CHAPTER V.

On certain forms of the Fundamental Equation of the Riemann Surface.

51. We have already noticed that the Riemann surface can be expressed in many different ways, according to the rational functions used as variables. In the present chapter we deal with three cases: the first, the hyperelliptic case (§§ 51—59), is a special case, and is characterised by the existence of a rational function of the second order; the second, which we shall often describe as that of Weierstrass's canonical surface (§§ 60—68), is a general case obtained by choosing, as independent variables, two rational functions whose poles are at one place of the surface: the third case referred to (§§ 69—71) is also a general case, which may be regarded as a generalization of the second case. It will be seen that both the second and third cases involve ideas which are in close connexion with those of the previous chapter. The chapter concludes with an account of a method for obtaining the fundamental integral functions for any fundamental algebraic equation whatever (§§ 73—79).

It may be stated for the guidance of the reader that the results obtained for the second and third cases (§§ 60—71) are not a necessary preliminary to the theory of the remainder of the book; but they will be found to furnish useful examples of the actual application of the theory.

52. We have seen that when p is greater than zero, no rational function of the first order exists. We consider now the consequences of the hypothesis of the existence of a rational function of the second order. Let ξ denote such a function; let c be any constant and α, β denote the two places where $\xi = c$, so that $(\xi - c)^{-1}$ is a rational function of the second order with poles at α, β. The places α, β cannot coincide for all values of c, because the rational function $d\xi/dx$ has only a finite number of zeros. We may therefore regard α, β as distinct places, in general. The most general rational function which has simple poles at α, β cannot contain more than two linearly entering arbitrary constants. For if such a function be $\lambda + \lambda_1 f_1 + \lambda_2 f_2 + \ldots, \lambda, \lambda_1, \ldots$ being arbitrary constants, each of the functions $f_1, f_2, \ldots$ must be of the second order at most and therefore actually of the second order: by choosing the constants so that the sum of the residues at α is zero, we can therefore

obtain a function infinite only at β, which is impossible*. Thus the most general rational function having simple poles at α, β is of the form $A (\xi - c)^{-1} + B$. Therefore, from the Riemann-Roch Theorem (Chapter III., § 37), $Q - q = p - (\tau + 1)$, putting $Q = 2$, $q = 1$, we obtain $p - (\tau + 1) = 1$; namely, the number of linearly independent linear aggregates

$$\Omega (x) = \lambda_1 \Omega_1 (x) + \dots + \lambda_p \Omega_p (x),$$

which vanish in the two places α, β is $p - 1$. Since α may be taken arbitrarily and c determined from it, and $p - 1$ is the number of these linear aggregates which vanish in an arbitrary place, we have therefore the result—*When there exists a function of the second order, every place α of the surface determines another place β: and the determination may be expressed by the statement that every linearly independent linear aggregate $\Omega (x)$ which vanishes in one of these places vanishes necessarily in the other.*

53. Conversely when there are two places α, β in which $p - 1$ linearly independent $\Omega (x)$ aggregates vanish, there exists a rational function having these two places for simple poles. To see this we may employ the formula of § 37, putting $Q = 2$, $\tau + 1 = p - 1$, and obtaining $q = 1$. Or we may repeat the argument upon which that result is founded, thus—Not every one of $\Omega_1 (x), \dots, \Omega_p (x)$ can vanish at α; let $\Omega_1 (\alpha)$ be other than zero. Since $p - 1$ linearly independent $\Omega (x)$ aggregates vanish in α, and, by hypothesis, $p - 1$ linearly independent $\Omega (x)$ aggregates vanish in both α and β, it follows that every $\Omega (x)$ aggregate which vanishes in α vanishes also in β. Hence each of the $p - 1$ aggregates

$$\Omega_2 (\alpha) \Omega_1 (x) - \Omega_1 (\alpha) \Omega_2 (x), \dots \dots, \Omega_p (\alpha) \Omega_1 (x) - \Omega_1 (\alpha) \Omega_p (x),$$

vanishes in β, namely, we have the $p - 1$ equations

$$\Omega_i (\alpha) \Omega_1 (\beta) - \Omega_1 (\alpha) \Omega_i (\beta) = 0, \quad (i = 2, 3, \dots, p).$$

Therefore the function

$$\Omega_1 (\beta) \Gamma_\alpha^x - \Omega_1 (\alpha) \Gamma_\beta^x$$

has each of its periods zero. Thus it is a rational function whose poles are at α and β: and $\Omega_1 (\beta)$ cannot be zero since otherwise the function would be of the first order.

Hence when there are two places at which $p - 1$ linearly independent $\Omega (x)$ aggregates vanish, there is an infinite number of pairs of places having the same character. For any pair of places the relation is reciprocal, namely, if the place α determine the place β, α is the place which is similarly determined by β: in other words, the surface has a reciprocal $(1, 1)$ correspondence with itself. It can be shewn by such reasoning as is employed in

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* By the equation $Q - q = p - (\tau + 1)$, if q were 2, $\tau + 1$ would be p, or all linear aggregates $\Omega(x)$ would vanish in the same places, which is impossible (Chap. II. § 21).

Chap. I. (p. 5), that if (x_1, y_1), (x_2, y_2) be the values of the fundamental variables of the surface at such a pair of places, each of x_1, y_1 is a rational function of x_2 and y_2, and that conversely x_2, y_2 are the same rational functions of x_1 and y_1.

54. We proceed to obtain other consequences of the existence of a rational function, ξ, of the second order. If the poles of ξ do not fall at finite distinct ordinary places of the surface, choose a function of the form $(\xi - c)^{-1}$, in accordance with the explanation given, for which the poles are so situated. Denote this function by z. Then* the function dz/dx has $2.2 + 2p - 2 = 2p + 2$ zeros at each of which z is finite. Denote their positions by $x_1, x_2, ..., x_{2p+2}$. If these are not all finite places we may, if we wish, suppose that, instead of x, such a linear function of x is taken that each of $x_1, ..., x_{2p+2}$ becomes a finite place. They are distinct places. For if the value of z at x_i be c_i, $z - c_i$ is there zero to the second order: that another place x_j should fall at x_i would mean that $z - c_i$ is there zero to higher than the second order, which is impossible because z is only of the second order. By the explanations previously given it follows that a linear aggregate $\Omega(x)$, which vanishes at any one of these places $x_1, ..., x_{2p+2}$, vanishes to the second order there. Hence there is no linear aggregate $\Omega(x)$ vanishing at p or any greater number of these places, for $\Omega(x)$ has only $2p - 2$ zeros. The general rational function which has infinities of the first order at the places $x_1, ..., x_{p+r}$ will therefore† contain a number of $q + 1$ of constants given by $p + r - q = p$, namely, will contain $r + 1$ constants. Such a function will therefore not exist when $r = 0$. In order to prove that a function actually infinite in the prescribed way does exist for all values of r greater than zero, it is sufficient, in accordance with §§ 23—27 (Chap. III.), to shew that there exists no rational function having $x_1, x_2, ..., x_i$ for poles of the first order for any value of i less than $p + 1$. Without stopping to prove this fact, which will appear *a posteriori*, we shall suppose r chosen so that a function of the prescribed character actually exists. For this it is certainly sufficient that r be as great as p‡. Denote the function by h, so that h has the form

$$h = \lambda + \lambda_1 \Sigma_1 + ... + \lambda_r \Sigma_r,$$

$\lambda, \lambda_1, ..., \lambda_r$ being arbitrary constants.

Let h, h' denote the values of h at the two places (x, y), (x', y'), where z has the same value. Then to each value of z corresponds one and only one value of $h + h'$, or $h + h'$ may be regarded as an uniform function of z: the infinities of $h + h'$ are clearly of finite order, so that $h + h'$ is a rational function of z. Consider now the function $(z - c_1)(z - c_2) ... (z - c_{p+r})(h + h')$.

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* Chap. I. § 6.

† Chap. III. § 37.

‡ Chap. III. § 27. For the need of the considerations here introduced compare § 37 of Chap. III.

Since h and h' are only infinite at places of the original surface at which z is equal to one or other of $c_1, \ldots, c_{p+r}$, this function is only infinite for infinite values of z. As it is a rational function of z, it must therefore be a polynomial in z of order not greater than $p + r$. Hence we may write

$$h + h' = (z, \ 1)_{p+r}/(z - c_1) \ldots (z - c_{p+r}).$$

But here the left hand is only infinite to the first order, at most, at any one of $c_1, \ldots, c_{p+r}$—and the denominator of the right hand is zero to the second order at such a place. Hence the numerator of the right hand must be zero at each of these places, and must therefore be divisible by the denominator. Thus $h + h'$ is an absolute constant, $= 2C$ say. From the equations

$$h \ = \lambda + \lambda_1 \Sigma_1 \ + \ldots + \lambda_r \Sigma_r,$$
$$h' = \lambda + \lambda_1 \Sigma'_1 + \ldots + \lambda_r \Sigma'_r,$$

we infer then that $\Sigma_i + \Sigma'_i$ is also a constant, $= 2C_i$ say: for h was chosen to be the most general function of its assigned character and the coefficients $\lambda, \ldots, \lambda_r$ are arbitrary. Thence we obtain

$$C = \lambda + \lambda_1 C_1 + \ldots + \lambda_r C_r.$$

We can therefore put

$$s = h - C = - s' = - (h' - C) = \lambda_1 (\Sigma_1 - C_1) + \ldots + \lambda_r (\Sigma_r - C_r),$$

so that s will be a function of the same general character as h, such however that $s + s' = 0$: in its expression the constants $\lambda_1, \ldots, \lambda_r$ are arbitrary, while the constants $C_1, \ldots, C_r$ depend on the choice made for the functions $\Sigma_1, \ldots, \Sigma_r$.

55. Consider now the two places α, α' at which z is infinite. Choose the ratios $\lambda_1 : \lambda_2 : \ldots : \lambda_r$ so that s is zero to the $(r - 1)$th order at α. This can always be done, and will define s precisely save for a constant multiplier, unless it is the case that when s is made to vanish to the $(r - 1)$th order at α, it vanishes, of itself, to a higher order. In order to provide for this possibility, let us assume that s vanishes to the $(r - 1 + k)$th order at α. Since $s' = - s$, s will also vanish to the $(r - 1 + k)$th order at α'. There will then be other $p + r - 2 (r - 1 + k)$, or $p - r + 2 - k$, zeros of s. From the manner of formation this number is certainly not negative. Consider now the function

$$f = (z - c_1) \ldots (z - c_{p+r}) \, s^2.$$

At the places where z is infinite f is infinite of order $p + r - 2 (r - 1 + k)$, or $p - r + 2 - 2k$ times. At the places, $x_1, \ldots, x_{p+r}$ where s is infinite, it is finite; each of the factors $z - c_1, \ldots, z - c_{p+r}$ is zero to the second order at the place where it vanishes. Since $s^2 = - ss'$, f is a symmetrical function of the values which s takes at the places where z has any prescribed value. Hence, by such reasoning as is previously employed, it follows that the func-

tion f is a rational integral polynomial in z of order $p - r + 2 - 2k$. Denote this polynomial by H. By consideration of the zeros of f it follows that the $2(p - r + 2 - 2k)$ zeros of the polynomial H are the zeros of s^2 which do not fall at α or α'. But since the sum of the values of s at the two places where z has any prescribed value is zero, it follows that s is zero at each of the places $x_{p+r+1}, \ldots, x_{2p+2}$. For each of these is formed by a coalescence of two places where z has the same value, and at each of them s is not infinite. Hence the polynomial H must be divisible by $(z - c_{p+r+1}) \ldots (z - c_{2p+2})$. Thus, as H is a polynomial of order $p - r + 2 - 2k$ in z, $p - r + 2 - 2k$ must be at least equal to $2p + 2 - (p + r)$ or to $p - r + 2$. Hence k is zero, and the value of H is determinate save for a constant multiplier. Supposing this multiplier absorbed in s we may therefore write

$$(z - c_1) \ldots (z - c_{p+r}) s^2 = (z - c_{p+r+1}) \ldots (z - c_{2p+2}) \qquad (A);$$

and s is determined uniquely by the conditions, (1) of being once infinite at $x_1, \ldots, x_{p+r}$, (2) of being $(r - 1)$ times zero at each of the places α, α' where z is infinite. Denote s, now, by s_{p+r}, and denote the function h from which we started, which was defined by the condition of being once infinite at each of $x_1, \ldots, x_{p+r}$, by h_{p+r}, and consider the function $(z - c_{p+r}) s_{p+r}$. *This* function is once infinite at each of $x_1, \ldots, x_{p+r-1}$, it is zero to the first order at x_{p+r}, and it is $r - 1 - 1, = r - 2$ times zero at each of the places α, α' where z is infinite. Hence the function

$$(z - c_{p+r}) s_{p+r} (A + A_1 z + \ldots + A_{r-2} z^{r-2}) + B,$$

wherein $B, A, A_1, \ldots, A_{r-2}$ are arbitrary constants, has the property of being once infinite at each of $x_1, \ldots, x_{p+r-1}$, and not elsewhere. It is then exactly such a function as would be denoted, in the notation suggested, by h_{p+r-1}, and it contains the appropriate number of arbitrary constants—and we can from it obtain a function s_{p+r-1}, having the property of being once infinite at each of $x_1, \ldots, x_{p+r-1}$ and vanishing $(r - 2)$ times at each of the places α, α' where z is infinite.

Ex. 1.   Determine s_{p+r-1} in accordance with this suggestion.

Ex. 2.   Prove that h_{p+r} is of the form $s_{p+r}(A + A_1 z + \ldots + A_{r-1} z^{r-1}) + B$.

Ex. 3.   Prove that h_{p+r+t} is of the form $\dfrac{s_{p+r}(A + A_1 z + \ldots + A_{r+t-1} z^{r+t-1})}{(z - c_{p+r+1}) \ldots (z - c_{p+r+t})} + B$.

Ex. 4.   Shew that the square root $\sqrt{\dfrac{(z - c_{p+r+1}) \ldots (z - c_{2p+2})}{(z - c_1) \ldots (z - c_{p+r})}}$ can be interpreted as an one-valued function on the original surface.

56.   The functions, z, s_{p+r} are defined as rational functions of the x, y of the original surface. Conversely x, y are rational functions of z, s_{p+r}. For* we have found a rational irreducible equation (A) connecting z and

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* See Chap. I. § 4.

s_{p+r} wherein the highest power of s_{p+r} is the same as the order of z. *Hence this equation (A) gives rise to a new surface, of two sheets, with branch places at $z = c_1, \ldots, c_{2p+2}$, whereon the original surface is rationally and reversibly represented.*

It is therefore of interest to obtain the forms of the fundamental integral functions and the forms of the various Riemann integrals for this new surface. It is clear that the function

$$(z - c_1) \ldots (z - c_{p+r}) s_{p+r} (z, 1)_{k-1},$$

where k is a positive integer, and $(z, 1)_{k-1}$ denotes any polynomial of order $k - 1$, is infinite only at the places α, α' where z is infinite, and in fact to order $p + r - (r - 1) + k - 1, = p + k$: and that, therefore, by suitable choice of the coefficients in another polynomial $(z, 1)_{p+k}$, we can find a rational function

$$(z - c_1) \ldots (z - c_{p+r}) s_{p+r} (z, 1)_{k-1} + (z, 1)_{p+k},$$

which is not infinite at α', and is infinite at α to any order, $p + k$, greater than p. Now, of rational functions which are infinite only at α, there are p orders for which the function does not exist*. Hence these must be the orders $1, 2, \ldots, p$.

Hence, of functions infinite only in one sheet at $z = \infty$, on the surface

$$(z - c_1) \ldots (z - c_{p+r}) s^2_{p+r} = (z - c_{p+r+1}) \ldots (z - c_{2p+2}),$$

that of lowest order is a function of the form

$$\eta = (z - c_1) \ldots (z - c_{p+r}) s_{p+r} + (z, 1)_{p+1},$$

which becomes infinite to the $(p + 1)$th order. Hence by Chapter IV. § 39, every rational function which becomes infinite only at the places $z = \infty$, can be expressed in the form

$$(z, 1)_\lambda + (z, 1)_\mu \, \eta,$$

and if the dimension of the function, namely, the number which is the order of its higher infinity at these places, be $\rho + 1$, λ and μ are such that

$$\rho + 1 \gtreqless \lambda, \quad \rho + 1 \gtreqless \mu + p + 1.$$

Therefore also, if $\sigma = (z - c_1) \ldots (z - c_{p+r}) s_{p+r} = \eta - (z, 1)_{p+1}$, in which case equation (A) may be replaced by the equation

$$\sigma^2 = (z - c_1)(z - c_2) \ldots (z - c_{2p+2}),$$

we have the result that all such functions can be also expressed in the form

$$(z, 1)_{\lambda'} + (z, 1)_{\mu'} \, \sigma,$$

with

$$\rho + 1 \gtreqless \lambda', \quad \rho + 1 \gtreqless \mu' + p + 1.$$

* Chap. III. § 28.

By means of this result, hitherto assumed, the forms for the various integrals given Chapter II., § 17, Chapter IV., § 46, are immediately obtainable by the methods of Chapter IV.

57. Or we can obtain the forms of the integrals of the first kind thus— Let v be such an integral. Consider the rational function

$$s_{p+r} (z - c_1) \dots (z - c_{p+r}) \frac{dv}{dz}.$$

It can only be infinite (1) where z is infinite (2) where $dz = 0$, that is at the branch places of the (s_{p+r}, z) surface. It is immediately seen that the latter possibility does not arise. Where z is infinite the function is infinite to the order $p + 1 - 2$, or $p - 1$. Hence it is an integral polynomial in z of order $p - 1$. Namely, the general integral of the first kind* is

$$\int \frac{(z, 1)_{p-1} \, dz}{(z - c_1) \dots (z - c_{p+r}) \, s_{p+r}}.$$

58. *Ex.* 1. A rational function h_{p-k}, infinite only at the places where $z = c_1, \dots, c_{p-k}$, contains $p - k - p + r + 1 + 1 = r + 2 - k$ arbitrary constants, where $r + 1$ is the number of coefficients in a general polynomial $(z, 1)_{p-1}$ which remain arbitrary after the prescription that $(z, 1)_{p-1}$ shall vanish at $c_1, \dots, c_{p-k}$. Prove this: and infer that $h_p, h_{p-1}, \dots$ do not exist.

Ex. 2. It can be shewn as in § 57 that at any ordinary place of the surface

$$\sigma^2 = (z - c_1) \dots (z - c_{2p+2}),$$

rational functions exist, infinite only there, of orders $p + 1, p + 2, \dots$: the gaps indicated by Weierstrass's theorem (Chapter III. § 28) come therefore at the orders $1, 2, \dots, p$. At a branch place, say at $z = c$, the gaps occur for the orders $1, 3, 5, \dots, (2p - 1)$. For, all other possible orders, which a rational function, infinite only there, can have, are expressible in one of the forms $2(p - k)$, $2p + 2r + 1$, $2p + 2r$, where k is a positive integer less than p, or zero, and r is a positive integer: and we can immediately put down rational functions infinite to these orders at the branch place $z = c$ and nowhere else infinite. Prove in fact that the following functions have the respective characters

$$\frac{(z, 1)_{p-k}}{(z - c)^{p-k}}, \quad \frac{(z, 1)_r \, \sigma + (z - c)(z, 1)_{p+r}}{(z - c)^{p+r+1}}, \quad \frac{(z, 1)_{p+r}}{(z - c)^{p+r}},$$

wherein $(z, 1)_{p-k}$, $(z, 1)_r$, $(z, 1)_{p+r}$ are polynomials of the orders indicated by their suffixes with arbitrary coefficients.

Shew further that the most general $\Omega(x)$ aggregate which vanishes $2p - 2k$ times at the branch place contains k arbitrary coefficients: and infer that the expressions given represent the most general functions of the prescribed character (see Chapter III. § 37).

Ex. 3. Prove for the surface

$$Ax^2 + Bxy + Cy^2 + Px^3 + Qx^2y + Rxy^2 + Sy^3 + a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4 = 0$$

that the function

$$z = \mu + \lambda x/y,$$

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* Cf. the forms quoted from Weierstrass. Forsyth, *Theory of Functions*, p. 456.

wherein λ and μ are arbitrary constants, is of the second order. And that there are six values of z for which the pairs of places at which z takes the same value, coincide, these places of coincidence being zeros of the function

$$2(Ax^2 + Bxy + Cy^2) + Px^3 + Qx^2y + Rxy^2 + Sy^3.$$

Prove further that a rational function which is infinite at these six places is given by

$$h = \frac{2(Ax^2 + Bxy + Cy^2) + P'x^3 + Q'x^2y + R'xy^2 + S'y^3}{2(Ax^2 + Bxy + Cy^2) + Px^3 + Qx^2y + Rxy^2 + Sy^3}$$

for arbitrary values of the constants P', Q', R', S'.

This function is, therefore, such a function as has been here called h_{p+r}: and since there are six places at which dz is zero, p is equal to 2 and r equal to 4.

Prove that the sum of the values of h at the two places other than $(0, 0)$ at which z has the same value is constant and equal to 2.

We may then proceed as in the text and obtain the transformed surface in the simple hyperelliptic form. But a simpler process in practice is to form the equation connecting z and h. Writing $k = h - 1$ and $Z = x/y$, prove that

$$k^2\{(PZ^3 + QZ^2 + RZ + S)^2 - 4(AZ^2 + BZ + C)(a_0Z^4 + a_1Z^3 + a_2Z^2 + a_3Z + a_4)\}$$
$$= \{(P' - P)Z^3 + (Q' - Q)Z^2 + (R' - R)Z + (S' - S)\}^2.$$

Hence, if the coefficient of k^2 on the left be written $(Z, 1)_6$, and we write

$$Y = [(P' - P)Z^3 + (Q' - Q)Z^2 + (R' - R)Z + (S' - S)]/k$$
$$= [2(Ax^2 + Bxy + Cy^2) + Px^3 + Qx^2y + Rxy^2 + Sy^3]/y^3,$$

we have

$$Y^2 = (Z, 1)_6,$$

which is the equation of the transformed surface. And, as remarked in the text, the transformation is reversible; verify in fact that x, y are given by

$$x = 2Z(AZ^2 + BZ + C)/[Y - (PZ^3 + QZ^2 + RZ + S)],$$
$$y = 2(AZ^2 + BZ + C)/[Y - (PZ^3 + QZ^2 + RZ + S)].$$

Hence any theorem referred to one form of equation can be immediately transformed so as to refer to the other form.

59. The equation

$$\sigma^2 = (z - c_1)(z - c_2) \ldots (z - c_{2p+2})$$

by which, as we have shewn, any hyperelliptic surface can be represented, contains $2p + 2$ constants, namely $c_1, c_2, \ldots, c_{2p+2}$. If we write $z = (ax + b)/(x + c)$ we introduce three new disposable constants; by suitable choice of these the equation of the surface can be reduced to a form in which there are only $2p - 1$ parametric constants. For instance if we put

$$(z - c_1)(c_3 - c_2)/(z - c_2)(c_3 - c_1) = x/(x - 1)$$

and then, further,

$$s = A\sigma(z - c_3)^{-p-1},$$

where the constant A is given by

$$A = (c_3 - c_1)^p(c_3 - c_2)^p/(c_1 - c_2)^{p+\frac{1}{2}}(c_3 - c_4)^{\frac{1}{2}}(c_3 - c_5)^{\frac{1}{2}} \ldots (c_3 - c_{2p+2})^{\frac{1}{2}},$$

the equation becomes

$$s^2 = x\,(x-1)\,(x-a_4)\,(x-a_5)\,\ldots\,(x-a_{2p+2}),$$

wherein

$$a_r = (c_2 - c_3)\,(c_r - c_1)/(c_1 - c_2)\,(c_3 - c_r),$$

and the right-hand side of the equation is now a polynomial of order $2p + 1$ only. Of its branch places three are now at $x = 0$, $x = 1$, $x = \infty$, and the values of x for the others are the parametric constants upon which the equation depends. It is quite clear that the transformation used gives s, x as rational function of σ, z. Thus

The hyperelliptic surface depends on $2p - 1$ *moduli only. Among the positions of the* $3p - 3$ *branch places upon which a general surface depends* (Chapter I. § 7), *there are, in this case,* $3p - 3 - (2p - 1) = p - 2$ *relations.*

Thus a surface for which $p = 2$ is hyperelliptic in all cases. There are in fact $(p - 1)\,p\,(p + 1) = 6$ places* for which we can construct a rational function of order 2 infinite only at the place.

A surface for which $p = 1$ is also hyperelliptic—but it is more than this (Chapter I. § 8), being susceptible of a reversible transformation into itself in which an *arbitrary* parameter enters.

Ex. 1. On the surface of six sheets associated with the equation

$$y^6 = x\,(x - a)\,(x - b)^4$$

there are four branch places, one at $(0, 0)$ where six sheets wind, and at $(a, 0)$ where six sheets wind, two at $(b, 0)$ at each of which three sheets wind. These count† in all as

$$w = 6 - 1 + 6 - 1 + 2\,(3 - 1) = 14.$$

Hence, by the formula

$$w = 2n + 2p - 2,$$

putting $n = 6$, we obtain $p = 2$.

Thus there exists a rational function ξ of the second order, and the surface can be reversibly transformed into the form $\eta^2 = (\xi, 1)_6$. In fact the function

$$\xi = \frac{x - b}{y}$$

is infinite to the first order at each of the branch places $(b, 0)$, $(a, 0)$ and is not elsewhere infinite.

To obtain the values of ξ at the branch places of the new surface, we may express either x or y in terms of ξ. Since there are two places at which ξ takes any value, each of x and y will be determined from ξ by a quadratic equation—which may reduce to a simple equation in particular cases. When ξ has a value such that the corresponding two places coincide, each of these quadratic equations will have a repeated root.

Now we have

$$\xi^6 = \frac{(x - b)^2}{x\,(x - a)} = \frac{y^2\,\xi^2}{(b + y\xi)\,(b - a + y\xi)}.$$

* Chap. III. § 31. † Forsyth, *Theory of Functions*, p. 349.

Hence

$$y^2 (\xi^6 - 1) - y\xi^5 (a - 2b) - b (a - b) \xi^4 = 0.$$

The condition then is

$$\xi^{10}(a - 2b)^2 + 4b (a - b)\xi^4 (\xi^6 - 1) = 0, \text{ or } \xi^4 [a^2 (\xi^6 - 1) + (a - 2b)^2] = 0.$$

The factor

$$a^2 (\xi^6 - 1) + (a - 2b)^2,$$

is equal to

$$[a^2 \{(x - b)^2 - x (x - a)\} + (a - 2b)^2 x (x - a)]/x (x - a),$$

which is immediately seen to be the same as

$$[x (a - 2b) + ab]/x (x - a)$$

or

$$\{[x (a - 2b) + ab] [x - b]^2/y^3\}^2.$$

Thus this factor gives rise to the six places at which $x = - ab/(a - 2b)$.   And if we put

$$\eta = [x (a - 2b) + ab][x - b]^2/y^3,$$

we obtain

$$\eta^2 = a^2 (\xi^6 - 1) + (a - 2b)^2,$$

which is then the equation associated with the transformed surface.

Then, from the equation

$$\eta\xi^{-3} = [x (a - 2b) + ab]/[x - b],$$

we obtain

$$x = [b\eta + ab\xi^3]/[\eta - \xi^3 (a - 2b)],$$
$$y = [2b (a - b) \xi^2]/[\eta - \xi^3 (a - 2b)],$$

which give the reverse transformation.

Ex. 2.   Prove for the surface

$$y^3 = x (x - a) (x - b)^2 (x - c)^2$$

that $p = 2$ and that the function

$$\xi = (x - b) (x - c)/y$$

is of the second order.   Prove further that

$$[a\xi^3 - b - c]^2 + 4bc (\xi^3 - 1) = \{[a - b - c) x^2 + 2bcx - abc]/x (x - a)\}^2$$

Hence shew that the surface can be transformed to

$$\eta^2 = [a\xi^3 - b - c]^2 + 4bc (\xi^3 - 1)$$

and that

$$x = [a^2\xi^3 + a\eta + 2bc - ab - ac]/[a\xi^3 + \eta + b + c - 2a],$$
$$y = 2\xi^2 [bc + a^2 - ab - ac][a^2\xi^3 + a\eta + 2bc - ab - ac]/[a\xi^3 + \eta + b + c - 2a]^2.$$

Ex. 3.   In the following five cases shew that $p = 2$, that ξ is a function of the second order, that in each case η^2 is either a quintic or a sextic polynomial in ξ, and obtain each of x and y as rational functions of ξ and η ;

(a) $y^{10} = x (x - a)^4 (x - b)^5$, $\xi = (x - a) (x - b)/y^2$, $\eta = \sqrt{a} \cdot (x - a)^2 (x - b)^3$

(β) $y^8 = x (x - a)^3 (x - b)^4$, $\xi = (x - a) (x - b)/y^2$, $\eta = \sqrt{a} \cdot (x - a)^2 (x - b)^3/y^5$

(γ) $y^5 = x (x - a) (x - b)^3$, $\xi = (x - b)/y$, $\eta = [x (a - 2b) + ab] [x - b]^2/y^3$

(δ) $y^6 = x^2 (x - a)^3 (x - b)^3 (x - c)^4$, $\xi = x(x - a)(x - b)(x - c)/y^2$, $\eta = cx (x - a)^2 (x - b)^2 (x - c)/y^3$

(ε) $y^4 = x (x - a)^2 (x - b)^2 (x - c)^3$, $\xi = (x - a) (x - b) (x - c)^2/y^2$, $\eta = c (x - a) (x - b) (x - c)/xy$.

Ex. 4. Shew that the surface

$$y^n = (x - a_1)^{n_1} \ldots (x - a_r)^{n_r}$$

can always be transformed to such form that $n_1, \ldots, n_r$ are positive integers whose sum is divisible by n : and in that form determine the deficiency of the surface. Shew also that, in that form, the only cases in which the deficiency is 2 are those given in Exs. 1, 2, 3. Prove that the cases in which $p = 1$ are*

$$y^6 = x(x - a)^2(x - b)^3, \quad y^3 = x(x - a)(x - b),$$
$$y^4 = x(x - a)(x - b)^2, \quad y^2 = x(x - a)(x - b)(x - c).$$

The results here given have been derived, with alterations, from the dissertation, E. Netto, *De Transformatione Aequationis $y^n = R(x)$* (Berlin, 1870, G. Schade).

The equation

$$y^n = (x - a_1)^{n_1} \ldots (x - a_r)^{n_r}$$

is considered by Abel, *Œuvres Complètes* (Christiania, 1881), vol. I., pp. 188, etc.

It is to be noticed that in virtue of Chapter IV. we are now in a position, immediately to put down the fundamental integrals for the surfaces considered in Examples 1, 2, 3.

60. Passing from the hyperelliptic case we resume now the consideration of the circumstances considered in Chapter III. §§ 28, 31—36.

Consider any place, c, of a Riemann surface : and consider rational functions which are infinite only at this place : all such functions will be denoted by symbols of the form g_N, the suffix N denoting the order of infinity of the function at the place.

Let g_a be the function of the lowest existing order. The suffixes of all other existing functions g_N can be written in the form $N = \mu a + i$, where $i < a$. Since there are only p orders for which functions of the prescribed character do not exist, all the values $i = 0, 1, \ldots, (a - 1)$ will arise. Let $\mu_i a + i$ be the suffix of the function of lowest order whose order is congruent to i for modulus a. We obtain thus a functions

$$g_a, \quad g_{\mu_1 a + 1}, \quad g_{\mu_2 a + 2}, \quad \ldots, \quad g_{\mu_{a-1} a + a - 1}.$$

Then, if g_{ma+i} be any other function that occurs, m cannot be less than μ_i, and a constant λ can be chosen so that $g_{ma+i} - \lambda g_a^{m - \mu_i} g_{\mu_i a + i}$, which is clearly a rational function infinite only at c, is not infinite to the order $\mu_i a + i$. Thus we have an equation of the form

$$g_{ma+i} = \lambda g_a^{m - \mu_i} g_{\mu_i a + i} + g_{\mu a + j},$$

wherein $\mu a + j$ is less than $ma + i$. Proceeding then similarly with $g_{\mu a + j}$, we clearly reach an equation of the form

$$g_{ma+i} = A + B g_{\mu_1 a + 1} + C g_{\mu_2 a + 2} + \ldots + K g_{\mu_{a-1} a + a - 1} \tag{i}$$

wherein the coefficients $A, B, \ldots, K$, whose number is a, are rational integral polynomials in g_a.

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* Cf. Forsyth, p. 486. Briot and Bouquet, *Théorie des Fonct. Ellipt.* (Paris, 1875), p. 390.

In particular, if g_r be any rational function whatever of the g_N functions, we have equations

$$g_r \quad = A_1 \quad + B_1 g_{\mu_1 a+1} \quad + \ldots\ldots + K_1 g_{\mu_{a-1} a+a-1}$$

$$g_r{}^2 \quad = A_2 \quad + B_2 g_{\mu_1 a+1} \quad + \ldots\ldots + K_2 g_{\mu_{a-1} a+a-1} \qquad \text{(ii)}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g_r^{a-1} = A_{a-1} + B_{a-1} g_{\mu_1 a+1} + \ldots\ldots + K_{a-1} g_{\mu_{a-1} a+a-1}.$$

61. If these equations, regarded as equations for obtaining $g_{\mu_1 a+1}, \ldots, g_{\mu_{a-1} a+a-1}$ in terms of g_a and g_r, be linearly independent, we can obtain, by solving, such results as

$$g_{\mu_i a+i} = Q_{i,1}(g_r - A_1) + Q_{i,2}(g_r{}^2 - A_2) + \ldots + Q_{i,a-1}(g_r^{a-1} - A_{a-1}),$$

wherein $Q_{i,1}, \ldots, Q_{i,a-1}$ are rational functions of g_a, which are not necessarily of integral form.

If however the equations be not linearly independent, there exist equations of the form

$$P_1(g_r - A_1) + P_2(g_r{}^2 - A_2) + \ldots + P_{a-1}(g_r^{a-1} - A_{a-1}) = 0$$

or say

$$P_{a-1} g_r^{a-1} + P_{a-2} g_r^{a-2} + \ldots + P_1 g_r + P = 0 \qquad \text{(iii)},$$

wherein $P_1, P_2, \ldots, P_{a-1}, P$ are integral rational polynomials in g_a. Denote the orders of these in g_a by $\lambda_1, \lambda_2, \ldots, \lambda_{a-1}, \lambda$ respectively; here P denotes the expression

$$P_1 A_1 + P_2 A_2 + \ldots + P_{a-1} A_{a-1}.$$

Then $P_k g^k$ is of order $a\lambda_k + rk$ at the place c of the surface. In order that such an equation as (iii) may exist, the terms of highest infinity at the place c must destroy one another: hence there must be such an equation as

$$a\lambda_k + rk = a\lambda_{k'} + rk',$$

and therefore

$$r/a = (\lambda_{k'} - \lambda_k)/(k - k').$$

Now k and k' are both less than a: this equation requires therefore that r and a have a common divisor.

62. Take now r *prime to* a; then it follows that the equations (ii) must be linearly independent. And in that case each of $g_{\mu_1 a+1}, \ldots, g_{\mu_{a-1} a+a-1}$ can be expressed rationally in terms of g_a and g_r, the expression being integral in g_r but not necessarily so in g_a.

Also by equation (i) it follows that every function infinite only at c is rationally expressible by g_a and g_r: and in particular that there is an equation of the form

$$L g_r^a + L_1 g_r^{a-1} + \ldots + L_{a-1} g_r + L_a = 0 \qquad \text{(iv)},$$

wherein L, L_1, ..., L_a are integral rational polynomials in g_a, of which however, since g_r is only infinite when g_a is infinite, L is an absolute constant. It follows from the reasoning given that the equation (iv) is irreducible, and therefore belongs to a new Riemann surface, wherein g_a and g_r are independent and dependent variables. Further, any rational function whatever on the original surface can be modified into a rational function which is infinite only at the place c, by multiplication by an integral polynomial in g_a of the form $(g_a - E_1)^{r_1} (g_a - E_2)^{r_2}.......$ Hence any rational function on the surface is expressible rationally by g_a and g_r. Hence the surface represented by (iv) is a surface upon which the original surface can be rationally and reversibly represented.

Since g_a^{-1} is zero to order a at the place where g_a is infinite, it is clear that the new surface is one for which there is a branch place at infinity at which all the sheets wind.

To every value of g_r there belong r places of the old surface, at which g_r takes this value, and therefore also, in general*, r values of g_a. Hence the highest power of g_a in equation (iv) is the rth, and this term does actually enter. While, because g_a only becomes infinite when g_r is infinite, the coefficient of the term g_a^r is a constant (and not an integral polynomial in g_r).

The equation (iv) is the generalization of that which is used in introducing what are called Weierstrass's elliptic functions, namely of the equation

$$g_3{}^2 - (4g_2{}^3 - a_2 g_2 - a_3) = 0.$$

This equation is satisfied by writing $g_2 = \wp(u)$, $g_3 = \wp'(u)$: it is a known fact that the poles of $\wp(u)$ are at one place (where $u = 0$). This is not true of the Jacobian function sn u.

63.   It follows from equation (i) that the functions

$$(1, g_{\mu_1 a+1}, ..., g_{\mu_{a-1} a+a-1})$$

form a fundamental set for the expression of rational functions infinite only at the place c of the surface, that is, a fundamental set for the expression of the integral rational functions of the surface (iv). And, defining the dimension D of such an integral function F as the lowest positive integer such that $g_a^{-D} F$ is finite at infinity on the surface (iv), in accordance with Chap. IV., § 39, it is clear that in the expression of an integral function by this fundamental system there arise no terms of higher dimension than the function to be expressed: this fundamental set is therefore entirely such an one as that used in Chapter IV. If k be the order of infinity of an integral function F, at the *single* infinite place of the surface (iv), it is obvious that the dimension of F is the least integer equal to or greater than $\dfrac{k}{a}$.

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*   * That is, for an infinite number of values of g_r.

64. We shall generally call the equation (iv) Weierstrass's canonical form; a certain interest attaches to the tabulation of the possible forms which the equation can have for different values of the deficiency p. It will be sufficient here to obtain these forms for some of the lowest values of p; it will be seen that the method is an interesting application of Weierstrass's gap theorem.

Take the case $p=4$, and consider rational functions which are only infinite at a single place c of a surface which is of deficiency 4. Such functions do not exist of all orders—there are four orders for which such functions do not exist; these four orders may be 1, 2, 3, 4, and this is the commonest case*, or they may fall otherwise. We desire to specify all the possibilities: their number is limited by the considerations—

(i) If functions of orders k_1, k_2, ... exist, say F_1, F_2, ..., then there exists a function of order $n_1 k_1 + n_2 k_2 + ...$, where n_1, n_2, ... are any positive integers. In fact $F_1^{n_1} F_2^{n_2}...$ is such a function.

(ii) The number of non-existent functions must be 4.

(iii) The highest order of non-existent function cannot be† greater than $2p - 1$ or 7.

It follows that a function of order 1 does not exist, and if a function of order 2 exists then a function of order 3 does not exist; for every positive integer can be written as a sum of integral multiples of 2 and 3.

Consider then first the case when a function of order 2 exists. Write down all positive integers up to $2p$ or 8. Draw‡ a bar at the top of the numbers 2, 4, 6, 8 to indicate that all functions of these orders exist—

$$1 \ \bar{2} \ 3 \ \bar{4} \ 5 \ \bar{6} \ 7 \ \bar{8} \qquad\qquad (a).$$

If then the functions of orders 5 or 7 existed there would need to be a gap beyond 8, which is contrary to the consideration (iii) above. Hence the non-existent orders are 1, 3, 5, 7. We have thus a verification of the results obtained earlier in this chapter (§ 58, Ex. 2).

Consider next the possibility that a function of order 3 exists, there being no function of order 2. If then a function of order 4 exists, the symbol will be

$$1 \ 2 \ \bar{3} \ \bar{4} \ 5 \ \bar{6} \ \bar{7} \ \bar{8},$$

a function of order 6 being formed by the square of the function of order 3, that of order 7 by the product of the functions of orders 3 and 4, and the function of order 8 by the square of the function of order 4. Thus there would need to be a gap beyond 8. Hence when a function of order 3 exists

* Chap. III. 31.

† Chap. III. § 34. Also Chap. III. § 27.

‡ Cf. Chap. III. § 26.

there cannot be one of order 4. If however functions of orders 3 and 5 exist the symbol would be

$$1\ 2\ \overline{3}\ 4\ \overline{5}\ \overline{6}\ 7\ \overline{8} \qquad (\beta),$$

the function of order 8 being formed by the product of the functions of orders 3 and 5. So far then as our conditions are concerned this symbol represents a possibility. Another is represented by the symbol

$$1\ 2\ \overline{3}\ 4\ 5\ \overline{6\ 7\ 8} \qquad (\gamma).$$

In this case however the existent integral function of order 8 is not expressible as an integral polynomial in the existent functions of orders 3 and 7.

When a function of order 3 exists there are no other possibilities; otherwise more than 4 gaps would arise.

Consider next the possibility that the lowest order of existent function is 4. Then possibilities are expressed by

$$1\ 2\ 3\ \overline{4\ 5\ 6}\ 7\ \overline{8} \qquad (\delta),$$

$$1\ 2\ 3\ \overline{4\ 5}\ 6\ \overline{7\ 8} \qquad (\epsilon),$$

$$1\ 2\ 3\ \overline{4}\ 5\ \overline{6\ 7\ 8} \qquad (\zeta),$$

as is to be seen just as before.

Finally, there is the ordinary case when no function of order less than 5 exists, given by

$$1\ 2\ 3\ 4\ \overline{5\ 6\ 7\ 8} \qquad (\eta).$$

For these various cases let a denote the lowest order of existent function and r the lowest next existent order prime to a. Then the results can be summarised in the table

$p=4$	a	r	Gaps at orders	Fundamental system of orders	Dimensions of functions of fundamental system	Sum of these dimensions	$p+a-1$	$\frac{1}{2}(a-1)(r-1)-p$
a	2	9	1, 3, 5, 7	0, 9	0, 5	5	5	0
β	3	5	1, 2, 4, 7	0, 5, 10	0, 2, 4	6	6	0
γ	3	7	1, 2, 4, 5	0, 7, 8	0, 3, 3	6	6	2
δ	4	5	1, 2, 3, 7	0, 5, 6, 11	0, 2, 2, 3	7	7	2
ϵ	4	5	1, 2, 3, 6	0, 5, 7, 10	0, 2, 2, 3	7	7	2
ζ	4	7	1, 2, 3, 5	0, 6, 7, 9	0, 2, 2, 3	7	7	5
η	5	6	1, 2, 3, 4	0, 6, 7, 8, 9	0, 2, 2, 2	8	8	6

That the seventh and eighth columns of this table should agree is in accordance with Chapter IV., § 41. The significance of the last column is explained in § 68 of this Chapter.

Similar tables can easily be constructed in the same way for the cases $p = 1$, 2, 3.

Ex. 1. Prove that for $p = 3$ the results are given by

$p=3$	a	r	Gaps at orders	Fundamental system of orders	Dimensions of functions of fundamental system	Sum of these dimensions	$p+a-1$
a	2	7	1, 3, 5	0, 7	0, 4	4	4
β	3	4	1, 2, 5	0, 4, 8	0, 2, 3	5	5
γ	3	5	1, 2, 4	0, 5, 7	0, 2, 3	5	5
δ	4	5	1, 2, 3	0, 5, 6, 7	0, 2, 2, 2	6	6

Ex. 2. Prove that for $p = 5$, 6, 7, 8, the possible cases in which the lowest existing function is of the third order are those denoted by the symbols

$$p=5 \begin{cases} 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; \overline{8 \; 9 \; 10} \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6 \; 7} \; 8 \; 9 \; \overline{10} \end{cases}$$

$$p=6 \begin{cases} 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; 9 \; \overline{10} \; \overline{11 \; 12} \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; \overline{8 \; 9} \; 10 \; \overline{11 \; 12} \end{cases}$$

$$p=7 \begin{cases} 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; \overline{9} \; 10 \; \overline{11 \; 12 \; 13 \; 14} \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; \overline{9} \; \overline{10} \; 11 \; \overline{12 \; 13 \; 14} \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; \overline{8} \; \overline{9} \; 10 \; \overline{11} \; \overline{12} \; 13 \; \overline{14} \end{cases}$$

$$p=8 \begin{cases} 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; \overline{9} \; 10 \; 11 \; \overline{12 \; 13 \; 14} \; \overline{15 \; 16} \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; \overline{9} \; 10 \; \overline{11 \; 12} \; 13 \; \overline{14 \; 15} \; 16 \\ 1 \; 2 \; \overline{3} \; 4 \; 5 \; \overline{6} \; 7 \; 8 \; \overline{9 \; 10} \; 11 \; \overline{12 \; 13} \; 14 \; \overline{15 \; 16} \end{cases}$$

65. We have already stated (Chap. IV. § 38) that when the fundamental set of integral functions are so far given that we know the relations expressing their products in terms of themselves, the form of an equation to represent the surface can be deduced. We give now two examples of how this may be done: these examples will be sufficient to explain the general method.

Take first the case $p = 4$, $a = 3$, $r = 7$. Denote the corresponding functions by g_3, g_7. In accordance with § 60 preceding, all integral functions can be expressed by means of g_3 and two functions g_7, g_8 whose orders are respectively $\equiv 1$ and 2 for modulus 3: in particular there are equations of the form

$$g_7{}^2 = g_8 (g_3, 1)_2 + g_7 (g_3, 1)_2 + (g_3, 1)_4$$
$$g_7 g_8 = g_8 (g_3, 1)_2 + g_7 (g_3, 1)_2 + (g_3, 1)_5$$
$$g_8{}^2 = g_8 (g_3, 1)_2 + g_7 (g_3, 1)_3 + (g_3, 1)_5$$

wherein $(g_3, 1)_2$ denotes an integral polynomial in g_3 of order 2 at most, the upper limit for the suffix being determined by the condition that no terms shall occur on the right of higher dimension than those on the left. Similarly for the other polynomials occurring here on the right.

Instead of g_7, g_8 we may clearly use any functions $g_7 - (g_3, 1)_2$, $g_8 - (g_3, 1)_2$. Choosing these polynomials to be those occurring on the right in the value of $g_7 g_8$, we may write our equations

$$g_7{}^2 = \alpha_2 g_8 + \beta_2 g_7 + \alpha_4, \quad g_8{}^2 = \gamma_2 g_8 + \alpha_3 g_7 + \alpha_5, \quad g_7 g_8 = \beta_5 \qquad \text{(A)},$$

where the Greek letters denote polynomials in g_3 of the orders given by their suffixes.

Multiplying the first and last equations by g_8 and g_7 respectively, and subtracting, we obtain

$$g_7 \beta_5 = g_8 \left(\alpha_2 g_8 + \beta_2 g_7 + \alpha_4 \right)$$
$$= \alpha_2 \left(\gamma_2 g_8 + \alpha_3 g_7 + \alpha_5 \right) + \beta_2 \beta_5 + g_8 \alpha_4,$$

and thence, since[*] 1, g_7, g_8 cannot be connected by an integral equation of such form,

$$\alpha_2 \gamma_2 + \alpha_4 = 0, \quad \alpha_2 \alpha_3 - \beta_5 = 0, \quad \alpha_2 \alpha_5 + \beta_2 \beta_5 = 0,$$

from which, as α_2 is not identically zero,—for then g_7 would satisfy a quadratic equation with rational functions of g_3 as coefficients—we infer

$$\alpha_5 + \beta_2 \alpha_3 = 0 \qquad \qquad \text{(B)}.$$

Similarly from the last two equations (A) we have

$$g_8 \beta_5 = g_7 \left(\gamma_2 g_8 + \alpha_3 g_7 + \alpha_5 \right)$$
$$= \gamma_2 \beta_5 + \alpha_3 \left(\alpha_2 g_8 + \beta_2 g_7 + \alpha_4 \right) + \alpha_5 g_7,$$

and thence

$$\beta_5 - \alpha_2 \alpha_3 = 0, \quad \alpha_3 \beta_2 + \alpha_5 = 0, \quad \gamma_2 \beta_5 + \alpha_3 \alpha_4 = 0,$$

so that, since α_3 cannot be zero—as follows from the second of equations (A)— we have

$$\gamma_2 \alpha_2 + \alpha_4 = 0 \qquad \qquad \text{(C)}.$$

The equations (B) and (C) have been formed by the condition that the equations (A) should lead to the same values for $g_7{}^2 g_8$ and $g_8{}^2 g_7$, however these latter products be formed from equations (A). We desire to shew that, conversely, these equations (B) and (C) are sufficient to ensure that any integral polynomial in g_7 and g_8 should have an unique value however it be formed from the equations (A). Now any product of powers of g_7 and g_8 is of one of the three forms g_7, $g_8{}^\mu$, $g_7 g_8 K$. In the first two cases it can be formed from equations (A) in one way only. In the third case let us suppose it proved that K has an unique value however it be derived from the equations (A);

* Chap. IV. § 43.

then to prove that $g_7 g_8 K$ has an unique value we require only to prove that $g_7 \cdot g_8 K = g_8 \cdot g_7 K$. Let K be written in the form $g_8 L + g_7 M + N$. Then the condition is that $g_7 (Lg_8{}^2 + Mg_7 g_8 + Ng_8)$ shall be equal to $g_8 (Lg_7 g_8 + Mg_7{}^2 + Ng_7)$. This requires only $g_7 \cdot g_8{}^2 = g_8 \cdot g_7 g_8$ and $g_7 \cdot g_7 g_8 = g_8 \cdot g_7{}^2$: and it is by these conditions that we have derived equations (B) and (C). Hence also $g_7 g_8 K$ has an unique value.

Thus every rational integral polynomial in g_7 and g_8 will, when the conditions (B), (C) are satisfied, have an unique value however it be formed from equations (A).

The equations (B) and (C) are equivalent to $\alpha_4 = -\alpha_2 \gamma_2$, $\beta_5 = \alpha_2 \alpha_3$, $\alpha_5 = -\alpha_3 \beta_2$, and lead to

$$g_7{}^2 = \alpha_2 g_8 + \beta_2 g_7 - \alpha_2 \gamma_2, \quad g_8{}^2 = \gamma_2 g_8 + \alpha_3 g_7 - \alpha_3 \beta_2, \quad g_7 g_8 = \alpha_2 \alpha_3.$$

Thence

$$g_7{}^2 = \alpha_2 \frac{\alpha_2 \alpha_3}{g_7} + \beta_2 g_7 - \alpha_2 \gamma_2,$$

or

$$g_7{}^3 - \beta_2 g_7{}^2 + \alpha_2 \gamma_2 g_7 - \alpha_2{}^2 \alpha_3 = 0,$$

which is the form of equation (iv) which belongs to the possibility under consideration.

The expression of the fundamental set of integral functions $1, g_7, g_8$ in terms of g_3 and g_7 is therefore

$$1, \ g_7, \ \frac{g_7{}^2 - \beta_2 g_7 + \alpha_2 \gamma_2}{\alpha_2}.$$

66. Take as another example the possibility ϵ, § 64 above, where $a = 4$, $r = 5$, the orders of non-existent functions being 1, 2, 3, 6. For a fundamental system of integral functions we may take $1, g_5, g_5{}^2, g_7$.

We have then such an equation as

$$g_5 g_7 = g_7 (g_4, \ 1)_1 + c g_5{}^2 + g_5 (g_4, \ 1)_1 + (g_4, \ 1)_3$$

where c is a constant: let this be written in the form

$$g_5 g_7 = \alpha_1 g_7 + g_5{}^2 + \beta_1 g_5 + \alpha_3,$$

the constant c being supposed absorbed in $g_5{}^2$.

Write h_5 for $g_5 - \alpha_1$ and h_7 for $g_7 - h_5 - \beta_1 - 2\alpha_1$.

Then

$$h_5 h_7 = \alpha_1{}^2 + \alpha_1 \beta_1 + \alpha_3.$$

Replacing now h_5, h_7 by the notation g_5, g_7 and $\alpha_3 + \alpha_1 \beta_1 + \alpha_1{}^2$ by α_3 we may write

$$g_5 g_7 = \alpha_3, \quad g_7{}^2 = \beta_3 + \alpha_2 g_5 + \alpha_1 g_5{}^2 + \beta_1 g_7, \quad g_5{}^3 = \gamma_3 + \beta_2 g_5 + \gamma_1 g_5{}^2 + \gamma_2 g_7.$$

B.

Hence the condition $g_5 . g_7{}^2 = g_5 g_7 . g_7$ requires

$$\alpha_3 g_7 = \beta_3 g_5 + \alpha_2 g_5{}^2 + \alpha_1 [\gamma_3 + \beta_2 g_5 + \gamma_1 g_5{}^2 + \gamma_2 g_7] + \beta_1 \alpha_3,$$

from which

$$\alpha_3 = \alpha_1 \gamma_2, \quad \beta_3 + \alpha_1 \beta_2 = 0, \quad \alpha_2 + \alpha_1 \gamma_1 = 0, \quad \alpha_1 \gamma_3 = -\beta_1 \alpha_3,$$

and thence

$$\alpha_1 \gamma_3 = -\beta_1 \alpha_1 \gamma_2, \text{ or if } \alpha_1 \text{ is not zero, } \gamma_3 = -\beta_1 \gamma_2.$$

Substituting this value for γ_3 and the value $g_7 = \alpha_3/g_5 = \alpha_1 \gamma_2/g_5$ in the expression for $g_5{}^3$ we obtain

$$g_5{}^3 = -\beta_1 \gamma_2 + \beta_2 g_5 + \gamma_1 g_5{}^2 + \alpha_1 \gamma_2{}^2/g_5$$

or

$$g_5{}^4 - \gamma_1 g_5{}^3 - \beta_2 g_5{}^2 + \beta_1 \gamma_2 g_5 - \alpha_1 \gamma_2{}^2 = 0,$$

which is then a form of the equation (iv) corresponding to the possibility (ϵ).

In this case the fundamental integral functions may be taken to be

$$1, \; g_5, \; g_5{}^2, \; (g_5{}^3 - \gamma_1 g_5{}^2 - \beta_2 g_5 + \beta_1 \gamma_2)/\gamma_2.$$

It is true in general, as in these examples, that the terms of highest order of infinity in the equation (iv) are the terms $g_a{}^r$, $g_r{}^a$. For there must be two terms (at least) of the highest order of infinity which occurs; and since r is prime to a, two such terms as $g_a{}^\lambda g_r{}^\mu$, $g_a{}^{\lambda'} g_r{}^{\mu'}$ cannot be of the same order of infinity.

Ex. 1. Prove that for $p=3$ the form of the equation of the surface in the case where $a=3$, $r=4$ is

$$g_4{}^3 + g_4{}^2 (g_3, 1)_1 + g_4 (g_3, 1)_3 + (g_3, 1)_4 = 0,$$

and shew that this is reducible to the form

$$y^3 + yx(x+a) + x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0,$$

x being of the form $Ag_3 + B$, y of the form $Cg_4 + Dg_3 + E$, A, B, C, D, E being constants.

Thus the surface depends on $3p - 4$ or 5 constants, at most.

Ex. 2. The reader who is acquainted with the theory of plane curves may prove that the homogeneous equation of a quartic curve which has a point of osculation, can be put into the form

$$\omega^3 \xi + \omega \xi \eta (\xi, \eta)_1 + (\xi, \eta)_4 = 0.$$

By putting $x = \eta/\xi$, $y = \omega/\xi$, this takes the form of the final equation of Example 1. Compare Chapter III. § 32.

Ex. 3. Prove that for $p=3$, the form of the equation of the surface in the case where $a=3$, $r=5$ is

$$g_5{}^3 + g_5{}^2 (g_3, 1)_1 + g_5 g_3 (g_3, 1)_2 + g_3{}^2 (g_3, 1)_3 = 0.$$

Ex. 4. Denoting the left hand of equation (iv) by $f(g_r, g_a)$, $\partial f/\partial g_r$ by $f'(g_r)$ and the operator

$$f'(g_r) \frac{d}{dg_a}$$

by D, prove that if g_m be any rational function which is infinite only where g_a and g_r are infinite, there exists an equation

$$X_0 D^{a-1} g_m + X_1 D^{a-2} g_m + \ldots\ldots + X_{a-1} g_m = 0,$$

where $X_0, \ldots\ldots, X_{a-1}$ are polynomials in g_a.

67. We have already in Chapter IV. referred to the fact that an integral function is not necessarily expressible integrally in terms of the coordinates x, y by which the equation of the surface is expressed, even though y be an integral function. The consideration of the Weierstrass canonical surface suggests interesting examples of integral functions which are not expressible integrally.

In order that an integral function g whose order is μ should be expressible as an integral polynomial in the coordinates g_a, g_r of the surface, in the form

$$g = g_a^m g_r^n + \ldots\ldots$$

it is necessary that there should be a term on the right hand whose order of infinity is the same as that of the function ; we must therefore have an equation of the form

$$\mu = ma + nr$$

wherein m, n are positive integers. Since a polynomial in g_a and g_r can be reduced by the equation of the surface until the highest power of g_r which enters is less than a, we may suppose n less than a.

This equation is impossible for any value of μ of the form $nr - ka$. And since herein k may be taken equal to any positive integer less than nr/a, the number of integers of this form, with any value of n, is $E(nr/a)$, or the greatest integer contained in the fraction nr/a. Hence on the whole there are

$$\sum_{n=1}^{a-1} E(nr/a)$$

orders of integral functions which are not expressible integrally by g_a and g_r.

Corresponding to any order which is not expressible in the form $nr - ka$, which is therefore of the form $nr + ma$, we can assign an integrally expressible integral function * namely $g_r^n g_a^m$: hence the p orders corresponding to which, according to Weierstrass's gap theorem, no integral functions whatever exist, must be among the excepted orders whose number we have proved to be

$$\sum_{n=1}^{a-1} E(nr/a) \quad \text{or} \dagger \quad \tfrac{1}{2}(a-1)(r-1).$$

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* Though it does not follow that every integral function whose order is of the form $nr + ma$ can be expressed wholly in integral form.

† If a right-angled triangle be constructed whose sides containing the right angle are respectively a and r, and the interior of the triangle be ruled by lines parallel to the sides

Hence the number of orders of actually existing integral functions which are not expressible integrally is

$$\tfrac{1}{2}(a-1)(r-1)-p.$$

In the table which we have given for $p=4$ (§ 64) the existing integral functions which are not expressible integrally are, for the case (γ), of orders 8 and 11 ; for the case (δ) of orders 6 and 11 ; for the case (ϵ) of orders 7 and 11 ; for the case (ζ) of orders 6, 9, 10, 13, 17 ; for case (η) of orders 7, 8, 9, 13, 14, 19. The reader can easily assign the numbers for the cases in which $p=3$.

Ex. 1. Prove that for the surface

$$g_5{}^3+g_5{}^2(g_3-c)+g_5g_3(g_3,\,1)_2+g_3{}^2(g_3,\,1)_3=0,$$

the function

$$g_7=g_5(g_5-c)/g_3$$

is an integral function which is not expressible as an integral polynomial in g_3 and g_5.

Ex. 2. Prove that for the surface

$$g_7{}^3+g_7{}^2\beta_2+g_7a_2\gamma_2+a_2{}^2a_3=0,$$

where

$$a_2=c\,(g_3-k_1)\,(g_3-k_2),$$
$$\beta_2=(g_3-k_1)f_1+b_1,$$

f_1 being of the first order in g_3, and c, b_1, k_1, k_2 being constants, the two following functions are integral functions not integrally expressible—

$$g_8=g_7(g_7+\beta_2)/a_2,\quad g_{11}=g_7(g_7+b_1)/(g_3-k_1).$$

68. The number $\tfrac{1}{2}(a-1)(r-1)-p$ is susceptible of another interpretation which is in close connexion with the last. Let the set of fundamental integral functions for the Weierstrass canonical surface be denoted by 1, G_1, G_2, ..., G_{a-1}. From the equations whereby 1, g_r, g_r^2, ..., g_r^{a-1} are expressed in terms of them we are able (Chapter IV., § 43) to deduce an equation

$$\Delta(1,\,g_r,\,...,\,g_r^{a-1})=\nabla^2.\,\Delta\,(1,\,G_1,\,G_2,\,...,\,G_{a-1}),$$

wherein $\Delta\,(1,\,g_r,\,...,\,g_r^{a-1})$ is formed as a determinant whose (i,j)th element is the sum of the values of g_r^{i+j-2} at the a places of the surface where g_a has the same value, and is therefore an integral polynomial in g_a, $\Delta(1,\,G_1,\,...,\,G_{a-1})$ is formed as a determinant whose (i,j)th element is the sum of the values of $G_{i-1}G_{j-1}$ for the same value of g_a, which also is an integral polynomial in

containing the right angle, and at unit distances from these sides and each other, so describing squares interior to the triangle, the number of angular points interior to the triangle is easily seen to be $\overset{a-1}{\underset{n=1}{\Sigma}} E\,(nr/a)$. On the other hand if the right-angled triangle be regarded as the half of a rectangle whose diagonal is the hypotenuse of the right-angled triangle, and the ruled lines be continued into the other half, it is easily seen that the total number of angular points of the squares interior to the whole rectangle is $(a-1)\,(r-1)$.

g_a, and ∇ is a determinant whose elements are those integral polynomials in g_a which arise in the expressions of $1, g_r, \ldots, g_r^{a-1}$ in terms of $1, G_1, \ldots, G_{a-1}$.

The determinant $\Delta (1, g_r, \ldots, g_r^{a-1})$ is the square of the product of all the differences of the values of g_r which correspond to any value of g_a. It therefore vanishes, for finite values of g_a, when and only when two of these are equal. If the form of the equation of the surface be denoted by $f(g_r, g_a) = 0$, this happens when, and only when, $\partial f / \partial g_r = 0$. Now $\partial f / \partial g_r$ is an integral polynomial in g_a and g_r, of order $a - 1$ in the latter. Regarded as a rational function on the surface it is only infinite when g_a and g_r are infinite. It follows from the fact (§ 66), that g_r^a is a term of the highest order of infinity which enters in the polynomial $f(g_r, g_a)$, that $\partial f / \partial g_r$ is infinite, at $g_a = \infty$, to an order $r(a-1)$. This is therefore the number of finite places on the surface at which $\partial f / \partial g_r$ vanishes. Hence we infer that the polynomial $\Delta (1, g_r, \ldots, g_r^{a-1})$ is of degree $r(a-1)$ in g_a.

Since there is a branch place at infinity counting for $(a-1)$ branch places, the polynomial $\Delta (1, G_1, \ldots, G_{a-1})$ is of order $2a + 2p - 2 - (a-1)$ $= a - 1 + 2p$ in g_a (§§ 48, 61).

Thus ∇ is of order

$$\tfrac{1}{2} \left[r(a-1) - (a-1+2p) \right],$$

that is, of order

$$\tfrac{1}{2} (r-1)(a-1) - p,$$

in g_a.

This interpretation of the degree of ∇ is of interest when taken in connexion with the theorem—Every integral function can be written in the form

$$(g_a, g_r)/(g_a, 1),$$

the numerator being an integral polynomial in g_a and g_r, and the denominator being an integral polynomial in g_a. *All the polynomials $(g_a, 1)$ thus occurring are divisors of the polynomial ∇.* See § 48 and § 88 Exs. ii, iii*.

When the factors of ∇ are all simple we may therefore expect to be able to associate each of them, as denominator, with an integral function which is not integrally expressible. In this connexion some indications are given in a paper, *Camb. Phil. Trans.* xv. pp. 430, 436. For Weierstrass's canonical surface see also a dissertation, De aequatione algebraica...in quandam formam canonicam transformata. G. Valentin. Berlin, 1879. (A. Haack.) Also Schottky, *Crelle*, 83. Conforme Abbildung...ebener Flächen.

69. The method which has been exemplified in §§ 65, 66 for the formation of the general form of the equation of a surface when the fundamental set of integral functions is given, is not limited to Weierstrass's canonical surface.

Take for instance any surface of three sheets, and let $1, g_1, g_2$ be any set

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* Cf. Harkness and Morley, *Theory of Functions*, p. 268, § 186.

of fundamental integral functions with the properties assigned in Chapter IV. § 42. Then there exist equations of the form

$$g_1 g_2 = \gamma + \beta\, g_1 + \alpha\, g_2$$
$$g_1{}^2 = \gamma_1 + \beta_1 g_1 + \alpha_1 g_2$$
$$g_2{}^2 = \gamma_2 + \alpha_2 g_1 + \beta_2 g_2$$

wherein the Greek letters denote polynomials in the independent variable of the surface, x, whose degrees are limited by the condition that no terms occur on the right of higher dimensions than those on the left.

Thus the dimension of β is not greater than that of g_2 and the dimension of α is not greater than that of g_1. Hence we may use $g_1 - \alpha$, $g_2 - \beta$ instead of g_1 and g_2 respectively, and so take the first equation in the form $g_1 g_2 = \gamma$, the form of the other equations being unaltered. As before, there are conditions that these equations should lead to unique values for every integral polynomial in g_1 and g_2, namely

$$g_2\,(\gamma_1 + \beta_1 g_1 + \alpha_1 g_2) = g_1 \gamma, \quad g_1\,(\gamma_2 + \alpha_2 g_1 + \beta_2 g_2) = g_2 \gamma.$$

These lead to the equations

$$\gamma = \alpha_1 \alpha_2, \quad \gamma_1 = -\alpha_1 \beta_2, \quad \gamma_2 = -\alpha_2 \beta_1,$$

and thence to

$$g_1{}^3 - \beta_1 g_1{}^2 + \alpha_1 \beta_2 g_1 - \alpha_1{}^2 \alpha_2 = 0$$
$$g_2{}^3 - \beta_2 g_2{}^2 + \alpha_2 \beta_1 g_2 - \alpha_2{}^2 \alpha_1 = 0. \tag{v}$$

Since every rational function can be represented rationally by x and g_1 and $g_2 = \alpha_1 \alpha_2 / g_1$, it follows that every rational function can be represented rationally by x and g_1. Hence the surface represented by the first of these two final equations is one upon which the original surface is rationally and reversibly represented. So also is the surface represented by the second of these equations.

The fundamental integral functions are derived immediately from the equation, being

$$1, \ g_1, \ (g_1{}^2 - \beta_1 g_1 + \alpha_1 \beta_2)/\alpha_1.$$

Ex. 1. Prove that the integrals of the first kind for the surface

$$f(g_1,\ x) = g_1{}^3 - \beta_1 g_1{}^2 + \alpha_1 \beta_2 g_1 - \alpha_1{}^2 \alpha_2 = 0$$

are given by

$$\int \frac{dx}{f'(g_1)}\, [(x,\ 1)^{\tau_1 - 1} g_1 + (x,\ 1)^{\tau_2 - 1} \alpha_1],$$

where $\tau_1 + 1$, $\tau_2 + 1$ are the dimensions of g_1 and g_2 and $f'(g_1) = \partial f / \partial g_1$.

Ex. 2. Prove that for the case quoted in Ex. i, § 40, Chapter IV, the form of the equation is, (i) when p is odd $= 2n - 1$, say,

$$g_n{}^3 - a_n g_n{}^2 + a_{n-1} a_{n+1} g_n - a^2{}_{n-1} a_{n+2} = 0,$$

where $a_{n-1}, a_n, a_{n+1}, a_{n+2}$ are polynomials in x of the orders indicated by their suffixes, (ii) when p is even $= 2n - 2$, say,

$$g_n{}^3 - a_n g_n{}^2 + \beta_n \delta_n g_n - \beta_n{}^2 \gamma_n = 0,$$

where $a_n, \beta_n, \gamma_n, \delta_n$ are polynomials in x of the nth order.

Ex. 3. Writing $g_1 = a_1 y$, the first of the equations (v) becomes

$$a_1 y^3 - \beta_1 y^2 + \beta_2 y - a_2 = 0. \tag{A}$$

If the dimensions of g_1 and g_2 be $\tau_1 + 1$, $\tau_2 + 1$, find the degrees of the polynomials a_1, β_1, a_2, β_2. And prove that if the positive quadrant of a plane of rectangular co-ordinates (x, y) be divided into squares whose sides are each 1 unit in length, and a convex polygon be constructed whose angular points are determined from this equation (A), by the rule that a term $x^r y^s$ in the equation determines the point (r, s) of the plane, then the number of angular points of the squares which lie within this polygon is p.

70. In obtaining the equation

$$g_1{}^3 - \beta_1 g_1{}^2 + a_1 \beta_2 g_1 - a_1{}^2 a_2 = 0 \tag{E}$$

we have spoken as if the original surface were of three sheets. *It is important to notice that this is not necessary.*

Suppose our given surface to be any surface for which a rational function of the third order, ξ, exists. Take c so that the poles of the function $(\xi - c)^{-1}$, which is also a function of the third order, are distinct ordinary places of the surface. So determined denote the function by x. Let a_1, a_2, a_3 denote these poles. Then just as in § 39 of Chapter IV. it can be shewn that there exist two rational functions g_1 and g_2, only infinite in a_1 and a_2, such that every rational function which is infinite only in a_1, a_2, a_3 can be expressed in the form

$$\gamma + a g_1 + \beta g_2,$$

wherein γ, a, β are integral polynomials in x whose degrees have certain upper limits determined by the condition of dimensions.

And as before we can obtain the equation (E). Further, if F be any rational function whatever and A_1, A_2, ... be the values of x at the places other than a_1, a_2, a_3 at which F becomes infinite, it is clearly possible to find a polynomial K of the form $(x - A_1)^{n_1} (x - A_2)^{n_2} \ldots$ such that KF only becomes infinite at a_1, a_2, a_3. Hence every rational function of the original surface can be expressed rationally by x and g_1.

Thus as x, g_1 are rational functions on the original surface, (E) represents a new surface upon which our canonical surface is rationally and reversibly represented. *And it is as much the proper normal form for surfaces upon which a rational function of the third order exists as is the equation*

$\sigma^2 = (z, 1)_{2p+2}$, *previously derived, for the hyperelliptic surfaces upon which a function of the second order exists.*

Ex. Obtain the hyperelliptic equation in this way.

71. In the same way we can obtain a canonical form for surfaces upon which a function of the fourth order exists. We can shew that there exist three functions g_1, g_2, g_3 satisfying such equations as

$$g_3{}^2 = a_1 g_1 + b_1 g_2 + c_1 g_3 + k_1$$
$$g_2 g_3 = a_2 g_1 \qquad\qquad + k_2$$
$$g_1 g_3 = a_3 g_1 + b_3 g_2 \qquad + k_3,$$

wherein the nine coefficients are integral polynomials in a rational function x, which is of the fourth order; and that the surface is rationally and reversibly representable upon a surface given by the equation

$$g_3{}^4 - g_3{}^3 (a_3 + c_1) - g_3{}^2 (a_2 b_3 + k_1 - a_3 c_1) + g_3 (a_1 k_3 - b_1 k_2 + a_2 b_3 c_1 + a_3 k_1)$$
$$+ a_1 b_3 k_2 + a_3 b_1 k_2 + a_2 b_3 k_1 = 0.$$

Ex. These coefficients $a_1, \ldots, k_3$ satisfy certain relations; prove that the conditions that $g_2 \cdot g_3{}^2 = g_2 g_3 \cdot g_3$, $g_1 \cdot g_3{}^2 = g_1 g_3 \cdot g_3$, $g_1 g_3 \cdot g_2 = g_2 g_3 \cdot g_1$ are that the following nine polynomials should be divisible by a polynomial Δ, whose value is $a_1{}^2 b_3 - a_3 a_1 b_1 - a_2 b_1{}^2$;

$$a_2 n_1 (a_1 b_3 - a_3 b_1) - b_1 (a_2 h_1 - a_1 k_2), \quad a_1 b_3 h_1 - b_1 (a_2 b_3 n_1 - a_1 k_3), \quad a_1 b_3 k_2 - a_2 b_1 k_3$$

$$- a_2{}^2 n_1 b_1 + a_1 a_2 h_1 + a_1{}^2 k_2, \quad -h_1 (a_1 a_3 + a_2 b_1) + a_1 (a_2 b_3 n_1 - a_1 k_3), \quad -k_2 (a_1 a_3 + a_2 b_1) + a_1 a_2 k_3$$

$$(h_1 + a_3 n_1)(a_1 b_3 - a_3 b_1) - b_1 (a_2 b_3 n_1 - b_1 k_3), \quad n_1 b_3 (a_1 b_3 - a_3 b_1) - b_3 (h_1 b_3 + b_1 k_3),$$

$$k_3 (a_1 b_3 - a_3 b_1) - b_1 b_3 k_2.$$

Herein $n_1 = a_3 - c_1$, $h_1 = a_2 b_3 - k_1$.

In fact if

$$g_1 g_2 = a_5 g_1 + b_5 g_2 + c_5 g_3 + k_5, \quad g_2{}^2 = a_4 g_1 + b_4 g_2 + c_4 g_3 + k_4, \quad g_1{}^2 = a_6 g_1 + b_6 g_2 + c_6 g_3 + k_6,$$

the results of the division of these nine polynomials by Δ are respectively

$$a_5, \ b_5, \ c_5, \ a_4, \ b_4, \ c_4, \ a_6, \ b_6, \ c_6,$$

while

$$k_4 = a_2 c_5 - c_1 c_4, \quad k_5 = n_1 c_5 + b_3 c_4, \quad k_6 = n_1 c_6 + b_3 c_5.$$

72. When the order of the independent function, denoted in §§ 69—71 by x, is known, and the dimensions of the fundamental integral functions in regard thereto, the general forms of the polynomial coefficients in the equations, whereby the products of pairs of these integral functions are expressed as linear functions of themselves, can be written down. And thence, if the necessary algebra (such as that indicated in the example of § 71), which serves to limit the forms of these polynomial coefficients, can be carried out, a canonical form of the equation of the surface can be deduced.

But the converse process may arise: when we are given a form of the fundamental equation associated with the surface, we may require to replace the given equation by one in which the dependent variable is one of the set of fundamental integral functions. More generally we may replace it by an equation in which the dependent variable is an integral function of the form

$$\eta = (x, 1)_\lambda + (x, 1)_{\lambda_1} g_1 + \ldots + (x, 1)_{\lambda_{n-1}} g_{n-1}.$$

This replacement possesses a high degree of interest (§ 88. Ex. iii). In either case it is necessary to be able to calculate the fundamental integral functions.

73. We give now sufficient explanation to enable the reader to calculate the expression of the fundamental integral functions for any given form of the fundamental equation associated with the Riemann surface. This equation may* be taken in the form

$$y^n + y^{n-1} a_1 + \dots + y a_{n-1} + a_n = 0, \qquad (A)$$

$a_1, \dots, a_n$ being integral polynomials in x; thus y is an integral function of x (§ 38).

The n values of any rational function, η, which arise for the same value of x, will be denoted by $\eta^{(1)}, \dots, \eta^{(n)}$ and called conjugate values; their sum will be denoted by $\Sigma \eta$. If any of the possible rational expressions of η be $\phi(x, y)/\psi(x, y)$, ϕ and ψ being integral polynomials in x and y, and if in the expression of $\eta^{(i)}$,

$$\eta^{(i)} = \phi(x, y^{(i)})/\psi(x, y^{(i)}),$$

we multiply numerator and denominator by the product of the $n-1$ values conjugate to $\psi(x, y^{(i)})$, the denominator will become an integral symmetric function of $y^{(1)}, \dots, y^{(n)}$, and can therefore be expressed by means of the equation (A), as an integral polynomial in x; and the numerator will take a form which can be expressed as an integral polynomial in x and $y^{(i)}$. Hence the value of any rational function, on the surface associated with the equation (A), can be expressed in the form

$$\eta = \frac{A + A_1 y + \dots + A_{n-1} y^{n-1}}{D}, \qquad (B)$$

$A, \dots, A_{n-1}, D$ denoting integral polynomials in x, with no common divisor.

Thus, to determine the expression of the fundamental integral functions, we may enquire what modification this general form undergoes when η is an integral function.

74. In the first place the denominator D must be such that D^2 is a factor of the integral polynomial† $\Delta(1, y, \dots, y^{n-1})$; so that D is capable only of a limited number of forms. For let $x - a$ be a factor of D, repeated r times, and write

$$A_i = (x-a)^r B_i + C_i, \qquad (i = 0, 1, \dots, (n-1))$$

wherein C_i is a polynomial of order less than r; since $A, \dots, A_{n-1}$ have no common divisor which divides D, not all of $C, C_1, \dots, C_{n-1}$ can be divisible by $x - a$. Then the function

$$\eta D/(x-a)^r - (B + B_1 y + \dots + B_{n-1} y^{n-1}), = (C + C_1 y + \dots + C_{n-1} y^{n-1})/(x-a)^r,$$

is an integral function, when η is an integral function, as appears from its first form of expression. Denote it by ζ.

Suppose C_i not divisible by $x - a$. From the equation†

$$\Delta(1, y, \dots, y^{i-1}, \zeta, y^{i+1}, \dots, y^{n-1}) = \nabla^2_i \Delta(1, g_1, \dots, g_{n-1}),$$

recalling the form of the determinant which is the square root of the left hand side, we infer

$$\frac{C_i^2}{(x-a)^{2r}} \Delta(1, y, \dots, y^{i-1}, y^i, y^{i+1}, \dots, y^{n-1}) = \nabla_i^2 \Delta(1, g_1, \dots, g_{n-1}).$$

Hence, save for sign,

$$\nabla/\nabla_i = (x-a)^r/C_i,$$

so that $(x-a)^r$ divides ∇.

Thus the first step in the determination of the integral functions is to put $\Delta(1, y, \dots, y^{n-1})$ into the form $u_1^{k_1} \dots u_r^{k_r}$, wherein $u_1, \dots, u_r$ are polynomials having only simple

* Chap. IV. § 38.   † Chap. IV. § 43.

factors. This can always be done by the rational process of finding the highest divisor common to $\Delta(1, y, \ldots, y^{n-1})$ and its differential coefficients in regard to x. It will include most cases of practical application if we further suppose all the linear factors of $\Delta(1, y, \ldots, y^{n-1})$ to be known*.

75. Suppose then that $x - a$ is a factor which occurs to at least the second order in $\Delta(1, y, \ldots, y^{n-1})$. Denote $x - a$ by u. By the solution of a system of linear equations, we can (below, § 78) find all the existing linearly independent expressions of the form

$$(a + a_1 y + \ldots + a_{n-1} y^{n-1})/u,$$

wherein $a, a_1, \ldots, a_{n-1}$ are constants, which represent integral functions. If the highest power of y actually entering be the same in two of these integral functions, say in ζ and ζ', we can use instead of ζ' a function of the form $\zeta' - \mu\zeta$, where μ is a certain constant. By continued application of this method of reduction we obtain, suppose, k integral functions, of the form

$$\zeta_r = (a' + a'_1 y + \ldots + a'_r y^r)/u, \qquad (C)$$

wherein, since these functions are linearly independent, k is less than n, and the values of r that occur are all different. These values of r that occur are among the sequence $1, 2, \ldots, (n-1)$; let s denote in turn all the $n - 1 - k$ other integers in this sequence. Put ζ_s for y^s. Consider now the set of integral functions

$$1, \zeta_1, \ldots, \zeta_{n-1}.$$

As before we can determine by the solution of a system of linear equations all the linearly independent functions of the form

$$(\beta + \beta_1 \zeta_1 + \ldots + \beta_{n-1} \zeta_{n-1})/u,$$

wherein $\beta, \beta_1, \ldots, \beta_{n-1}$ are constants, which are integral functions ; and, as before, we can choose them so that the ζ's of highest suffix which occur shall not be the same in any two of these integral functions. Then in place of $1, \zeta_1, \ldots, \zeta_{n-1}$ we obtain a set $1, \xi_1, \ldots, \xi_{n-1}$, wherein ξ_r is ζ_r unless there be an integral function of the form

$$(\beta' + \beta'_1 \zeta_1 + \ldots + \beta'_r \zeta_r)/u, \qquad (D)$$

wherein the ζ of highest suffix occurring is ζ_r, in which case ξ_r denotes this function.

Then we enquire whether there are any integral functions of the form

$$(\gamma + \gamma_1 \xi_1 + \ldots + \gamma_{n-1} \xi_{n-1})/u,$$

$\gamma, \ldots, \gamma_{n-1}$ being constants. If there are, the process is to be continued†. If there are none, let v denote any other linear factor occurring in $\Delta(1, y, \ldots, y^{n-1})$ to at least the second order. Then, as for the set $1, y, \ldots, y^{n-1}$, we investigate what linearly independent integral functions exist of the form

$$(a + a_1 \xi_1 + \ldots + a_{n-1} \xi_{n-1})/v,$$

and continue the process for v as for u : and afterwards for all other repeated factors of $\Delta(1, y, \ldots, y^{n-1})$.

76. When these processes are completed, we shall obtain a set of integral functions

$$1, \eta_1, \ldots, \eta_{n-1},$$

such that there exists no integral function of the form

$$(a + a_1 \eta_1 + \ldots + a_{n-1} \eta_{n-1})/(x - c),$$

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* In the work below, if u be a polynomial of order r, it is necessary to suppose $a, a_1, \ldots, a_k$ to be polynomials of order $r - 1$.

† The number of steps is finite, by § 74.

wherein $a, \ldots, a_{n-1}$ are constants, for *any* value of c. It is obvious now from the successive definitions (C), (D), ... of the sets $(1, \zeta_1, \ldots, \zeta_{n-1})$, $(1, \xi_1, \ldots, \xi_{n-1})$, ..., $(1, \eta_1, \ldots, \eta_{n-1})$, that every power of y can be represented in the form

$$y^i = v + v_1 \eta_1 + \ldots + v_{n-1} \eta_{n-1},$$

wherein $v, v_1, \ldots, v_{n-1}$ are integral polynomials in x. Hence every integral function can be written in the form

$$\eta = (E + E_1 \eta_1 + \ldots + E_{n-1} \eta_{n-1})/F,$$

wherein $E, \ldots, E_{n-1}, F$ are integral polynomials in x without common divisor. If now $x - c$ be a factor of F and we write

$$E_i = (x - c) G_i + a_i, \quad i = 0, 1, 2, \ldots, (n-1),$$

a_i being a constant, the function

$$\eta F/(x - c) - [G + G_1 \eta_1 + \ldots + G_{n-1} \eta_{n-1}] = (a + a_1 \eta_1 + \ldots + a_{n-1} \eta_{n-1})/(x - c)$$

is an integral function, as appears from the form of the left-hand side. By the property of the set $1, \eta_1, \ldots, \eta_{n-1}$ there is no integral function having the form of the right-hand side, unless each of $a, a_1, \ldots, a_{n-1}$ be zero.

Hence each of $E, \ldots, E_{n-1}$ are divisible by $x - c$. By successive steps of this kind it can be shewn that every integral function can be written in the form

$$\eta = H + H_1 \eta_1 + \ldots + H_{n-1} \eta_{n-1}, \tag{E}$$

wherein $H, H_1, \ldots, H_{n-1}$ are *integral* polynomials in x.

77. But in order that the set $1, \eta_1, \ldots, \eta_{n-1}$ should be such a fundamental set as $1, g_1, \ldots, g_{n-1}$, used in Chap. IV., there must be no terms occurring on the right-hand side here, which are of higher dimension than η. We prove now that this requires a further reduction in the forms of $1, \eta_1, \ldots, \eta_{n-1}$, which is of a kind precisely analogous to the reductions already described.

Let $\sigma + 1$ be the dimension of η, ρ_i the order, and therefore also the dimension of the polynomial H_i (§ 76) and $\sigma_i + 1$ the dimension of η_i; we suppose $\sigma_1 \not> \sigma_2 \not> \ldots \not> \sigma_{n-1}$; then

$$\eta/x^{\sigma+1} = \ldots + (H_i x^{-\rho_i})(\eta_i/x^{\sigma_i+1}) x^{\rho_i + \sigma_i - \sigma} + \ldots.$$

Putting $x = 1/\xi$, $h = \eta/x^{\sigma+1}$, $h_i = \eta_i/x^{\sigma_i+1}$, $H_i x^{-\rho_i} = (1, \xi)_{\rho_i}$, an integral polynomial in ξ, this equation is

$$h = \ldots + (1, \xi)_{\rho_i} h_i/\xi^{\rho_i + \sigma_i - \sigma} + \ldots.$$

If now in equation (E) a term arises of higher dimension than η, one of the integers

$$\rho - (\sigma + 1), \ldots, \rho_i + \sigma_i - \sigma, \ldots$$

is greater than zero. In that case let $r + 1$ be the greatest of these integers. Then we can write

$$\xi^r h = (\ldots + (1, \xi)_{m_i} h_i + \ldots)/\xi,$$

wherein the symbols $(1, \xi)_{m_i}$ denote integral polynomials in ξ. Putting

$$(1, \xi)_{m_i} = \xi K_i + a_i, \quad (i = 0, 1, 2, \ldots, n-1),$$

wherein a_i is a constant, we have

$$\xi^r h - (K + K_1 h_1 + \ldots + K_{n-1} h_{n-1}) = (a + a_1 h_1 + \ldots + a_{n-1} h_{n-1})/\xi.$$

Herein the left hand is a function which is not infinite when x is infinite. Hence,

when the set $1, \eta_1, ..., \eta_{n-1}$ are such that the condition of dimensions* is not satisfied, there exist functions of the form

$$(a + a_1 h_1 + ... + a_{n-1} h_{n-1})/\xi,$$

i.e. of the form

$$x\, [a + a_1\, \eta_1/x^{\sigma_1+1} + ... + a_{n-1}\, \eta_{n-1}/x^{\sigma_{n-1}+1}],$$

wherein $a, ..., a_{n-1}$ are constants which are not infinite when ξ is zero or x is infinite.

In virtue of their definition the functions $h_1, ..., h_{n-1}$ are not infinite when x is infinite, and are therefore infinite only when x is zero or ξ infinite. We may therefore regard them as integral functions of ξ. And since there exists no integral function of the form η_i/x, the dimensions of $h_1, ..., h_{n-1}$ as functions of ξ are $\sigma_1 + 1, ..., \sigma_{n-1} + 1$.

As before determine a set of linearly independent functions of the form

$$(a + a_1 h_1 + ... + a_{n-1} h_{n-1})/\xi,$$

$a, ..., a_{n-1}$ being constants, which are not infinite when $\xi = 0$, choosing them so that the h of highest suffix which occurs is not the same in any two of the functions. Let the function wherein the h of highest suffix is h_r be denoted by k_r, so that k_r is of the form

$$k_r = (\mu + \mu_1 h_1 + ... + \mu_r h_r)/\xi.$$

Then

$$k_r x^{\sigma_r} = x^{\sigma_r+1}\, (\mu + \mu_1 \eta_1/x^{\sigma_1+1} + ... + \mu_r \eta_r/x^{\sigma_r+1})$$

is a function which is not infinite when $x = 0$, as appears from the form of the right-hand side ; it is therefore an integral function of x, and since k_r is not infinite when x is infinite it is an integral function of x whose dimension is only σ_r. Denote it by G_r. Then η_r can be expressed in the form

$$\eta_r = -\frac{1}{\mu_r}\, [\mu x^{\sigma_r+1} + \mu_1 \eta_1 x^{\sigma_r-\sigma_1} + ... + \mu_{r-1}\eta_{r-1} x^{\sigma_r-\sigma_{r-1}} - G_r], \qquad (\text{F})$$

and in the right hand no term occurs of higher dimension than that of η_r, while G_r is of less dimension than η_r. If then there be m functions such as k_r, m of the functions $\eta_1, ..., \eta_{n-1}$ can be expressed in the form (F) in terms of the remaining $n - 1 - m$ functions of $\eta_1, ..., \eta_{n-1}$ and m functions G_r ; the sum of the dimensions of these m functions G_r is less by m than that of the dimensions of the functions η_r which they replace. Denoting the functions among $\eta_1, ..., \eta_{n-1}$ which are not thus replaced by functions G, also by the symbol G, for the sake of uniformity, every integral function is expressible in the form

$$(x,\, 1)_\lambda + (x,\, 1)_{\lambda_1}\, G_1 + ... + (x,\, 1)_{\lambda_{n-1}}\, G_{n-1},$$

and the sum of the dimensions of $G_1, ..., G_{n-1}$ is less by m than the sum of the dimensions of $\eta_1, ..., \eta_{n-1}$.

If now in this expression of integral functions by $G_1, ..., G_{n-1}$ any terms can arise which are of higher dimension than the functions to be expressed, we can similarly replace the set $G_1, ..., G_{n-1}$ by another set whose dimensions have a still less sum.

Since no integral function can have a less dimension than 1, the sum of the dimensions of the functions whereby integral functions are expressed, cannot be diminished below $n - 1$. We shall therefore arrive at length at a set $g_1, ..., g_{n-1}$ of integral functions, in terms of which all integral functions can be expressed so that the condition of dimensions is satisfied.

It is this system which it was our aim to deduce.

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* Chap. IV. § 39.

Ex. For the surface associated with the equation $y^2 = (x, 1)_{2p+2}$ all integral functions can in fact be represented in the form $(x, 1)_\lambda + (x, 1)_{\lambda_1} \eta_1$, where $\eta_1 = y + x^m$. If $m > p+1$ the dimension of η_1 is m. In order to ascertain whether the condition of dimensions is satisfied we enquire whether there exist any functions of the form $x[a + a_1(y + x^m)/x^m]$, wherein a, a_1 are constants, which are finite for $x = \infty$, namely whether $[a + a_1(y\xi^m + 1)]/\xi$ can be an integral function of ξ.

Shew that this can only be the case when $a + a_1 = 0$. Putting $k_r = [-a + a_1(y\xi^m + 1)]/\xi$ it is clear that $k_r x^{m-1} = a_1 y$. Thus all integral functions can be represented in the form $(x, 1)_\lambda + (x, 1)_{\lambda_1} y$. Shew that the condition of dimensions is now satisfied.

78. There is one part of the process given here which has not been explained. Let $\eta_1, \ldots, \eta_{n-1}$ be integral functions, and let u denote a linear function of the form $x - c$. It is required to find all possible functions of the form

$$(a + a_1 \eta_1 + \ldots + a_{n-1} \eta_{n-1})/u,$$

wherein $a, \ldots, a_{n-1}$ are constants, which are not infinite when $u = 0$. We suppose $\eta_1, \ldots, \eta_{n-1}$ to be such that the product of every two of them is expressible in the form $v + v_1 \eta_1 + \ldots + v_{n-1} \eta_{n-1}$, $v, \ldots, v_{n-1}$ being integral polynomials in x; this condition is always satisfied in the actual case under consideration.

The integral function $H = a + a_1 \eta_1 + \ldots + a_{n-1} \eta_{n-1}$ will satisfy an equation of the form

$$(H - H^{(1)}) \ldots (H - H^{(n)}) = H^n + K_1 H^{n-1} + \ldots + K_{n-1} H + K_n = 0,$$

wherein K_i is an integral polynomial in $a, \ldots, a_{n-1}$ of the ith order; K_i is also an integral polynomial in x. In order that H/u be an integral function it is sufficient that K_i be divisible by u^i, and when H/u is an integral function these n conditions will always be satisfied. And it is easy to see that if S_i denote the sum of the ith powers of the n values of H which arise for any value of x, these conditions may be replaced by the conditions that S_i be divisible by u_i. It is clear that it may not be an easy matter to obtain the values of $a, \ldots, a_{n-1}$, which satisfy the conditions thus expressed.

But in fact these conditions can be reduced to a set of linear congruences, and eventually to a set of linear equations for $a, \ldots, a_{n-1}$. We shall not give here the proof of this reduction*, but give the resulting equations. For in many practical cases we can obtain the results, geometrically or otherwise, in a much shorter way.

Let

$$\frac{1}{n}, \quad \frac{1}{n-1}, \quad \ldots, \quad \epsilon, \quad \epsilon', \quad \ldots, \quad 1$$

denote in order of magnitude all the positive rational numerical fractions not greater than unity, whose denominators are not greater than n; each being in its lowest terms. Let $\eta_1, \ldots, \eta_r$ denote any linearly independent integral functions. Let Σ denote the sum of the n values of a function which arise for any value of x. Determine all the possible sets of values of the constants $a, a_1, \ldots, a_r$ such that the congruence

$$\Sigma (a + a_1 \eta_1 + \ldots + a_r \eta_r)(c + c_1 \eta_1 + \ldots + c_r \eta_r) \equiv 0 \quad (\text{mod. } u)$$

is satisfied for all values of the quantities $c, c_1, \ldots, c_r$. Substituting in the left hand the value of x for which $u = 0$ and equating separately to zero the coefficients of $c, c_1, \ldots, c_r$, we obtain $r + 1$ linear *equations* for the constants $a, a_1, \ldots, a_r$. By these equations we can

* Which is given by Hensel, *Acta Math.* 18, pp. 284—292. His use of homogeneous variables is explained below Chap. VI. § 85. But it is unessential to the theory of the reduction referred to.

express a certain number* of a, a_1, ..., a_r in terms of the others ; denoting these others by β_1, ..., β_s the function $a + a_1\eta_1 + ... + a_r\eta_r$ takes the form $\beta_1\zeta_1 + ... + \beta_s\zeta_s$, wherein ζ_1, ..., ζ_s are *definite* linear functions of 1, η_1, ..., η_r with constant coefficients, and the equations in question are then satisfied for *all constant values* of β_1, ..., β_s. We *associate*† the functions ζ_1, ..., ζ_s with the first term $\frac{1}{n}$ of the series of fractions specified above. We proceed thence to deduce a set of integral functions associated with the next term of the series, $\frac{1}{n-1}$.

But in order to be able to describe the successive processes in as few words as possible, let us assume we have obtained a set of integral functions ξ_1, ..., ξ_m which in the sense employed are *associated with*‡ the fraction ϵ of the series, and wish to deduce a set of functions associated with the next following fraction of the series, ϵ'. Put down the congruence

$$\Sigma \, (\gamma_1\xi_1 + ... + \gamma_m\xi_m)(e_1\xi_1 + ... + e_m\xi_m)^{i-1} \equiv 0 \quad (\text{mod. } u^{|i\epsilon'|}).$$

Herein γ_1, ..., γ_m denote constants, i denotes in turn all positive integers not greater than n which are exact multiples of the denominator of the fraction ϵ, so that $i\epsilon$ is an integer, $|i\epsilon'|$ denotes the least integer which is not less than $i\epsilon'$, and, for any proper value of i, the congruence is to be satisfied for *all values* of the quantities e_1, ..., e_m. It will be found in practice that the left-hand side divides by $u^{|i\epsilon'|-1}$ for all values of γ_1, ..., γ_m, e_1, ..., e_m. If we carry out the division, then, in the result, substitute the value of x which makes $u = 0$, and equate separately to zero the coefficients of the $\binom{m}{i-1}$ products of e_1, ..., e_m which enter on the left, we shall have this number of linear equations for γ_1, ..., γ_m. Solving these, and thereby expressing as many as possible of γ_1, ..., γ_m in terms of the remaining, which we may denote by γ_1', ..., $\gamma_{m'}'$, $\gamma_1\xi_1 + ... + \gamma_m\xi_m$ will take a form $\gamma_1'\xi_1' + ... + \gamma_{m'}'\xi_{m'}'$, wherein γ_1', ..., $\gamma_{m'}'$ are arbitrary constants, and ξ_1', ..., $\xi_{m'}'$ are definite linear functions of ξ_1, ..., ξ_m. We say that ξ_1', ..., $\xi_{m'}'$ are associated with the fraction ϵ'.

This process is to be continued beginning with the case when $\epsilon = \frac{1}{n}$ and ending with the case when $\epsilon' = 1$. The functions associated with the last term, 1, of the series of fractions, say G_1, ..., G_k, are all the functions of the form $a + a_1\eta_1 + ... + a_{n-1}\eta_{n-1}$, wherein a, a_1, ..., a_{n-1} are constants, which are such that G_1/u, ..., G_k/u are finite when $u = 0$.

For the case $n = 3$, of a surface of three sheets, the series is $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1. The successive congruences may therefore be denoted by

$$(S_2) \equiv 0 \, (\text{mod. } u), \; (S_3) \equiv 0 \, (\text{mod. } u^2), \; (S_2) \equiv 0 \, (\text{mod. } u^2), \; (S_3) \equiv 0 \, (\text{mod. } u^3),$$

wherein (S_i) denotes such an expression as $\Sigma \, (\gamma_1\xi_1 + ... + \gamma_m\xi_m)(e_1\xi_1 + ... + e_m\xi_m)^{i-1}$.

In fact 3 is the only integer not greater than 3 such that $3 . \frac{1}{3}$ is integral and $|3 . \frac{1}{2}| = 2$. And 2 is the only integer not greater than 3 such that $2 . \frac{1}{2}$ is integral and $|2 . \frac{2}{3}| = 2$; finally 3 is the only integer such that $3 . \frac{2}{3}$ is integral, and $|3 . 1| = 3$.

For a surface of four sheets the fractions are

$$\tfrac{1}{4}, \; \tfrac{1}{3}, \; \tfrac{1}{2}, \; \tfrac{2}{3}, \; \tfrac{3}{4}, \; 1.$$

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* At most, and in general, equal to r.

† In a certain sense the functions ζ_1, ..., ζ_s are all divisible by $u^{\frac{1}{n}}$.

‡ Divisible by x^ϵ, in a sense.

We therefore have

ϵ	ϵ'	i such that $i\epsilon$ =integral	$|i\epsilon'|$	congruence
0	$\frac{1}{4}$	$i=2$	1	$(S_2)\equiv 0$ (mod. u)
$\frac{1}{4}$	$\frac{1}{3}$	$i=4$	$|\frac{4}{3}|=2$	$(S_4)\equiv 0$ (mod. u^2)
$\frac{1}{3}$	$\frac{1}{2}$	$i=3$	$|\frac{3}{2}|=2$	$(S_3)\equiv 0$ (mod. u^2)
$\frac{1}{2}$	$\frac{2}{3}$	$i=2$	$|\frac{4}{3}|=2$	$(S_2)\equiv 0$ (mod. u^2)
		$i=4$	$|\frac{8}{3}|=3$	$(S_4)\equiv 0$ (mod. u^3)
$\frac{2}{3}$	$\frac{3}{4}$	$i=3$	$|\frac{9}{4}|=3$	$(S_3)\equiv 0$ (mod. u^3)
$\frac{3}{4}$	1	$i=4$	$|4|=4$	$(S_4)\equiv 0$ (mod. u^4)

It must be borne in mind that the results of the solution of each of the seven congruences of the sequence in the right-hand column, are here supposed to be substituted in the next one: so that, for instance, the fourth congruence here may be quite other than a slightly harder case of the first congruence.

Ex. Prove that for a surface of five sheets the congruences are, in order,

(1) $(S_2)\equiv 0$ (, u); (2) $(S_5)\equiv 0$ (, u^2); (3) $(S_4)\equiv 0$ (, u^2); (4) $(S_3)\equiv 0$ (, u^2); (5) $(S_5)\equiv 0$ (, u^3); (6) $(S_2)\equiv 0$ (, u^2); (7) $(S_4)\equiv 0$ (, u^3); (8) $(S_5)\equiv 0$ (, u^4); (9) $(S_3)\equiv 0$ (, u^3); (10) $(S_4)\equiv 0$ (, u^4); (11) $(S_5)\equiv 0$ (, u^5).

79. *Ex.* i. Prove for the equation $y^4=x^2(x-1)$ that $\Delta(1, y, y^2, y^3)=-256\,x^6(x-1)^3$.

Shew that the equations

$$\Sigma\,(a+a_1 y+a_2 y^2+a_3 y^3)^i\equiv 0\ (\text{mod.}\ (x-1)^i),$$

where a, a_1, a_2, a_3 are constants, and i is in turn equal to 1, 2, 3, 4, are only satisfied by $a=a_1=a_2=a_3=0$.

Shew that the equations

$$\Sigma\,(\beta+\beta_1 y+\beta_2 y^2+\beta_3 y^3)^i\equiv 0\ (\text{mod.}\ x^i),$$

where $\beta, \beta_1, \beta_2, \beta_3$ are constants, and i is in turn equal to 1, 2, 3, 4, require $\beta=\beta_1=0$ and leave β_2 and β_3 arbitrary. Hence $\dfrac{y^2}{x}$, $\dfrac{y^3}{x}$ are the only integral functions of the form

$$(a+a_1 y+a_2 y^2+a_3 y^3)/x.$$

Shew that the equations

$$\Sigma\left(\gamma+\gamma_1 y+\gamma_2\frac{y^2}{x}+\gamma_3\frac{y^3}{x}\right)^i\equiv 0\ (\text{mod.}\ x^i)$$

require $\gamma=\gamma_1=\gamma_2=\gamma_3=0$.

Prove that the dimensions of $1, y, \dfrac{y^2}{x}, \dfrac{y^3}{x}$ are 0, 1, 1, 2. Prove then that there is no function of the form

$$\left(\delta+\delta_1\frac{y}{x}+\delta_2\frac{y^2}{x^2}+\delta_3\frac{y^3}{x^3}\right)x,$$

which is finite for x infinite.

Hence $1, y, \dfrac{y^2}{x}, \dfrac{y^3}{x}$ are a fundamental system such as $1, g_1, g_2, g_3$ in Chap. IV.; and the deficiency of the surface is $1+1+2-(4-1)=1$.

Ex. ii.   In partial illustration of Hensel's method of reduction consider the case of the equation

$$y^3 - 3xy^2 + 3yx (x-1) + x^2 (x-1)^2 (9x^3 + 7x^2 + 5x + 3) = 0,$$

for which the sums of the powers of y are given by

$$s_1 = 3x, \quad s_2 = 3x^2 + 6x, \quad s_3 = -27x^7 + 33x^6 + 3x^3 + 18x^2,$$
$$s_4 = -108x^8 + 132x^7 + 3x^4 + 36x^3 + 18x^2.$$

The determinant $\Delta (1, y, y^2)$ is divisible by x^3 and by $(x-1)^2$, as appears on calculation. By forming the equation satisfied by y^2/x it appears that y^2/x is an integral function. Denote it by η.   We consider now what functions exist of the form

$$(a + a_1 y + a_2 \eta)/(x-1),$$

wherein a, a_1, a_2 are constants, which are integral functions.

The congruence $(S_2) = \Sigma (a + a_1 y + a_2 \eta) (c + c_1 y + c_2 \eta) \equiv 0$ (mod. $\overline{x-1}$) leads, considering the coefficients of c, c_1, c_2 separately, to the congruences

$$3a + a_1 s_1 + a_2 \frac{s_2}{x} \equiv 0 \ (\ , x-1), \ as_1 + a_1 s_2 + a_2 s_3/x \equiv 0 \ (\ , x-1), \ a \frac{s_2}{x} + a_1 \frac{s_3}{x} + a_2 \frac{s_4}{x^2} \equiv 0 \ (\ , x-1),$$

and therefore to the equations

$$3a + 3a_1 + 9a_2 = 0, \ 3a + 9a_1 + 27a_2 = 0, \ 9a + 27a_1 + 81a_2 = 0,$$

which give $a = 0$, $a_1 = -3a_2$, and shew that the only function of the kind required is, save for a constant multiplier,

$$(\eta - 3y)/(x-1).$$

The other three congruences reduce then to conditions for this function; for example, the congruence $(S_3) \equiv 0 \ (\ , x^2)$ becomes

$$\Sigma \left[\frac{y^2}{x(x-1)} - \frac{3y}{x-1} \right]^3 \equiv 0 \ (\ , x^2).$$

But in fact, if we write $g = (y^2 - 3xy)/x (x-1)$, $A = 9x^3 + 7x^2 + 5x + 3$, we immediately find from the original equation that

$$g^3 + 6g^2 - 3g (Ax - 3) + A^2 x (x-1) + 9Ax = 0,$$

so that g is an integral function.

Apply the method to shew that y^2/x is the only integral function of the form $(a + a_1 y + a_2 y^2)/x$.

Prove that the dimensions of the functions

$$1, y, (y^2 - 3xy)/x (x-1)$$

are respectively 0, 3, 3.

Putting $x = 1/\xi$, $y/x^3 = h$, examine whether there exists any integral function of ξ of the form

$$[a + a_1 h + 3a_2 (h^2 - 3\xi^2 h)/\xi (1 - \xi)]/\xi,$$

and deduce the fundamental integral functions.

The deficiency of the surface is $3 + 3 - (3 - 1) = 4$.