), p. 22, "on an apparent contradiction in the Theory of Curves" and the references there given, which include a reference to a paper by Euler of date 1748. For further consideration of curves in space see Appendix I. to the present volume.

of sufficiently high order, r, there are a certain number, P, which are determined by prescribing the others, *P being independent of r.*

We take first the case of the curve in three dimensions, defined as the complete intersection of two surfaces of orders m and n, say $U_m = 0$, $U_n = 0$. The curve is here supposed to be of the most general kind possible, having only such singularities as those considered in Salmon, *Solid Geometry* (Dublin, 1882, p. 291). For instance the surfaces $U_m = 0$, $U_n = 0$ are not supposed to touch; for at such a place the curve would have a double point. We prove that if $r > m+n-4$, all but $\frac{1}{2}mn(m+n-4)+1$ of the intersections of the curve $U_m = 0$, $U_n = 0$ with a surface of order r, $U_r = 0$, are determined by prescribing the others, whose number is

$$rmn - \tfrac{1}{2}mn(m+n-4) - 1.$$

For when, firstly, $r > m+n-1$, the intersections of $U_r = 0$ with the curve are the same as those of a surface

$$U_r - U_m V_{r-m} - U_n V_{r-n} - U_m U_n V_{r-m-n} = 0,$$

wherein V_{r-m}, V_{r-n}, V_{r-m-n} are general polynomials whose highest aggregate order in the coordinates is that given by their suffixes. Hence, in analogy with the argument given in § 81, it may at first sight appear that, of the $(r+3, 3)$ coefficients in U_r, we can reduce a certain number, K, given by

$$K = (r-m+3, 3) + (r-n+3, 3) + (r-m-n+3, 3),$$

to zero, by using the arbitrary coefficients in V_{r-m}, V_{r-n}, V_{r-m-n}. This however is not the case. For if W_{r-m-n}, T_{r-m-n} denote general polynomials, of the orders of their suffixes, we can write the modified equation of the surface of order r in the form

$$U_r - U_m(V_{r-m} - U_n W_{r-m-n}) - U_n(V_{r-n} - U_m T_{r-m-n}) \\ - U_m U_n (V_{r-m-n} - W_{r-m-n} - T_{r-m-n}) = 0.$$

Now, whatever be the values assigned to the coefficients in W_{r-m-n}, T_{r-m-n}, the coefficients in $V_{r-m-n} - W_{r-m-n} - T_{r-m-n}$ are just as arbitrary as those of V_{r-m-n}. And we may use the coefficients in W_{r-m-n}, T_{r-m-n} to reduce $(r-m-n+3, 3)$ of the coefficients in each of the polynomials

$$V_{r-m} - U_n W_{r-m-n},\quad V_{r-n} - U_m T_{r-m-n},$$

to zero.

Hence the K equations by which we should reduce the number of effective coefficients in U_r to $(r+3, 3) - K$, are really unaltered when $2(r-m-n+3, 3)$ of the disposeable quantities entering therein, are put equal to zero. Thus we may conclude, that so far as the intersections of U_r with the curve are concerned, its coefficients are effectively

$$(r+3, 3) - (r-m+3, 3) - (r-n+3, 3) + (r-m-n+3, 3)$$

in number. Provided the linear equations reducing the others to zero are independent, what we prove is that the number of effective coefficients is certainly not more than this.

This number can immediately be seen to be equal to

$$rmn - \tfrac{1}{2}mn\,(m + n - 4).$$

Hence, we cannot arbitrarily prescribe more than $rmn - \tfrac{1}{2}mn\,(m + n - 4) - 1$ of the intersections of $U_r = 0$ with the curve.

This result is obtained on the condition that $r > m + n - 1$. If $r = m + n - 1$, $m + n - 2$ or $m + n - 3$, the number of effective coefficients in U_r cannot be more than in the polynomial

$$U_r - U_m V_{r-m} - U_n V_{r-n},$$

namely, than

$$(r + 3,\ 3) - (r - m + 3,\ 3) - (r - n + 3,\ 3).$$

By the previous result this number is equal to

$$rmn - \tfrac{1}{2}mn\,(m + n - 4) - (r - m - n + 3,\ 3),$$

and $(r - m - n + 3,\ 3)$, $= (r - m - n + 1)(r - m - n + 2)(r - m - n - 3)/3\,!$, vanishes when $r = m + n - 1$, $m + n - 2$, or $m + n - 3$. Hence the result obtained holds provided $r > m + n - 4$.

If we denote the number $\tfrac{1}{2}mn\,(m + n - 4) + 1$ by P, the result is, that when $r > m + n - 4$, we cannot prescribe more than $mnr - P$ of the intersections of the curve $U_m = 0$, $U_n = 0$ with a surface of order r; the prescription of this number of independent points determines the remaining intersections.

Corollary. Hence it follows, when $(r + 3,\ 3) - 1 \geqq rmn - P + 1$, that a surface of order r described through $rmn - P + 1$ quite general points of the curve, will entirely contain the curve. Hence, in general, the curve lies upon $(r + 3,\ 3) - rmn + P - 1$ linearly independent surfaces of order r, r being greater than $m + n - 4$.

Ex. i. For the curve of intersection of two quadric surfaces, $P = 1$; every surface of order r drawn through $4r$ quite arbitrary points of the curve entirely contains the curve; the $4r$ intersections of a surface of order r, which does not contain the curve, are determined by $4r - 1$ of them. When $r = 2$, the number $(r + 3,\ 3) - rmn + P - 1$ is equal to 2. This is the number of linearly independent quadric surfaces containing the curve.

Ex. ii. For the curve of intersection of a quadric surface with a cubic surface, $P = 4$; of the $6r$ intersections of the curve with a surface whose order r is > 1, $6r - 4$ determine the others. The number $(r + 3,\ 3) - rmn + P - 1$ is equal to 1 when $r = 2$, and equal to 5 when $r = 3$; thus, as previously found, the curve lies on one quadric surface and on five linearly independent cubic surfaces; the number, for any value of r, is in agreement with the result of § 111.

116. In regard to the intersections, with the curve, of a surface of order $m + n - 4$, such a surface has effectively not more coefficients than are contained in the polynomial

$$U_{m+n-4} - U_m V_{n-4} - U_n V_{m-4},$$

for arbitrary values of the coefficients in V_{n-4} and V_{m-4}. Here we firstly suppose $m > 3$, $n > 3$.

Now we can prove, as before, that

$$(m+n-1, 3)-(n-1, 3)-(m-1, 3)=\tfrac{1}{2}mn(m+n-4)+1, \ =P.$$

Hence, also when $m > 3$ and $n=3$, 2 or 1,

$$(m+n-1, 3)-(m-1, 3), \ =\tfrac{1}{2}mn(m+n-4)+1+(n-1)(n-2)(n-3)/6,$$

is equal to P, and the number of effective coefficients in a polynomial $U_{m+n-4}-U_n V_{m-4}$, wherein the coefficients in V_{m-4} are arbitrary, is as before equal to P. Similarly for other cases.

Hence P is the number of coefficients in a polynomial U_{m+n-4}, which are effective so far as the intersections of the curve with the surface $U_{m+n-4}=0$ are concerned; in other words, $P-1$ of the intersections determine the others. The total number of intersections is $mn(m+n-4)$, $=2P-2$.

The analogy of these polynomials of order $m+n-4$ with the ϕ-polynomials in the case of a plane curve is obvious.

117. If now, the homogeneous coordinates of the points of the curve in space being denoted by X_1, X_2, X_3, X_4, the symbol $[i, j]$ denote the Jacobian $\partial(U_m, U_n)/\partial(X_i, X_j)$, and $(X_1+dX_1, X_2+dX_2, X_3+dX_3, X_4+dX_4)$ denote a point of the curve consecutive to (X_1, X_2, X_3, X_4), it follows from the equations

$$\frac{\partial U_m}{\partial X_1}dX_1+\frac{\partial U_m}{\partial X_2}dX_2+\frac{\partial U_m}{\partial X_3}dX_3+\frac{\partial U_m}{\partial X_4}dX_4=0$$
$$=X_1\frac{\partial U_m}{\partial X_1}+X_2\frac{\partial U_m}{\partial X_2}+X_3\frac{\partial U_m}{\partial X_3}+X_4\frac{\partial U_m}{\partial X_4},$$

and the similar equations holding for U_n, that the ratios

$$X_2dX_3-X_3dX_2 : X_3dX_1-X_1dX_3 : X_1dX_2-X_2dX_1 : X_1dX_4$$
$$-X_4dX_1 : X_2dX_4-X_4dX_2 : X_3dX_4-X_4-dX_3,$$

are the same as the ratios

$$[1, 4] : [2, 4] : [3, 4] : [2, 3] : [3, 1] : [1, 2];$$

each of these rows is in fact constituted by the coordinates of the tangent line of the curve. If then u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 denote any quantities whatever, and, in each of these rows, we multiply the elements respectively by

$$u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1,\ u_1v_4-u_4v_1,\ u_2v_4-u_4v_2,\ u_3v_4-u_5v_3,$$

and add the results, we shall obtain for the first row

$$\Sigma(u_2v_3-u_3v_2)(X_2dX_3-X_3dX_2)=udv-vdu,$$

where

$$u=u_1X_1+u_2X_2+u_3X_3+u_4X_4,\ du=u_1dX_1+u_2dX_2+u_3dX_3+u_4dX_4, \text{ etc.,}$$

and, for the second row we shall obtain the determinant

$$\begin{vmatrix} u_1\ , & u_2\ , & u_3\ , & u_4 \\ v_1\ , & v_2\ , & v_3\ , & v_4 \\ \frac{\partial U_m}{\partial X_1}, & \frac{\partial U_m}{\partial X_2}, & \frac{\partial U_m}{\partial X_3}, & \frac{\partial U_m}{\partial X_4} \\ \frac{\partial U_n}{\partial X_1}, & \frac{\partial U_n}{\partial X_2}, & \frac{\partial U_n}{\partial X_3}, & \frac{\partial U_n}{\partial X_4} \end{vmatrix},$$

which we may denote by (uvU_mU_n).

From the proportionality of the elements of the two rows considered, it follows, therefore, that the ratio $(udv - vdu)/(uvU_mU_n)$ is independent of the values of the quantities $u_1, \ldots, v_4$. This ratio is of degree

$$-(m-1+n-2-2) = -(m+n-4)$$

in the homogeneous coordinates; namely, if X_1, X_2, X_3, X_4 be replaced by $\rho X_1, \rho X_2, \rho X_3, \rho X_4$, the ratio will be multiplied by $\rho^{-(m+n-4)}$. Hence, if U_{m+n-4} be any polynomial of degree $m+n-4$, the product

$$U_{m+n-4}(udv - vdu)/(uvU_mU_n)$$

is a functional differential, independent of the arbitrary factor of the homogeneous coördinates.

The integral,

$$\int U_{m+n-4} \frac{udv - vdu}{(uvU_mU_n)},$$

can only be infinite at the places where the curve is intersected by the surface $(uvU_mU_n) = 0$: if $u = 0$, $v = 0$ be regarded as the equations of planes, this equation expresses that the straight line $u = 0$, $v = 0$, is intersected by the tangent line of the curve at the point (X_1, X_2, X_3, X_4). The differential

$$udv - vdu, \ = \Sigma\,(u_2v_3 - u_3v_2)(X_2dX_3 - X_3dX_2),$$

is zero, to the second order, when the line $u = 0 = v$ is intersected by the tangent line, whose coordinates are $X_2dX_3 - X_3dX_2$, etc. Hence the ratio $(udv - vdu)/(uvU_mU_n)$ is never infinite, and the integral above is finite for all points of the curve.

Hence*, since U_{m+n-4} contains P terms, *we can obtain P everywhere-finite algebraical integrals.*

The same result is obtained if $u_1, \ldots, v_4$ be polynomials in the coordinates, $u_1, \ldots, u_4$ being of the same degree, and $v_1, \ldots, v_4$ of the same degree.

* As stated, we are considering a curve without singular points. If the curve had a double point, the polynomial (uvU_mU_n) would vanish at that point, for all values of $u_1, \ldots, v_4$. We could then prescribe $U_{m+n-4} = 0$ to pass through the double point, thus obtaining a reduction of one in the number of finite integrals. Etc.

Ex. i. For a plane curve of order n, without multiple points, prove similarly that we can obtain p finite algebraical integrals in the form

$$\int \phi_{n-3}\,(udv-vdu)/(uvf),$$

where $f(x_1, x_2, x_3)=0$ is the homogeneous equation of the curve, $u=u_1x_1+u_2x_2+u_3x_3$, etc., and (uvf) denotes a determinant of three rows.

Ex. ii. Shew that a surface of order $m+n-4+\mu$ which vanishes in all but two of the intersections of the curve in space with a surface of order μ, $U_\mu=0$, is of the form

$$\psi=\lambda U+(\lambda_1 V_1+\ldots+\lambda_P V_P)\, U_\mu=0,$$

where $\lambda, \lambda_1, \ldots, \lambda_P$ are arbitrary; and that an integral of the third kind is of the form

$$\int \frac{\psi}{U_\mu}\, \frac{udv-vdu}{(uvU_mU_n)}.$$

118. Retaining still the convention that $u=0$, $v=0$ are the equations of planes, let $u'=0$, $v'=0$ be the equations of other planes whose line of intersection does not coincide with the line $u=0=v$.

From the equations

$$zu-v=0,\ su'-v'=0,\ U_m=0,\ U_n=0,$$

wherein z, s have any values, we can eliminate the coordinates of the points of the curve in space, and obtain a rational equation, $(s, z)=0$, with which we may associate a Riemann surface*. To any point of the curve corresponds a single point, $z=v/u$, $s=v'/u'$, of the Riemann surface; to any point of the Riemann surface will in general correspond conversely only one point of the curve in space. Hence the Riemann surface will have mn sheets, the places, at which z has any value, being those which correspond to the places, on the curve in space, at which the plane $zu-v=0$ intersects this curve. Thus the Riemann surface will have $2mn+2p-2$ branch places, p being the deficiency of the surface. These are the places where dz is zero of the second order. Thus they correspond to the places, on the curve in space, where $udv-vdu$ is zero to the second order. We have seen that these are given as the intersections of this curve with the surface $(uvU_mU_n)=0$, of order $m+n-2$; their number is therefore $mn\,(m+n-2)=2mn+2P-2$. Hence the number P, obtained for the curve in space, is equal to the deficiency p of the Riemann surface with which it is reversibly related. The same result can be proved when u, v are polynomials of any, the same, order, and u', v' are polynomials of any, the same, order.

And from the reversibility of this transformation it follows that the everywhere-finite integrals for the Riemann surface are the same as those here obtained for the curve in space.

* We may of course interpret the equation as that of a plane curve; a particular case is that in which this curve is a central projection of the space curve.

Ex. Prove that if e_1, e_2, e_3 be such that $e_1+e_2+e_3=0$,

$$(b-c)(c-a)(a-b)=(b-c)(a-d)/(e_2-e_3)=(c-a)(b-d)/(e_3-e_1)=(a-b)(c-d)/(e_1-e_2),$$

the points of the curve $aX^2+bY^2+cZ^2+dT^2=0$, $X^2+Y^2+Z^2+T^2=0$ can be expressed in terms of two quantities, x, y, satisfying the equation $y^2=4(x-e_1)(x-e_2)(x-e_3)$, in the form $T : X : Y : Z$

$$=y : \sqrt{b-c}\,[(x-e_1)^2-(e_1-e_2)(e_1-e_3)] : \sqrt{c-a}\,[(x-e_2)^2-(e_2-e_3)(e_2-e_1)] : \sqrt{a-b}\,[(x-e_3)^2-(e_3-e_1)(e_3-e_2)].$$

Find x, y in terms of X, Y, Z, T in the form

$$[e_1(e_2-e_3)X/\sqrt{b-c}+e_2(e_3-e_1)Y/\sqrt{c-a}+e_3(e_1-e_2)Z/\sqrt{a-b}]/x$$
$$=(e_2-e_3)X/\sqrt{b-c}+(e_3-e_1)Y/\sqrt{c-a}+(e_1-e_2)Z/\sqrt{a-b}=2(e_2-e_3)(e_3-e_1)(e_1-e_2)T/y.$$

See Mathews, *London Math. Soc.* t. xix. p. 507.

119. As already remarked we have considered here only the case of a non-singular curve in space which is completely defined as the intersection of two algebraical surfaces. For this case the reader may consult Jacobi, *Crelle*, t. 15 (1836), p. 298; Plücker, Crelle, t. 16, p. 47; Clebsch, *Crelle*, t. 63, p. 229; Clebsch, *Crelle*, t. 64, p. 43; Salmon, *Solid Geometry* (Dublin, 1882), p. 308; White, *Math. Annal.* t. 36, p. 597; Cayley, *Collected Works, passim.* For the more general case, in connexion however with an extension of the theory of this volume to the case of two *independent* variables, the following, inter alia, may be consulted: Noether, *Math. Annal.* t. 8 (1873), p. 510; Clebsch, *Comptes Rendus de l'Acad. des Sciences*, t. 67, July—December, 1868, p. 1238; Noether, *Math. Annal.* t. 2, p. 293, and t. 29, p. 339 (1887); Valentiner, *Acta Math.* t. ii. p. 136 (1883); Halphen, *Journal de l'École Polyt.* t. lii. (1882), p. 1; Noether, *Abh. der Akad. zu Berlin* (1882); Cayley, *Collected Works*, Vol. v. p. 613, etc.; and Picard, *Liouv. Journ. de Math.* 1885, 1886 and 1889.

Ex. i. Prove that

$$(r+k,\,k)-\sum_1(r+k-m_i,\,k)+\sum_2(r+k-m_i-m_j,\,k)-\dots+(-)^{k-1}(r+k-m_1-\dots-m_{k-1},\,k)$$
$$=rm_1m_2\dots m_{k-1}-\tfrac{1}{2}m_1m_2\dots m_{k-1}(m_1+m_2+\dots+m_{k-1}-k-1),$$

where (r,μ) denotes $r(r-1)\dots(r-\mu+1)/\mu!$, $m_1, \dots, m_{k-1}$, k are any positive integers, r is a positive integer greater than $m_1+m_2+\dots+m_{k-1}-k-1$, $\sum_1$ denotes a summation extending to all the values $i=1, 2, \dots, (k-1)$, $\sum_2$ denotes a summation extending to every pair of two unequal numbers chosen from the series $m_1, m_2, \dots, m_{k-1}$, and so on. Hence infer that of the intersections of a general curve in space of k dimensions, which is determined as the complete locus common to $k-1$ algebraic surfaces of orders $m_1, m_2, \dots, m_{k-1}$, with a surface of order r, all but

$$\tfrac{1}{2}m_1m_2\dots m_{k-1}(m_1+m_2+\dots+m_{k-1}-k-1)+1$$

are determined by the others. The result is known to hold for $k=2$. We have here been considering the case $k=3$.

Ex. ii. With the notation and hypotheses employed in Salmon's *Solid Geometry* (1882), Chap. XII. (p. 291) (see also a note by Cayley, *Quarterly Journal*, t. VII., or *Collected Works*, Vol. v. p. 517), where m is the degree of a curve in space, n is its class, namely the number of its osculating planes which pass through an arbitrary point, r is its rank, namely the number of its tangents which intersect an arbitrary line, α is the number of osculating planes containing four consecutive points of the curve, β the number of points through which four consecutive planes pass, x the number of points of intersections of non-consecutive tangents which lie in an arbitrary plane, y the number of planes containing two non-consecutive tangents which pass through an arbitrary point, h the number of chords of the curve which can be drawn through an arbitrary point, g the number of lines of intersection of two non-consecutive osculating planes which lie in an arbitrary plane, ϑ the number of tangent lines of the curve which contain three consecutive points, prove, by using Plücker's equations (Salmon, *Higher Plane Curves*, 1879, p. 65) for the plane curve traced on any plane by the intersections, with this plane, of the tangent lines of the curve in space, that the equations hold,

$$(1)\quad n=r(r-1)-2x-3m-3\vartheta, \qquad (3)\quad r=n(n-1)-2g-3a,$$

$$(2)\quad a=3r(r-2)-6x-8(m+\vartheta), \qquad (4)\quad m+\vartheta=3n(n-2)-6g-8a,$$

$$p_1-1=\tfrac{1}{2}r(r-3)-x-m-\vartheta=\tfrac{1}{2}n(n-3)-g-a \qquad \text{(A)},$$

p_1 being the deficiency of this plane curve.

Prove further, by projecting the curve in space from an arbitrary point, and using Plücker's equations for the plane curve in which the cone of projection is cut by an arbitrary plane, the equations

$$(5)\quad r=m(m-1)-2h-3\beta, \qquad (7)\quad m=r(r-1)-2y-3(\vartheta+n),$$

$$(6)\quad \vartheta+n=3m(m-2)-6h-8\beta, \qquad (8)\quad \beta=3r(r-2)-6y-8(\vartheta+n),$$

$$p_2-1=\tfrac{1}{2}m(m-3)-h-\beta=\tfrac{1}{2}r(r-3)-y-n-\vartheta \qquad \text{(B)},$$

p_2 being the deficiency of this plane curve.

From the equations (1) and (7) we can infer $n-m=3n-3m-2(x-y)$, and therefore

$$y+n=x+m.$$

Hence $p_1=p_2$.

Ex. iii. For the non-singular curve which is the complete intersection of two algebraic surfaces of orders μ, ν, prove (cf. Salmon, *Solid Geometry*, pp. 308, 309) that in the notation of Ex. ii. here,

$$\beta=0,\ m=\mu\nu,\ r=\mu\nu(\mu+\nu-2),\ h=\tfrac{1}{2}\mu\nu(\mu-1)(\nu-1).$$

Hence, by the equations (B) of Ex. ii. prove that, now,

$$p_1=p_2=\tfrac{1}{2}\mu\nu(\mu+\nu-4)+1.$$

This is the number we have denoted by P.

Ex. iv. Denoting the number $p_1=p_2$, in Ex. ii., by p, prove from equations (5) and (B) that

$$6(p-1)=m(m-7)-2h+2r=3(r+\beta-2m).$$

Hence shew that if, through a curve C of order m, lying on a surface S of order μ, we draw a surface of order ν, cutting the surface S again in a curve C' of order m', and if p, p' denote the values of p for these curves C, C' respectively, then

$$m'(\mu+\nu-4)-(2p'-2)=m(\mu+\nu-4)-(2p-2)$$

(see Salmon, pp. 311, 312). Shew that each of these numbers is equal to the number, i, of points in which the curves C, C' intersect, and interpret geometrically the relation

$$i+r+\beta=m(\mu+\nu-2).$$

Ex. v. If in Ex. iv. a surface ϕ of order $\mu+\nu-4$ be drawn through $(\mu+\nu-4)m'-p'+1$, or $i-1+p'$, of the points of the curve C', prove that, so far as its intersections with the curve C are concerned, the surface ϕ contains effectively p terms. Prove further that ϕ contains the curve C' entirely.

Ex. vi. Prove that a surface of order $\mu+\nu-4$ passing through $i-1$ of the intersections of the curves C, C', in Ex. iv., will pass through the other intersection.

Ex. vii. An example of the case in Ex. iv. is that in which $\mu=2$, $\nu=2$, $m=3$, $m'=1$. Then C' is a straight line and $p'=0$: hence p is given by $-2=2p-2$. Hence, for the cubic curve of intersection of two quadrics having a common generator, $p=0$. And in fact coordinate planes can be chosen so that the homogeneous coordinates of the points of the cubic can be expressed in the form

$$X : Y : Z : T=1 : \theta : \theta^2 : \theta^3,$$

θ being a variable parameter. For instance (using Cartesian coordinates) the polar planes of a fixed point $(X'Y'Z')$ in regard to quadrics confocal with $X^2/a+Y^2/b+Z^2/c=1$ are the osculating planes of such a cubic curve, the coordinates of whose points are expressible in the form

$$XX'=(a+\lambda)^3/(a-b)(a-c),\quad YY'=(b+\lambda)^3/(b-c)(b-a),\quad ZZ'=(c+\lambda)^3/(c-a)(c-b),$$

λ being a variable parameter.

Ex. viii. For the quintic curve of intersection of a quadric and a cubic surface having a common generator we obtain, from Ex. iv., putting $m'=1$, $p'=0$, $m=5$, that $p=2$; the results of Exx. iv., v., vi. can be immediately verified for this curve; further, if the surfaces be taken to be $yU-zV=0$, $yS-zT=0$, where U, V are of the first degree in x, y, z and S, T of the second degree, and we put $y=z\xi$, $x=z\eta$, we obtain

$$z(\eta\alpha_1+\alpha_2)=\lambda_1,\quad z^2(\eta^2\beta_1+\eta\beta_2+\beta_3)+z(\eta\gamma_1+\gamma_2)+\delta_1=0,$$

where the Greek letters α_1, α_2,... denote polynomials in ξ of the degrees of their suffixes. Hence, if σ be defined by the equation,

$$\lambda_1\sigma=2\eta(\lambda_1^2\beta_1+\lambda_1\alpha_1\gamma_1+\delta_1\alpha_1^2)+\lambda_1^2\beta_2+\lambda_1(\alpha_1\gamma_2+\alpha_2\gamma_1)+2\delta_1\alpha_1\alpha_2,$$

we obtain $\sigma^2=(\xi, 1)_6$; ξ, σ are rational functions of x, y, z and x, y, z are rational functions of ξ, σ.

Ex. ix. Prove that if the sextic intersection of a cubic surface and a quadric surface, break up into a quartic curve and a curve of the second order, the numbers p, p' for these curves are $p=1$, $p'=0$ or $p=0$, $p'=-1$ according as the curve of the second order is a plane curve or is two non-intersecting straight lines.

Ex. x. In analogy with Ex. iv., shew that the deficiencies of two non-singular plane curves of orders m, m' are connected by the equation

$$m(m+m'-3)-(2p-2)=mm'=m'(m+m'-3)-(2p'-2),$$

and further in analogy with Ex. v. that if a plane curve, of order $m+m'-3$, be drawn through $(m+m'-3)m'-p'+1$ independent points of the curve of order m', only $p-1$ of its intersections with the curve of order m can be prescribed.

Further indications of the connexion of the theory of curves in space with the subject of this chapter will be found in Appendix I.